

Beyond Worst-Case Budget-Feasible Mechanism Design

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Abstract

Motivated by large-market applications such as crowdsourcing, we revisit the problem of budget-feasible mechanism design under a “small-bidder assumption”. Anari, Goel, and Nikzad (2018) gave a mechanism that has optimal competitive ratio $1 - 1/e$ on worst-case instances. However, we observe that on many realistic instances, their mechanism is significantly outperformed by a simpler open clock auction by Ensthaler and Giebe (2014), although the open clock auction only achieves competitive ratio $1/2$ in the worst case. Is there a mechanism that gets the best of both worlds, i.e., a mechanism that is worst-case optimal and performs favorably on realistic instances? To answer this question, we initiate the study of *beyond worst-case budget-feasible mechanism design*.

Our first main result is the design and the analysis of a natural mechanism that gives an affirmative answer to our question above:

- We prove that *on every instance*, our mechanism performs at least as good as all uniform mechanisms, including Anari, Goel, and Nikzad’s and Ensthaler and Giebe’s mechanisms.
- Moreover, we empirically evaluate our mechanism on various realistic instances and observe that it *beats* the worst-case $1 - 1/e$ competitive ratio *by a large margin* and compares favorably to both mechanisms mentioned above.

Our second main result is more interesting in theory: We show that in the semi-adversarial model of *budget-smoothed analysis*, where the adversary designs a single worst-case market for a distribution of budgets, our mechanism is optimal among all (including non-uniform) mechanisms; furthermore our mechanism guarantees a strictly better-than- $(1 - 1/e)$ expected competitive ratio for any non-trivial budget distribution *regardless of the market*. (In contrast, given any bounded range of budgets, we can construct a single market where Anari, Goel, and Nikzad’s mechanism achieves only $1 - 1/e$ competitive ratio for every budget in this range.) We complement the positive result with a characterization of the worst-case markets for any given budget distribution and prove a fairly robust hardness result that holds against any budget distribution and any mechanism.

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1 Introduction

The budget-feasible mechanism design problem was introduced by Singer [20] and has become a core problem in algorithmic mechanism design [8, 9, 3, 21, 6, 10, 11, 13, 5, 7, 18, 22, 23, 15, 2, 14, 1, 12, 16, 4]. We will use microtask crowdsourcing as a running example for this problem (see Section 2.1 for a formal setup): An employer (buyer) on a crowdsourcing platform (market I) such as Mechanical Turk or Microworkers is given a fixed budget B ,



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and is looking to acquire some services from a set of workers (sellers) $[n]$. Each worker i can perform a microtask (provide a service) that has a utility u_i to the employer at an incurred cost c_i to the worker himself. The employer's total utility is $\sum_{i \in W} u_i$ for the services provided by each subset of workers $W \subseteq [n]$. As is common in the literature, the employer knows the workers' utilities u_i 's (e.g. by grading their work ex-post, or using the worker's rating on previous tasks), but does not know their private costs c_i 's. Moreover, in large-market applications like microtask crowdsourcing, it is often very natural to make a small-bidder assumption: the cost of each worker is a small fraction of the employer's total budget¹.

The objective of budget-feasible mechanism design is to design a truthful mechanism that maximizes the employer's total utility while keeping the total payment to the workers within the budget. Roughly speaking, a truthful mechanism makes sure the workers honestly report their private costs c_i 's by providing them incentives (payments) and decides which subset of services the employer will get (allocation), and we want the mechanism to maximize the total utility of the services allocated to the employer, under the constraint that the total payment does not exceed B .

Without any incentive constraints (i.e., the workers' costs are public, and the employer only needs to pay a worker's cost to get the worker's service), this becomes the well-known knapsack problem. Therefore, it is standard to consider the following performance metric for a mechanism: the ratio between the utility achieved by the mechanism and the optimal utility of the knapsack problem without incentive constraints *in a worst-case market and for a worst-case budget*. This metric is called the worst-case competitive ratio, and a mechanism is α -competitive if its worst-case competitive ratio is $\geq \alpha \in [0, 1]$.

Research in budget-feasible mechanism design has been focusing on designing (polynomial-time) mechanisms that achieve optimal worst-case competitive ratio. Under the small-bidder assumption, [2] gave a $(1 - 1/e)$ -competitive mechanism and characterized the worst-case instances² for which any mechanism can only achieve at most $1 - 1/e$ competitive ratio.

Although this optimal result provides a satisfactory answer with respect to worst-case competitive ratio, our quest to design even better mechanisms does not come to an end. Indeed, recall that worst-case competitive ratio measures a mechanism's performance in worst-case market given worst-case budget. Such worst-case market and worst-case budget rarely appear in practice. Even if we are given an typical-case market and/or an typical-case budget, a mechanism that achieves optimal worst-case competitive ratio could (potentially) perform as bad as on the worst-case instance. We probably would not prefer such worst-case optimal mechanism over other mechanisms that perform much better on the typical-case instances. To make this point more concrete, consider the following extremely simple instance:

► **Example 1.** The buyer has a budget $B = n$, and each of n seller's services has a cost 1 to the seller himself and a utility 1 to the buyer.

For the simple instance in Example 1 (which satisfies small-bidder assumption), simply offering a payment of 1 to each seller extracts full utility, but [2]'s worst-case optimal mechanism only obtains a $(1 - 1/e)$ -fraction. Moreover, instead of identical sellers' costs, consider a more natural variant of Example 1, where the sellers' costs are sampled i.i.d. from a natural distribution (e.g., Gaussian/uniform/exponential/mixture distribution). Our numerical simulation shows that [2]'s mechanism only obtains close-to- $(1 - 1/e)$ fraction of the optimal utility for these natural instances, while a simple open clock auction [10], that is equivalent

¹ Other important applications where this assumption is natural include allocation of R&D subsidies by government agencies and emission reduction auctions [2].

² An instance is specified by a market I and a budget B .

to setting a single uniform price, obtains significantly larger fractions (see Table 1). Of course the open clock auction also extracts the full utility for the simple instance in Example 1. However, it is known that the open clock auction is suboptimal in the worst case: [2] exhibits a simple example for which the open clock auction has worst-case competitive ratio $1/2$.

Is there a mechanism that gets the best of both worlds, i.e., a mechanism that is worst-case optimal and “performs favorably” on every instance (not just in the worst case)?

By “perform favorably”, we mean that the mechanism should achieve utility at least as good as a large class of mechanisms. Which class of mechanism should we consider as an appropriate benchmark? At least, we want this class to include the previous mechanisms of [2] and [10]. The most ambitious class is obviously the class of all the mechanisms, but as we now explain, it is unfair to compare with this class. Consider the mechanism in the following example:

► **Example 2.** Consider an arbitrary instance (I, B) specified by market I and budget B , which becomes a knapsack problem when sellers’ costs are public, and the optimal solution (i.e., the optimal subset of sellers’ services) to this knapsack problem always exists. Now we hard-code the market I in the following mechanism: When given an input instance (I', B') (assume for simplicity³ that I' has the same number of sellers as I , but the sellers’ costs and utilities in I' can be arbitrarily different from I), this mechanism reads nothing from input except the budget B' , and it always non-uniformly offers each seller, who is in the optimal knapsack solution of instance (I, B') , a posted price that is equal to this seller’s cost in I , and offers nothing to the remaining sellers.

Although the mechanism in Example 2 is silly (because it always decides the allocation and payments according to I regardless of the actual market I' it is facing), it is a well-defined non-uniform posted price mechanism that is truthful and budget-feasible. Even though we expect this mechanism to perform poorly in general, it is optimal for the specific market I that is hard-coded in it, and there is no way we can compete with such unreasonable mechanism on instance (I, B) . In order to exclude such mechanisms while including the mechanisms of [2] and [10], we restrict our attention to the class of all the uniform mechanisms (for now⁴). Roughly speaking, a mechanism is uniform if the distributions of normalized offers is essentially the same for all the sellers (see Section 2.1.1 for the exact definition).

It is noteworthy that unlike algorithm design, where one can combine two algorithms by taking the best solution outputted by these algorithms, naively combining two mechanisms in such way typically does not result in a truthful mechanism, which motivates us to search for a new mechanism that satisfies the desiderata in our main question. With the above motivation, we initiate the study of *beyond worst-case budget-feasible mechanism design*, and we also make the small-bidder assumption given its wide applicability in practice (see [2, Section 10]). In the next two subsections, we give an overview of our results. In terms of the significance, we believe the first main result (instance optimality), which compares our new mechanism with uniform mechanisms, is more significant from the practical perspective, and the techniques are arguably not complicated and hence can be applied in practice. The second main result (budget-smoothed analysis), which compares our new mechanism with the general (possibly non-uniform) mechanisms is more interesting from the theoretical perspective. We believe these two results complement each other, and we hope these results

³ This is without loss of generality, because otherwise the mechanism could use an arbitrary mapping from sellers in I to sellers in I' .

⁴ In our results for budget-smoothed analysis, we will compare to all (possibly non-uniform) mechanisms.

could encourage researchers in the broad area of mechanism design to examine the worst-case optimal mechanisms for their mechanism design problems through beyond-worst-case lens and design even better mechanisms with improved beyond-worst-case performance.

1.1 Main result I: instance optimality

The first result of this paper is the design and (theoretical and empirical) analysis of a new natural mechanism. We prove that our new mechanism performs at least as good as any uniform mechanism on every instance (Theorem 11).

► **Theorem 3** (Instance-optimality against uniform mechanisms).

We give a computationally efficient, truthful and strictly budget-feasible randomized mechanism, that, on every instance of the budget-feasible mechanism design problem with additive buyer’s utility function and small sellers, achieves $\geq (1 - o(1))$ of the expected utility of any uniform mechanism.

Moreover, we empirically evaluate our mechanism on many realistic instances and observe that it beats the worst-case $1 - 1/e$ competitive ratio by a large margin.

Empirical analysis. Specifically, we compare the performance of our mechanism, the open clock auction [10], and [2]’s mechanism on synthetic instances (see Section 3.4 for details). We observe that our mechanism and the open clock auction outperform [2]’s worst-case optimal mechanism on all synthetic instances by a large margin. In the instances where the distribution of sellers’ cost-per-utility is multi-modal⁵, our mechanism outperforms both other mechanisms significantly (recall that we indeed prove that it is always optimal).

Our mechanism in a nutshell. An idealized version of our mechanism, where we know the market statistics (i.e., the empirical distribution of sellers’ types⁶), has the following nice interpretation: each seller is independently offered one of two possible prices, and can choose to accept or reject the offer she receives. Knowing the market statistics is a reasonable assumption in many cases in practice, e.g. when the buyer has access to historical bids. In general, when the statistics are not known, we can randomly partition the sellers into two subsets, and compute prices for each half based on market statistics estimated from truthful reporting of costs from the other half.

The main novelty of our mechanism is the design of its idealized version – a greedy-type uniform “mechanism” (Mechanism 1), which can be interpreted as a probabilistic combination of at most two uniform prices per utility. We prove that this greedy “mechanism” is instance-optimal compared to all the uniform mechanisms by a neat greedy exchange argument. We are surprised that despite being such a natural “mechanism” (from the information-theoretic point of view), Mechanism 1 has never been studied in the literature to our best knowledge.

In the random partitioning step for estimating the market statistics, the technical part is how to control the noise caused by random partitioning (overly large noise could ruin the budget feasibility of the mechanism without giving up a significant fraction of utility). Thanks to the simple form of our idealized mechanism, we are able to succinctly discretize the space of candidate allocation rules. Moreover, in order to upper bound the influence of

⁵ We note that multi-modal sellers’ distribution is possible in the real world. For example, in an international market, the average cost-per-utility of sellers in a country could differ from that in another country because of the difference of resources/technology between different countries.

⁶ A seller’s type is specified by his cost and utility.

each individual seller during the random partitioning, we truncate the allocation rule, which does not lose much utility because of small-bidder assumption. By a careful probabilistic analysis, we show that combining these techniques is sufficient to approximate our idealized greedy mechanism within negligible error. Therefore, the approximate version of the greedy mechanism, which is our final mechanism (Mechanism 2), is (nearly) optimal on every instance compared to any uniform mechanism.

1.2 Main result II: budget-smoothed analysis

We have shown that empirically our mechanism’s performance on realistic instances is much better than the $1 - 1/e$ competitive ratio on the “worst-case instance”, which suggests that optimality on the “worst-case instance” is a weak notion that fails to capture better-than-worst-case performance. We also have shown that our mechanism beats all the uniform mechanisms on “every instance”, but as we explained before, we restricted our attention to the class of uniform mechanisms, because it is unreasonable to compare with the class of non-uniform mechanisms on “every instance”, which suggests that optimality on “every instance” is somewhat too strong if we hope to compare our mechanism with the more general non-uniform mechanisms.

Thus in addition to our first result, we strike a reasonable middle ground between “worst-case instance” and “every instance” by examining our mechanism under the *budget-smoothed analysis* framework recently introduced in [19] in the context of submodular maximization. This framework gives a reasonable notion of beyond-worst-case instances that allows us to theoretically compare our mechanism to all the (even non-uniform) mechanisms.

Briefly (see the formal definition in Section 2.2), the budget-smoothed analysis framework is a semi-adversarial model: We first pick a budget distribution and a mechanism, and then the adversary, who knows the mechanism and the budget distribution, chooses a single worst-case market, and finally we sample a budget from the distribution and measure the mechanism’s *expected competitive ratio* (formally defined in Section 2.2) on the adversarially chosen market, where the expectation is over the randomness of the budget distribution and (potentially) the mechanism itself. (The motivation of the budget-smoothed analysis in the context of budget-feasible mechanism design deserves an in-depth discussion, which we defer to Section A in the full version due to the interest of space.)

We show the following fundamental results in the budget-smoothed analysis model.

Optimal mechanism and worst-case markets for any budget distribution. We prove that our mechanism is optimal (see Definition 5) among all the (not necessarily uniform) mechanisms on the worst-case market for any budget distribution⁷ (Theorem 13), and moreover, the expected competitive ratio of our mechanism is guaranteed to be strictly better than $1 - 1/e$ for every nontrivial budget distribution regardless of the market⁸ (Theorem 4.3 in the full version). In contrast, given any bounded range of budgets, we construct a *single* market where [2]’s worst-case optimal mechanism cannot beat the worst-case $1 - 1/e$ competitive ratio *for any budget in this range* (Theorem D.2 in the full version), which exhibits a strong separation between our mechanism and [2]’s mechanism.

⁷ It is particularly interesting that our mechanism, which does not require any knowledge of the budget distribution, is optimal even when compared with the mechanisms that know the budget distribution. In other words, our mechanism intrinsically adjusts itself to the budget distribution optimally.

⁸ Moreover, in Section E of the full version, we formulate a (non-convex) mathematical program that computes the expected competitive ratio on the worst-case market for any given budget distribution. We solve this program for various distributions and observe nonnegligible improvement over $1 - 1/e$.

Our proof of the optimality result is conceptually appealing: We observe that once we fix an arbitrary budget distribution, determining the worst-case market and the optimal mechanism is a min-max game between the adversary and the mechanism designer, in which the adversary tries to give a market that minimizes the mechanism's performance, and the mechanism designer hopes to design a mechanism that performs best on the adversarially chosen market. We analytically solve the equilibrium for this min-max program, and the solution comes with a characterization of the worst-case markets for any given budget distribution (Theorem 13).

Robust hardness result against any budget distribution. On the negative side, we prove a robust hardness result that shows for any budget distribution and any mechanism, there is a market on which the mechanism's expected competitive ratio is bounded away from 1 (specifically, at most 0.854 – see Theorem 4.4 in the full version). In comparison, the previous worst-case hardness result [2] is very sensitive to budget perturbation: If we perturb (i.e., multiply) the budget of the worst-case instance by a significant factor like 2.5, then simply setting a single uniform price (i.e., [10]'s open clock auction) will achieve 100% of the optimal utility.

2 Preliminaries

2.1 Problem setup

In the budget-feasible mechanism design (a.k.a., procurement auction) problem with additive utility, there is a market I consisting of one buyer and n sellers, and each seller i has an item with a public utility $u_i \in \mathbb{R}_{\geq 0}$ and a private cost $c_i \in \mathbb{R}_{\geq 0}$. The buyer has a budget $B \in \mathbb{R}_{\geq 0}$ and wants to buy items from the sellers. The goal of the budget-feasible mechanism design problem is to design a truthful mechanism that maximizes buyer's total utility while keeping the total payment to sellers within the budget, which we now explain more formally.

Truthful mechanisms. A mechanism takes as input the buyer's budget B , the sellers' public utilities u_i 's and the private costs⁹ c_i 's reported by the sellers, and then outputs which items should be allocated to the buyer and how much the buyer should pay to each seller. Formally, the output of a (randomized) mechanism, i.e., allocation and payments, can be represented by¹⁰ an *allocation function* $g : \mathbb{R}_{\geq 0}^n \rightarrow [0, 1]^n$ and a *payment function* $Q_g : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}^n$, where g takes the sellers' cost-per-utility $\gamma_i := c_i/u_i$'s as input $\vec{\gamma}$, and outputs (the expectation of) the fraction of each item that is allocated to the buyer (and hence, the expected utility the buyer gets from seller i is the i -th coordinate of the output of g , which we denote by $g(\vec{\gamma})_i$, times u_i , and the expected cost of seller i is $g(\vec{\gamma})_i \cdot c_i$), and Q_g takes the same input and outputs the associated (expected) payment-per-utility for each item (namely, the expected payment to seller i is the i -th coordinate of the output of Q_g , which we denote by $Q_g(\vec{\gamma})_i$, times u_i).

A deterministic mechanism is *truthful* if reporting the true γ_i always maximizes the net profit for each seller $i \in [n]$, namely, for any $\gamma_{-i} \in \mathbb{R}_{\geq 0}^{n-1}$ (where γ_{-i} denotes all the γ_j 's except γ_i), for all $z \in \mathbb{R}_{\geq 0}$,

⁹ We note that there are mechanisms that do not directly ask the sellers to report their costs such as clock auctions. However, our definition is without loss of generality by the revelation principle.

¹⁰ If the mechanism is deterministic, g and Q_g output the deterministic allocations and deterministic payments, respectively, and if the mechanism is randomized, they output the expected allocations and expected payments.

$$Q_g(\gamma_i, \gamma_{-i})_i \cdot u_i - g(\gamma_i, \gamma_{-i})_i \cdot c_i \geq Q_g(z, \gamma_{-i})_i \cdot u_i - g(z, \gamma_{-i})_i \cdot c_i. \quad (1)$$

In general, a mechanism can be randomized, and a *randomized mechanism* is simply a distribution of deterministic mechanisms. In this light, we say a randomized mechanism is *truthful-in-expectation* if reporting the true γ_i only maximizes seller i 's net profit in expectation over the randomness of the mechanism, i.e., Eq. (1) holds in expectation for the randomized mechanism.

The celebrated Myerson's lemma [17] asserts that (i) an allocation function g can be implemented as a truthful-in-expectation mechanism if and only if g is *monotone*, i.e., for all $i \in [n]$ and any $\gamma_{-i} \in \mathbb{R}_{\geq 0}^{n-1}$, $g_i(\cdot) := g(\cdot, \gamma_{-i})_i$ is a non-increasing function, and (ii) there exists a unique payment function Q_{g_i} associated with g_i , which is given by $Q_{g_i}(\gamma) := \gamma \cdot g_i(\gamma) + \int_{\gamma}^{\infty} g_i(z) dz$.

Budget feasibility. Note that we want the randomized mechanisms to *strictly satisfy* the budget constraint, i.e., every deterministic mechanism in the support of the distribution has to satisfy the following budget constraint

$$\sum_{i \in [n]} Q_{g_i}(\gamma_i) u_i \leq B,$$

and our proposed randomized mechanism will indeed strictly satisfy the budget constraint.

The goal of budget-feasible mechanism design is to design a (randomized) mechanism, that is (truthful/truthful-in-expectation) and budget-feasible, to maximize the buyer's (expected) total utility $\sum_{i \in [n]} g_i(\gamma_i) u_i$.

If the sellers' costs are public, the problem becomes the well-known knapsack problem, and we call the optimal utility of this knapsack problem the *non-IC* (i.e., without the incentive compatible constraints) optimal utility. The standard performance measure for a (randomized) mechanism \mathcal{M} on the instance (I, B) is the *competitive ratio*, i.e. the ratio $\mathcal{R}_{\mathcal{M}}(I, B)$ between the (expected) total utility (over \mathcal{M} 's randomness) achieved by \mathcal{M} and the non-IC optimal utility.

Finally, we make a *small-bidder assumption* [2]: for budget B , we require that each seller's cost is at most $o(B)$.

2.1.1 Further important concepts

Uniform mechanism. We call a mechanism¹¹ with allocation function g *uniform* if given any γ_i 's, there exists a 1-dimensional allocation function $f : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$ such that for all $i \in [n]$, it holds that $g_i(\cdot) = f(\cdot)$. Otherwise, we call the mechanism *non-uniform*.

Fractional versus indivisible. We mentioned that the allocation function specifies the fraction of item purchased from each seller. This makes sense when the item is fractional, e.g., the item is the time of a worker. However, there are settings where the items are indivisible, and then, the image of an allocation function should be $\{0, 1\}^n$ instead. Under small-bidder assumption, an indivisible item procurement problem can be reduced to a fractional problem. Specifically, there is a rounding procedure from [2, Supplemental Material, Section 7] that we can directly apply.

¹¹For example, the idealized versions of [2]'s mechanism (Mechanism 3 in the full version of our paper), [10]'s mechanism and our mechanism (Mechanism 1) are all uniform mechanisms.

► **Lemma 4** ([2, Supplemental Material, Section 7]). *Let $\tilde{x}_1, \dots, \tilde{x}_n$ be the fractional allocations and $\tilde{p}_1, \dots, \tilde{p}_n$ be the associated payments. Under small-bidder assumption, there is a rounding procedure that outputs integral allocations x_1, \dots, x_n and payments p_1, \dots, p_n , which achieves approximately the same expected utility as the fractional allocations, while preserving individual rationality, truthfulness in expectation, and strict budget feasibility.*

Henceforth, given this reduction, **unless specified otherwise, we only consider divisible items** in this paper, and the results apply to indivisible items as well.

2.2 Budget-smoothed analysis

Budget-smoothed analysis is a semi-adversarial model introduced in [19] in the context of submodular optimization. In our setting, given any fixed distribution of budgets \mathcal{D} , the performance metric for a mechanism \mathcal{M} in the budget-smoothed analysis is the \mathcal{D} -budget-smoothed competitive ratio: the worst possible ratio between the utility achieved by \mathcal{M} and the non-IC optimum in expectation (over budget distribution and mechanism's randomness), i.e.,

$$\min_I \mathbb{E}_{B \sim \mathcal{D}} [\mathcal{R}_{\mathcal{M}}(I, B)],$$

where $\mathbb{E}_{B \sim \mathcal{D}} [\mathcal{R}_{\mathcal{M}}(I, B)]$ is the *expected competitive ratio* of \mathcal{M} on market I for budget distribution \mathcal{D} . Fixing an arbitrary budget distribution \mathcal{D} , the goal of the mechanism designer is to design a mechanism \mathcal{M} that achieves optimal \mathcal{D} -budget-smoothed competitive ratio, and hence, we have a max-min game between the mechanism designer and the adversary

$$\max_{\mathcal{M}} \min_I \mathbb{E}_{B \sim \mathcal{D}} [\mathcal{R}_{\mathcal{M}}(I, B)].$$

In other words, we are interested in the expected outcome of the following budget-smoothed analysis game:

Budget-smoothed analysis game

1. Fix a distribution of budgets \mathcal{D} . The mechanism designer, who knows the budget distribution \mathcal{D} , picks a mechanism^a \mathcal{M} .
2. The adversary, who knows the budget distribution \mathcal{D} and the mechanism \mathcal{M} chosen by the mechanism designer, chooses a worst-case market^b I (sellers' costs and utilities).
3. Then, a budget B is drawn at random from \mathcal{D} .
4. Finally, the mechanism designer runs \mathcal{M} on the instance (I, B) (and compare the performance to the non-IC optimum).

^a Note that the mechanism designer knows \mathcal{D} and hence is allowed to choose a mechanism \mathcal{M} that is tailored to \mathcal{D} , i.e., the mechanism designer knows \mathcal{D} and then specifies what \mathcal{M} does for each budget B in the support of \mathcal{D} as she likes. Interestingly, as we will show later, our optimal mechanism does not need any knowledge of \mathcal{D} .

^b Note that the adversary chooses the market I after knowing \mathcal{D} and \mathcal{M} . For example, if the mechanism designer chose the silly mechanism in Example 2 that hard-codes some market I_1 , this mechanism will perform poorly in the budget-smoothed analysis game, because the adversary can choose a completely different market I_2 after observing the mechanism chosen by the mechanism designer.

► **Definition 5.** A mechanism \mathcal{M}^* is worst-case optimal for a budget distribution \mathcal{D} if for any other mechanism \mathcal{M} , $\min_I \mathbb{E}_{B \sim \mathcal{D}} [\mathcal{R}_{\mathcal{M}^*}(I, B)] \geq \min_I \mathbb{E}_{B \sim \mathcal{D}} [\mathcal{R}_{\mathcal{M}}(I, B)]$.

We refer the interested readers to [19] for the original motivation and naming of the budget-smoothed analysis model. In our context, we can think of the \mathcal{D} -budget-smoothed competitive ratio as the average competitive ratio of multiple employers operating in the worst-case market I with different budgets (the empirical distribution of their budgets is \mathcal{D}), and the employers' budgets can easily vary by an order of magnitude because of different sizes of business (as in the Microworkers example in Table 3 of the full version). Given such budget distribution supported on a wide range, even if the market I is worst-case, the “average” employers who use an “average” budget could potentially enjoy a competitive ratio that is significantly better than the worst-case optimal competitive ratio $1 - 1/e$ (and by Markov inequality, most employers achieve strictly better-than- $(1 - 1/e)$ competitive ratio).

3 Instance-optimality against uniform mechanisms

In this section, we first derive a uniform “mechanism” in the complete-information setting, where the sellers' private costs are known. To be precise, the complete-information uniform “mechanism” applies a single monotone allocation function and the associated Myerson's payment function to all the sellers and guarantees strict budget-feasibility just like a normal uniform budget-feasible mechanism, and the only caveat is that to compute the allocation function, the complete-information uniform “mechanism” needs to know all the sellers' costs. This complete-information uniform “mechanism” is essentially a greedy procedure. Then, we show that this greedy “mechanism” is *instance-optimal* compared to all the uniform budget-feasible mechanisms. That is, for every market and every budget, compared to all the uniform budget-feasible mechanisms that satisfy Myerson's characterization of truthful-in-expectation mechanisms, the greedy “mechanism” achieves the optimal buyer's utility.

Apparently, this greedy “mechanism” by itself is not very useful, since we eventually want a normal mechanism that works in the setting where sellers' private costs are hidden. Therefore, we design a randomized mechanism to approximate the greedy “mechanism”, i.e., our randomized mechanism is nearly as good as the greedy “mechanism” on every instance. On a high level, this is done by first randomly partitioning the market into two halves, and then applying our greedy “mechanism” on each half to get an allocation function and the associated payment function, and finally applying the allocation and payment function we get from one half to the other half in a sequential fashion until certain budget threshold is met.

3.1 Greedy is an instance-optimal uniform “mechanism”

In this subsection, we describe the greedy “mechanism” GREEDY in the complete information setting, where the sellers' private costs are given, and prove that it is instance-optimal compared to all the uniform truthful-in-expectation budget-feasible mechanisms. The pseudocode of GREEDY is given in Mechanism 1.

It works as follows – Suppose that the sellers are grouped and sorted according to their cost-to-utility ratio c/u . GREEDY searches for the best monotone allocation function (and given the allocation function, the payment is determined by Myerson's lemma). It does this iteratively. In each iteration, suppose that it has bought all the items from the sellers with cost-to-utility ratio at most c_{i-1}/u_{i-1} , then it will choose the sellers whose c/u ranges from $(c_{i-1}/u_{i-1})_+$ to c_j/u_j for some $j \geq i$, and simultaneously increase the fraction bought from these sellers until either they are fully purchased or the budget is exhausted.

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We now explain how GREEDY selects the next j in each iteration. For each candidate seller k , it computes the *marginal utility per marginal payment* (denoted by $e_{i,k}$) achieved by simultaneously increasing the fraction of items purchased from all the buyers in i, \dots, k . GREEDY then greedily selects j to be the index that maximizes the marginal utility per marginal payment.

■ Mechanism 1 GREEDY.

Input : (c_i, u_i) for $i \in [n]$, B .

- 1 Merge the sellers with equal cost-to-utility $\gamma_i := \frac{c_i}{u_i}$ into one seller by summing up their costs and utilities, and let u_0 be the utility of the merged seller with cost 0 ($u_0 = 0$ if there is no such seller) and let $\gamma_0 = 0, f(0) = 1$. Sort all non-zero-cost merged sellers such that the γ_i are non-decreasing, and let n' be the number of non-zero-cost merged sellers;
- 2 $i \leftarrow 1$;
- 3 **while** $i \leq n'$ **do**
- 4 Choose $j \in \{i, i+1, \dots, n'\}$ that maximizes $e_{i,j}$ ($e_{i,j}$ is defined by Eq. (3));
- 5 **if** $B > q_{i,j}^{\max}$ ($q_{i,j}^{\max}$ is defined in Eq. (2)) **then**
- 6 Let $f(\gamma) = 1$ for all $\gamma \in (\gamma_{i-1}, \gamma_j]$ and $B = B - q_{i,j}^{\max}$;
- 7 $i = j + 1$;
- 8 **else**
- 9 Let $f(\gamma) = \frac{B}{q_{i,j}^{\max}}$ for all $\gamma \in (\gamma_{i-1}, \gamma_j]$ and **break**;
- 10 **end**
- 11 **end**
- 12 **for** each original seller $i \in [n]$ **do**
- 13 Purchase $f(\gamma_i)$ fraction of seller i 's item and pay $u_i Q_f(\gamma_i)$, where Q_f is the payment rule corresponding to f given by Myerson's lemma, i.e.,

$$Q_f(\gamma) = \gamma \cdot f(\gamma) + \int_{\gamma}^{\infty} f(z) dz;$$
- 14 **end**

► **Theorem 6.** *For divisible items, GREEDY decides the allocation and the payment for all the sellers using a single monotone allocation function and the associated Myerson's payment function, and it is strictly budget-feasible, and moreover, on every instance, GREEDY achieves buyer's utility no less than any uniform truthful-in-expectation budget-feasible mechanism.*

Proof. First, observe that f in GREEDY is a non-increasing function, and we apply the same f, Q_f to all the sellers in GREEDY. In each iteration of the while loop, suppose we increase the allocation function f 's value over $(\gamma_{i-1}, \gamma_j]$ from zero to certain $f(\gamma_j)$, the payment-per-utility $Q_f(\gamma) = \gamma \cdot f(\gamma) + \int_{\gamma}^{\infty} f(z) dz$ should also increase for every $\gamma \leq \gamma_j$. Specifically, for every $\gamma \leq \gamma_{i-1}$, the $\gamma \cdot f(\gamma)$ part does not change, but the $\int_{\gamma}^{\infty} f(z) dz$ part increases from zero to $\int_{\gamma_{i-1}}^{\gamma_j} f(\gamma_j) dz = f(\gamma_j) \cdot (\gamma_j - \gamma_{i-1})$, and thus, $Q_f(\gamma)$ increases by $f(\gamma_j) \cdot (\gamma_j - \gamma_{i-1})$. For every $\gamma \in (\gamma_{i-1}, \gamma_j]$, the $\gamma \cdot f(\gamma)$ part increases from zero to $\gamma \cdot f(\gamma_j)$, and the $\int_{\gamma}^{\infty} f(z) dz$ part increases from zero to $\int_{\gamma}^{\gamma_j} f(\gamma_j) dz = f(\gamma_j) \cdot (\gamma_j - \gamma)$, and thus, $Q_f(\gamma)$ increases by $f(\gamma_j) \cdot \gamma_j$ in total. Since the total utility of the sellers with cost-per-utility at most γ_{i-1} is $\sum_{0 \leq l \leq i-1} u_l$, and the total utility of the sellers with cost-per-utility in $(\gamma_{i-1}, \gamma_j]$ is $\sum_{i \leq l \leq j} u_l$, it follows that the additional payment the mechanism makes in this iteration is $q_{i,j} := f(\gamma_j) \cdot (\gamma_j - \gamma_{i-1}) \cdot \sum_{0 \leq l \leq i-1} u_l + f(\gamma_j) \cdot \gamma_j \cdot \sum_{i \leq l \leq j} u_l$, which is at most (equal when $f(\gamma_j) = 1$)

$$q_{i,j}^{\max} := (\gamma_j - \gamma_{i-1}) \cdot \sum_{0 \leq l \leq i-1} u_l + \gamma_j \cdot \sum_{i \leq l \leq j} u_l, \quad (2)$$

and hence, the if condition in GREEDY ensures the budget feasibility. Moreover, observe that the additional utility GREEDY gains in this iteration is $v_{i,j} := f(\gamma_j) \cdot \sum_{i \leq l \leq j} u_l$. Therefore, the ratio between the additional utility we gain and the additional price we pay, when we increase $f(\gamma)$ uniformly for all $\gamma \in (\gamma_{i-1}, \gamma_j]$, is

$$e_{i,j} := \frac{v_{i,j}}{q_{i,j}}. \quad (3)$$

In each iteration, GREEDY selects the best j that maximizes $e_{i,j}$. Now we show the instance optimality using a greedy exchange argument. Consider any other monotone allocation rule g and suppose γ_{i+1} is the smallest among all the sellers' γ 's such that $g(\gamma_{i+1}) \neq f(\gamma_{i+1})$. (Such γ_{i+1} cannot be 0 because otherwise, letting $g(0) = 1$ cannot increase the payment or decrease the utility for g .) Now we show how to make g more consistent with f without decreasing its achieved utility.

Case (i): $g(\gamma_{i+1}) > f(\gamma_{i+1})$. Then $f(\gamma_{i+1}) < 1$ since $g(\gamma_{i+1}) \leq 1$. We now argue that $f(\gamma_i) = 1$. By our choice of γ_{i+1} , $f(\gamma_i) = g(\gamma_i) \geq g(\gamma_{i+1}) > f(\gamma_{i+1})$, and given that $f(\gamma_i) > f(\gamma_{i+1})$, GREEDY prefers the items before $i+1$. Therefore it will not start buying the $(i+1)$ -th item until those items are exhausted. Moreover, $f(\gamma_{i+1})$ must be strictly positive, because otherwise, f does not spend as much budget as g . Hence indeed $f(\gamma_i) = 1$.

Hence GREEDY must have chosen the best $e_{i+1,k}$ for some $k > i+1$, where the inequality is due to the budget feasibility of g . (Indeed, if $k = i+1$, then there is enough budget for GREEDY to increase $f(\gamma_{i+1})$ to $g(\gamma_{i+1})$, since g is budget-feasible.) Let $\gamma \geq \gamma_{i+1}$ denote the largest cost-per-utility such that $g(\gamma) > 0$. We can assume $\gamma = \gamma_l$ for some $l \geq i+1$ because otherwise we can truncate the extra part of g while preserving its utility. Note that GREEDY guarantees that $e_{i+1,k} \geq e_{i+1,l'}$ for all $i+1 \leq l' \leq l$. Hence, if we decrease g over $(\gamma_i, \gamma_l]$ to 0 and use the saved budget to uniformly increase g over $(\gamma_i, \gamma_k]$, the resulting utility cannot decrease.

Case (ii): $g(\gamma_{i+1}) < f(\gamma_{i+1})$. Suppose that GREEDY chose the best e_{i_1, i_2} for some $i_1 \leq i+1 \leq i_2$. Therefore, f is a constant on $(\gamma_{i_1-1}, \gamma_{i_2}]$, and by monotonicity of g and our assumption that γ_{i+1} is the first place where two allocation functions differ, it follows that f is strictly larger than g on $(\gamma_{i_1-1}, \gamma_{i_2}]$. Since GREEDY guarantees that $e_{i_1, i_2} \geq e_{i_1, j}$ for any $j \geq i_1$, we can decrease g on $(\gamma_{i_1-1}, +\infty)$ simultaneously and use the saved budget to uniformly increase g on $(\gamma_{i_1-1}, \gamma_{i_2}]$, which can not decrease the achieved utility. We keep doing this unless g reaches 1 on $(\gamma_{i_1-1}, \gamma_{i'}]$ for some $i' \leq i_2$. Then, either f is 1 on $(\gamma_{i_1-1}, \gamma_{i_2}]$, and hence, g becomes more consistent with f , or f is < 1 on this interval, in which case, we can decrease g on $(\gamma_{i_1-1}, +\infty)$ to 0 and use the saved budget to uniformly increase g on $(\gamma_{i_1-1}, \gamma_{i_2}]$. ◀

Since [2] showed for a single budget, there is a uniform mechanism (also with knowledge of all c_i 's) that has worst-case competitive ratio $1 - 1/e$ (and there is a matching hardness result), Theorem 6 implies that GREEDY has worst-case competitive ratio $1 - 1/e$.

3.2 Greedy allocation rule: a lottery of two posted prices

Before we present the final randomized mechanism, we observe some nice properties of GREEDY which will help us analyze GREEDY in a more intuitive way. The key observation, which follows directly from the design of GREEDY, is that the allocation rule of GREEDY has a simple form that can be fully characterized by three parameters¹²:

► **Observation 7.** *The allocation rule f of GREEDY is a (≤ 2) -step function, i.e., there exists some (t, p_1, p_2) where $t \in [0, 1)$ and¹³ $0^- \leq p_1 \leq p_2$,*

$$f\left(\frac{c}{u}\right) = \begin{cases} 1 & \frac{c}{u} \leq p_1 \\ t & p_1 < \frac{c}{u} \leq p_2, \\ 0 & \frac{c}{u} > p_2 \end{cases}$$

and we say that f is characterized by (t, p_1, p_2) .

Observation 7 allows us to think of the allocation rule of GREEDY as a lottery (distribution) of at most two posted prices:

► **Observation 8.** *Given an allocation rule f of GREEDY that is characterized by (t, p_1, p_2) where $t \in [0, 1)$ and $0^- \leq p_1 \leq p_2$, consider the following randomized posted-price mechanism:*

For each seller, the buyer independently tosses a (biased) random coin and offers this seller either (i) a payment-per-utility p_2 with probability t ; or (ii) a payment-per-utility p_1 with probability $1 - t$. Then, each seller can accept the offer (give the item to the buyer and receive the payment) or leave.

The above randomized posted-price mechanism, which is a lottery of two posted prices, has the same allocation function as f in expectation.

Proof. Let \bar{f} denote the expected allocation function of the above randomized posted-price mechanism. Now we show that \bar{f} is equivalent to f . First, a seller with a cost-per-utility $\frac{c}{u} \leq p_1$ will accept either offer p_1 or p_2 (because both payments-per-utility are no less than his cost-per-utility), and hence $\bar{f}(\frac{c}{u}) = 1$. On the other hand, a seller with a cost-per-utility $\frac{c}{u} \in (p_1, p_2]$ will only accept offer p_2 (because only p_2 is no less than his cost-per-utility), and hence $\bar{f}(\frac{c}{u}) = \Pr[p_2 \text{ is offered}] = t$. Finally, a seller with a cost-per-utility $\frac{c}{u} > p_2$ will not accept either of the offer (because both payments-per-utility are below his cost-per-utility), and hence $\bar{f}(\frac{c}{u}) = 0$. ◀

The same allocation function obviously achieves the same total utility in expectation, and moreover, by Myerson's lemma, it also makes the same total payment in expectation. Therefore, Observation 8 provides a more intuitive way to calculate the total utility and the total payment for GREEDY (using the posted prices rather than explicitly using the allocation rule of GREEDY and Myerson's payment rule), which we formalize in the following observation:

¹²One might notice that this characterization actually captures a strictly more general class of allocation rules than just the possible outputs of GREEDY. This is for the convenience of analysis later, and we will call any allocation rule that can be characterized in this way a "greedy allocation rule".

¹³ 0^- denotes a strictly negative number that is arbitrarily close to 0.

► **Observation 9.** *Given an allocation rule f of GREEDY that is characterized by (t, p_1, p_2) where $t \in [0, 1)$ and $0^- \leq p_1 \leq p_2$, for any subset of sellers $S \subseteq [n]$, let $U_f(S)$ and $B_f(S)$ denote the total utility and the total payment respectively when we apply f to the sellers in S , and let $U_p(S)$ and $B_p(S)$ denote the total utility and total payment respectively when we offer a posted price (payment-per-utility) $p \in \mathbb{R}_{\geq 0}$ to the sellers in S (and each seller can accept the offer or leave). Then, we have that*

$$\begin{aligned} U_f(S) &= (1-t)U_{p_1}(S) + tU_{p_2}(S), \\ B_f(S) &= (1-t)B_{p_1}(S) + tB_{p_2}(S), \end{aligned} \tag{4}$$

and moreover, for all $p \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} B_p(S) &= pU_p(S), \\ U_p(S) &= \sum_{i \in S \text{ s.t. } c_i/u_i \leq p} u_i. \end{aligned} \tag{5}$$

Proof. Eq. (4) follows immediately by Observation 8 and the discussion above, and Eq. (5) follows by definition of the posted-price mechanism. ◀

We remark that Observation 9 makes it easier to prove multiplicative concentration inequalities for $U_f(S)$ and $B_f(S)$ when S is a random subset of $[n]$ (specifically, by Eq. (4) and Eq. (5), both $U_f(S)$ and $B_f(S)$ can be written as non-negative linear combination of $U_{p_1}(S)$ and $U_{p_2}(S)$, and thus, it suffices to prove multiplicative concentration inequalities for $U_{p_1}(S)$ and $U_{p_2}(S)$).

3.3 Approximating greedy via random sampling

We have shown that GREEDY is instance-optimal compared to all the uniform mechanisms in Theorem 6, but it requires the knowledge of private costs. In this subsection, we present a proxy of GREEDY called RANDOM-SAMPLING-GREEDY, which uses random sampling¹⁴ to approximate the distribution of private costs, and in Theorem 11, we will show that this randomized mechanism strictly satisfies the budget constraint and with high probability achieves almost the same utility as GREEDY.

Before that, we introduce two subroutines that will be applied in RANDOM-SAMPLING-GREEDY. The first subroutine handles an edge case of a small subset T of sellers with exceptionally high utility. The second subroutine adjusts the price p_1 to a new price \hat{p}_1 , to handle an edge case where $U_{p_1}([n] \setminus T)$ is very small. Intuitively, after those adjustments, the utility of any individual seller, who is not in T and has a cost-per-utility at most \hat{p}_1 , is tiny relative to $U_{\hat{p}_1}([n] \setminus T)$. Therefore when S is a uniformly random subset of $[n] \setminus T$, $U_{\hat{p}_1}(S)$ is concentrated around its expectation w.h.p. (We will show this formally in the analysis of RANDOM-SAMPLING-GREEDY in the full version.)

Pre-purchasing the most valuable items. The first subroutine, which will be the first step of RANDOM-SAMPLING-GREEDY, is pre-purchasing the items of highest utilities. By the small-bidder assumption, each seller's cost is $o(B)$. Thus, for an arbitrarily large integer

¹⁴ An alternative method often used in the literature is for every seller, computing the prices for the market excluding this seller and then offering the computed prices to this seller. We remark there exist instances for which this method violates budget-feasibility when applied to our idealized mechanism. Besides, random partitioning is much more computationally efficient than this alternative method.

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constant C , we can pre-purchase the top C items of highest utilities by making a payment $\epsilon_1 B/C$ to each of the C sellers, and the remaining budget is $(1 - \epsilon_1)B$. Henceforth, we let T denote the set of the top- C items and let $U(T)$ denote their total utility.

Truncating a greedy allocation rule. We let $\eta > 0$ be a parameter which we use for this truncation step (later we will choose η to be an arbitrarily small constant and then choose C such that ηC is arbitrarily large). Suppose we are given an allocation rule f of GREEDY that is characterized by (t, p_1, p_2) where $t \in [0, 1)$ and $0^- \leq p_1 \leq p_2$. We let \hat{f} denote the truncated allocation rule of f . Specifically, \hat{f} is characterized by (t, \hat{p}_1, p_2) , and \hat{p}_1 is defined as follows

$$\hat{p}_1 := \begin{cases} 0^- & U_{p_1}([n] \setminus T) < \frac{U(T)}{\eta C} \\ p_1 & U_{p_1}([n] \setminus T) \geq \frac{U(T)}{\eta C} \end{cases}.$$

That is, we get \hat{f} by decreasing the value of f over $[0, p_1]$ to t if $U_{p_1}([n] \setminus T)$ is less than $\frac{U(T)}{\eta C}$ (recall that $U_{p_1}([n] \setminus T)$ is the total utility of the sellers in $[n] \setminus T$ whose cost-per-utility is at most p_1 by Observation 9).

We observe that applying truncation will not significantly decrease (relative to $U(T)$) the utility attained by the allocation rule:

► **Observation 10.** For all $S \subseteq [n]$, $U_f(S) - U_{\hat{f}}(S) \leq \frac{U(T)}{\eta C}$.

Proof. By our design of the truncation step and Observation 9, $U_f(S) - U_{\hat{f}}(S) = (1 - t)(U_{p_1}(S) - U_{\hat{p}_1}(S)) \leq U_{p_1}(S) - U_{\hat{p}_1}(S)$. Moreover, by definition of \hat{p}_1 , $U_{p_1}(S) - U_{\hat{p}_1}(S) = 0$ if $U_{p_1}(S) \geq \frac{U(T)}{\eta C}$, and obviously $U_{p_1}(S) - U_{\hat{p}_1}(S) \leq U_{p_1}(S) < \frac{U(T)}{\eta C}$ if otherwise. ◀

The Random-Sampling-Greedy mechanism

Now we present RANDOM-SAMPLING-GREEDY (Mechanism 2) and its theoretical guarantee (Theorem 11). The analysis of RANDOM-SAMPLING-GREEDY (the proof of Theorem 11), which is rather technical but still interesting, is deferred to Section C in the full version for the interest of space.

■ Mechanism 2 RANDOM-SAMPLING-GREEDY.

Input : (c_i, u_i) for $i \in [n]$, B , and parameters $\epsilon_1, \delta_1, \eta, C$.

- 1 Buy the items from the top C sellers T of highest utilities and pay each of them $\epsilon_1 B/C$;
- 2 Partition the other sellers $[n] \setminus T$ into X and Y uniformly at random;
- 3 (Virtually, aka without making actual allocations or payments) run GREEDY mechanism on X and Y with budget $\frac{(1-\delta_1)B}{2}$, separately, and get the resulting allocation rules f_X, f_Y ;
- 4 Truncate f_X, f_Y using parameter η and get \hat{f}_X, \hat{f}_Y and their associated payment rules \hat{Q}_X, \hat{Q}_Y ;
- 5 In an arbitrary order, sequentially apply \hat{f}_X, \hat{Q}_X to the sellers in Y until we spend $\frac{(1-\epsilon_1)B}{2}$ on Y , and then sequentially apply \hat{f}_Y, \hat{Q}_Y to the sellers in X until we spend $\frac{(1-\epsilon_1)B}{2}$ on X ;

► **Theorem 11.** *For divisible items, under the small-bidder assumption, for every $\epsilon > 0$, there exists sufficiently small $\delta_1, \eta > 0$ and sufficiently large C such that, RANDOM-SAMPLING-GREEDY is truthful-in-expectation and strictly budget-feasible and with high probability achieves utility at least $(1 - \epsilon)$ -fraction of the utility attained by GREEDY.*

3.4 Numerical simulation on synthetic instances

We compare the performance of GREEDY, RANDOM-SAMPLING-GREEDY with [2]’s mechanism AGN, and the best cutoff rule with proper tie breaking CUTOFF (i.e., [10]’s open clock auction) on synthetic datasets, where the market has 1000 sellers, each of whom has unit utility and cost sampled from various distributions (negative cost is rounded to 0), and the buyer’s budget is 20000. When we run RANDOM-SAMPLING-GREEDY for this instance, we simply set $\epsilon_1, \delta_1, \eta, C$ to 0 (these constants were only used to prove asymptotically high probability bounds). The results are summarized in Table 1 (for each cost distribution, we take the average and the standard deviation of the results of 100 runs). We observe that GREEDY always dominates other mechanisms since it is instance-optimal uniform mechanism, and RANDOM-SAMPLING-GREEDY (RS-GREEDY) is usually almost as good as GREEDY with only a small difference due to random sampling (as illustrated in Figure 1, this difference goes to 0 when the size of market increases, which matches Theorem 11). Moreover, on all the synthetic instances, GREEDY and RS-GREEDY beat the worst-case $1 - 1/e$ competitive ratio by a large margin, while AGN only obtains close-to- $(1 - 1/e)$ competitive ratio. On the other hand, for unimodal distributions, CUTOFF often performs well, but for multi-modal distributions, it is significantly outperformed by GREEDY and RS-GREEDY. This matches our intuition:

► **Example 12.** Consider the instance where all n sellers have unit utilities, and $n/2$ sellers have costs 0 and $n/2$ sellers have costs 1, and the buyer has budget n . The best cutoff rule (i.e., setting a best uniform price-per-utility for all sellers) only gets the $n/2$ sellers with zero cost and hence achieves competitive ratio $1/2$, while GREEDY can choose tuple $(t = 1/2, p_1 = 0, p_2 = 1)$ and achieve competitive ratio $3/4$.

■ **Table 1** Competitive ratios achieved by different mechanisms on synthetic datasets.

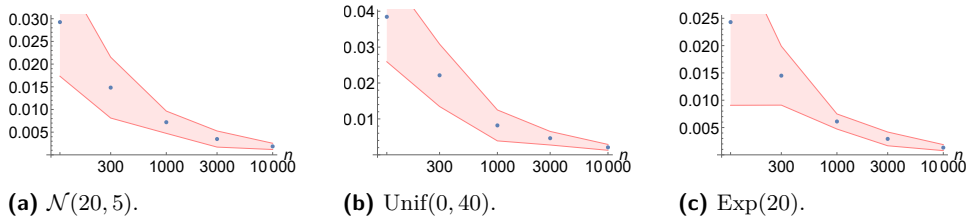
CUTOFF	AGN	GREEDY	RS-GREEDY
0.816 ± 0.004	0.632 ± 0.001	0.818 ± 0.004	0.81 ± 0.006
0.709 ± 0.005	0.633 ± 0.003	0.711 ± 0.004	0.702 ± 0.006
0.74 ± 0.008	0.663 ± 0.006	0.743 ± 0.008	0.736 ± 0.009
0.69 ± 0.003	0.633 ± 0.002	0.726 ± 0.003	0.718 ± 0.005
0.68 ± 0.009	0.634 ± 0.003	0.712 ± 0.006	0.706 ± 0.007

Each row contains the results for a distinct cost distribution. The distributions from top to bottom are $\mathcal{N}(20, 5)$, $\text{Unif}(0, 40)$, $\text{Exp}(20)$, $\frac{1}{2}\mathcal{N}(10, 3) + \frac{1}{2}\mathcal{N}(30, 3)$, $\frac{1}{3}\mathcal{N}(5, 3) + \frac{1}{3}\mathcal{N}(20, 3) + \frac{1}{3}\mathcal{N}(35, 3)$.

4 Budget-smoothed analysis

In this section, we analyze our mechanism in the budget-smoothed analysis framework. Our main results of budget-smoothed analysis are:

- Our mechanism obtains near-optimal budget-smoothed competitive ratio for any budget distribution when compared to all (possibly non-uniform) mechanisms (Theorem 13).



■ **Figure 1** This figure shows that the difference between competitive ratios achieved by GREEDY and RS-GREEDY (y -axis) diminishes when the market size n (x -axis) increases. In each subplot, the market has n sellers, each of whom has unit utility and cost sampled from a distinct distribution (negative cost is rounded to 0), and the buyer’s budget is $20n$. Each datapoint in the plot is the average of 20 runs, and the shaded area captures one standard deviation.

- Our mechanism obtains strictly better than $1 - 1/e$ budget-smoothed competitive ratio on any non-trivial budget distribution (Theorem 4.3 in the full version). In Section E of the full version, we also formulate a (non-convex) mathematical program that computes the budget-smoothed competitive ratio for any given budget distribution. We solve this program for various distributions and observe non-negligible improvement over $1 - 1/e$.
- Given any bounded range of budgets, there is a single market on which, simultaneously for every budget in the range, [2]’s mechanism obtains only $1 - 1/e$ competitive ratio (Theorem D.2 in the full version).
- Our mechanism (and hence all possibly non-uniform mechanisms by Theorem 13) has budget-smoothed competitive ratio bounded away from 1 (specifically, at most 0.854) for any budget distribution (Theorem 4.4 in the full version).

4.1 Greedy is optimal for any budget distribution

In this subsection, we analyze the budget-smoothed competitive ratio of our complete-information “mechanism” GREEDY for any budget distribution. We show that GREEDY is optimal for any budget distribution –even among non-uniform mechanisms– and the ratio goes beyond $1 - 1/e$ when there are multiple budgets in the support of the budget distribution. These results extend to RANDOM-SAMPLING-GREEDY due to Theorem 11. We will characterize the worst Bayesian market for truthful-in-expectation uniform mechanisms, where n sellers have the same utility, and their costs are drawn from a continuous distribution. This characterization can be viewed as a generalization of the worst-case instance¹⁵ in [2]. Then, we argue that this Bayesian market is as hard for truthful-in-expectation non-uniform mechanisms. But before that, we explain why the continuous cost distribution and equal utilities are not restrictions, i.e., for an arbitrary market, we can construct a Bayesian market with continuous cost distribution and equal utilities that exhibits the same hardness for truthful-in-expectation uniform mechanisms as the original market.

4.1.1 From arbitrary market to Bayesian market

Given an market I with n sellers of utilities u_i ’s and costs c_i ’s, we first construct a Bayesian market I_1 with a discrete distribution. The market I_1 has $M \cdot \sum_i u_i$ sellers¹⁶, where M is a sufficiently large number. Let D_1 be a distribution over $\{\frac{c_i}{M u_i} \mid i \in [n]\}$ such that the

¹⁵ J.Z. wants to thank Nima Anari for an inspiring discussion of the worst-case instance in [2].

¹⁶ Without loss of generality, we assume that $M \cdot \sum_i u_i$ is an integer.

probability of $\frac{c_i}{Mu_i}$ is $u_i/(\sum_j u_j)$. Each seller has utility $\frac{1}{M}$, and his cost is drawn from D_1 . We need to verify two things: (i) the non-IC optimal utilities of the knapsacks for I and I_1 are almost equal, and (ii) the best achievable utilities by uniform mechanisms for I and I_1 are also almost the same. For (ii), it suffices to consider GREEDY because of Theorem 6. The reason both of these hold is that the optimal utility and the best achievable utility only depend on the cost-to-utility ratio $\frac{c}{u}$'s and the total utility of the sellers with the same $\frac{c}{u}$, and if M is sufficiently large, with high probability these quantities do not change much in I_1 compared to I .

Next, we construct a Bayesian market I_2 with the same setup as I_1 but a continuous distribution for sellers' costs. To this end, consider the CDF of D_1 , which is some step function $F(c)$, we can approximate each step in F arbitrarily well by a logistic function and glue them together such that the CDF is differentiable. For the same reason as above, the best achievable competitive ratio of a uniform mechanism for I_2 is approximately equal to that for I_1 .

4.1.2 Characterizing the worst Bayesian market

► **Theorem 13.** *For any distribution \mathcal{D} over any m budgets $B_1 < B_2 \dots < B_m$, let $F(c)$ be the CDF of the distribution of costs of the worst¹⁷ Bayesian market for \mathcal{D} . Then, the following hold:*

- Consider the plot of $cF(c)$ with respect to $F(c)$. $cF(c)$ is a piecewise-linear function of $F(c)$ with at most m non-zero linear pieces, and has non-decreasing slope.
- For each budget, the utility-maximizing allocation rule for this market is a uniform cutoff rule, namely, $f(c/u) = \mathbb{1}(c/u \leq c^*/u^*)$ for some c^*/u^* .
- GREEDY is worst-case optimal for budget distribution \mathcal{D} (see Definition 5) compared to all the truthful-in-expectation (not necessarily uniform) mechanisms. (Note that the optimality also holds for RANDOM-SAMPLING-GREEDY due to Theorem 11.)

Proof. From the previous discussion, it suffices to consider a Bayesian market, where n sellers have the same utility, and their costs are drawn from a continuous distribution, the CDF of which is some continuous F . The following min-max program computes the cost distribution that gives the worst expected competitive ratio for budget distribution \mathcal{D} against uniform allocation rules (later we will show that non-uniform rules are not any better for the worst distribution),

$$\begin{aligned} & \inf_F \sup_{f_1, \dots, f_m} \sum_{i=1}^m \Pr[B_i] \cdot \frac{F(0) + \int_{0^+}^{\infty} f_i(c) dF(c)}{\tau_i} \\ \text{s.t. } & \forall i \in [m], \quad \underbrace{\int_0^{\tau_i} c dF(c)}_{\text{total payment of non-IC optimum}} = \underbrace{Q_{f_i}(0) \cdot F(0) + \int_{0^+}^{\infty} Q_{f_i}(c) dF(c)}_{\text{Myerson's payment for } f_i} = B_i, \\ & \forall i \in [m], \forall c \geq 0, f_i(c) \in [0, 1], \\ & \text{and } F \text{ is a continuous CDF,} \end{aligned}$$

where $\Pr[B_i]$ is the probability of B_i according to \mathcal{D} , f_i is the allocation function for i -th budget, and τ_i 's denote the expected non-IC optimal utility for the corresponding budgets, and we hard-code in the program those τ_i 's which result in the worst expected ratio. Although we did not require f_i 's to be monotone here, later we will show that if we add the monotonicity

¹⁷By "worst for \mathcal{D} ", we mean it minimizes the best possible expected competitive ratio that is achievable by any mechanism given budget distribution \mathcal{D} .

constraint, the worst ratio does not change. Also, note that we only require budget feasibility in expectation for the allocation function, and hence, the optimality of GREEDY will hold even among ex ante budget-feasible mechanisms. We should have restricted the non-IC optimal solution to be ex post budget-feasible, but this is fine, because as market size n grows, with high probability, the budget spent in the optimal solution is concentrated around its expectation, and cutting the budget slightly does not decrease the optimal utility much (see Lemma B.5 in the full version).

Now we derive that

$$\begin{aligned}
 \int_{0^+}^{\infty} Q_f(c) dF(c) &= \int_{0^+}^{\infty} \left(f(c) \cdot c + \int_c^{\infty} f(x) dx \right) dF(c) \quad (\text{Myerson's payment identity}) \\
 &= \int_{0^+}^{\infty} f(c) \cdot c dF(c) + \left(F(c) \cdot \int_c^{\infty} f(x) dx \right) \Big|_{0^+}^{\infty} \\
 &\quad - \int_{0^+}^{\infty} F(c) d \left(\int_c^{\infty} f(x) dx \right) \quad (\text{Integration by parts}) \\
 &= \int_{0^+}^{\infty} f(c) \cdot c dF(c) - F(0) \cdot \int_{0^+}^{\infty} f(x) dx \\
 &\quad + \int_{0^+}^{\infty} f(c) \underbrace{F(c) dc}_{= \frac{F(c)}{F'(c)} dF(c)} \\
 &= \int_{0^+}^{\infty} f(c) \cdot \left(c + \frac{F(c)}{F'(c)} \right) dF(c) - F(0) \cdot Q_f(0).
 \end{aligned}$$

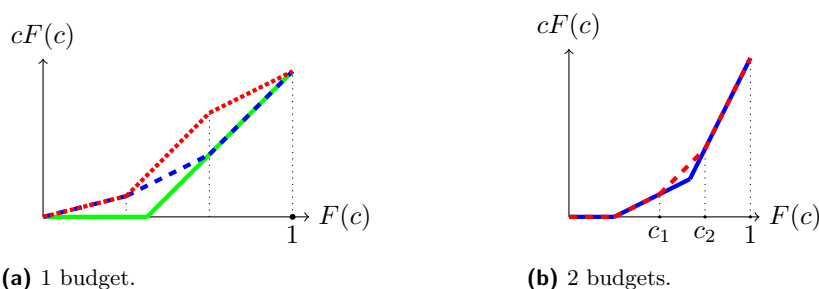
Therefore, the maximization problem in the min-max program can be seen as a fractional knapsack (where $dF(c)$ is the value of an item c , and $c + \frac{F(c)}{F'(c)}$ is its weight per value), and the best allocation function should choose the c 's with small $c + \frac{F(c)}{F'(c)} = \frac{d(cF(c))}{dF(c)}$. We now prove several structural properties about the plot (curve) of $cF(c)$ with respect to $F(c)$.

Any feasible curve should have non-decreasing slope from the origin. Notice that $\frac{d(cF(c))}{dF(c)}$ is the slope of this curve at $F(c)$, and c is the slope of the line from the origin to the point of the curve at $F(c)$. Any feasible curve should have non-decreasing slope from the origin since $F(c)$ is non-decreasing in c and vice versa.

Now we show the structural result about the worst-case F for one budget, and later we will extend to many budgets.

The curve is piecewise-linear w.l.o.g. Given a feasible curve for some F , we discretize the smooth curve into a piecewise-linear curve. The discretization is sufficiently fine-grained such that the slope of the curve and the slope to the origin at each point are close to those of the original curve, and there are only finitely many non-differentiable points. Hence the min-max program is still valid, and its result does not change much. It suffices to consider such piecewise-linear curves.

Worst-case curve has non-decreasing slope. Our first observation is that the worst-case curve should have non-decreasing slope. If it does not, we can re-order the linear pieces according to their slopes, and the re-ordering preserves the probability mass of c 's with any fixed slope. Hence the best allocation rule makes the same utility as before. Meanwhile, the slope to the origin at each c can only become smaller than that before re-ordering. The budget spent by the non-IC optimal solution to get the same utility as before is the integration of the slope from origin from 0 to some τ , which can only decrease. Therefore,



■ **Figure 2** **On the left:** Starting from an arbitrary piecewise linear curve (red dotted), we can re-order its pieces to get blue dashed curve and then again into the green solid curve. These steps only make the market worse. The worst F (green solid) for one budget is “ReLU shaped”. **On the right:** The worst F for 2 (m respectively) budgets should have at most 2 (m respectively) non-zero linear pieces. Consider the optimal allocation functions for two budgets, which are cutoff rules, if neither cutoff lies in (c_1, c_2) , then changing the red dashed curve into the blue solid curve makes the market worse.

the re-ordering can only decrease the competitive ratio of the best allocation rule. This is illustrated in Figure 2 (a), when we re-order the red dotted curve and get the blue dashed curve.

A claim following from the non-decreasing slope is that the best allocation rule should be a cutoff rule.

▷ **Claim 14.** Utility-maximizing allocation function for a convex $F(c)$ -to- $cF(c)$ curve is a cutoff rule.

Proof of Claim 14. Indeed, since the best allocation rule comes from solving the fractional knapsack problem we mentioned above and the value per weight (equal to slope) is non-decreasing, the solution should be $f(c) = 1$ for all $c \leq c_1$, and $f(c) = t < 1$ for all $c_1 < c \leq c_2$, for some $c_1 < c_2$. This rule can be seen as a probabilistic combination of two cutoff rules, i.e., with some probability α offer cutoff price c_1 and offer c_2 otherwise, and the expected utility and payment are $\alpha F(c_1) + (1 - \alpha)F(c_2)$ and $\alpha c_1 F(c_1) + (1 - \alpha)c_2 F(c_2)$ respectively. Consider the c_3 such that $F(c_3) = \alpha F(c_1) + (1 - \alpha)F(c_2)$, because the curve is convex, $c_3 F(c_3) \leq \alpha c_1 F(c_1) + (1 - \alpha)c_2 F(c_2)$. Hence the cutoff rule at c_3 makes the same utility but spends no more than the probabilistic rule, and the claim follows. ◁

Worst-case curve for one budget is a ReLU function. Next, we argue that the worst-case (convex) curve for one budget should be a ReLU function, i.e., it is zero at first and then becomes a linear function. Suppose otherwise, we let c^* be the cutoff price of the best allocation rule. We can draw a line between $(F(c^*), c^*F(c^*))$ and the $F(c)$ -axis with slope equal to the slope of the worst curve at $(F(c^*), c^*F(c^*))$. Consider the ReLU curve whose non-zero linear part is this line. Notice that $(F(c^*), c^*F(c^*))$ does not change and is still optimal, and hence the optimal utility achievable by any allocation rule does not change. Meanwhile, the slope from origin at each c can only become smaller, and therefore, the budget spent by the non-IC optimal solution to get the same utility as before can only decrease. This is illustrated in Figure 2 (a), where we change the blue dashed curve into the green solid curve. Furthermore, the part of the original curve after $(F(c^*), c^*F(c^*))$ has slope larger than the slope at this point, and decreasing this part to the line with the slope at this point only decreases the spent budget for the non-IC optimal solution and does not change the result of the best allocation rule. The final curve is a ReLU, and the best achievable competitive ratio only gets worse.

Characterizing the worst-case curve for many budgets. Now we show that the worst-case curve for m budgets has at most m non-zero linear pieces by generalizing the above argument. Consider the m optimal cutoff rules for m budgets respectively, if there are more than m non-zero linear pieces, then there is one piece from some $(F(c_1), c_1F(c_1))$ to some other $(F(c_2), c_2F(c_2))$ such that the open interval (c_1, c_2) does not contain any optimal cutoff. If this is not the last piece of the curve, we can extend the piece before $(F(c_1), c_1F(c_1))$ upwards and the piece after $(F(c_2), c_2F(c_2))$ downwards until they intersect, which decreases the number of linear pieces. Similar to the one budget case, this step does not change the payments of optimal cutoff rules and can only decrease the payments of non-IC optimal solutions, and therefore, this only makes the market worse. This is illustrated in Figure 2 (b), where we change the red curve to the blue. If it is the last piece of the curve, namely $F(c_2) = 1$, then we can simply extend the piece before $(F(c_1), c_1F(c_1))$ upwards until it hits 1 horizontally. The argument in this case is analogous.

Adding monotonicity constraints to the min-max program. Next, we explain why restricting f_i 's to be monotone does not change the optimal value to the min-max program. As we argued above, the best allocation rules for the worst distribution F^* are cutoff rules f_i^* , which are monotone. Since $(F^*, \{f_i^* \mid i \in [m]\})$ is an equilibrium of the min-max program without monotonicity constraints, $(F^*, \{f_i^* \mid i \in [m]\})$ is obviously also an equilibrium of the min-max program with monotonicity constraints. Notice that the min-max program with monotonicity constraints satisfies the conditions of Sion's minimax theorem (see Lemma B.6 in the full version). Hence the optimal value to this program is equal to the objective value at this equilibrium.

Non-uniform allocation rule is not better. We show that for a Bayesian market that matches our characterization, non-uniform rules do not outperform uniform rules. Consider a general (possibly non-uniform) mechanism where each seller i has its own allocation rule $A_{c_{-i}}^{(i)}$. Now let $P_{c_{-i}}^{(i)}(c) = 1 - A_{c_{-i}}^{(i)}(c)$. An implementation of $A_{c_{-i}}^{(i)}$ is sampling cutoff prices from the distribution whose CDF is $P_{c_{-i}}^{(i)}$. To see this, the probability that the item of price c_i is bought is $1 - P_{c_{-i}}^{(i)}(c_i) = A_{c_{-i}}^{(i)}(c_i)$, and the expected payment is

$$\begin{aligned} \int_{c_i}^{c_{\max}} c dP_{c_{-i}}^{(i)}(c) &= cP_{c_{-i}}^{(i)}(c)|_{c_i}^{c_{\max}} - \int_{c_i}^{c_{\max}} P_{c_{-i}}^{(i)}(c)dc \\ &= c_{\max} - c_i P_{c_{-i}}^{(i)}(c_i) - \int_{c_i}^{c_{\max}} P_{c_{-i}}^{(i)}(c)dc \\ &= c_{\max} - c_i(1 - A_{c_{-i}}^{(i)}(c_i)) - \int_{c_i}^{c_{\max}} (1 - A_{c_{-i}}^{(i)}(c))dc \\ &= c_i A_{c_{-i}}^{(i)}(c_i) + \int_{c_i}^{c_{\max}} A_{c_{-i}}^{(i)}(c)dc, \end{aligned}$$

which is exactly the Myerson payment corresponding to $A_{c_{-i}}^{(i)}$. Since the allocation rule $A_{c_{-i}}^{(i)}$ can be seen as a probabilistic combination of cutoff rules, as we have shown before, by convexity of the $F(c)$ -to- $(cF(c))$ curve (see Claim 14), there is a cutoff rule $p_{c_{-i}}^{(i)}$ that achieves the same utility but with less or equal payment compared to $A_{c_{-i}}^{(i)}$. Moreover, $p_{c_{-i}}^{(i)}$'s together can be seen as a probabilistic cutoff rule that depends on random variable c_{-i} , and hence again by the same argument, there is a cutoff rule p_i that does as good as the random $p_{c_{-i}}^{(i)}$ for seller i . Finally, for the same reason, the uniform cutoff rule p such that $F(p) = \frac{1}{n} \sum_{i=1}^n F(p_i)$ is as good as the non-uniform rule p_i 's.

Since GREEDY uses the best monotone uniform allocation rule for any instance by Theorem 6, the min-max program with additional monotonicity constraint solves for the worst-case expected competitive ratio for GREEDY. Thus, the observation in the above paragraph implies that GREEDY is worst-case optimal compared to all the truthful-in-expectation (even non-uniform) mechanisms. ◀

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