Garland's Technique for Posets and High Dimensional Grassmannian Expanders

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Abstract

Local to global machinery plays an important role in the study of simplicial complexes, since the seminal work of Garland [11] to our days. In this work we develop a local to global machinery for general posets. We show that the high dimensional expansion notions and many recent expansion results have a generalization to posets. Examples are fast convergence of high dimensional random walks generalizing [2, 14], an equivalence with a global random walk definition, generalizing [6] and a trickling down theorem, generalizing [20].

In particular, we show that some posets, such as the Grassmannian poset, exhibit qualitatively stronger trickling down effect than simplicial complexes.

Using these methods, and the novel idea of *posetification* to Ramanujan complexes [18, 19], we construct a constant degree expanding Grassmannian poset, and analyze its expansion. This it the first construction of such object, whose existence was conjectured in [6].

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1 Introduction

High dimensional expanders are at the focus of an intensive recent study. What is a high dimensional expansion phenomenon? We argue that the high dimensional expansion phenomenon is the situation when certain global properties of a high dimensional object, such as fast convergence of random walks, are determined by certain properties of its marginals, i.e., by local properties of its "links", which are "small" substructures that compose it and correspond to local neighborhoods of the global object.

This philosophy, that the global behaviour of a simplicial complex is determined by the behaviour of its local links, was present in many recent researches in mathematics and computer science [2, 7, 10–14, 20]. This yoga was initiated in the pioneering work of Garland [11], who used, in the language of this paper, expansion of links in simplicial complexes to deduce vanishing of cohomologies, and since then had found many applications. This local to global philosophy for simplicial complexes is by now well studied. Developing such a theory beyond simplicial complexes and exploiting its consequences to the study of high dimensional expansion of more general objects is the first main goal of this work.

A more general framework in which one can study high dimensional expansion is the framework of general posets as was suggested by [6, 17] etc. The most notable example of an expanding poset, which is not a simplicial complex, is the Grassmannian Poset. It was recently studied in relation to proving the 2-to-1 games conjecture [8, 9, 16, 17] . However, in this more general setting what does it mean for the object to be a high dimensional expander? Can we still adopt the yoga of local to global behaviour in the general case?

The work of [6] defined high dimensional expansion as a *global* property which roughly occurs when two certain related random walks defined on the poset (the up-down walk and the down-up walk) are "close" to each other with respect to a natural norm. [6] show, under some assumptions, that this global definition coincides with an earlier (two-sided) local definition in the case of simplicial complexes.

The focus of our work is to define high dimensional expansion for general posets as a local to global property. Namely, we say that a general poset is a high dimensional expander, iff all its links, which are posets of lower dimensions, possess certain expansion properties. We further show that in such a case there is a way to deduce global properties of the poset, such as fast convergence of random walks on the poset, by local expansion properties of its links. This is the philosophy we advocate in this work: For this we need to explain what do we mean by general posets; we should then define their links; and then we have to show that expanding links imply that many global properties of the poset are dictated by the local properties of its links.

A central challenge is to find the correct *axioms* a poset should satisfy in order to have a certain property of interest. We do it in three levels of generality: structural axioms (see below in this section) and axioms on weights which generalize the former axioms.We show that with the correct axiomatizations the main expansion theorems of [2, 6, 14, 20] generalize. These results can serve as a tool box for future works on the subject.

Among these generalizations, notable is the generalization of Oppenheim's Trickling down [20]. In this generalization different posets exhibit qualitatively different behaviours. For some posets, such as the Grassmannian poset, the expansion *improves* while going down the links.

Finally, we use the tools we develop in this work to construct a family of expanding high dimensional bounded degree posets that are not simplicial complexes. This is the first construction of such an object, whose existence was previously conjectured in [6]. This construction is achieved by sparsifying the Grassmannian poset using a high dimensional expander.

1.1 Posets; Links, Random Walk operators and regularity properties

We start by briefly recalling what are posets, and their basic properties (see also Section 2).

1.1.1 Posets and Graded Posets

A *poset* $(P, <)$ is a set P, together with a binary order relation $<$. We say b *covers* a if $a < b$ and there is no intermediate c with $a < c < b$. A subposet is a subset of a poset, endowed with the restriction of the binary relation $\leq$.

Two useful examples of posets are simplicial complexes and Grassmannian poset.

The simplicial complex Poset. For a given set S, all its subsets form a poset with respect to the containment order $\subseteq$. Any subposet of such a poset, for an arbitrary underlying set S, is called a *simplicial complex*.

The Grassmannian Poset. Let V be a space over a field $\mathbb{F}$, the collection of its subspaces forms a poset, again with respect to containment. The poset is finite when $\mathbb{F}$ is finite and $\dim_{\mathbb{F}}(V) < \infty$. A *Grassmannian poset* is any subposet of this poset, for an arbitrary V.

A *graded poset* is a triple $(P, <, \rho)$ such that $(P, <)$ is a poset, together with a *rank function* (See Section 2.2). For simplicial complexes the rank of a set A can be taken to be $\rho(A) = |A| - 1$. For the Grassmannian poset, we can also define a rank by putting $\rho(U) = \dim_{\mathbb{F}}(U) - 1$.

We put $P(i) = \rho^{-1}(i)$, and $C^i = \mathbb{R}^{P(i)}$, the space of real functions on $P(i)$. We write $\mathbf{1} \in C^i$ for the constant function 1. The *rank of* P is the maximal d for which $P(d) \neq \emptyset$. A graded poset is said to be *pure* if there exists d such that for every element $x \in P$ there exists $y \in P(d),\ x \leq y$. In this case $P(d)$ is the set of maximal elements. Throughout this work all graded posets we consider, unless specified differently, are assumed to be finite and pure.

1.1.2 Weighted Posets and Weighted Random Walks

A *weighted graded poset* is a triple $(P, <, \rho, \mathrm{m}, \mathrm{p})$ where $(P, <, \rho)$ is a graded poset, together with a *weight function* $\mathrm{m} : P \to \mathbb{R}_+$, and transition probabilities $\mathrm{p} : P \times P \to \mathbb{R}_{\geq 0}$ which satisfy some relations (for the exact definition, and for other definitions in this subsection, see Subsection 2.3). A weight scheme in which the transition probabilities $\mathrm{p}_{x \to y} = \frac{1}{\#\{z | z \text{ covered by } x\}}$ is called *standard*, and a graded poset with such a weight scheme is called *standard graded (weighted) poset*. The weight function endows C^i with a natural inner-product $\langle \cdot, \cdot \rangle$. The associated norm is denoted $\| \cdot \|$. The "standard" transitions $p_{x \to y}$ are only for y covered by x, other transitions are zero.

1.1.2.1 The Up, Down Operators and the associated Random walks

The data of weights and transition probabilities allows defining the *Up* and *Down* operators. The Up operator U_k maps $C^k \to C^{k+1}$, while the down operator D_k maps C^k to C^{k-1}. Both mappings are defined using the structure constants of the weighted poset.

Combining these two operators we obtain two random walks which are defined on the weighted posets: The *up-down* random walk operator, $M_k^+ = D_{k+1}U_k$ corresponds to the random walk on $P(k)$ defined as follows: given a $x \in P(k)$, choose randomly (according to the parameters of the weighted poset) $y \in P(k+1)$ such that y covers x and then choose randomly (according to the parameters of the weighted poset again) $z \in P(k)$, that is covered by y.

The *down-up* random walk operator, $M_k^- = U_{k-1}D_k$ is similarly defined, only that now starting from $x \in P(k)$, we first choose randomly $z \in P(k-1)$ that is covered by x, and then choose randomly $y \in P(k)$ such that y covers z.

We also define the lth *Adjacency operator* $A_l : C^l \to C^l$, which is a non lazy version of the up-down walk. We write $A = A_0$, for the zeroth adjacency operator (also called the adjacency operator). We normalize these operators so that the largest eigenvalue is 1.

1.1.3 Links and induced weight functions

Let P be a poset, and $x \in P$. The subposet P_x made of x and all elements $y > x$, is called the *link of* x. If P is graded, then so is P_x, and the induced rank function ρ_x is given by

$$\rho_x(y) = \rho(y) - \rho(x) - 1.$$

If P is in addition weighted, then we can also induce the weight function and transition probabilities (for exact definitions see Section 2.4). We write C_x^i for the space of real functions on P_x, and write $\langle -,- \rangle_x, \| - \|_x$ for the induced inner products and norms, we can also define the operator $D_{x,i}, U_{x,i}$ as above, only with $\mathrm{m}_x, \mathrm{p}_x$ instead of m, p. The *localization* is the linear map from C^i to $C_x^{i-\rho(x)-1}$ which maps f to f_x, defined by $f_x(y) = f(y)$ for $y \in P_x$.

1.1.3.1 Basic Localization

The starting point of what is now known as Garland's technique, is the observation [11] that the global inner products $\langle f, g \rangle$, $\langle Df, Dg \rangle$, $\langle Uf, Ug \rangle$, in the case of simplicial complexes, can be written as sums of local inner products over links. In this work we investigate to which extent these results can be generalized to more general posets.

▶ **Proposition 1.1** (Basic Localization Property). *Let P be a graded weighted poset of rank d. Let $-1 \leq k < l \leq d$, and $f, g \in C^l$. Then*

1. $\langle f, g \rangle = \sum_{x \in P(k)} m(x) \langle f_x, g_x \rangle_x$.
2. $\langle D_l f, D_l g \rangle = \sum_{x \in P(k)} m(x) \langle D_{x,l-k-1} f_x, D_{x,l-k-1} g_x \rangle_x$.

1.1.4 Regularity properties of posets

Instead of providing the rather technical assumptions on the weights and transition probabilities of the weighted posets which give rise to the different localizations, we describe regularity properties of the structure of posets which are important special cases of those localization assumptions, and are the motivation for them. For more details on the regularity properties, see Subsection 2.5. In the body of the article we shall describe the more general assumptions, and verify that they indeed generalize these structural properties.

1.1.4.1 Lower regularity

A graded poset P is said to be *lower regular at level i* for $i > -1$, if there exists a constant N_i^-, such that every $x \in P(i)$, covers exactly N_i^- elements of $P(i-1)$. The poset is *lower regular* if it is lower regular at level i, for every $i > -1$.

1.1.4.2 Middle regularity

A graded poset P is said to be *middle regular at level i*, $i > -1$, if there exists a constant N_i^{mid}, such that for any $x > z$ with $x \in P(i+1), z \in P(i-1)$ there are precisely N_i^{mid} elements $y \in P(i)$ which cover z and are covered by x. P is *middle regular* if it is middle regular at level i for each $i > -1$.

1.1.4.3 $\wedge \to \vee$ regularity

P is *$\wedge \to \vee$ regular at level i*, $i > -1$, if there is a constant $N_i^{\wedge \to \vee}$, such that for any $y_1, y_2 \in P(i)$ which are covered by an element $x \in P(i+1)$ there are precisely $N_i^{\wedge \to \vee}$ elements $z \in P(i-1)$ covered by both. P is *$\wedge \to \vee$ regular* if it has this property at each at level $i > -1$.

$\curlywedge$ regularity. P is *$\curlywedge$ regular* if there exists a constant $R^{\curlywedge} = R^{\curlywedge}(P)$ such that for each $u \in P(2)$, $y_1 \neq y_2 \in P(0)$, with $u > y_1, y_2$, there are exactly $R^{\curlywedge}$ elements $z \in P(1)$ satisfying $y_1, y_2 \lhd z \lhd u$.

A poset is *regular* if it is lower, middle and $\wedge \to \vee$ regular. A poset P is $2-$*skeleton regular* if it is lower regular at levels $1, 2$, middle regular at level 1, and $\curlywedge$ regular.

One can also consider local regularity properties, which means that the links are also required to be regular, and in a uniform way (which may depend on the rank). For example P is locally $2-$skeleton regular if for all $s \in P(\leq d-3)$, P_s is $2-$skeleton regular with regularity structure constants which depend only on the level of s.

1.1.4.4 Regularity of the simplicial complex poset and the Grassmannian poset

A simplicial complex is regular and $2-$skeleton regular. The regularity constants are $N_i^- = i+1$, $N_i^{\text{mid}} = 2$, $N^{\wedge\to\vee} = 1$, $R^{\curlywedge} = 1$. Also a Grassmannian poset over $\mathbb{F}_q$ is regular and $2-$skeleton regular, with constants $N_i^- = [i+1]_q$, $N_i^{\text{mid}} = q+1$, $N^{\wedge\to\vee} = 1$, $R^{\curlywedge} = 1$.

1.1.5 Definitions of expanding posets

We will study different notions of expanding posets. A poset is *connected* if M_0^+ induces an irreducible Markov chain.

The following global definition of an expanding poset was given in [6].

▶ **Definition 1** (Eposet - Global expanding poset)**.** *A poset P of rank d is a λ-global eposet if for all $1 \leq j \leq d-1$ there exist constants r_j, δ_j such that*

$$\| D_{j+1}U_j - \delta_j U_{j-1}D_j - r_j Id_{C^j} \| \leq \lambda.$$

In this work we suggest an alternative, local to global definition of an expanding poset, generalizing the local to global definition that was studied for simplicial complexes.

▶ **Definition 2** (One sided local spectral expander)**.** *Let P be a standard, weighted, graded poset of rank d. Then P is called one-sided λ-local spectral expanding poset if P and any link P_x, for $x \in P(i)$, $i \leq d-2$, are connected, and the non trivial eigenvalues of the adjacency matrix of the link P_x, for every $x \in P(i)$, $i \leq d-2$, are upper bounded by λ.*

▶ **Definition 3** (Two sided local spectral expander)**.** *Let P be a standard, weighted, graded poset of rank d. Then P is called two-sided $[\nu, \lambda]$-local spectral expanding poset if P and any link P_x, for $x \in P(i)$, $i \leq d-2$ are connected, and the non trivial eigenvalues of the adjacency matrix of the link P_x, for every $x \in P(i)$, $i \leq d-2$, lie in $[\nu, \lambda]$.*

1.1.5.1 Local spectral expansion in the non standard setting

The current definition of local spectral expansion in posets assumes that the adjacency matrix is diagonalizable with real eigenvalues. Moreover, for most applications we need assume that this matrix is self adjoint. This happens automatically in the standard setting, but it does not need to happen more generally. An alternative way to define the local spectral expansion of posets can be by using the spectral gap of $M_{x,0}^+$, $x \in P(\leq d-2)$, which is always self adjoint with respect to the natural inner product induced from the weights, and is positive semi definite. Such a definition does not assume a standard weight scheme. It should be noted that analyzing the spectrum of M^+ is not equivalent to analyzing the spectrum of the adjacency matrix. It is equivalent when P is standard and locally lower regular at level 1. Then the spectra of M^+ and of the adjacency matrix differ by some scaling and shifting. In this work we chose to define local spectral expansion according to the spectra of the link adjacency matrices, but many of the tools we have developed apply also for the alternative choice.

1.2 Local to global theorems for general posets

For the simplicial complex poset the following three theorems form a cornerstone in the study of local to global properties, and have led to several recent breakthroughs in computer science, such as counting bases of matroids [4] and optimal mixing in Glauber dynamics [3, 5].

- Random walks convergence from one-sided expanding links: One sided local expansion in all links imply fast convergence of all high dimensional random walks [2, 14].
- Trickling-down Theorem: If all top most links expand, and the complex and its links are connected, then the global underlying graph of the complex expands [20].
- Equivalence between global random walk convergence and two-sided expanding links: All links are two-sided expanders iff at each degree the Up-Down walk operator is almost identical to Down-Up walk operator [6].

In what follows we generalize the above theorems for general posets which satisfy the previously mentioned localization assumptions (or in particular the previously defined regularity properties) and whose links are sufficiently expanding. Not only do we obtain generalizations of the above local to global theorems for general posets, but we also observe that different posets behave qualitatively different under these generalization. The different behaviour might have desired effects. One notable example is the Trickling Down theorem. This theorem for simplicial complexes, that was known prior to our work, bounds the expansion of the global complex by the expansion of its links, but in general the global expansion tends to be inferior to the expansion of the links. We show that for general posets with certain regularity conditions, that occur e.g, in the Grassmannian poset, the expansion of the global poset is **superior** to that of the links. This plays an important role in our construction of bounded degree expanding posets that are not simplicial.

1.2.1 Fast Random walk convergence from one sided local expansion for posets

Our first main theorem is that we get fast mixing of random walks on general posets from local **one**-sided spectral expansion in links. This is the first result of mixing of random walks for general posets that relies only on one-sided local expansion in links. Previous works that discussed expanding posets defined them to be expanding using global properties of their random walks [1, 6]. In particular these previous works relied on **two** sided global expansion bounds.

For simplicity we state the theorem for standard regular posets, although it holds for any poset (not even standard) satisfying the more general Up Localization property.

▶ **Theorem 1.2.** *Let P be a standard regular poset which is one sided λ local spectral expander with λ small enough then:*

$$\| \langle M_k^+ f, f \rangle \| \leq (\max_{j \leq k} \{a_{k,j}\} + \lambda (\max_{j \leq k} \frac{N_j^{mid} - 1}{N_j^{mid}}) \sum_{i=0}^{k} b_{k,i}) \| f \|^2,$$

with the constants

$$a_{l,r} = 1 - \frac{N_r^-}{N_{l+1}^-} \prod_{j=r}^{l} \frac{N_j^{mid} - 1}{N_j^{\wedge \to \vee}}, \quad b_{l,r} = \frac{N_r^- N_r^{mid}}{N_{l+1}^- N_r^{\wedge \to \vee}} \prod_{j=r+1}^{l} \frac{N_j^{mid} - 1}{N_j^{\wedge \to \vee}}.$$

1.2.1.1 Implications for Random walks in simplicial complexes and in the Grassmannian poset

For one-sided $\lambda-$local expanding simplicial complexes we recover a result of [14]: In particular for any $f \in C^k,\ f \perp \mathbf{1}$

$$\langle M_k^+ f, f\rangle \leq (\frac{k+1}{k+2} + \frac{k+1}{2}\lambda) \parallel f \parallel,$$

so that the largest eigenvalue of $M_k^+|_{C_0^k}$ is at most $\frac{k+1}{k+2} + \frac{k+1}{2}\lambda$, which is [14, Theorem 5.4].

For one-sided $\lambda-$local expanding Grassmannian posets we get for any $f \in C^k,\ f \perp \mathbf{1}$

$$\langle M_k^+ f, f\rangle \leq (\frac{[k+1]_q}{[k+2]_q} + S(k,q)\lambda) \parallel f \parallel .$$

where $S(k,q) = \sum_{i=0}^{k} \frac{q^{k-i+1}[i+1]_q}{[k+2]_q}$. Note that as q grows, $S(k,q) = k+1+O(\frac{1}{q})$, and $\frac{[k+1]_q}{[k+2]_q} \to \frac{1}{q}$.

1.2.1.2 Random walks from sequence of spectral gaps on links

We generalize the recent result of [2], which bounds the second eigenvalue of M_k^+ by the sequence of bounds μ_i on the second eigenvalues of the adjacency matrices of level i links, to general posets. Again, for the clarity of the introduction we state the result in the special case of standard regular posets.

▶ **Theorem 1.3.** *Let P be a standard regular poset. Suppose that for all i, the second eigenvalue of any adjacency matrix of a level i link is bounded from above by μ_i. Then*

$$\lambda_2(M_k^+) \leq 1 - \prod_{j=-1}^{k-1} \frac{N_{j+2}^- - 1}{N_{j+2}^-} \prod_{j=-1}^{k-1} (1-\mu_j).$$

1.2.2 Equivalence between global RW convergence and two-sided expanding links

Our next main theorem is a generalization of theorem of [6] showing that for simplicial complexes global random walks expansion is equivalent to two-sided local spectral expansion in links.

The equivalence between two-sided local spectral expanders and eposets proven in [6] was for simplicial complexes and decomposable posets. Now, that we have defined local spectral expanding posets, and developed general local to global machinery we can generalize the equivalence between two-sided spectral expanders and eposets to a more general setting. We are following similar ideology to the one studied by [6] and applies it to our more general setting.

▶ **Theorem 1.4.** *Suppose P is standard and regular poset then P is two-sided λ-local-spectral expander iff it is $O(\lambda)$-eposet.*

1.2.3 Trickling Down theorem from local expansion for posets

We generalize the notable Trickling-Down theorem of Oppenheim [20]. Finding the correct axiomatization that allows this generalization is more tricky than in the previous situations. But it results with higher gain. We show that different posets behave qualitatively different

under the trickling down process, and we learn a surprising fact: Unlike the simplicial complex case, where the bound for expansion deteriorates as we go down the links, for posets with large local lower regularity, such as the Grassmanian poset, the global expansion is improving over the links.

Again we state the theorem in this section under simplifying assumptions, we also write only the "one-step" version of the theorem, leaving the full theorem with the repetitive application to the full version of this paper [15].

▶ **Theorem 1.5.** *Let P be a standard graded weighted poset. Suppose that P is $2-$skeleton regular, with constants N_1^-, N_2^-, N_1^{mid} and $R^{\perp}$. Assume in addition that P and any link P_x, for $x \in P(0)$, are connected, and that the non trivial eigenvalues of the adjacency matrix of the link P_x lie in $[\nu, \mu]$. Then*

$$\frac{\frac{\nu}{N_1^- - 1} - \frac{(N_2^- N_1^- - N_1^{mid}) R^{\perp}(R^{\perp}-1)}{(N_1^{mid}-1)(N_1^{mid})^2(N_1^- - 1)^2}}{1-\nu} \leq \lambda \leq \frac{\frac{\mu}{N_1^- - 1} - \frac{(N_2^- N_1^- - N_1^{mid}) R^{\perp}(R^{\perp}-1)}{(N_1^{mid}-1)(N_1^{mid})^2(N_1^- - 1)^2}}{1-\mu}.$$

1.2.3.1 Implications for Trickling down for simplicial complexes and for Grassmannian poset

When P is a simplicial complex we obtain the following upper and lower bound on the non trivial eigenvalues of the adjacency matrix: $\frac{\nu}{1-\nu} \leq \lambda \leq \frac{\mu}{1-\mu}$, reproducing the result of [20]. Moving to the Grassmannian poset, we obtain the following upper and lower bound on the non trivial eigenvalues of the adjacency matrix: $\frac{\nu}{q(1-\nu)} \leq \lambda \leq \frac{\mu}{q(1-\mu)}$.

The map $x \mapsto \frac{x}{q(1-x)}$, for $q > 1$, has two fixed points, 0 and $\frac{q-1}{q}$. The former is attractive and the latter is repulsive. Thus, if we have a rank d locally connected Grassmannian poset, whose $d-2$ links have all their nontrivial eigenvalues in $[\nu, \mu]$, then if $\mu < \frac{q-1}{q}$, the non trivial eigenvalues become smaller in absolute value as we consider links of elements of lower and lower ranks. For simplicial complexes such phenomenon existed only for the negative eigenvalues. These ideas, and $\frac{q-1}{q}$ being a critical value for the trickling process, will play a role in the analysis of Section 3 below.

1.3 A construction of non simplicial bounded degree expanding posets

Recall that the Ramanujan complexes [18, 19] are bounded degree expanding simplicial complexes. They can be seen as a sparsification of the complete simplex. It was asked by [6] whether a similar sparsification exists for the Grassmannian poset, and can the expansion parameter be arbitrary small. Namely, whether it is possible to construct a subposet of the Grassmannian that is expanding and is of bounded degree.

In Section 3 we provide an explicit sparsification of the complete Grassmannian, via a process which we call *posetification*. We prove, using the tools developed in this work, that it is indeed expanding, with expansion parameters of magnitude $\frac{q-1}{q}$, partially answering the question of [6]. To the best of our knowledge, this is the first proof of existence, and the first known construction, of a bounded degree expanding subposet of the complete Grassmannian.

The posetification process which we describe below uses the sparsification for simplicial complexes obtained by a Ramanujan complex, to sparsify the the complete Grassmannian. The analysis of expansion relies heavily on the general trickling down we develop, and on the fact it amplifies expansion when one goes down the links, as opposed to the simplicial case. We then use the other tools we developed to bound the spectra of random walk operators on it.

▶ **Theorem 1.6** (Informal, For formal see Corollary 3.2)**.** *Let q be a prime power and d a natural number. Let X be a $d-$dimensional Ramanujan complex of thickness*[1] *Q, where Q is large enough as a function of q, d. Then the posetification V_X is a bounded degree global expanding, and a two-sided local spectral expanding Grassmannian poset. The second largest eigenvalue of each link is at most $\frac{q-1}{q} + o(1)$, while the lowest eigenvalue is at least $-\frac{1}{q}$.*

1.4 Plan of the paper

Section 2 provides basic definitions for graded weighted posets, including those of links, localizations and regularity properties, and proves basic localization results a-la-Garland [11]. In Section 3 we construct a bounded degree expanding Grassmannian poset. Due to space limitations the Random Walk theorems , The Tricking down theorem and the equivalence theorem between local to global expansion definitions for posets are excluded from this short conference version.

2 Weighted graded posets: definitions, regularity properties and the basic decomposition

We begin by reviewing the standard definition of partially ordered set (poset), and the slightly less standard notions of graded poset, and weighted graded posets. We then describe some additional properties that many posets share, and will play a role in our study.

2.1 Posets

A *poset* $(P, \leq)$ is a set P together with a binary relation $\leq$ which satisfies

1. *Reflexivity* $\forall a \in P,\ a \leq a$.
2. *Antisymmetry* If $a \leq b$ and $b \leq a$ then $a = b$.
3. *Transitivity* If $a \leq b$ and $b \leq c$ then $a \leq c$.

We write $a < b$ (a is strictly less that b) if $a \leq b$ and $a \neq b$. a is *covered* by b (equivalently, b covers a) if $a < b$ but there is no intermediate c with $a < c < b$. In this case we write $a \lhd b$. We can similarly define the relations $\geq, >, \rhd$. For $z \in P$ we write $N^-(z)$ for the number of elements covered by z. A *minimal element for a set of elements* $\{a_\alpha\}_\alpha$ is an element b in the set, which is smaller or equal all other elements in the set. A *minimal element* is a minimal element for all the elements in the poset. *Maximal elements* are similarly defined.

The poset is finite if the underlying set is. A subposet is a subset of a poset, endowed with the restriction of the binary relation $\leq$.

A *chain* from y to x is a tuple $c = (c_1, \ldots, c_m)$ where each $c_i \in P,\ c_1 = x, c_m = y$ and $c_i < c_{i+1}$ for all i. A chain is maximal if for all i, $c_i \lhd c_{i+1}$. $\mathcal{C}_{y \to x}$ denotes the collection of chains from y to x.

▶ **Example 2.1.** *Examples of posets are the $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ together with the standard orders. The former has a unique minimal element* 1*. The others do not have. Neither have a maximal element.*

For a given set S, all its subsets form a poset with respect to the containment order $\subseteq$. This poset is finite precisely when $|S| < \infty$. The unique minimal element here is $\emptyset$, and the unique maximal is S. Any subposet of such a poset, for an arbitrary underlying set S is called

[1] the thickness of the Ramanujan complex is the number of top cells a given codimension 1 cell touches, minus 1

a simplicial complex. *The subposet obtained by restricting to subsets of size at most k has the sets of size k as maximal elements. We are thinking of subposets as downword closed with regard to the original poset.*

Let M be a matroid, then its independent sets form a poset with respect to inclusion. This poset has the empty set as the unique minimal element, maximal elements are the bases, but usually there is no unique maximal element. This poset can naturally be considered as a simplicial complex whose underlying set is (the self independent) elements of M, and whose simplices correspond to independent sets.

Let V be a space over a field $\mathbb{F}$, the collection of its subspaces forms a poset, again with respect to containment. The poset is finite when $\mathbb{F}$ is finite and $dim_{\mathbb{F}}(V) < \infty$. The unique minimal element here is the 0 *vector space, and the unique maximal is V. A* Grassmannian poset *is any subposet of this poset, for an arbitrary V. The subposet obtained by restricting to sub vector spaces of dimension at most k has the vector subspaces of dimension k as maximal elements.*

2.2 Graded posets

A *graded poset* is a triple $(P, \leq, \rho)$ such that $(P, \leq)$ is a poset, together with a *rank function* $\rho : P \to \mathbb{Z}_{\geq -1}$ which is an order preserving map. We further impose that

1. If $a \lhd b$ then $\rho(b) = \rho(a) + 1$.
2. There is a single element $\star \in \rho^{-1}(-1)$.

We put $P(i) = \rho^{-1}(i)$, and $C^i = \mathbb{R}^{P(i)}$, the space of real functions on $P(i)$. We write $\mathbf{1} \in C^i$ for the constant function 1. The *rank of P* is the maximal d for which $P(d) \neq \emptyset$. A graded poset is *pure* if there exists d such that for every element $x \in P$ there exists $y \in P(d)$, $x \leq y$. In this case $P(d)$ is the set of maximal elements. The *ith skeleton*, denoted $P(\leq i)$, is the subposet $\rho^{-1}([-1, i])$. Throughout this work, all graded posets we consider are finite and pure.

▶ **Example 2.2.** *Returning to Example 2.1, for a given set S define a rank function $\rho : 2^S \to \mathbb{Z}_{\geq -1}$ by putting $\rho(A) = |A| - 1$. If S is a matroid, we may use the same rank function, restricted to independent sets. For the vector spaces example we can define a rank by putting $\rho(U) = \dim_{\mathbb{F}}(U) - 1$.*

2.3 Weighted graded posets

We now turn to consider weighted posets, following [6]. A *weighted poset* is a triple $(P, \leq, \mathrm{m}, \mathrm{p})$ where $(P, \leq)$ is a poset, together with a *weight function* $\mathrm{m} : P \to \mathbb{R}_+$, and transition probabilities $\mathrm{p} : P \times P \to \mathbb{R}_{\geq 0}$ which satisfy

1. $\mathrm{p}_{y \to x} > 0$ if and only if $y \rhd x$.
2. $\forall y$, $\sum_{x \lhd y} \mathrm{p}_{y \to x} = 1$.
3. For any $x \in P$ which is not maximal,

$$\mathrm{m}(x) = \sum_{y \rhd x} \mathrm{p}_{y \to x} \mathrm{m}(y). \tag{1}$$

Thus, m is determined by its values on maximal elements and the transition probabilities. In particular, any choice of weights m for maximal elements of P, can be extended to a weight function m satisfying

$$\mathrm{m}(x) = \sum_{y \rhd x} \frac{\mathrm{m}(y)}{N^-(y)},$$

i.e. all transition probabilities from y equal $\frac{1}{N^-(y)}$. We call such a weight scheme a standard weight scheme, and the weighted poset is called a *standard weighted poset*.

We define $U : \mathbb{R}^P \to \mathbb{R}^P$ to be the Markovian transition operator,

$$\forall g \in \mathbb{R}^P,\ y \in P,\ (Ug)(y) = \sum_{x \lhd y} \mathrm{p}_{y \to x} g(x).$$

We similarly define $D : \mathbb{R}^P \to \mathbb{R}^P$

$$\forall f \in \mathbb{R}^P,\ x \in P,\ (Df)(x) = \sum_{y \rhd x} \frac{\mathrm{p}_{y \to x} \mathrm{m}(y)}{\mathrm{m}(x)} f(y).$$

For a maximal chain $c = (c_1, \ldots, c_m)$ from y to x we set

$$\mathrm{p}(c) = \prod_{i=1}^{m-1} \mathrm{p}_{c_{i+1} \to c_i}.$$

A *weighted graded (pure) poset* (called *measured poset* in [6]) of rank d is a weighted and graded poset P such that $\mathrm{m}(\star) = 1$. We write

$$\mathrm{m}_i = \mathrm{m}|_{P(i)},\ D_i = D|_{C^i} : C^i \to C^{i-1},\ U_i = U|_{C^i} : C^i \to C^{i+1}.$$

▶ **Observation 2.3.** *Let P be a (finite, pure) graded weighted poset of rank d. Then for any $k \le l \le d$, and $x \in P(k)$,*

$$m(x) = \sum_{y \in P(l)} \sum_{c \in \mathcal{C}(y \to x)} p(c) m(y).$$

Consequently, each m_i is a probability distribution. In addition, for all $y \in P(l)$,

$$\sum_{x \in P(k)} \sum_{c \in \mathcal{C}(y \to x)} p(c) = 1. \tag{2}$$

The weight function m induces a non degenerate inner product on C^i given by

$$\forall f, g \in C^i,\ \langle f, g \rangle = \sum_{x \in P_i} \mathrm{m}(x) f(x) g(x).$$

We write $\| - \|$ for the induced norm.

▶ **Lemma 4.** *Under the above assumptions, U_i is dual to D_{i+1}.*

Proof.

$$\begin{aligned}
\langle f, U_i g \rangle &= \sum_{x \in P(i+1)} \mathrm{m}(x) f(x) (U_i g)(x) = \\
&= \sum_{x \in P(i+1)} \mathrm{m}(x) f(x) \sum_{y \lhd x} \mathrm{p}_{x \to y} g(y) = \\
&= \sum_{y \in P(i)} \mathrm{m}(y) \left(\sum_{x \rhd y} \frac{\mathrm{m}(x) \mathrm{p}_{x \to y}}{\mathrm{m}(y)} f(x) \right) g(y) = \langle D_{i+1} f, g \rangle.
\end{aligned}$$

◀

▶ **Lemma 5.** *Whenever defined,* $D_l\mathbf{1} = U_l\mathbf{1} = \mathbf{1}$. *In addition, for every* $f \in C^0$

$$D_0(f) = \langle f, \mathbf{1}\rangle.$$

Proof. By the definition of the transition probabilities,

$$(U_l\mathbf{1})(x) = \sum_{y \lhd x} \mathrm{p}_{x\to y} = 1,$$

and by (1)

$$(D_l\mathbf{1})(x) = \sum_{y \rhd x} \frac{\mathrm{m}(y)\mathrm{p}_{y\to x}}{\mathrm{m}(x)} = 1.$$

When $\star$ is the unique minimal element, for every $y \in P(0)$, $\mathrm{p}_{y\to\star} = 1$ and $\mathrm{m}(\star) = 1$. Thus,

$$\forall f \in C^0, \quad D_0 f(\star) = \sum_{y \rhd \star} \mathrm{m}(y) f(y) = \langle f, \mathbf{1}\rangle. \qquad \blacktriangleleft$$

2.3.1 The upper and lower walks

Using the operators $(D_i, U_i)_i$ we can define natural random walks on $P(i)$. The $i-$th lower walk is the random walk on C^i, $i \geq 0$ induced by $M_i^- = U_{i-1}D_i$, the $i-$th upper walk is the random walk on C^i, for i less than the rank of P, is random walk induced by $M_i^+ = D_{i+1}U_i$.

▶ **Lemma 6.** *Consider* $f \in C^i$. *The upper and lower walks can be described as follows.*

$$(M_i^+(f))(y) = \left(\sum_{z \rhd y} \frac{m(z)p_{z\to y}^2}{m(y)}\right) f(y) + \sum_{x\in P(i),\ x\neq y} \left(\sum_{z \rhd x,y} \frac{m(z)p_{z\to x}p_{z\to y}}{m(y)}\right) f(x). \tag{3}$$

$$(M_i^-(f))(y) = \sum_{x\in P(i)} \left(\sum_{z \lhd x,y} \frac{p_{x\to z}p_{y\to z}}{m(z)}\right) m(x)f(x). \tag{4}$$

Proof.

$$(D_{i+1}U_i(f))(y) = \sum_{z\rhd y} \frac{\mathrm{m}(z)\mathrm{p}_{z\to y}}{\mathrm{m}(y)}(U_i(f))(z) =$$
$$= \sum_{z\rhd y} \frac{\mathrm{m}(z)\mathrm{p}_{z\to y}}{\mathrm{m}(y)} \sum_{x\lhd z} \mathrm{p}_{z\to x} f(x) = \sum_{x\in P(i)} \left(\sum_{z\rhd x,y} \frac{\mathrm{m}(z)\mathrm{p}_{z\to x}\mathrm{p}_{z\to y}}{\mathrm{m}(y)}\right) f(x).$$

The last equality is obtained by changing the order of summation. (3) is obtained from the last expression by separating into the cases $x = y$ and $x \neq y$. Similarly,

$$(U_{i-1}D_i(f))(y) = \sum_{z\lhd y} \mathrm{p}_{y\to z}(D_i f)(z) =$$
$$= \sum_{z\lhd y} \mathrm{p}_{y\to z} \sum_{x\rhd z} \left(\frac{\mathrm{m}(x)\mathrm{p}_{x\to z}}{\mathrm{m}(z)}\right) f(x) = \sum_{x\in P(i)} \left(\mathrm{m}(x) \sum_{z\lhd x,y} \frac{\mathrm{p}_{x\to z}\mathrm{p}_{y\to z}}{\mathrm{m}(z)}\right) f(x). \qquad \blacktriangleleft$$

▶ **Observation 2.4.** M_k^+, M_{k+1}^- *have the same non-zero eigenvalues, including multiplicities.*

Indeed, this is always the case for two operators of the form AA^*, A^*A, and

$$M_k^+ = D_{k+1}U_k,\ M_{k+1}^- = U_k D_{k+1},\ D_{k+1} = U_k^*.$$

The multiplicity of the zero eigenvalue can be different.

▶ **Example 2.5.** *In case P is a standard weighted graded poset, a single step of the lower random walk from y is obtained by choosing a uniformly random z covered by y, and then choosing x which covers z, according to the transition probabilities $\frac{m(x)}{N^-(x)m(z)}$. A single step of the upper random walk from y is obtained by choosing a random z which covers y, with probability $\frac{m(z)}{N^-(z)m(y)}$, and then choosing uniformly a random element $x \lhd z$.*

An immediate corollary of Lemma 5 is

▶ **Corollary 2.6.** *If P has a unique minimum $\star \in P(-1)$ then*

$$M_0^+ M_0^- = M_0^-.$$

▶ **Definition 7.** *We define the* l-th adjacency matrix *of P by its action on $f \in C^l$:*

$$(A_l(f))(y) = \sum_{x \neq y} \left(\sum_{z \rhd x,y} \frac{m(z)p_{z\to x}p_{z\to y}}{(1-p_{z\to y})m(y)} \right) f(x). \tag{5}$$

[2] *The zeroth adjacency matrix is sometimes referred as the adjacency matrix, and we sometimes write only A.*

$\mathbf{1} \in C^l$ is always a eigenvector for eigenvalue 1 of the adjacency matrix. The eigenvalues of eigenvectors which are not constant are called *non trivial eigenvalues*. If the weight scheme is standard the adjacency operator is self-adjoint. If, in addition, each $z \in P(l+1)$ covers the same number of elements $N^-(z) = N_{l+1}^-$ (this property will be called 'lower regular' in what follows), then A_l is related to M_l^+ by

$$A_l = \frac{N_{l+1}^- - 1}{N_{l+1}^-} M_l^+ + \frac{1}{N_{l+1}^-} Id_{C^l}. \tag{6}$$

2.4 Links and localization

Let P be a poset, and $x \in P$. We will call the subposet made of all elements $y \geq x$ the *link of* x and we shall denote it by P_x. If P is graded, then also P_x, and the induced rank function ρ_x is given by

$$\rho_x(y) = \rho(y) - \rho(x) - 1.$$

If P is graded, weighted (finite and pure), then we can also induce the weight function and transition probabilities by putting

$$\mathrm{m}_x(y) = \frac{\mathrm{m}(y)}{\mathrm{m}(x)} \sum_{c \in \mathcal{C}(y\to x)} \mathrm{p}(c), \tag{7}$$

[2] In what follows, whenever we work with the $l-$th adjacency matrix we shall implicitly assume that any $z \in P(l+1)$, for covers at least two elements, otherwise the adjacency matrix is ill defined. This assumption is valid in almost all interesting applications.

$$(\mathrm{p}_x)_{z\to y} = \frac{\mathrm{p}_{z\to y}\sum_{c\in\mathcal{C}(y\to x)}\mathrm{p}(c)}{\sum_{c\in\mathcal{C}(z\to x)}\mathrm{p}(c)}. \tag{8}$$

We need to verify that the properties of transition probabilities and weights hold for the induced weights and probabilities. We will verify Property (1), the other properties are straight forward:

$$\begin{aligned}\sum_{z\rhd y}\mathrm{m}_x(z)(\mathrm{p}_x)_{z\to y} &= \sum_{z\rhd y}\frac{\mathrm{m}(z)}{\mathrm{m}(x)}\sum_{c\in\mathcal{C}(z\to x)}\mathrm{p}(c)\frac{\mathrm{p}_{z\to y}\sum_{c\in\mathcal{C}(y\to x)}\mathrm{p}(c)}{\sum_{c\in\mathcal{C}(z\to x)}\mathrm{p}(c)}\\ &= \sum_{z\rhd y}\frac{\mathrm{m}(z)\mathrm{p}_{z\to y}}{\mathrm{m}(x)}\sum_{c\in\mathcal{C}(y\to x)}\mathrm{p}(c)\\ &= \frac{\mathrm{m}(y)}{\mathrm{m}(x)}\sum_{c\in\mathcal{C}(y\to x)}\mathrm{p}(c) = \mathrm{m}_x(y),\end{aligned}$$

the one before last passage used (1) for m, and the last passage is just the definition of m_x.

It is also straightforward to observe that the localization of a standard graded weighted poset is "standard in rank 1" in the sense that-

▶ **Observation 2.7.** *If P is a standard weighted graded poset of rank d, then for any $x \in P(\leq d-2)$, and any $z \in P_x(1)$, for any $y \in P_x(0)$, with $y \lhd z$*

$$(p_x)_{z\to y} = \frac{1}{|\{w \in P_x(0) | w \lhd z\}|}.$$

We write $C^i_x = C^i(P_x)$ for the space of real functions on P_x, and write $\langle -,-\rangle_x, \|-\|_x$ for the induced inner products and norms, we can also define the operator $D_{x,i}, U_{x,i}$ as above, only with $\mathrm{m}_x, \mathrm{p}_x$ instead of m, p. The *localization* is the linear map from C^i to $C_x^{i-\rho(x)-1}$ which maps f to f_x, defined by $f_x(y) = f(y)$ for $y \in P_x$.

A poset P is *connected* if the $0-$th upper walk is irreducible, i.e. for any $y, z \in P(0)$, there exists a power $j > 0$ such that the z entry of $(M_0^+)^j e_y$ is non zero, where e_y is the vector whose w component is δ_{yw}. P is *locally connected* if P and all of its proper links, for any x of rank at most $d-2$, are connected.

The following simple but powerful proposition generalized a classic result by Garland [11] to our setting. It provides a decomposition of inner products and of inner products of the operator D to sum of local terms.

▶ **Proposition 2.8.** *Let P be a graded weighted poset of rank d. Let $-1 \leq k < l \leq d$, and $f, g \in C^l$. Then*

1.

$$\langle f, g\rangle = \sum_{x\in P(k)} m(x)\langle f_x, g_x\rangle_x. \tag{9}$$

2.

$$\langle D_l f, D_l g\rangle = \sum_{x\in P(k)} m(x)\langle D_{x,l-k-1}f_x, D_{x,l-k-1}g_x\rangle_x. \tag{10}$$

Proof. For (9),

$$\begin{aligned}
\sum_{x\in P(k)} \mathrm{m}(x)\langle f_x, g_x\rangle_x &= \sum_{x\in P(k)} \mathrm{m}(x) \sum_{y\in P_x(l-k-1)} \mathrm{m}_x(y) f_x(y) g_x(y) \\
&= \sum_{x\in P(k)} \mathrm{m}(x) \sum_{y\in P_x(l-k-1)} \left(\frac{\mathrm{m}(y)}{\mathrm{m}(x)} \sum_{c\in\mathcal{C}(y\to x)} \mathrm{p}(c) \right) f(y)g(y) \\
&= \sum_{x\in P(k)} \sum_{y\in P_x(l-k-1)} \left(\mathrm{m}(y) \sum_{c\in\mathcal{C}(y\to x)} \mathrm{p}(c) \right) f(y)g(y) \\
&= \sum_{y\in P(l)} \mathrm{m}(y) f(y) g(y) \left(\sum_{x\in P(k)} \sum_{c\in\mathcal{C}(y\to x)} \mathrm{p}(c) \right) \\
&= \sum_{y\in P(l)} \mathrm{m}(y) f(y) g(y) = \langle f, g\rangle,
\end{aligned}$$

the first three equalities and the last one follow from the definitions, the fourth one from changing order of summation and the fifth one from (2).

For (2),

$$\begin{aligned}
&\sum_{z\in P(k)} \mathrm{m}(z)\langle D_{z,l-k-1} f_z, D_{z,l-k-1} g_z\rangle_z \\
&= \sum_{z\in P(k)} \mathrm{m}(z) \sum_{x\in P_z(l-k-2)} \mathrm{m}_z(x) \sum_{y_1,y_2\rhd x} \frac{\mathrm{m}_z(y_1)(\mathrm{p}_z)_{y_1\to x}}{\mathrm{m}_z(x)} f_z(y_1) \frac{\mathrm{m}_z(y_2)(\mathrm{p}_z)_{y_2\to x}}{\mathrm{m}_z(x)} g_z(y_2) \\
&= \sum_{y_1,y_2\in P(l)} f(y_1)g(y_2) \sum_{x\in P(l-1),\ x\lhd y_1,y_2} \sum_{z\in P_k,\ z<x} \mathrm{m}(z) \frac{\mathrm{m}_z(y_1)(\mathrm{p}_z)_{y_1\to x}\mathrm{m}_z(y_2)(\mathrm{p}_z)_{y_2\to x}}{\mathrm{m}_z(x)} \\
&= \sum_{y_1,y_2\in P(l)} f(y_1)g(y_2) \sum_{\substack{x\in P(l-1),\\ x\lhd y_1,y_2}} \sum_{z\in P_k,\ z<x} \mathrm{m}(z) \frac{\mathrm{m}(z)}{\mathrm{m}(x)\sum_{c\in\mathcal{C}(x\to z)}\mathrm{p}(c)} \cdot \\
&\quad \cdot \frac{\mathrm{m}(y_1)}{\mathrm{m}(z)} \sum_{c\in\mathcal{C}(y_1\to z)} \mathrm{p}(c) \frac{\mathrm{p}_{y_1\to x}\sum_{c\in\mathcal{C}(x\to z)}\mathrm{p}(c)}{\sum_{c\in\mathcal{C}(y_1\to z)}\mathrm{p}(c)} \frac{\mathrm{m}(y_2)}{\mathrm{m}(z)} \sum_{c\in\mathcal{C}(y_2\to z)} \mathrm{p}(c) \frac{\mathrm{p}_{y_2\to x}\sum_{c\in\mathcal{C}(x\to z)}\mathrm{p}(c)}{\sum_{c\in\mathcal{C}(y_2\to z)}\mathrm{p}(c)} \\
&= \sum_{y_1,y_2\in P(l)} \mathrm{m}(y_1)f(y_1)\mathrm{m}(y_2)g(y_2) \sum_{x\in P(l-1),\ x\lhd y_1,y_2} \frac{1}{\mathrm{m}(x)} \mathrm{p}_{y_1\to x}\mathrm{p}_{y_2\to x} \sum_{z\in P_k,\ z<x} \sum_{c\in\mathcal{C}(x\to z)} \mathrm{p}(c) \\
&= \sum_{y_1,y_2\in P(l)} \mathrm{m}(y_1)f(y_1)\mathrm{m}(y_2)g(y_2) \sum_{x\in P(l-1),\ x\lhd y_1,y_2} \frac{1}{\mathrm{m}(x)} \mathrm{p}_{y_1\to x}\mathrm{p}_{y_2\to x} \\
&= \sum_{x\in P(l-1)} \mathrm{m}(x) \sum_{y_1\rhd x} \frac{\mathrm{m}(y_1)\mathrm{p}_{y_1\to x}}{\mathrm{m}(x)} f(y_1) \sum_{y_2\rhd x} \frac{\mathrm{m}(y_2)\mathrm{p}_{y_2\to x}}{\mathrm{m}(x)} g(y_2) = \langle D_l f, D_l g\rangle.
\end{aligned}$$

the third equality uses the definitions of the localizations of the weights and transition probabilities, the second before last uses (2). ◀

2.5 Regularity properties and their basic consequences

In practice, many posets which appear in the literature and in applications have more structure. In Subsection 1.1.4 we described several *structural regularity properties* possessed by many posets of interest. We now study the most basic properties of such posets. We will see throughout this article that these additional structural properties allow generalizing many non trivial results from the class of simplicial complexes to more general graded weighted

posets. More precisely, in later sections we will generalize these structural properties to properties of weighted graded posets. We will show that the more general properties yield strong spectral consequences. Then we will see that *standard* weighted graded posets which possess some of the structural regularity, also possess corresponding more general properties of the weight schemes, hence their consequences.

Recall the regularity properties of Subsection 1.1.4. When P is lower regular, middle and $\wedge \to \vee$ regular, the corresponding structure constants are not independent. Indeed, for a given $z \in P(l+1)$, the number of triples (x, y, s) with $x \neq y$, $z \rhd x, y$, and $x, y \rhd s$ is, on the one hand $N^-_{l+1}(N^-_{l+1} - 1)N^{\wedge\to\vee}_l$, if we first count x, y and then s. But if we first count x, then s and then y we obtain $N^-_{l+1}N^-_l(N^{\mathrm{mid}}_l - 1)$. Thus

$$\frac{N^-_l(N^{\mathrm{mid}}_l - 1)}{(N^-_{l+1} - 1)N^{\wedge\to\vee}_l} = 1. \tag{11}$$

▶ **Example 2.9.** *If P is a simplicial complex then it is lower regular with $N^-_i = i + 1$, it is middle regular with $N^{mid}_i = 2$ and it is $\wedge \to \vee$ regular with $N^{\wedge\to\vee} = 1$. If P is a Grassmannian poset, where the ground field $\mathbb{F}_q$ has q elements, then it is lower regular with $N^-_i = [i+1]_q$, it is middle regular with $N^{mid}_i = q+1$ and it is $\wedge \to \vee$ regular with $N^{\wedge\to\vee} = 1$.*

Note that for a poset P which is lower regular at levels $1, 2$ and middle regular at level 1, for every $u \in P(2)$,

$$|\{x \in P(0) | x < u\}| = \frac{N^-_2 N^-_1}{N^{\mathrm{mid}}_1}. \tag{12}$$

Indeed, there are $N^-_2 N^-_1$ chains of length 2 descending from u, and by the definition of middle regularity, they are grouped into groups of size N^{mid}_1 of chains having the same endpoints. Thus, the number of different descendants of u is $\frac{N^-_2 N^-_1}{N^{\mathrm{mid}}_1}$.

If P is $2-$skeleton regular then

$$R^{\curlywedge} = \frac{N^{\mathrm{mid}}_1(N^-_1 - 1)}{\frac{N^-_2 N^-_1}{N^{\mathrm{mid}}_1} - 1}. \tag{13}$$

Indeed, given $P(2) \ni u > y \in P(0)$ we count pairs of the form

$$\{(x, z) \in P(0) \times P(1) | x \neq y, u \rhd z \rhd x, y\}.$$

On the one hand, if we first count x, and then z, we obtain, by (12) $\frac{N^-_2 N^-_1}{N^{\mathrm{mid}}_1} - 1$ choices for x, and then, by definition $R^{\curlywedge}$ choices of z. All together this number is $(\frac{N^-_2 N^-_1}{N^{\mathrm{mid}}_1} - 1)R^{\curlywedge}$. On the other hand, we can first choose z, there are N^{mid}_1 ways to do that. Then, there are $N^-_1 - 1$ choices for an element $x \neq y$ covered by z. The total number is then $N^{\mathrm{mid}}_1(N^-_1 - 1)$. Comparing the expressions, (13) follows.

For any poset property Q, e.g. regularity, lower regularity at level 1, etc., we say that P *is locally* Q if P, and all its links on which it is possible to verify property Q possess property Q, and if Q depends on parameters, e.g. N^-_i, then the parameters of Q at the link depend only on $\rho(s)$. For example, being middle regular is equivalent to being locally level 1 lower regular.

Another example is that a graded poset P of rank d is *locally* $\curlywedge$ *regular* if there exist constants $R^{\curlywedge}_i$, $-1 \leq i \leq d-3$, such P and all its links P_s, $s \in P(\leq d-3)$ are $\curlywedge$ regular, and moreover, the constants depend only on the level, in the sense that $R^{\curlywedge}(P_s) = R^{\curlywedge}_{\rho(s)}$.

A graded poset P is 2−skeleton regular if it is lower regular at levels $1, 2$, middle regular at level 1, and $\perp$ regular. Similarly it is locally 2−skeleton regular if for all $s \in P(\leq d-3)$, P_s is 2−skeleton regular with uniform structure constants. The lower, middle and $\wedge \to \vee$ regularity constants at a link of level i will be denoted $N^-_{i,1},\ N^-_{i,2},\ N^{\text{mid}}_{i,1}$, respectively.

▶ **Example 2.10.** *Simplicial complexes and Grassmannian posets are locally* $\perp$ *regular with* $R^{\perp}_i = 1$ *for all* i. *They are also locally* 2−*skeleton regular, where for both types of posets*

$$N^-_{i,1} = N^-_1,\ N^-_{i,2} = N^-_2,\ N^{mid}_{i,1} = N^{mid}_1,\ \forall i.$$

2.6 Expansion notions for posets

For a diagonalizable $n \times n$ matrix M with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, we write $\lambda(M) = \max\{\lambda_2, |\lambda_n|\}$.

We shall now provide local and global definitions of expansion in posets.

We first define local expansion notions.

▶ **Definition 8.** *Let* P *be a standard*[3]*, weighted, graded poset of rank* d. P *is called* one-sided λ-local spectral expander *if* P *and any link* P_x, *for* $x \in P(\leq d-2)$ *are connected, and the non trivial eigenvalues of the adjacency matrix of the link* P_x, *for every* $x \in P(\leq d-2)$, *are bounded by* λ.

▶ **Definition 9.** *Let* P *be a standard, weighted, graded poset of rank* d. P *is called a* two-sided $[\nu, \lambda]$-local spectral expander *if* P *and any link* P_x, *for* $x \in P(\leq d-2)$ *are connected, and the non trivial eigenvalues of the adjacency matrix of the link* P_x, *for every* $x \in P(\leq d-2)$ *lie in* $[\nu, \lambda]$.

In [6] the following definition of expansion was given, and they termed posets which satisfy this definition *eposet*. We shall call these posets *global eposets*, to distinguish them from the local ones defined above, and because the definition is via a global criterion.[4]

▶ **Definition 10.** *A weighted graded poset* P *of rank* d *is a* λ-global eposet *if for all* $1 \leq j \leq d-1$ *there exist constants* r_j, δ_j *such that*

$$\| D_{j+1}U_j - \delta_j U_{j-1}D_j - r_j Id_{C^j} \| \leq \lambda.$$

3 Posetification and constructions of sparse expanding posets

Let S be a set, and X a set of subsets of S, which includes the empty set, and is closed under taking subsets. Such sets X are in natural bijection with simplicial complexes on the vertex set S, and the poset structure on X is the containment order. Suppose we are given a poset P and an order preserving association $I \mapsto P_I \in P,\ I \in X$. Write P_X for the *posetification of* X which is the subposet of P defined by

$$P_X = \{y \in P | \exists I \in X \text{ s.t. } y \leq P_I\}.$$

[3] In this definition and the next one we require P to be standard so that the adjacency matrices will be self adjoint, and will thus be diagonalizable with real eigenvalues.

[4] Note that this definition does not assume that the adjacency matrix of the links have real eigenvalues hence it applies also to non standard posets.

When X is endowed with a standard weight scheme the posetification is also naturally endowed by a standard scheme, defined by putting $\mathrm{m}(V_I) = \mathrm{m}(I)$, for any maximal simplex I.

It turns out that when the simplicial complex X is an expander, then P_X inherits expansion properties which depend on the expansion properties of X and of P.

We illustrate this idea in the following theorem, which, to the best of our knowledge, is the first example of a sparse expanding Grassmannian poset.

▶ **Theorem 3.1.** *Let X be a local spectral expander of dimension d, on the vertex set $[n]$. Suppose that X is locally connected*[5]*, and the links of all its $d-2$-dimensional cells are regular bipartite expanders, with a second eigenvalue bounded by ϵ. Let V be a vector space over $\mathbb{F}_q$ with basis $e_1, \ldots, e_n$, and to each $I \subseteq [n]$ associate the vector space*

$$V_I = span\{v_i | i \in I\} \subseteq V.$$

Then the posetification V_X is a Grassmannian poset which satisfies:

1. *It is a subposet of the Grassmannian poset on V.*
2. *Suppose that each cell of X of dimension k, for any $k \geq 0$, is contained in at most Q cells of dimension $k+1$. Then for every element of V_X of rank $k \geq 0$, the number of elements of rank $k+1$ which cover it is upper bounded by a function of q, Q and d (independent of n).*
3. *The second largest eigenvalue of each link is at most $\frac{q-1}{q} + f(\epsilon, d, q)$, where f is a function which tends to 0 as $\epsilon \to 0$, while the lowest eigenvalue is at least $-\frac{1}{q}$.*

Proof. The first two items are clear. For the third, we first prove that for any $x \in (V_X)_{d-2}$, all non trivial eigenvalues of the link adjacency matrix A_x lie in the interval $[-\frac{1}{q}, \frac{q-1+\epsilon}{q}]$.

There are three cases. The first is when x is not contained in any V_I with $|I| = d$, hence it is contained in a single V_I for $|I| = d+1$. The second is that $x \subset V_I$ for $I \subseteq [n],\ |I| = d$, but $x \neq V_J$ for any $J \subset [n]$ with $|J| = d-1$. The last case is that $x = V_I$ for some subset $I \subseteq [n],\ |I| = d-1$.

The first case is the easiest. In this case the link of x is equivalent to the Grassmannian poset of $\mathbb{F}_q^2$, meaning the $((V_X)_x)(0)$ are the $q+1$ lines in a vector space isomorphic to $\mathbb{F}_q^2$. The corresponding adjacency matrix is the normalized adjacency matrix of K_{q+1}, the complete graph on $q+1$ elements, and its eigenvalues are 1, with multiplicity 1, and $-\frac{1}{q}$ with multiplicity q.

Turning to the second case, suppose x is contained in precisely p $d+1$-spaces $V_{I_1}, \ldots, V_{I_p}$ with $\bigcap_h I_h = I$. The link of x is equivalent to the subposet of $\mathbb{F}_q\{e_0, \ldots, e_p\}$, where $\{e_0, \ldots, e_p\}$ are independent generators, with

$$((V_X)_x)_1 = \{\mathrm{span}(e_0, e_1), \mathrm{span}(e_0, e_2), \ldots, \mathrm{span}(e_0, e_p)\},$$

and $((V_X)_x)(0)$ are all the lines contained in these 2−spaces.

The adjacency matrix is the normalized adjacency matrix of a bouquet of p K_{q+1}. This is a $(pq+1) \times (pq+1)$ matrix with 0 diagonal, all non diagonal entries in the last row are $\frac{1}{pq}$, while all non diagonal entries in the last column are $\frac{1}{q}$. For any different $0 \leq i, j < pq$, the (i,j) entry is 0 if $\lfloor \frac{i}{q} \rfloor \neq \lfloor \frac{j}{q} \rfloor$, and otherwise it is $\frac{1}{q}$. This matrix has the constant vector as the eigenvector of 1.

[5] we remind the reader that 'locally connected' means that each link P_x for $x \in P(\leq d-2)$, including $P = P_\star$, is connected.

It has two other families of eigenvectors: $v_{a,b}$, $a = 0, \ldots, p-1$, $b \in [q-1]$, all of whose entries are 0, except the aq entry which is 1 and the $aq+b$ entry which is -1. These are eigenvectors for $-\frac{1}{q}$. The other family v_i, $i \in [p-1]$ is a family of eigenvectors for $\frac{q-1}{q}$. All entries of v_i are 0, except the first q which are 1 and the entries $iq, iq+1, \ldots, iq+q-1$ which are -1. By considering the support of the vectors in each family it is straight forward to verify that they are linearly independent. These families span two orthogonal spaces, which are also orthogonal to the constant vectors. Thus, these families, and the eigenvector of 1 amount to pq independent eigenvalues. Their sum is

$$1 + p(q-1)\frac{-1}{q} + (p-1)\frac{q-1}{q} = \frac{1}{q}.$$

Trace considerations show that the remaining eigenvalue is also $-\frac{1}{q}$.

We now turn to the last case. Now x is a vector space of dimension $d-1$. It corresponds to a $d-2$ cell of X. Let $G = (V, E)$ be the link of this cell, which is a graph. By assumptions it is bipartite, connected, and the second eigenvalue is at most ϵ. Similarly to the previous cases, the adjacency matrix of the link of x is the normalized adjacency matrix of the graph G' obtained from G as follows The vertex set of G' is $V \cup (E \times [q-1])$. There is an edge between $v, u \in V$ if they were connected in G, there is an edge between and $(e,i), (e,j)$, $i, j \in [q-1]$, and there is an edge between $v, (e,i)$ if e is an edge of v in G. G' is a connected graph on $m + (q-1)|E|$ vertices, where $m = |V|$. We now write three families of eigenfunctions for the normalized adjacency matrix of G'.

1. $H_1(G, \mathbb{Z})$, the first homology of G, is of rank $|E| - m + 1$, and is generated by simple cycles of even length in G, by bipartiteness. Let $C_1, \ldots, C_{|E|-m+1}$ be simple cycles representing such generators. Simple induction on the construction of the cycles allows assuming that each C_i contains an edge e_i such that $e_i \notin C_j$ for $j < i$. For each C_i order its edges starting from e_i in any way which agrees with one of its two cyclic orders. Let c_i to be the vector whose (e,j) entries are 1, if e is an edge at an even place in C_i, according to this order, they are -1 if e is located in an odd place, and all other entries are 0. These vectors are eigenfunctions for $\frac{q-2}{q}$.
2. For every $e \in E$, $i \in [q-2]$ set $f_{e,i}$ to be the vector all of whose entries are 0, but the (e,i) and $(e, q-1)$ which are $1, -1$ respectively. These $(q-2)|E|$ vectors are eigenfunctions for $\frac{-1}{q}$.
3. Let $g_1, \ldots, g_m$ be the eigenfunctions of G, ordered according to the order $1 = \lambda_1 > \lambda_2 \geq \ldots > \lambda_m = -1$, of the corresponding eigenvalues. Define $g_m^a \in \mathbb{R}^{V \cup (E \times [q-1])}$ by

$$g_k^a|_V = g, \quad g_k^a(e,i) = a(g_k(v) + g_k(u)), \quad \text{for } e = \{u, v\}.$$

Applying the adjacency operator of G' to g_k^a we obtain

$$(A_{G'} g_k^a)(v) = (\frac{\lambda_k}{q} + \frac{a}{q}(q-1)(1+\lambda_k)) g_k^a(v),$$

$$(A_{G'} g_k^a)(e,i) = (\frac{1}{q} + \frac{a(q-2)}{q})(g_k(v) + g_k(u)), \quad \text{for } e = \{u, v\}.$$

Note that g_m assigns opposite values to neighboring vertices, hence g_m^a is independent of a and it is easy to see that its eigenvalue is $\frac{-1}{q}$. We write g_m^- for g_m^a. In order for g_k^a, $k \neq m$, to be an eigenfunction we must have

$$(\frac{\lambda_k}{q} + \frac{a}{q}(q-1)(1+\lambda_k))a = \frac{1}{q} + \frac{a(q-2)}{q}, \tag{14}$$

and in this case the eigenvalue will be $\frac{\lambda_k}{q} + \frac{a}{q}(q-1)(1+\lambda_k)$. (14) is equivalent to

$$a^2(q-1)(1+\lambda_k) + a(\lambda_k - q + 2) - 1 = 0.$$

This quadratic equation has two different solutions, $a = \frac{1}{1+\lambda_k}$, and $a = -\frac{1}{q-1}$. We denote by g_k^+ the eigenfunction for the former solution, and by g_k^- the eigenfunction for the latter. The eigenvalue of g_k^+ is $\lambda_k^+ = \frac{q-1+\lambda_k}{q}$, while the eigenvalue of g_k^- is $\lambda_k^- = -\frac{1}{q}$. Altogether we wrote $2m - 1 + |E| - m + 1 + |E|(q-2) = (q-1)|E| + m$. We claim that these vectors are a complete set of eigenfunctions. Since they are of the correct cardinality, it is enough to prove linear independence. In addition, it is enough to prove the linear independence inside the sets of eigenfunctions for the same eigenvalue.

Regarding the eigenvalue $-\frac{1}{q}$, the vectors $g_1^-, \ldots, g_m^-$ are linearly independent, as can be seen from their restriction to the $V-$entries, and using the independence of $g_1, \ldots, g_m$. They are orthogonal to the elements $f_{e,i},\ e \in E,\ i \in [q-2]$, hence independent of them as well. The latter vectors are also easily seen to be independent.

Regarding the eigenvalue $\frac{q-2}{q}$, the elements c_i are linearly independent. Indeed, assume towards contradiction that $\sum a_i c_i = 0$, for some scalars a_i, not all are 0. Let i^* be the maximal index with a non zero coefficient. When evaluating on $(e_{i^*}, 1)$ we obtain, on the one hand $(\sum a_i c_i)(e_{i^*}, 1) = 0$. On the other hand, by the choice of the edges e_i, $(\sum a_i c_i)(e_{i^*}, 1) = a_{i^*} c_{i^*}(e_{i^*}, 1) = \pm a_{i^*}$, which is a contradiction. Since no $\lambda_k = -1$, for $k \neq m$, no $\lambda_k^{\pm} = \frac{q-2}{q}$, so there are no more $\frac{q-2}{q}$ eigenfunctions.

We are left only with $g_k^+,\ k \in [m-1]$. These have eigenvalues different from the previous ones we considered, and again they can be seen to be linearly independent by restricting to V and using the independence of $g_1, \ldots, g_{m-1}$.

Note that for any $w \in [-\frac{1}{q}, 0]$, the map $w \mapsto Tw = \frac{w}{q(1-w)}$ satisfies $Tw \in [w, 0]$. Similarly, T maps $[0, \frac{q-1}{q}]$ onto itself. For $w > \frac{q-1}{q}$, $Tw > w$, but, since $\frac{q-1}{q}$ is a fixed point of T, one has

$$T^{d-2}(\frac{q-1+\epsilon}{q}) = \frac{q-1}{q} + f(\epsilon, d, q),$$

where, for fixed d, q, $f \to 0$ as $\epsilon \to 0$, by the continuity of T.

Thus, by applying the Tricking down theorem with the parameters of the Grassmannian,all non trivial eigenvalues of the adjacency matrix of each link are seen to be at most $\frac{q-1}{q} + f(\epsilon, d, q)$. ◀

The Ramanujan complexes constructed in [18,19] are $d-$dimensional simplicial complexes which are obtained as arithmetic quotients of Bruhat-Tits buildings of dimension d. They are locally connected, their $d-2-$dimensional links are bipartite graphs and, and all non trivial eigenvalues of the adjacency matrices of the links are bounded from above by $O(\frac{1}{\sqrt{Q}})$, where Q, the *thickness* of the Ramanujan complex is the number of top cells touched by a given codimension 1 cell, minus 1. For any Q_0 and given d there exist d dimensional Ramanujan complexes with $Q \geq Q_0$.

▶ **Corollary 3.2.** *Let q be a prime power, and let X be a $d-$dimensional Ramanujan complex of thickness Q, large enough as a function of q, d. Then the posetification V_X is a two sided bounded degree expanding Grassmannian poset. The second largest eigenvalue of each link is at most $\frac{q-1}{q} + o(1)$, while the lowest eigenvalue is at least $-\frac{1}{q}$.*

In [6] it was conjectured that there exist high dimensional bounded degree expanding Grassmannian posets for any bound $\mu > 0$. This corollary proves this conjecture for $\mu = \frac{q-1}{q} + o(1)$.

The question for general μ still awaits for an answer. Based on the trickling down for Grassmannians, if one can find a bounded degree Grassmannian complex of high enough dimension such that the second eigenvalue of the top links is bounded from above by $\lambda < \frac{q-1}{q} - c$, for some positive c, lower skeletons will be $\mu-$local spectral Grassmannian expanders, for arbitrary small μ (there is a trade off between c, μ, the dimension and how low one should go).

Regarding how large must Q be, observe that the derivative of the transformation $x \mapsto Tx$, from the end of the previous proof, at $(q-1)/q$ is q. Thus, for ϵ small enough, as a function of q, d,

$$T^{d-2}(\frac{q-1+\epsilon}{q}) = \frac{q-1}{q} + q^{d-2}\epsilon + o(\epsilon).$$

Since for Ramanujan complexes $\epsilon = O(\frac{1}{\sqrt{Q}})$ in order to obtain expansion we need $Q \gg q^{2d-4}$.

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