



# Making Auctions Robust to Aftermarkets

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## Abstract

A prevalent assumption in auction theory is that the auctioneer has full control over the market and that the allocation she dictates is final. In practice, however, agents might be able to resell acquired items in an aftermarket. A prominent example is the market for carbon emission allowances. These allowances are commonly allocated by the government using uniform-price auctions, and firms can typically trade these allowances among themselves in an aftermarket that may not be fully under the auctioneer's control. While the uniform-price auction is approximately efficient in isolation, we show that speculation and resale in aftermarkets might result in a significant welfare loss. Motivated by this issue, we consider three approaches, each ensuring high equilibrium welfare in the combined market. The first approach is to adopt smooth auctions such as discriminatory auctions. This approach is robust to correlated valuations and to participants acquiring information about others' types. However, discriminatory auctions have several downsides, notably that of charging bidders different prices for identical items, resulting in fairness concerns that make the format unpopular. Two other approaches we suggest are either using posted-pricing mechanisms, or using uniform-price auctions with anonymous reserves. We show that when using balanced prices, both these approaches ensure high equilibrium welfare in the combined market. The latter also inherits many of the benefits from uniform-price auctions such as price discovery, and can be introduced with a minor modification to auctions currently in use to sell carbon emission allowances.

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## 1 Introduction

There is a vast literature in economics and computer science analyzing the welfare properties of auctions at equilibrium. A common assumption in this literature is that the auction participants are the *consumers* who derive value from using the goods they purchase. However, in many settings, buyers may have the option to resell their winnings in outside markets that are beyond the purview of the auction designer. Resale possibilities can change behavior in the primary auction, such as by encouraging speculation. The resulting distortions can reduce final welfare even after accounting for any gains from post-auction trade.

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This raises the possibility that mechanisms that appear approximately efficient in isolation might actually generate very poor outcomes if agents anticipate aftermarket trade opportunities and adjust their strategies accordingly. Further complicating the issue, it may be that neither the auction designer nor the auctioneer have any sway over the format of the secondary market(s), and may not even be aware that they exist. We therefore ask: how robust are different auction formats to the distortion effects of aftermarket trading? Can existing theoretical analyses of auction efficiency at equilibrium be adapted to this setting?

**Application: Markets for Carbon Emission Allowances.** Our planet is warming at an alarming rate. The Swiss Re Institute, the research arm of the reinsurance company Swiss Re based in Zurich, Switzerland, estimated in an April 2021 report [33] on the impact of climate change that “The world stands to lose close to 10% of total economic value by mid-century if climate change stays on the currently-anticipated trajectory.” Due to these dire circumstances, the United Nations made combating climate change and its impacts one of 17 sustainable development goals [36].

Current efforts are largely focused on imposing limits on greenhouse gas emissions. This makes the “right to emit” a scarce resource. Emission allowances are allocated via markets, with the goal of distributing them to the industries that can provide the highest value to society per unit of emission. A large fraction of these allowances in major markets, such as the EU and California, are distributed via auctions on a monthly or quarterly basis. The EU Emissions Trading System (EU ETS), for instance, allocated 57% of allowances via auction between 2013 and 2020.<sup>2</sup> Each of these auctions is typically run using a single-round sealed-bid uniform-price auction. This format was adopted to reflect the priorities of the EU commission which requires, according to Article 10(4) of DIRECTIVE 2003/87/EC, that auctions are designed to ensure transparency, equitable informational and procedural access, and that “participants do not undermine the operation of the auctions.”<sup>3</sup>

The uniform-price auction is known to be approximately efficient at equilibrium when run in isolation [6]. However, emission allowances can also be traded via unregulated (or only partially regulated) secondary markets. These secondary markets are not fully controlled by the primary auctioning bodies and can take many forms – bilateral trade, brokered trade, and exchanges, to name a few. The existence of these secondary markets is concerning because they can distort outcomes, but they are also unavoidable absent extreme regulation.

What is the impact of these secondary markets on the primary market? There are certainly potential benefits to secondary markets, such as providing simpler market access to smaller firms who do not feel confident participating in the primary auction. But distortion effects may be present as well. According to [10], the auction clearing prices closely track the mean of the best-ask and best-bid prices on the EEX spot secondary market. Furthermore, both prices rise steadily month-over-month. These facts, taken together, suggest there is room for agents to speculate by buying allowances in the auctions and reselling them at a later time in the secondary markets. As noted in prior work of Quemin and Pahle [27], “regulators are currently ill-equipped to appraise the beneficial and detrimental facets of speculation, and proper warning systems are wanting.” Their work provides a diagnostic toolkit to assess the degree and impact of speculation in these markets. In our work, we ask whether the design of the primary market itself can defend against detrimental speculative behavior. Namely, *are the welfare guarantees of certain auction formats (approximately) robust to reallocation by arbitrary secondary markets? Can small changes to currently-used auctions achieve such robustness?*

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<sup>2</sup> See the article from EU ETS [11].

<sup>3</sup> See the legal document from [13].

**A Model of Aftermarkets.** Motivated by emission allowance markets, our primary model is a multi-unit auction in which each item corresponds to an allowance for one unit of emission. However, we note that most of our results extend to a richer class of combinatorial auctions; see Section 2.2. In the multi-unit model, agents (buyers/bidders) have decreasing marginal value for units, representing their value for using (consuming) the allowances. These valuation functions are private knowledge but drawn from known distributions. Items (allowances) acquired in the auction can be resold in a secondary market (aftermarket). To distinguish a secondary market from a general mechanism, we impose some mild conditions on the form these markets can take. Specifically, we assume that these are *trade mechanisms*: mechanisms that are budget balanced<sup>4</sup> and do not force participation.

Since agents anticipate the secondary market, the potential for resale can change behavior in the primary auction. Our equilibrium notion is perfect Bayesian equilibrium (PBE).<sup>5</sup> Roughly speaking, each agent is assumed to behave rationally in the secondary market given the auction outcome and her beliefs about other agents (subgame perfection), and her belief must be consistent with observed outcomes of the auction (Bayesian updating). Moreover, each agent is forward-looking and bids at equilibrium in the auction given their beliefs about what will occur in aftermarket trading. One subtlety is that behavior in the secondary market can depend on the information released after the primary auction, such as whether bids are publicly observed. We want results that are robust to this choice, so we allow an arbitrary revelation of signals correlated with the auction bids and outcomes before the secondary market begins. We assume that agents are fully aware of the secondary market (and what information they'll learn about the primary auction outcome) when participating in the primary auction. We call the resulting mechanism that combines the primary auction and the aftermarket trade mechanism the *combined market*. We seek conditions on the design of the primary market that guarantee high welfare in every equilibrium allocation in the combined market.

**Aftermarkets can Substantially Reduce Welfare.** It may seem counter-intuitive that secondary trade mechanisms can reduce overall welfare. Indeed, given the outcome of the primary market, as trade is voluntary, any trade in the secondary market only Pareto improves the utilities of all agents. The problem is that the agents, being aware of the existence of the secondary market, adjust their strategies in the primary market. In particular, aftermarkets create opportunities for strategically acquiring items in the primary market for the sole purpose of opportunistically reselling them later. In principle this can lead to lower welfare overall.

To formally illustrate the problem, we show that even a uniform-price auction can suffer arbitrarily large loss of welfare due to the presence of a secondary market. Importantly, we insist that the equilibria we construct avoid weakly dominated strategies. Indeed, low-welfare equilibria are already known to exist for uniform-price auctions in isolation and thus also in combined markets, but these equilibria rely on overly aggressive bids that seem not predictive

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<sup>4</sup> Our results will hold not only for trade mechanisms that are strongly budget balanced (net payment of 0), but also for weakly budget balanced mechanisms (mechanisms that never lose money).

<sup>5</sup> All of the positive results in this paper actually hold for all Bayes-Nash equilibria (and thus, in particular, for all perfect Bayesian equilibria). We choose the stronger notion of PBE as our solution concept to make the negative results more convincing, as we discuss later.

and hence cannot be viewed as negative results; these equilibria are eliminated by excluding weakly dominated strategies.<sup>6</sup> In contrast, we establish that in the presence of a secondary market, even equilibria that exclude weakly dominated strategies can have vanishing welfare.

► **Theorem (informal).** *There exist instances of a uniform-price auction of  $m$  items followed by a trade mechanism such that the expected welfare obtained in some perfect Bayesian equilibrium is only a  $\frac{1}{\Omega(\log(m))}$  fraction of the optimal expected welfare. This is true even when restricting to equilibria that avoid weakly dominated strategies.*

What drives such bad equilibria? Intuitively, if an individual participant wins many items in the auction, they can use their market power to distort prices in the secondary market and increase their own revenue at the expense of efficiency. This incentivizes a speculator to win many items in the auction. In particular, it can be perfectly rational for a speculator to bid aggressively at auction to ensure they have many items to work with in the secondary market. Moreover, this type of aggressive resale strategy will not necessarily be resisted by the other agents since bidders are incentivized to keep their bids low to reduce the price of items they are already winning. In Section 3 we present an example illustrating all of these effects in a perfect Bayesian equilibrium in undominated strategies. Our example needs only a very simple form of secondary market with dominant strategies, in which a reseller makes take-it-or-leave-it price offers.

**Auction Formats that are Robust to Aftermarkets.** The example above shows that the uniform-price auction is susceptible to the presence of an aftermarket. We argue however that some market formats do have robust welfare guarantees in combined markets. First, we show that *smooth* auctions maintain their welfare guarantees in such environments. Smoothness is a technical condition introduced by Roughgarden [28] in the context of proving worst-case guarantees on welfare properties of equilibria. We formally define smoothness in Section 4, but for now it is enough to know that if an auction format is  $(\lambda, \mu)$ -smooth for some  $\lambda \in (0, 1]$  and  $\mu \geq 1$ , then any equilibrium will generate at least a  $\lambda/\mu$  fraction of the optimal expected welfare [35]. As noted by Syrgkanis and Tardos [35], smoothness can be thought of as a sort of approximate First Welfare Theorem,<sup>7</sup> whereby any loss in efficiency due to a reduced allocation to one bidder can always be partially offset by the payment of another bidder. Our result shows that any efficiency guarantee proven for an auction in isolation using smoothness will immediately extend to any equilibrium of the combined market, no matter what trade mechanism is used in the secondary market.

► **Theorem (informal).** *For any combined market consisting of a  $(\lambda, \mu)$ -smooth auction followed by a trade mechanism, the expected welfare of any Bayes-Nash equilibrium (and thus, in particular, of any perfect Bayesian equilibrium) is at least  $(\lambda/\mu)$  times the optimal welfare.*

We also show that this guarantee continues to hold even when agent valuations can be arbitrarily correlated, and even if participants can choose to acquire costly information about others' types in advance of the auction (such as a potential speculator investigating market forecasts).

<sup>6</sup> For example, suppose that one agent bids infinitely high on all units, and all other bidders bid 0. The first agent then wins all items and pays nothing, regardless of the valuations. This is technically a Bayes-Nash equilibrium (and such a BNE exists even in the dominant-strategy Second Price Auction). However, since the first agent's bid in this example is dominated by bidding her true marginal values, this equilibrium does not survive the elimination of weakly dominated strategies.

<sup>7</sup> Informally, the First Welfare Theorem states that when prices clear the market, the allocation is socially efficient.

Uniform-price auctions are not  $(\lambda, \mu)$ -smooth for any constants  $\lambda$  and  $\mu$ ; so the theorem does not apply for such auctions. Unlike for smooth auctions, their welfare can be severely reduced in the presence of a secondary market (as we mentioned above). Thus, current emission allowances markets are not robust to speculation opportunities created by after-markets. However, discriminatory-price auctions, in which each buyer pays her marginal bid for each unit she wins, are  $(1 - 1/e, 1)$ -smooth. The smooth discriminatory-price auction therefore guarantees  $1 - 1/e$  fraction of the optimal welfare in the combined market.<sup>8</sup>

Is the discriminatory price auction a viable solution for allocating carbon allowances? Unfortunately, this auction format has some downsides. First, bidding is rather challenging and highly depends on distributional knowledge by the bidders. Second, discriminatory auctions might be perceived as unfair since identical goods are sold for different prices, creating envy between buyers. Finally, winners typically realize in retrospect that they could lower their payment by lowering their bids, creating regret. These issues could discourage participation in the auction, and indeed such concerns have been cited as reasons why this auction format was not adopted by the EU ETS [12].

**An Alternative Solution: Posted Prices.** Motivated by these concerns, we also show that one can achieve robustness to secondary market distortions in another way. Instead of running an auction, one could use *posted prices*: make a quantity of items available at a declared price and allow buyers to purchase (in an arbitrary order) while supplies last. This combines the fixed-price feature of a carbon tax with the quantity restriction of an auction.

In general, posted-price mechanisms are not robust to secondary markets. We show by way of example that even if a posted-price mechanism achieves high welfare on its own, this welfare can be significantly decreased by speculation that occurs at equilibrium in the presence of a secondary market. This motivates us to focus on a particular form of posted-price mechanism: those that use *balanced prices*, which are set proportional to the expected average welfare generated in the efficient allocation. We define balanced prices formally in Section 5.1. It is known that balanced prices yield strong welfare guarantees for many allocation problems, including multi-unit auctions [14, 7]. Specifically, if prices are  $(\alpha, \beta)$ -balanced for  $\alpha, \beta \geq 1$ , then the expected welfare obtained when buyers purchase sequentially is at least a  $1/(1 + \alpha\beta)$  fraction of the optimum [7]. We prove that the welfare guarantee from balanced prices continues to hold at any equilibrium given any arrival order of the buyers even in the presence of a secondary market.<sup>9</sup>

► **Theorem (informal).** *For any combined market consisting of a posted-price mechanism with  $(\alpha, \beta)$ -balanced prices followed by a trade mechanism, the expected welfare of any Bayes-Nash equilibrium (and thus, in particular, of any perfect Bayesian equilibrium) is at least  $1/(1 + \alpha\beta)$  times the optimal welfare.*

For our setting of selling identical items to buyers with decreasing marginal values, there is a per-item price that is  $(1, 1)$ -balanced, and therefore guarantees at least half of the expected optimal welfare even in the presence of a secondary market.

<sup>8</sup> In the special case of a single item, the discriminatory-price auction reduces to the first-price auction. In asymmetric buyers setting, Jin and Lu [19] show that the equilibrium welfare of first-price auction can be at most  $1 - \frac{1}{e^2}$  fraction of the optimal when there are no aftermarkets. In the full version online, we show that when there are two symmetric buyers, any Bayes-Nash equilibrium of the combined market created by any trade mechanism that follows the first-price auction, is efficient.

<sup>9</sup> Unlike with smooth auctions, this result does not extend to correlated value distributions or to settings where buyers can acquire additional information before the auction opens. For example, it is problematic if the buyers become more informed than the designer who set the prices.

**A Proposal: Balanced Reserves.** To this point we have described two methods for allocating items that achieve robust welfare guarantees in the presence of secondary markets. One is to use a smooth mechanism, such as a discriminatory auction, and the other is to find a balanced price and sell items at that price while supplies last. We have already discussed potential drawbacks of the former solution, but the latter has its own set of practical challenges: for one thing, it is a dramatic change relative to the uniform-price auctions typically used; for another, if the government underestimates demand and sets its price too low, this could encourage a rush where buyers race to purchase items at the moment they become available, resulting in low welfare, buyer frustration, and a perception of unfairness. As it turns out, one can address these issues and obtain all of the benefits of balanced posted prices with a small tweak to a uniform-price auction – the introduction of appropriately-chosen reserves. Reserve prices are already common in many emission auctions, such as the one administered by the California Air Resources Board [1], in the form of price floors. We show that an appropriate choice of reserve prices guards against welfare loss in combined markets: namely, one can augment a uniform-price auction with a per-allowance reserve price with bounded welfare loss, by setting the reserve to be the balanced per-allowance price one would use in a posted-price mechanism. We prove in Section 6 that the expected welfare at any equilibrium of this auction is at least half of the expected optimal welfare, and this guarantee persists in the presence of an arbitrary secondary market. We view this as a practical solution that can be implemented with minimal effort: as long as the government can estimate just one statistic, the expected average social value of a carbon allowance, they can mitigate the impact of speculation and other equilibrium effects of resale by employing an appropriately-determined auction reserve. We further show that the welfare guarantees degrade gracefully as one adjusts the prices, meaning that unavoidable misspecifications in the price determination will have a modest effect on the welfare guarantees.

## 1.1 Additional Related Work

**Equilibria of Combined Markets.** The challenges in analyzing equilibria in combined markets was acknowledged in Haile [18] due to the fact that there exist endogenously induced common value components in the auction. In the simple single-item setting with winner posting prices as secondary markets,<sup>10</sup> Hafalir and Krishna [16] characterized the equilibrium behavior of the agents in the combined market, and Hafalir and Krishna [17] adopted the characterization to show that the expected welfare of the first-price auction with secondary markets may decrease by a multiplicative factor of  $2e/(2e - 1)$ . In addition to the above discussions, there are many papers discussing various properties of the resale model in the economics literature, including but not limited to the observation of bid shading in the auction [26], and the revenue ranking of the simple auctions [21]. See the survey of Susan [32] for more discussions on the equilibrium properties of the resale model. Finally, there are several recent papers focusing on designing optimal mechanisms when the seller has no control over the secondary market. Carroll and Segal [2] show that second price auction with reserve prices is the robustly revenue optimal mechanisms with unknown resale opportunities. Dworzak [8] considers the design of information released to the secondary markets and show that the information structure that induces truthful behaviors are cutoff rules. He also provides sufficient conditions for simple information structure to be optimal.

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<sup>10</sup>In this model, the authors also assume that no information, especially the bids, are revealed in the secondary market to avoid the ratchet effect.

**Sequential Auctions.** One closely related line of theoretical work is price of anarchy for sequential auctions, which also study subgame perfect equilibrium outcomes [34, 25]. Leme, Syrgkanis, and Tardos [25] illustrate that although price of anarchy of the sequential composition of first-price auction is small for unit-demand agents, the result breaks for agents with submodular valuations, and the price of anarchy can be unbounded in the latter case. In contrast, our results indicate that for submodular valuations, a simultaneous first-price auction followed by any trade mechanism will have constant price of anarchy for the combined market. The main difference that allows us to handle combinatorial auctions in sequential auction format is that all items are sold only in the first market, and the secondary market is only providing the platform for agents to retrade the items, rather than selling items sequentially, with each item sold once in one of the auctions. Recently, Eden, Feldman, Talgam-Cohen, and Zviran [9] bound the price of anarchy when each agent is subject to an externality from the allocation of the other agents. The authors motivate the externality by the resale model where those resale behaviors are assumed to be fixed exogenously, which is substantially different from our model where agents behaviors depend on the format of the combined market.

**Price of Anarchy.** As discussed earlier, our techniques leverage smoothness and balanced pricing. For multi-unit auctions in particular, this theory has been used to derive equilibrium welfare bounds for different auction formats in isolation [6, 24], and we extend this analysis to settings with aftermarkets. The balanced pricing framework is a general approach for designing posted-price mechanisms in a broad class of allocation problems [20, 14, 7]. These constructions employ the theory of Prophet inequalities to bound the welfare obtained when buyers sequentially purchase their preferred bundles at the proposed prices, which are calculated using the distribution of buyer values. Similar to smoothness, we extend the existing analysis to show that the welfare guarantees attainable through balanced pricing extend to settings with aftermarkets.

**Carbon Markets.** There is a rich literature exploring market and regulation-based techniques for reducing emissions and their effectiveness. Here we discuss a small sampling of this literature, referring the reader to many excellent overviews such as [3] or [5] for further details. Weitzman [37] asks whether it is better to control emissions via imposing standards (quantity regulation) or charging taxes (price regulation), and notes that prices tend to fare better when the social cost of emissions is close to linear, whereas quantity regulation can be preferable in the face of uncertainty when marginal costs are variable. We note that taxes share many similarities with the posted-pricing mechanism we study. Cramton and Kerr [4], in turn, propose selling emission allowances in an auction (they suggest an ascending auction). Their paper explicitly suggests these allowances be tradeable in aftermarkets to maximize liquidity. Goldner, Immorlica, and Lucier [15] study uniform-price auctions with price floors and ceilings, a common mechanism in practice, and prove welfare guarantees under certain conditions in the absence of aftermarkets. Our paper complements these by exploring the interplay of these aftermarkets and the primary auction and stating conditions under which welfare guarantees extend to the combined market.

## 2 Preliminaries

### 2.1 The Basic Setting

For clarity we begin by describing a basic model focused on a multi-unit auction of identical items. Our results actually apply to a more general model of combinatorial auctions and different information structures (including the ability to purchase information about aggregate demand); we describe these extensions in Section 2.2.

#### 2.1.1 The Allocation Problem

A seller initially holds a set  $M$  of  $m$  identical items to be allocated among a set  $\mathcal{N}$  of  $n$  buyers. A feasible allocation is a profile  $\mathbf{x} = (x_1, \dots, x_n)$ , where  $x_i \in [m]$  is the number of items obtained by agent  $i$  and  $\sum_i x_i \leq m$ . We write  $\mathcal{X}$  for the set of feasible allocations.

Buyer  $i$  has a private valuation function  $v_i: [m] \rightarrow \mathbb{R}_{\geq 0}$  where  $v_i(x_i)$  denotes buyer  $i$ 's value for obtaining  $x_i$  items, normalized so that  $v_i(0) = 0$ . We emphasize that this is a consumption value. Valuations are assumed to have non-increasing marginal valuations: for each  $j \geq 1$ ,  $v_i(j) - v_i(j-1)$  is non-negative and weakly decreasing in  $j$ . We will sometimes refer to  $v_i$  as the type of agent  $i$ . We write  $\Theta = \times_i \Theta_i$  for the set of valuation profiles. We assume that  $v_i$  is sampled independently from a known distribution  $F_i$ , and denote the prior product distribution over the valuations by  $F = \times_i F_i$ . The utility of agent  $i$  given allocation  $x_i$  and total payment  $p_i$  is  $u_i(x_i, p_i) = v_i(x_i) - p_i$ . Buyers are assumed to be risk-neutral and seek to maximize expected utility.

The *welfare* of an allocation  $\mathbf{x} \in \mathcal{X}$  when the valuations are  $\mathbf{v}$  is defined to be  $\text{Wel}(\mathbf{v}, \mathbf{x}) = \sum_i v_i(x_i)$ . For any valuation profile  $\mathbf{v}$ , let  $\text{Wel}(\mathbf{v}, \mathcal{X}) = \sup_{x \in \mathcal{X}} \text{Wel}(\mathbf{v}, x)$  be the optimal (highest) welfare given the valuation functions  $\mathbf{v}$  and feasibility constraint  $\mathcal{X}$ . We say an allocation is *efficient* if it achieves the optimal welfare. Let  $\text{Wel}(F, \mathcal{X}) = \mathbf{E}_{\mathbf{v} \sim F}[\text{Wel}(\mathbf{v}, \mathcal{X})]$  be the expected optimal welfare. When  $\mathcal{X}$  is clear from the context, we omit it in the notation and use  $\text{Wel}(\mathbf{v})$ ,  $\text{Wel}(F)$  to denote the optimal welfare and expected optimal welfare, respectively.

#### 2.1.2 Mechanisms

Agents can acquire items by participating in an auction then trading among themselves in a secondary market. We will formally describe both the auction and the secondary market as *mechanisms*. Formally, a mechanism  $\mathcal{M} = (x^{\mathcal{M}}, p^{\mathcal{M}}) : A \rightarrow \Delta(\mathcal{X} \times \mathbb{R}^n)$  is defined by an *allocation rule*  $x^{\mathcal{M}} : A \rightarrow \Delta(\mathcal{X})$  and a *payment rule*  $p^{\mathcal{M}} : A \rightarrow \mathbb{R}^n$ , where  $A = \times_i A_i$  and  $A_i$  is the action space of agent  $i$  in the mechanism. Thus, for action profile  $\mathbf{a} = (a_1, a_2, \dots, a_n) \in (A_1, A_2, \dots, A_n) = A$  the *outcome* of the mechanism is the (randomized) allocation  $x^{\mathcal{M}}(\mathbf{a})$ , and each agent  $i$  is charged (in expectation) a *payment* of  $p_i^{\mathcal{M}}(\mathbf{a}) \geq 0$ . The utility of agent  $i$  with valuation  $v_i$  when participating in the mechanism  $\mathcal{M}$  in which agents take actions  $\mathbf{a} \in A$  is  $u_i(\mathcal{M}(\mathbf{a})) = v_i(x^{\mathcal{M}}(\mathbf{a})) - p_i^{\mathcal{M}}(\mathbf{a})$ .

A mechanism  $\mathcal{M}$  with valuation distribution  $F$  defines a game. A *strategy*  $\sigma_i : v_i \rightarrow \Delta(a_i)$  for agent  $i$  is a mapping from her valuation  $v_i$  to a distribution over her actions. With slight abuse of notation denote by  $\sigma_{-i}(v_{-i})$  the profile of actions taken by agents other than  $i$  when each  $j \neq i$  has valuation  $v_j$ . A strategy  $\sigma_i$  is a *best response* for agent  $i$  given strategies of the others  $\sigma_{-i}$  if for any strategy  $\sigma'_i$  it holds that  $\mathbf{E}[u_i(\mathcal{M}((\sigma_i(v_i), \sigma_{-i}(v_{-i}))) \geq \mathbf{E}[u_i(\mathcal{M}((\sigma'_i(v_i), \sigma_{-i}(v_{-i})))]$  for every valuation  $v_i$ , where the expectation is over the valuations of the other agents as well as any randomness in the mechanism and strategies. A



profile of strategies  $\sigma = (\sigma_1, \dots, \sigma_n)$  is a *Bayesian Nash equilibrium* (BNE) for mechanism  $\mathcal{M}$  with distribution  $F$ , if for every agent  $i$ , strategy  $\sigma_i$  is a best response for agent  $i$  given strategies of the others  $\sigma_{-i}$ .

By slightly overloading the notation, we also denote  $\text{Wel}(\mathcal{M}, \sigma, F)$  as the expected welfare obtained in mechanism  $\mathcal{M}$  using equilibrium strategy profile  $\sigma$ . Let the *price of anarchy of mechanism  $\mathcal{M}$  within the family of distributions  $\mathcal{F}$*  be

$$\text{PoA}(\mathcal{M}, \mathcal{F}) = \sup_{F \in \mathcal{F}} \frac{\text{Wel}(F)}{\inf_{\sigma \in \text{BNE}(F, \mathcal{M})} \{\text{Wel}(\mathcal{M}, \sigma, F)\}}$$

where  $\text{BNE}(F, \mathcal{M})$  is the set of Bayesian Nash equilibria given distributions  $F$  and mechanism  $\mathcal{M}$ .

**Auctions.** We can describe multi-unit auctions as mechanisms. For example, in most common multi-unit auctions, an action of bidder  $i$  is a *bid* of the form  $(a_{i1}, \dots, a_{im}) \in A_i = \mathbb{R}_{\geq 0}^m$  representing her  $m$  marginal values, with  $a_{i1} \geq \dots \geq a_{im} \geq 0$ . The agents simultaneously declare these bids to the auctioneer. The  $n \times m$  received marginal bids are then sorted from largest to smallest, and the  $m$  identical items are greedily allocated to the bidders of the  $m$  highest marginal bids (breaking ties arbitrarily). The two most common auction formats, uniform and discriminatory, then differ in how payments are calculated:

- *discriminatory auction:* Each agent pays her winning bids. That is, if agent  $i$  wins  $x_i$  items, then she pays her highest  $x_i$  marginal bids:  $p_i = \sum_{j=1}^{x_i} a_{ij}$ .
- *uniform-price auction:* A common price  $p$  per unit is chosen, and each agent pays  $p$  for each license won. This price  $p$  is taken to lie between the  $(m)$ -th highest marginal bid and the  $(m+1)$ -st highest (within the  $n \times m$  reported marginals). In other words,  $p$  lies between the highest losing bid and the lowest winning bid. In this paper we focus on the case where  $p$  is the highest losing bid.

We will also be interested in a simple form of posted-price mechanism. In the multi-unit setting, the auctioneer selects in advance a price  $p$  per item. The buyers are then approached sequentially in a fixed order. Each buyer can choose to buy as many items as desired, up to the amount remaining, at a price of  $p$  per item. That is, if the buyers purchase in the order  $1, 2, \dots, n$ , then each bidder  $i$  can choose any non-negative  $x_i \leq m - \sum_{j < i} x_j$  and pays  $px_i$ . Once all items are sold the mechanism ends.

**Secondary Markets.** Informally, a secondary market allows users to trade items that they obtained from the auction. For example, one might imagine that agents could offer to sell their items at a certain price, and other agents might choose to purchase at the suggested price (or not). Similar to the auction, we will model the secondary market as a mechanism. The starting point of the secondary market is the allocation picked by the auction, which is publicly revealed. The secondary market is therefore parameterized by an allocation  $x \in \mathcal{X}$ , which we think of as the auction outcome. We will tend to use  $\mathcal{M}^2$  to refer to secondary market mechanisms, and in a slight abuse of notation we will write  $\mathcal{M}^2(\mathbf{a}; \mathbf{x}) = (x^{\mathcal{M}^2}(\mathbf{a}; \mathbf{x}), p^{\mathcal{M}^2}(\mathbf{a}; \mathbf{x}))$  for the allocation and payment rules of a secondary market  $\mathcal{M}^2$  as a function of the initial allocation  $\mathbf{x} \in \mathcal{X}$ .<sup>11</sup>

<sup>11</sup>In some secondary market formats it is more natural to think of actions being chosen sequentially rather than simultaneously. E.g., in the example above, a seller first chooses a price then buyers choose whether to purchase. One can model this by having a buyer's "action" be a mapping from all possible observations (e.g., prices) to a realized action (e.g., whether to buy).

To capture our intuitive notion of a secondary market, we will introduce two mechanism properties that we will assume in secondary markets we consider. First, we assume voluntary participation, which informally means that each agent can choose not to participate. More formally, voluntary participation requires that each agent has an “opt-out” action that guarantees their utility is not reduced by the secondary market.

► **Definition 1.** A secondary market  $\mathcal{M}^2$  satisfies voluntary participation if for each agent  $i$ , all valuations  $v_i$ , and all feasible allocations  $\mathbf{x}$ , there exists an action  $a_i^*$  such that, for any action profile  $\mathbf{a}_{-i}$  of the other agents,  $v_i(x_i) \leq u_i(\mathcal{M}^2((a_i^*, \mathbf{a}_{-i}); \mathbf{x}))$ .

We argue that this condition is quite mild. For example, if the secondary market is one in which license holders can suggest take-it-or-leave-it prices, and trade happens if another user agrees to trade at that price, then a license holder might decide not to make an offer (“not participate”), and other agents can decide to decline any offer made (again, “not participate”). Each agent can therefore ensure that her utility in the secondary market is the same as the utility obtained from the initial allocation  $\mathbf{x}$ .

We also assume that our secondary market satisfies weak budget balance, which means that it does not run a deficit with respect to payments.

► **Definition 2.** A secondary market  $\mathcal{M}^2$  satisfies weak budget balance if  $\sum_i p_i^{\mathcal{M}^2}(\mathbf{a}; \mathbf{x}) \geq 0$  for any action profile  $\mathbf{a}$  and feasible allocation  $\mathbf{x} \in \mathcal{X}$ .

A mechanism that satisfies both voluntary participation and weak budget balance is called a *voluntary-non-subsidized-trade mechanism*, or a *trade mechanism* for short.

### 2.1.3 The Combined Market

We are finally ready to formally model our setting of an auction followed by a secondary market. We model this scenario as a two-round game  $\mathcal{G}$  that consists of two mechanisms,  $\mathcal{M}^1$  and  $\mathcal{M}^2$ , run sequentially. We refer to  $\mathcal{M}^1$  as the *auction* and  $\mathcal{M}^2$  as the *secondary market*.

In the first round of the game, the agents participate in the auction  $\mathcal{M}^1$ . We denote the action space of  $\mathcal{M}^1$  by  $A^1 = \times_i A_i^1$ . The agents simultaneously choose actions  $a^1 \in A^1$ , resulting in outcome  $x^{\mathcal{M}^1}(a^1)$  and payments  $(p_i^{\mathcal{M}^1}(a^1))_i$ . Each agent observes the outcome of the auction and her own payment.

The second round then starts and the agents participate in the secondary market  $\mathcal{M}^2$ . The allocation  $x^{\mathcal{M}^1}(a^1)$  from the auction is used as the initial allocation in the secondary market. We denote the action space of  $\mathcal{M}^2$  by  $A^2 = \times_i A_i^2$ . Note that the action spaces of the two mechanisms can be different, but they share a common set of feasible allocations.

To summarize, the timing of the two-round game  $\mathcal{G}(\mathcal{M}^1, \mathcal{M}^2)$  proceeds as follows:

1. Each agent  $i$  picks an action  $a_i^1 \in A_i^1$  simultaneously. Mechanism  $\mathcal{M}^1$  runs on actions  $a^1$ .
2. Each agent  $i$  observes  $x^{\mathcal{M}^1}(a^1)$  and  $p_i^{\mathcal{M}^1}(a^1)$ .
3. Each agent  $i$  picks an action  $a_i^2 \in A_i^2$  simultaneously. Mechanism  $\mathcal{M}^2$  runs on actions  $a^2$ , starting from allocation  $x^{\mathcal{M}^1}(a^1)$ .
4. The total payoff to agent  $i$  in the combined market is  $u_i(\mathcal{M}^2(a^2; x^{\mathcal{M}^1}(a^1))) - p_i^{\mathcal{M}^1}(a^1)$ .<sup>12</sup>

Note that an instance of the two-round game  $\mathcal{G}(\mathcal{M}^1, \mathcal{M}^2)$  naturally corresponds to a combined mechanism  $\mathcal{M}^C$  which we denote by  $\mathcal{M}^C = \mathcal{G}(\mathcal{M}^1, \mathcal{M}^2)$ , in which an action has two components: (1) an action  $a_i^1 \in A_i^1$  for the auction mechanisms, and (2) a mapping for each agent  $i$  from the tuple of (allocation, payment) from the auction into an action for the secondary market.

<sup>12</sup>Note that this expression includes the payments from both the auction and the secondary market, as the secondary market transfers are included in the utility term.

**Equilibria.** The notions of BNE and PoA extend to a combined mechanism  $\mathcal{M}^C$  as before. We note that since an action of  $\mathcal{M}^C$  encodes actions for both markets, the definition of BNE does not require that agents are best-responding in the secondary market given the auction’s outcome. This enables strategies with non-credible threats like “refuse to trade with any competitors in the secondary market, even if it is beneficial to do so, unless they let me win at least 7 items in the auction.” For multi-round games like ours, it is standard to refine the BNE solution concept and instead consider the more restrictive class of perfect Bayesian equilibria (PBE). We formally define this equilibrium notion in the full version online. Roughly speaking, a PBE requires (a) subgame perfection, where behavior in the secondary market is always rational given any auction outcome, and (b) that agents accurately update their beliefs after the auction outcome and behave in accordance with those beliefs in the secondary market. As it turns out, our positive results about welfare hold at *any* BNE, whether or not they satisfy these requirements. So in particular our welfare bounds also hold for any PBE as well. Moreover, each example we use to illustrate a negative result will not only be a BNE, but rather also be a PBE. In fact, the perfect Bayesian equilibria we consider will also satisfy the stronger conditions of sequential equilibria; see the full version online for further discussion.

## 2.2 Extensions

So far we have focused on the simpler setting for ease of notation and to more directly connect to the application of allocating emission licenses, but we will now describe two generalizations of the model. Most of our positive results in the remainder of the paper will actually be proven for this generalized setting.<sup>13</sup>

**Combinatorial Allocation and Multiple Items.** In the basic model, the items to be allocated are all identical, so each buyer is concerned only with the number of items she obtains. More generally, we can consider a *combinatorial auction* scenario where there is a set of (possibly different) goods to allocate and each buyer has a value for each possible combination of goods. We then interpret an allocation  $x_i$  to buyer  $i$  as a subset of the available goods,  $x_i \subseteq M$ . An allocation profile  $\mathbf{x} = (x_1, \dots, x_n)$  is then feasible if no item is double-allocated, meaning that  $x_i \cap x_j = \emptyset$  for all  $i \neq j$ . The basic model is the special case where all of the items are identical and agents have non-increasing marginal values for the items. This generalization also captures scenarios where items of different types being sold alongside each other, such as licenses that apply to different calendar years or that permit different forms of emissions. The natural generalization of “non-increasing marginal values” is then that agent valuations are submodular, meaning that  $v_i(S) + v_i(T) \geq v_i(S \cap T) + v_i(S \cup T)$  for all sets of items  $S, T \subseteq M$ .

**Post-Auction Information Revealed.** In the basic model, after the auction but before the secondary market, each agent observes the outcome of the auction and her own auction payment. More generally, each agent might also observe some additional information about the auction before the secondary market begins. For example, it may be that all agents’ payments are revealed, or it might be that all bids are made public. Formally, we can think

<sup>13</sup>In particular, our welfare bounds in Sections 4 and 5 apply in this generalized setting, assuming that agent valuations are submodular (which is a natural generalization of the assumption that agents have non-increasing marginal values).

of each buyer  $i$  as observing a private signal  $s_i \in S_i$  after the auction that can be correlated with  $A^1$ ,  $x^{\mathcal{M}^1}(a^1)$ , and  $(p_i^{\mathcal{M}^1}(a^1))_i$ . In fact, we could also allow these signals to be correlated with the valuations of the agents, which allows the agents to receive additional information that even the auction has no direct access to. We can write  $\Gamma$  for the (possibly randomized) mapping from  $\mathbf{v}$ ,  $a^1$ ,  $x^{\mathcal{M}^1}(a^1)$ , and  $(p_i^{\mathcal{M}^1}(a^1))_i$  to the profile of signals  $(s_1, \dots, s_n)$  that the agents receive after the auction. Under this generalization we would include  $\Gamma$  in the description of the combined mechanism, so that  $\mathcal{M}^C = \mathcal{G}(\mathcal{M}^1, \Gamma, \mathcal{M}^2)$ . In the basic setting the agents receive no signals, so we can think of  $\Gamma$  as being the empty mapping that always returns a null signal.

### 3 Welfare Loss and Secondary Markets

In this section we present an example showing that even for uniform-price auctions (which has high welfare in every equilibrium when runs in isolation), the presence of a secondary market can induce a perfect Bayesian equilibrium with low expected welfare.

► **Theorem 3.** *There exists a valuation distribution  $F$ , a combined market  $\mathcal{M}^C$  consisting of a uniform-price auction followed by a trade mechanism, and a PBE  $\sigma$  that avoids undominated strategies such that  $\text{Wel}(\mathcal{M}^C, \sigma, F) \leq \frac{1}{\Omega(\log(m))} \text{Wel}(F)$ .*

The equilibrium  $\sigma$  that we use to exhibit Theorem 3 will have additional nice properties, such as avoiding weakly dominated strategies.<sup>14</sup> Before discussing this in detail, we first describe the example.

► **Example 4.** There are  $m > 3$  units to be allocated and 3 agents named  $A, B$  and  $C$ . Agent  $A$  has marginal value 2 for the first unit, value uniformly sample from  $[1, 1.5]$  for the second unit, and 0 for any subsequent units. Agent  $B$  has the following distribution over valuations. She always has marginal value 2 for the first unit acquired, then a value  $z_B > 0$  for each subsequent unit acquired. Here  $z_B$  is a random variable drawn from a distribution with CDF  $F_B(z) = 1 - \frac{1}{1+(2m-1)z}$  for  $z \in [0, 1)$  and  $F_B(z) = z - \frac{1}{2m}$  for  $z \in [1, 1 + \frac{1}{2m}]$ . Note that given buyer value with distribution  $F$ , the unique revenue maximizing price is  $p = 1$  with expected revenue of  $p \cdot \Pr[z_B \geq p] = 1/(2m)$  per-unit. Agent  $C$  has value 0 for any number of units; we refer to agent  $C$  as a *speculator*.

The primary auction is a uniform-price auction with standard bidding. In the secondary market, the speculator  $C$  can put some or all of the items that she has acquired in the auction up for sale, at a take-it-or-leave-it price of her choice. Agent  $A$  has the first opportunity to purchase any (or all) of the items made available by  $C$ . Then agent  $B$  has the option to purchase any items that are still available.

We now describe a particular choice of bidding strategies in the primary auction and agent behavior in the secondary market. We will then prove that these form a perfect Bayesian equilibrium in the combined market. In the auction, agents  $A$  and  $B$  each bid 2 for exactly a single unit and 0 for the rest of the units. The speculator  $C$  bids 1 for  $m - 2$  units and 0 for the rest of the units. Then, in the secondary market, the speculator offers all the units she has acquired for the price of 1. Agent  $A$  buys one unit, and agent  $B$  buys all  $m - 3$  remaining units if  $z_B \geq 1$ , and nothing otherwise. Note that under this behaviour the price in the auction is 0. Agent  $C$  has expected utility of  $1 + (m - 3)/(2m)$ , as she makes 1 from selling to  $A$ , and  $(m - 3)/(2m)$  in expectation from selling to  $B$  (selling the remaining  $m - 3$  items to  $B$  when  $z_B \geq 1$ ).

<sup>14</sup>This extends a “no-overbidding” refinement commonly used when considering auctions in isolation.

Consider the social welfare obtained in the combined market. The total expected welfare obtained under this behavior is at most 6 (as  $2 + 1.5 + 2 + \frac{1}{2m} \cdot (m - 3) \cdot (1 + \frac{1}{2m}) \leq 6$ ), whereas the optimal expected welfare is at least  $4 + (m - 2)\mathbb{E}[z]$ . It is easy to compute that  $\mathbb{E}[z] \geq \int_0^1 \frac{1}{1+(2m-1)z} dz = \Theta((\log m)/m)$ , and hence the optimal expected welfare is  $\Theta(\log m)$ . Thus, if this behavior occurs at equilibrium, this implies that the price of anarchy for this combined market is  $\Theta(\log m)$ , growing unboundedly large with  $m$ .

To complete the proof of Theorem 3 we must show that this behavior forms a perfect Bayesian equilibrium. To see this, first note that the secondary market is dominance solvable: agents  $A$  and  $B$  should always accept utility-improving trades offered by agent  $C$ , and hence it is a dominant strategy for agent  $C$  to offer a revenue-maximizing price. So one can equivalently think of the combined market as a one-shot auction game where payoffs of the primary auction take into account the outcomes of the secondary market, which are unambiguous (up to zero-measure ties that do not impact utilities). We can then show that the bidding strategies described above form an equilibrium of this implied one-shot game. Moreover, those bidding strategies are not weakly dominated, again thinking of them as strategies in an implied single-shot game.<sup>15</sup> The formal proof of Proposition 5 appears in the full version online.

► **Proposition 5.** *The above behaviour in the combined market forms a perfect Bayesian equilibrium. Moreover, no agent is using a weakly dominated strategy in the primary auction with respect to the payoffs implied by the secondary market.*

**Discussion.** Let's interpret this example. One thing to notice is that, at equilibrium, the speculator is placing a very high bid for a very large number of items; much higher than the revenue she obtains in the secondary market. Of course, the speculator can afford these items because of the low price, but is this a "reasonable" strategy? At first glance it seems that this sort of behavior is a form of bullying that should be excluded by removing weakly dominated strategies (similar to overbidding in a second-price auction). But we note that, at equilibrium, the speculator behaves as a monopolist in the secondary market. She generates a modest amount of revenue from each license sold to agent  $B$  (namely,  $1/(2m)$  each), plus a large amount of revenue (revenue of 1) by selling a license to agent  $A$ . However, this sale to agent  $A$  can only occur if agent  $A$  obtains fewer than 2 items at auction. This creates an extra incentive for the speculator  $C$  to obtain many items, to prevent agent  $A$  from obtaining a second one. This can cause the items to appear complementary to speculator  $C$ , depending on the bidding behavior of agent  $A$ : obtaining one fewer items could dramatically reduce  $C$ 's utility if that one license is won by agent  $A$  instead. This rationalizes the overbidding that occurs at equilibrium in the primary auction, where the speculator makes an effective bid much higher than her obtained revenue in the secondary market. It is for this reason that the overbidding behavior of the speculator is not weakly dominated.

Another implication of speculator  $C$ 's monopolistic behavior is that she posts a high price that distorts the allocation to agent  $B$ . Although agent  $B$  has high expected welfare for the goods that agent  $C$  holds, the speculator  $C$  maximizes revenue by setting the probability of trade very low and significantly reducing welfare.

<sup>15</sup> Strategy  $\sigma_i$  is said to be weakly dominated by strategy  $\sigma'_i$  if  $\sigma'_i$  results in weakly better utility for agent  $i$  for any actions that could be taken by the other agents, and strictly better utility in at least one instance.

Finally, we note that bidders  $A$  and  $B$  are systematically under-bidding in this equilibrium. This is driven by demand reduction effects, where the bidders are strictly incentivized to underbid in order to keep prices low. Importantly, this behavior is not driven by indifference, and is undominated.

In this example a single speculator is obtaining nearly all of the items in the auction, and the bound is logarithmic in the number of items he acquires. In the full version online, we give a modified example of a market and PBE in weakly undominated strategies in which no single agent obtains more than a  $\gamma$  fraction of the items in the auction, and the welfare gap is  $\Omega(\log(\gamma m))$ . Thus, the lower bound does not hinge on a single speculator becoming a monopolist in the aftermarket, but rather degrades gracefully with the market power that is attained by any single speculator. In particular, the inefficiency persists even though multiple speculators obtain a large quantity of items, and there is no one speculator monopolizing the aftermarket.

#### 4 Price of Anarchy via Smooth Framework

In the previous section we saw that the expected welfare of a uniform-price auction may decrease catastrophically when there is a secondary market, even in “natural” equilibria that are sequentially rational and avoid weakly dominated strategies. Can the welfare loss due to aftermarkets be bounded for other auction formats? Unfortunately, explicitly characterizing the associated welfare loss is a laborious task: it requires one to construct and analyze the equilibria in the combined market, which can depend on the agents’ distributions in subtle ways.

In this section, we circumvent the challenges of explicitly characterizing all equilibrium strategies by showing that while adding a secondary market might harm welfare, the worst-case welfare guarantees of several classical mechanisms (including discriminatory auctions) will not decrease in the combined market, as long as the auction mechanism satisfies certain smoothness properties. In other words, while the equilibrium welfare may decrease in particular market instances, worst-case guarantees are retained for smooth mechanisms. The following definition captures the notion of smoothness we require.

► **Definition 6** ([35]). *Auction  $\mathcal{M}$  with action space  $A$  is  $(\lambda, \mu)$ -smooth for  $\lambda > 0$  and  $\mu \geq 1$ , if for any valuation profile  $\mathbf{v}$ , there exists action distributions  $\{D_i(\mathbf{v})\}_{i \in [n]}$  such that for any action profile  $a \in A$ ,*

$$\sum_{i \in [n]} \mathbf{E}_{a'_i \sim D_i(\mathbf{v})} [u_i(\mathcal{M}(a'_i, a_{-i}))] \geq \lambda \cdot \text{Wel}(\mathbf{v}) - \mu \cdot \text{Rev}(a; \mathcal{M})$$

It is known that a smooth auction in isolation achieves approximately optimal welfare at any equilibrium.

► **Proposition 7** ([29, 35]). *Let  $\mathcal{F}^\Pi$  be the family of all possible product type distributions. If a mechanism  $\mathcal{M}$  is  $(\lambda, \mu)$ -smooth for  $\lambda > 0$  and  $\mu \geq 1$ , then the price of anarchy of  $\mathcal{M}$  within the family of distributions  $\mathcal{F}^\Pi$  is at most  $\frac{\mu}{\lambda}$ , i.e.,  $\text{PoA}(\mathcal{M}, \mathcal{F}^\Pi) \leq \frac{\mu}{\lambda}$ .*

We now show the main result of this section: if a smooth auction is followed by a secondary market that satisfies voluntary participation and weak budget balance, then the combined market is smooth as well, and hence the price of anarchy is bounded for product type distributions.

► **Theorem 8.** *Let  $\mathcal{F}^\Pi$  be the family of all possible product type distributions. For any signaling protocol  $\Gamma$  and any trade mechanism  $\mathcal{M}^2$  in the secondary market, if an auction mechanism  $\mathcal{M}$  is  $(\lambda, \mu)$ -smooth for  $\lambda \in (0, 1]$  and  $\mu \geq 1$ , the combined mechanism  $\mathcal{M}^C = \mathcal{G}(\mathcal{M}, \Gamma, \mathcal{M}^2)$  is  $(\lambda, \mu)$ -smooth. Thus, the price of anarchy of  $\mathcal{M}^C$  within the family of distributions  $\mathcal{F}^\Pi$  for the combined market is at most  $\frac{\mu}{\lambda}$ , i.e.,  $\text{PoA}(\mathcal{M}, \mathcal{F}^\Pi) \leq \frac{\mu}{\lambda}$ .*

The proof of Theorem 8 is given in the full version online. Here we comment on the implications of the theorem. First, note that the welfare bound does not depend on the details of the information revelation structure  $\Gamma$  or the trade mechanism  $\mathcal{M}^2$  adopted in the secondary market, so the bounds hold for any choice of each. Moreover, our reduction framework does not require refinements on the equilibrium such as sequential equilibrium in the combined market to show that the price of anarchy is small – the result holds for *any* Bayes-Nash equilibrium.

We also note that Theorem 8 extends directly to settings with multiple secondary markets executed sequentially, with any information released between each market. This is because combining a smooth auction with a trade mechanism results with a new smooth mechanism (with the same parameters), which we can now view as a smooth auction to be combined with the next trade mechanism.

Theorem 8 establishes a robust welfare guarantee for a combined market as long as the initial auction is  $(\lambda, \mu)$ -smooth. For multi-unit allocation problems (such as in our basic setting), it is known that the discriminatory price auction is  $(1 - 1/e, 1)$ -smooth [6]. We therefore obtain the following corollary.

► **Corollary 9.** *Consider any multi-unit auction setting with non-increasing marginal values where the agents' valuations are distributed independently. Let  $\mathcal{M}^C$  be a combined mechanism that runs the discriminatory price auction followed by an arbitrary signaling protocol  $\Gamma$  and trade mechanism  $\mathcal{M}^2$ . Then at any BNE of  $\mathcal{M}^C$  the expected welfare is at least  $(1 - 1/e)$  times the expected optimal welfare.*

As we know from Section 3, a similar welfare bound does not hold for uniform-price auctions. This is because, unlike the discriminatory price auction, the uniform-price auction is not  $(\lambda, \mu)$ -smooth for any positive constants  $\lambda$  and  $\mu$ .<sup>16</sup>

We can also apply Theorem 8 to other smooth auctions for more general allocation problems, such as submodular combinatorial auctions. This can capture, for example, a scenario where different license types are being auctioned off simultaneously. See the full version online for further details on the implied welfare bounds.

In the remainder of this section we will extend Theorem 8 in two ways. In Section 4.1 we show that if the auction  $\mathcal{M}$  satisfies a stronger notion of smoothness known as semi-smoothness, the price of anarchy of the combined market is bounded even for correlated type distributions. It turns out that the discriminatory auction is  $(1 - 1/e, 1)$ -semi-smooth, so the welfare bound from Corollary 9 applies even if agent valuations are correlated. Second, in Section 4.2 we show that our welfare bound continues to hold even if agents are allowed to purchase signals correlated with the value realizations of agents in advance of the auction. For example, this captures settings in which a speculator could invest in market research before participating in the auction.

<sup>16</sup>The uniform-price auction does satisfy a relaxed version of smoothness: it is weakly  $(1 - 1/e, 1)$ -smooth, which implies a constant welfare bound in isolation as long as agents avoid weakly dominated “overbidding” strategies [35, 6]. In contrast, our example in Section 3 shows that eliminating dominated strategies is not sufficient to provide good welfare guarantees when this auction is part of a combined market.

#### 4.1 Extension: Correlated Valuations

As Proposition 7 is proven only for independent value distributions, Theorem 8 likewise applies only to product distributions. As it turns out, we can extend Theorem 8 to derive similar results for correlated distributions based on semi-smoothness [23].

► **Definition 10** ([23, 30]). *Auction  $\mathcal{M}$  with action space  $A$  is  $(\lambda, \mu)$ -semi-smooth for  $\lambda > 0$  and  $\mu \geq 1$ , if for any valuation profile  $\mathbf{v}$ , there exists action distributions  $\{D_i(v_i)\}_{i \in [n]}$  such that for any action profile  $a \in A$ ,*

$$\sum_{i \in [n]} \mathbf{E}_{a'_i \sim D_i(v_i)} [u_i(\mathcal{M}(a'_i, a_{-i}); v_i)] \geq \lambda \text{Wel}(\mathbf{v}) - \mu \text{Rev}(a; \mathcal{M})$$

The main difference between the definition of semi-smooth and smooth is that for each agent  $i$ , the deviating action distribution  $D_i(v_i)$  in semi-smooth only depends on her private valuation  $v_i$ , not the entire valuation profile  $\mathbf{v}$ .

► **Proposition 11** ([23]). *If a mechanism  $\mathcal{M}$  is  $(\lambda, \mu)$ -semi-smooth for  $\lambda > 0$  and  $\mu \geq 1$ , then the price of anarchy of  $\mathcal{M}$  within the family of all distributions  $\mathcal{F}$  is at most  $\frac{\mu}{\lambda}$ , i.e.,  $\text{PoA}(\mathcal{M}, \mathcal{F}) \leq \frac{\mu}{\lambda}$ .*

Similarly to Theorem 8, we next show that combining a  $(\lambda, \mu)$ -semi-smooth auction with any signaling protocol and any trade mechanism happening aftermarkets, the resulting mechanism in the combined market has small price of anarchy for arbitrary distributions.

► **Theorem 12.** *Let  $\mathcal{F}$  be the family of all possible type distributions. For any signaling protocol  $\Gamma$  and any trade mechanism  $\mathcal{M}^2$  in the secondary market, if a mechanism  $\mathcal{M}$  is  $(\lambda, \mu)$ -semi-smooth for  $\lambda \in (0, 1]$  and  $\mu \geq 1$ , the combined mechanism  $\mathcal{M}^C = \mathcal{G}(\mathcal{M}^1, \Gamma, \mathcal{M}^2)$  is  $(\lambda, \mu)$ -semi-smooth. Thus, the price of anarchy of  $\mathcal{M}$  within the family of distributions  $\mathcal{F}$  for the combined market is at most  $\frac{\mu}{\lambda}$ , i.e.,  $\text{PoA}(\mathcal{M}, \mathcal{F}) \leq \frac{\mu}{\lambda}$ .*

The proof of Theorem 12 is essentially identical to Theorem 8 (up to replacing  $D_i(\mathbf{v})$  by  $D_i(v_i)$ ) and hence omitted here. We can now use results regarding semi-smooth auction from the literature to prove that the price of anarchy of the corresponding combined markets is bounded.

► **Proposition 13** ([6]). *For multi-unit auctions with non-increasing marginal values, the discriminatory auction is  $(1 - 1/e, 1)$ -semi-smooth.<sup>17</sup>*

#### 4.2 Extension: Acquiring Additional Information

We consider the extension where agents can acquire costly information about other agents' private types before the auction starts. This captures the application where speculators gather information on the demands in the carbon market, and use the acquired information to improve their utilities through buying items in the auction and reselling them more expensively in the secondary market. In general this could have a negative impact on the equilibrium welfare of the combined market. In this section, we show that if the designer uses smooth auctions, then the welfare guarantees we obtained in Theorems 8 and 12 hold even when agents can acquire costly information.

<sup>17</sup>In de Keijzer et al [6], the authors only explicitly state that the discriminatory auction is smooth. However, their construction directly implies that the discriminatory auction is semi-smooth.



Information is captured by a signal from the types of the others to some signal space. Specifically, for each agent  $i$ , let  $\Psi_i$  be the set of feasible signal structures for agent  $i$ . Any signal structure  $\psi_i \in \Psi_i$  is a mapping from the opponents' valuations  $\mathbf{v}_{-i}$  to a distribution over the signal space  $S_i$ . Note that the information acquisition is potentially costly, i.e., there is a non-negative cost  $c_i(\psi_i; v_i)$  for any  $\psi_i \in \Psi_i$  and any type  $v_i$  of  $i$ . Let  $\bar{\psi}$  be the signal that acquires no information with zero cost. We assume that  $\bar{\psi} \in \Psi_i$  for any agent  $i$ . In our model, both the cost function  $c_i$  and the set  $\Psi_i$  of any agent  $i$  are common knowledge among all agents.

In the following theorem, we extend Theorem 8 and show that the price of anarchy for smooth auctions in the combined market is bounded when agents can acquire information on the competitors.

► **Proposition 14.** *Let  $\mathcal{F}^\Pi$  be the family of all possible product type distributions. For any set of signals  $\Psi$  and any cost function  $c$ , if mechanism  $\mathcal{M}$  is  $(\lambda, \mu)$ -smooth for  $\lambda \in (0, 1]$  and  $\mu \geq 1$ , then the price of anarchy of  $\mathcal{M}$  within the family of distributions  $\mathcal{F}^\Pi$  for the combined market with information acquisition is at most  $\frac{\mu}{\lambda}$ , i.e.,  $\text{PoA}(\mathcal{M}, \mathcal{F}^\Pi) \leq \frac{\mu}{\lambda}$ .*

The proof of Proposition 14 is provided in the full version online. Similarly, one can extend Theorem 12 for all family of all possible type distributions when the mechanism  $\mathcal{M}$  is  $(\lambda, \mu)$ -semi-smooth. As the proof is similar to the proof of Proposition 14, we omit it.

## 5 Welfare Guarantees under Posted Pricing

In the previous section we showed that the welfare guarantees derived from smooth auctions, such as discriminatory price auctions, are robust to the presence of a secondary market. Such smooth auctions have the advantage of being agnostic to the prior distributions from which agent valuations are drawn. A downside is that bidding in such auctions can be quite complex: constructing an optimal bidding strategy requires sophisticated reasoning and the ability to predict market conditions. If the designer (i.e., government) has a sense of the market conditions, then a tempting alternative to running a smooth auction is to sell a fixed quantity of items at a pre-specified unit price, while supplies last. Such posted-price mechanisms have the advantage of being very simple to participate in, since each potential buyer can simply purchase her utility-maximizing bundle of the remaining items at the given prices.

A recent literature on static posted pricing and Prophet Inequalities has illustrated that such pricing methods can provide strong welfare guarantees in a variety of allocation problems, even when the order of buyer arrival is adversarial. See [22] for a recent survey. For example, consider the special case where there is a single indivisible good to be sold and each buyer's value is drawn independently. It is known that if there is no secondary market, then setting a single take-it-or-leave-it price and (while the item is still available) letting buyers in sequence each choose whether to purchase, guarantees half of the expected maximum. Does this guarantee still hold in the presence of a secondary market?

As it turns out, the answer depends on how the fixed price is selected. The canonical solution to the single-item problem, based on the classic Prophet inequality, chooses price  $p^* = \sup\{p : \Pr[\max_i v_i > p] \geq 1/2\}$ . That is, the median of the distribution over maximum values. This choice of price guarantees half of the expected optimal welfare with no secondary market [31], but the following example shows that this is no longer the case in the presence of a secondary market.<sup>18</sup>

<sup>18</sup>The guarantee of half of the expected maximum welfare with no secondary market requires some care in the case that a value is precisely equal to  $p$ . But this occurs with probability 0 in our example, so our negative result holds regardless of how this is handled.

► **Example 15.** There is a single item to be sold and two potential buyers. The first buyer has value  $v_1$  drawn uniformly from  $[0, 1]$ . The second buyer has value  $v_2$  equal to 0 with probability  $1 - \epsilon < 1$ , and with the remaining probability  $\epsilon$  the value  $v_2$  is set equal to  $\epsilon^{-1}$  times a random variable drawn from the equal-revenue distribution capped at  $H$ . That is, with probability  $\epsilon > 0$ ,  $v_2 = \epsilon^{-1}z$  where  $z$  is drawn from a distribution with CDF  $F(z) = \frac{z-1}{z}$  for  $z \in [1, H]$  and  $F(H) = 1$ , and with the remaining probability  $v_2 = 0$ .

The efficient allocation gives the good to buyer 2 whenever  $v_2 > 0$ , leading to an expected welfare of at least  $\epsilon \times \epsilon^{-1} \times \mathbf{E}[z] = \Theta(\log H)$ .

Note that the median price is  $p^* = \sup\{p : \Pr[\max_i v_i > p] \geq 1/2\} = \frac{1}{2(1-\epsilon)}$ . Suppose we offer this price to each buyer in sequence, starting with buyer 1. If there is no secondary market, then the first buyer will purchase only if  $v_1 \geq p^*$ , which occurs with probability less than 1/2. The item is therefore available for purchase for the second buyer with probability at least 1/2, leading to an expected welfare of  $\Omega(\log(H))$ . This mechanism therefore obtains a constant fraction of the optimal welfare when running in isolation.

Now suppose that the posted-price mechanism is followed by a secondary market in which the winning buyer (if any) can post a take-it-or-leave-it price offer to the losing buyer. In this case the first buyer would always prefer to purchase the item at price  $p^* = \frac{1}{2(1-\epsilon)}$ , and then offer to resell it to the second buyer in the secondary market at price  $p' = \epsilon^{-1}H$ . Note that this choice of  $p'$  is revenue-maximizing, assuming that no extra information about buyer valuations is revealed between the auction and the secondary market, and obtains expected revenue  $1 \geq v_1$ . This is therefore a sequential equilibrium. The expected welfare at this equilibrium is  $O(1)$ , since  $v_1 = O(1)$  and  $\Pr[v_2 \geq \epsilon^{-1}H] = \epsilon H^{-1}$ . Taking  $H$  sufficiently large leads to an arbitrarily large welfare gap.

This example shows that the pricing strategy based the median of maximum values can lead to significant incentives for a low-value buyer to purchase with the intention to resell. Then, due to monopolist distortions, significant welfare is subsequently lost in the secondary market.

Another approach to setting static posted prices is based on so-called “balanced prices” [20, 7]. In the single-item example described above, this corresponds to setting a price equal to  $\frac{1}{2}\mathbf{E}[\max_i v_i]$ . This approach likewise guarantees half of the optimal welfare for the single-item prophet inequality problem [20]. We will show that, unlike Example 15, this guarantee continues to hold even in the presence of a secondary market. This is true not only for single-item auctions, but for the broader broad class of multi-unit auctions, and even to combinatorial allocation problems.

To formalize this claim, we make use of a general definition of balanced prices due to Dutting, Feldman, Kesselheim and Lucier [7]. We will state this definition in a general combinatorial auction setting in which the set of items  $M$  are not necessarily identical. A pricing function  $p$  assigns to each set of items  $x \subseteq M$  a price  $p(x) \geq 0$ . For example, this function might assign a price to each individual item and set  $p(x)$  to be the sum of item prices, or more generally  $p(x)$  might assign arbitrary prices to each bundle.<sup>19</sup>

First some notation. Write  $\text{OPT}(\mathbf{v}, S)$  for the welfare-optimal allocation of the items in  $S \subseteq M$  when the valuations are  $\mathbf{v}$ , and denote  $\text{OPT}(\mathbf{v}) = \text{OPT}(\mathbf{v}, M)$ . In a slight abuse of notation, write  $M \setminus \mathbf{x}$  for  $M \setminus (\cup_i x_i)$ , the set of items that are unallocated in feasible solution  $\mathbf{x}$ . Then, in particular,  $\text{OPT}(\mathbf{v}, M \setminus \mathbf{x})$  is the welfare-optimal allocation of the items that are unallocated in  $\mathbf{x}$ .

<sup>19</sup>This definition naturally extends to fractional or randomized allocations, but in order to keep notation simple, in this section we will restrict attention to deterministic allocations. Theorem 18 can be extended to fractional or randomized allocations with the appropriate notational adjustments.

The following is a definition from [7] specialized to our setting.

► **Definition 16.** For a given valuation realization  $\mathbf{v}$ , a pricing function  $p$  is  $(\alpha, \beta)$ -balanced if, for any pair of allocations  $\mathbf{x}, \mathbf{x}'$  that are disjoint and jointly feasible (i.e., they allocate disjoint sets of items), we have

- $\sum_{i \in \mathcal{N}} p(x_i) \geq \frac{1}{\alpha} (\mathbf{v}(\text{OPT}(\mathbf{v})) - \mathbf{v}(\text{OPT}(\mathbf{v}, M \setminus \mathbf{x})))$ ,
- $\sum_{i \in \mathcal{N}} p(x'_i) \leq \beta \cdot \mathbf{v}(\text{OPT}(\mathbf{v}, M \setminus \mathbf{x}))$

That is, prices  $p$  are balanced if the price paid for any allocation  $\mathbf{x}$  is at least the loss in optimal welfare due to losing the items in  $\mathbf{x}$  (up to a factor of  $\alpha$ ). Secondly, the total price of any allocation  $\mathbf{x}'$  that remains feasible after  $\mathbf{x}$  is removed, is at most  $\beta$  times the optimal welfare achievable using the items not allocated in  $\mathbf{x}$ . Note that  $\mathbf{x}'$  need not be the welfare-optimal allocation of the items in  $M \setminus \mathbf{x}$ .

Note that this definition of balanced prices is with respect to a particular realization  $\mathbf{v}$  of agent types. We will use this to construct a static pricing rule (i.e., prices that are independent of realizations) by taking an expectation over types. Formally, we say that a posted-price mechanism uses  $(\alpha, \beta)$ -balanced prices if it (a) defines an  $(\alpha, \beta)$ -balanced price function  $p^{\mathbf{v}}$  for all valuation profiles  $\mathbf{v}$ , then (b) sets its actual static price function  $p$  according to  $p(x) = \frac{\alpha}{1+\alpha\beta} E_{\mathbf{v}}[p^{\mathbf{v}}(x)]$ . That is, prices are set by taking expectations of the type-specific prices over the buyer types and multiplying by the constant  $\frac{\alpha}{1+\alpha\beta}$ . For example, in the single-item prophet inequality setting, a  $(1, 1)$ -balanced price for a given realization of values  $\mathbf{v}$  is the price  $p^{\mathbf{v}} = \max_i v_i$ . The appropriate choice of posted price for the prophet inequality is then  $\frac{1}{2} \mathbf{E}[\max_i v_i]$ , the expectation of the balanced prices times  $\frac{1}{2}$ .

It is known that in the absence of a secondary market, using balanced prices leads to a strong welfare guarantee.

► **Proposition 17** ([7]). Fix the valuation distributions  $F = \times_i F_i$ . If a posted-price mechanism  $\mathcal{M}$  uses  $(\alpha, \beta)$ -balanced prices for  $\alpha, \beta \geq 1$ , then the expected welfare obtained by  $\mathcal{M}$  is at least  $\frac{1}{1+\alpha\beta}$  times the expected optimal welfare.

As we now show, this result extends to any equilibrium in the combined market setting with an arbitrary trade mechanism being used as a secondary market. This theorem applies to multi-unit and combinatorial auction allocation problems.

► **Theorem 18.** Fix the valuation distributions  $F = \times_i F_i$ . For any signaling protocol  $\Gamma$  and any trade mechanism  $\mathcal{M}^2$  in the secondary market, if a posted-price mechanism  $\mathcal{M}^1$  uses  $(\alpha, \beta)$ -balanced prices for  $\alpha, \beta \geq 1$ , then the expected welfare obtained by the combined mechanism  $\mathcal{M}^C = \mathcal{G}(\mathcal{M}^1, \Gamma, \mathcal{M}^2)$  at any Bayesian-Nash equilibrium is at least  $\frac{1}{1+\alpha\beta}$  times the expected optimal welfare.

Our argument is an adaptation of a proof method due to Dutting, Feldman, Kesselheim and Lucier [7]. The details of the proof is given in the full version online.

## 5.1 Application: Balanced Prices for Carbon Markets

We can apply Theorem 18 to any allocation problem for which balanced prices exist. For example, it is known that one can design  $(1, 1)$ -balanced item prices for submodular combinatorial auctions [14]. A multi-unit auction with weakly decreasing marginal values is a special case of a submodular combinatorial auction in which all items are identical. We can therefore conclude from Theorem 18 the existence of item prices that guarantee half of the expected optimal welfare in our model of a carbon license market with an arbitrary aftermarket.<sup>20</sup>

<sup>20</sup>The approximation ratio is tight even for the single-item setting [20].

One thing to note is that in the standard construction for submodular combinatorial auctions, there is no guarantee that all items will be assigned the same price even if the items are identical. We next argue that in fact there always exist  $(1, 1)$ -balanced prices that are identical across items, so in fact there is only a single price that can be interpreted as a per-unit price offered to all prospective buyers.

Our construction is as follows. For each valuation profile  $\mathbf{v}$ , calculate the optimal allocation  $\mathbf{x}^*(\mathbf{v})$  by greedily allocating in order of highest marginal value. Define  $w^\mathbf{v} \triangleq \frac{1}{m} \sum_i v_i(x_i^*(\mathbf{v})) \geq 0$  to be the average per-unit welfare of this optimal allocation. We use the average per-unit welfare  $w^\mathbf{v}$  as a price for each unit of the item, so in particular all units have the same price. That is, the price to acquire  $k$  units is  $p^\mathbf{v}(k) = w^\mathbf{v} \times k$ . We claim that this choice of unit prices is  $(1, 1)$ -balanced.

▷ **Claim 19.** For a multi-unit allocation problem, for any profile  $\mathbf{v}$  of valuations with non-increasing marginal values, the price function  $p^\mathbf{v}(x) = w^\mathbf{v} \cdot |x|$  where  $w^\mathbf{v} = \frac{1}{m} \sum_i v_i(x_i^*(\mathbf{v}))$  is  $(1, 1)$ -balanced.

The proof of the claim is provided in the full version online.

Given this choice of  $(1, 1)$ -balanced prices, Theorem 18 then implies that setting a static price per unit that equals to half of the expected per-unit welfare,  $\frac{1}{2m} \mathbf{E}[\sum_i v_i(x_i(\mathbf{v}))]$ , guarantees half of the optimal expected welfare in any equilibrium, under any secondary market implementable as a trade mechanism and for any set of information revealed between the posted-price mechanism and the secondary market.

► **Example 20.** Recall the lower bound market example in Section 3, which illustrated a low-welfare equilibrium in a uniform auction with an aftermarket. In that example, the welfare-maximizing allocation always allocates two goods to bidder  $A$  and the remaining  $m - 2$  goods to bidder  $B$ , for a total expected welfare of  $\Theta(\log m)$ . In this example, Theorem 18 together with the  $(1, 1)$ -balanced prices we present in Claim 19 would set a per-unit price of  $E[v(\text{OPT}(v))]/(2m) = \Theta((\log m)/m)$  and allow bidders to purchase as many units as desired at this price, while supplies last. Theorem 18 implies that, at this price, any purchasing behavior at equilibrium will achieve at least half of the expected optimal welfare. Intuitively, the price is high enough to discourage the speculator agent  $C$  from buying more than two items (since at most two items can be sold to agent  $A$ , and any items sold to agent  $B$  generate an expected revenue of at most  $1/m$  each, which is less than the auction per-unit reserve) but also low enough that agent  $B$  will nearly always decide to purchase most of the available items.

## 6 Best of Both Worlds: Uniform-Price Auction with Reserves

We realize that moving from an auction to a posted price mechanism is a radical change in many markets, including the carbon allowances market. We thus consider the problem of finding a minimal change to a uniform-price auction which will still take care of the problem of significant efficiency loss that can happen as a result of combining the auction with a secondary market (as illustrated in the example presented in Section 3). As we now show, a sufficient change is to add to the uniform-price auction a per-unit reserve price based on balanced prices. Any marginal bids strictly less than this reserve price are ignored, but bids exactly equal to the reserve are allowed. Formally, in the following theorem we show that the welfare bound from Theorem 18 continues to hold when we add an appropriately chosen per-unit reserve price to the uniform-price auction. The proof of Theorem 21 is given in the full version online.

► **Theorem 21.** *For a multi-unit allocation problem with non-increasing marginal values sampled independently from  $F = \times_i F_i$ , consider mechanism  $\mathcal{M}^1$  that is a uniform-price auction subject to a per-unit reserve price of  $p = \frac{1}{2m} E_{\mathbf{v} \sim F}[\mathbf{v}(\text{OPT}(\mathbf{v}))]$ . Then for any signaling protocol  $\Gamma$  and any trade mechanism  $\mathcal{M}^2$  in the secondary market, and at any Bayesian Nash equilibrium of the combined market  $\mathcal{M}^C = \mathcal{G}(\mathcal{M}^1, \Gamma, \mathcal{M}^2)$ , the expected welfare is at least half of the expected optimal welfare.*

**Estimation Errors and Calculating Reserve Prices.** The reserve price in Theorem 21 depends on the expected optimal welfare attainable by allocating items. As it turns out, the welfare guarantee in Theorem 21 is robust to mistakes in the prices. This robustness was noted by Feldman, Gravin and Lucier [14] for posted-price mechanisms in submodular combinatorial auctions, and the argument extends to our setting without change. Specifically, for any  $\epsilon > 0$ , if instead of setting reserve  $p = \frac{1}{2m} E[\mathbf{v}(\text{OPT}(\mathbf{v}))]$  we set a reserve price  $p'$  such that  $|p' - p| < \epsilon$ , then the expected welfare of the resulting combined market is at least half of the expected optimal welfare less  $m\epsilon$ , where recall that  $m$  is the total number of items for sale. To see why, recall the proof of Theorem 21 and note that lowering a license's price by  $\epsilon$  reduces the revenue obtained by at most  $\epsilon$ , and increasing a license's price by  $\epsilon$  reduces buyer surplus from purchasing that item by at most  $\epsilon$ . Collecting up these additive losses leads to a total loss of at most  $m\epsilon$ . A formal statement and proof are deferred to the full version online.

One implication of this robustness result is that if one sets reserve prices based on a noisy estimate of  $E[\mathbf{v}(\text{OPT}(\mathbf{v}))]$ , such as obtained by samples or historical data, then the welfare guarantee one obtains will degrade in proportion to the estimation error.

► **Corollary 22.** *For a multi-unit allocation problem with non-increasing marginal values sampled independently from  $F = \times_i F_i$ , suppose that the seller has access to an estimate  $\psi$  to the optimal welfare such that with probability at least  $1 - \delta$ ,  $\psi \in [(1 - \epsilon)\mathbf{v}(\text{OPT}(\mathbf{v})), (1 + \epsilon)\mathbf{v}(\text{OPT}(\mathbf{v}))]$  for constants  $\delta, \epsilon \in [0, 1]$ . Consider the uniform-price auction subject to a per-unit reserve price of  $p = \frac{\psi}{2m}$ . Then with probability at least  $1 - \delta$ , for any trade mechanism, and at any Bayesian Nash equilibrium of the combined market, the expected welfare is at least  $\frac{1-\epsilon}{2}$  fraction of the expected optimal welfare.*

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