Tower-Complete Problems in Contraction-Free Substructural Logics

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Abstract

We investigate the non-elementary computational complexity of a family of substructural logics without contraction. With the aid of the technique pioneered by Lazić and Schmitz (2015), we show that the deducibility problem for full Lambek calculus with exchange and weakening ($\mathbf{FL_{ew}}$) is not in ELEMENTARY (i.e., the class of decision problems that can be decided in time bounded by an elementary recursive function), but is in PR (i.e., the class of decision problems that can be decided in time bounded by a primitive recursive function). More precisely, we show that this problem is complete for TOWER, which is a non-elementary complexity class forming a part of the fast-growing complexity hierarchy introduced by Schmitz (2016). The same complexity result holds even for deducibility in BCK-logic, i.e., the implicational fragment of $\mathbf{FL_{ew}}$. We furthermore show the TOWER-completeness of the provability problem for elementary affine logic, which was proved to be decidable by Dal Lago and Martini (2004).

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1 Introduction

The term "substructural logic" [14, 31, 34] is an umbrella term for a family of logics that limit the use of some of the structural rules. Substructural logics encompass a wide range of non-classical logics (e.g., intuitionistic, classical, relevance, paraconsistent and multi-valued logics), and thus are discussed in several distinct areas close to mathematical logic such as philosophy, linguistics and computer science. In any of those research fields, one of the major topics is to settle the computational complexity of the *provability problem* for a logic, i.e., the problem of whether a given formula is provable in the logic. There are many seminal papers concerning this subject; see, e.g., [3, 21, 26, 28, 33, 36, 37, 41].

It is no surprise that a more general problem can be considered for a given logic. The *deducibility problem* for the logic asks for a given finite set Φ of formulas and a given formula A whether A is provable in the logic augmented with Φ as a set of non-logical axioms. In the setting of classical and intuitionistic logic, the notion of deducibility is reduced to provability via the deduction theorem. As a result, the deducibility problem for intuitionistic (resp. classical) propositional logic is complete for PSPACE (resp. coNP) as with the provability problem. On the other hand, since most of substructural logics do not admit the deduction theorem, there is no guarantee that these two problems are inter-reducible to each other. For this reason, it is important to distinguish them in the framework of substructural logic. In fact, some substructural logics have a critical "gap" between the complexity of provability and the complexity of deducibility. The so-called *Lambek calculus* is a striking example of such logics. Buszkowski [6] showed that its deducibility is undecidable, but later on Pentus [33] proved that its provability is NP-complete.

Main contribution

This paper aims at clarifying the non-elementary computational complexity of deducibility in contraction-free substructural logics. So far, such a topic has not been sufficiently explored while some earlier papers investigated a non-primitive recursive complexity of weakening-free substructural logics (i.e., relevance logics); see [26, 36, 41].

Full Lambek calculus with exchange and weakening ($\mathbf{FL_{ew}}$), i.e., intuitionistic logic without the rule of contraction, is one of the most basic contraction-free logics. The deducibility problem for this logic is known to be decidable, thanks to the finite embeddability property of FL_{ew}-algebras shown by Blok and van Alten [4, 5]. However, its exact complexity remained open. Hence the following natural questions arise:

- *Is there a primitive recursive algorithm – i.e., one whose runtime is bounded by a primitive recursive function – for the deducibility problem for* $\mathbf{FL_{ew}}$*?*
- *If so, is there an elementary recursive algorithm – i.e., one whose runtime is bounded by a tower of exponentials of fixed height – for the problem?*

We answer "yes" to the first question, but provide a negative answer to the second question. To be precise, we show that the problem is actually complete for the class Tower (Corollary 23). This class forms a part of the *fast-growing complexity hierarchy* introduced by Schmitz [35], and roughly speaking, is located between Elementary (i.e., the class of problems decidable in elementary time) and PR (i.e., the class of problems that can be solved in time bounded by a primitive recursive function). As a consequence, it turns out that the aforementioned "gap" also lies between provability and deducibility in $\mathbf{FL_{ew}}$; the provability problem for $\mathbf{FL_{ew}}$ is Pspace-complete, cf. [21].

We stress that the same holds even when almost all the logical connectives are removed from $\mathbf{FL_{ew}}$. We also prove that the deducibility problem for *BCK-logic* [22, 32], i.e., the implicational fragment of $\mathbf{FL_{ew}}$, is Tower-complete (Corollary 23). This is in sharp contrast to the NP-completeness of provability in BCK-logic (Corollary 3).

Proof overview

To show the Tower-membership of deducibility in $\mathbf{FL_{ew}}$, we prove that there are reductions:

(1) from deducibility in $\mathbf{FL_{ew}}$ to provability in a variant of intuitionistic affine logic (denoted by $\mathbf{ILZW'}$),

(2) from the provability problem for $\mathbf{ILZW'}$ to the lossy reachability problem for *alternating branching vector addition systems with states* (ABVASS, for short).

The first reduction is quite similar to the one used in the famed proof of the undecidability of propositional linear logic by Lincoln, Mitchell, Scedrov and Shankar [28]. The second reduction is substantially inspired by Lazić and Schmitz [26]. Due to the Tower-completeness of lossy reachability in ABVASS, shown in [26], we obtain the membership in Tower of deducibility in $\mathbf{FL_{ew}}$.

In order to show the Tower-hardness, we introduce the notion of !-*prenex implicational sequent*. It is a slight modification of !-prenex sequents which Terui introduced in his PhD thesis [38]. We prove the Tower-hardness of a restricted version of the provability problem for intuitionistic affine logic, which asks whether a given !-prenex implicational sequent is provable in intuitionistic affine logic. We obtain the desired result by showing that this problem can be reduced into deducibility in $\mathbf{FL_{ew}}$.

Provability (type inhabitation) in elementary affine logic

As a by-product resulting from our methods, we provide the precise complexity of provability (not of deducibility) in *propositional elementary affine logic* [1]. Its name comes from the fact that it characterizes elementary recursive computation in the paradigm of proofs-as-programs; see also [11, 17]. Although this logic is seemingly just an extension of BCK-logic (and $\mathbf{FL_{ew}}$) by a sort of modal storage operator, it is exploited for a variety of purposes, e.g., to characterize the class P and the exponential time hierarchy [2], to formulate a consistent naive set theory with a rich computational power [39].

In most situations, elementary affine logic is treated as a type system rather than a purely logical system. Accordingly, as with many other type systems, some decision problems can be considered, i.e., typability, type checking and type inhabitation (provability). For instance, Coppola and Martini [8] showed the decidability of typability in the $\{\multimap, !\}$-fragment of intuitionistic elementary affine logic.

On the other hand, of particular interest to us is the provability problem for elementary affine logic. Dal Lago and Martini [10] showed that provability in a classical variant of elementary affine logic is decidable. However, there are no known upper and lower bounds for that problem. We refine and extend the existing decidability result by showing the Tower-completeness of some variants of elementary affine logic (Section 6.2). It should be noted that such a non-elementary aspect of elementary affine logic does not conflict with its elementary character that comes from the proofs-as-programs correspondence.

Organization of this paper

In the next section, we review various contraction-free logics in a step-by-step manner, and define a useful translation from classical affine logic into intuitionistic affine logic. A large portion of Sections 3 and 4 is taken from [26, Section 3]. In Section 3, we summarize some basic notions involved in ABVASSs. Section 4 is devoted to a short discussion about the existing complexity results which are crucial in proving the main claims in Sections 5 and 6. We prove the main results in Sections 5 and 6. In Section 7, we conclude the paper with some remarks on the complexity status of other substructural logics.

Proofs omitted due to space limitations appear in the full version of this paper (`https://arxiv.org/abs/2201.02496`).

2 Contraction-free substructural logics

2.1 Sequent calculi for contraction-free substructural logics

For convenience, we start with the formal definition of a sequent calculus for *intuitionistic affine logic with bottom*, denoted by **ILZW**. It is merely the extension of Troelstra's **ILZ** by the rule of left-weakening, cf. [26, 40]. The *language* $\mathcal{L}$ of **ILZW** contains logical connectives $\&, \oplus, \otimes, \multimap$ of arity 2, ! of arity 1, and $\mathbf{1}, \top, \bot, \mathbf{0}$ of arity 0. We fix a countable set of propositional variables $V = \{p, q, r, \ldots\}$. An *intuitionistic $\mathcal{L}$-formula* is built from propositional variables using connectives in $\mathcal{L}$. For brevity, parentheses in formulas are omitted when confusion is unlikely. Throughout this paper, metavariables $A, B, C, \ldots$ range over formulas and $\Gamma, \Delta, \Sigma, \ldots$ over finite multisets of formulas. By abuse of notation, we

$$\frac{}{A \vdash A}\ (\mathrm{Init}) \qquad \frac{}{\vdash \mathbf{1}}\ (\mathbf{1}\mathrm{R}) \qquad \frac{}{\bot \vdash}\ (\bot\mathrm{L})$$

$$\frac{}{\Gamma \vdash \top}\ (\top\mathrm{R}) \qquad \frac{}{\mathbf{0}, \Gamma \vdash \Pi}\ (\mathbf{0}\mathrm{L}) \qquad \frac{\Gamma \vdash A \qquad A, \Delta \vdash \Pi}{\Gamma, \Delta \vdash \Pi}\ (\mathrm{Cut})$$

$$\frac{\Gamma \vdash \Pi}{\mathbf{1}, \Gamma \vdash \Pi}\ (\mathbf{1}\mathrm{L}) \qquad \frac{\Gamma \vdash}{\Gamma \vdash \bot}\ (\bot\mathrm{R}) \qquad \frac{\Gamma \vdash A \qquad B, \Delta \vdash \Pi}{A \multimap B, \Gamma, \Delta \vdash \Pi}\ (\multimap\mathrm{L})$$

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \multimap B}\ (\multimap\mathrm{R}) \qquad \frac{A, B, \Gamma \vdash \Pi}{A \otimes B, \Gamma \vdash \Pi}\ (\otimes\mathrm{L}) \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ (\otimes\mathrm{R})$$

$$\frac{A, \Gamma \vdash \Pi}{A \,\&\, B, \Gamma \vdash \Pi}\ (\&\mathrm{L1}) \qquad \frac{B, \Gamma \vdash \Pi}{A \,\&\, B, \Gamma \vdash \Pi}\ (\&\mathrm{L2}) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\ (\&\mathrm{R})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\ (\oplus\mathrm{R1}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\ (\oplus\mathrm{R2}) \qquad \frac{A, \Gamma \vdash \Pi \qquad B, \Gamma \vdash \Pi}{A \oplus B, \Gamma \vdash \Pi}\ (\oplus\mathrm{L})$$

$$\frac{A, \Gamma \vdash \Pi}{!A, \Gamma \vdash \Pi}\ (!\mathrm{D}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!\mathrm{P}) \qquad \frac{\Gamma \vdash \Pi}{A, \Gamma \vdash \Pi}\ (\mathrm{W}) \qquad \frac{!A, !A, \Gamma \vdash \Pi}{!A, \Gamma \vdash \Pi}\ (!\mathrm{C})$$

Figure 1 Inference rules of **ILZW**; A, B range over intuitionistic $\mathcal{L}$-formulas and Γ, Δ range over finite multisets of intuitionistic $\mathcal{L}$-formulas, and Π ranges over stoups.

simply write A for the singleton of a formula A. The multiset sum of Γ and Δ is denoted by Γ, Δ. We write $!\Gamma$ for the multiset obtained by prefixing each formula in Γ with exactly one $!$. An *intuitionistic $\mathcal{L}$-sequent* is an expression of the form $\Gamma \vdash \Pi$, where Γ is a finite multiset of intuitionistic $\mathcal{L}$-formulas, and Π is a stoup, i.e., either an intuitionistic $\mathcal{L}$-formula or the empty multiset ε. We always denote an intuitionistic $\mathcal{L}$-sequent of the form $\varepsilon \vdash \Pi$ (resp. $\Gamma \vdash \varepsilon$) by $\vdash \Pi$ (resp. $\Gamma \vdash$). The sequent calculus for **ILZW** consists of the inference rules depicted in Figure 1. A *proof* of a sequent $\Gamma \vdash \Pi$ in **ILZW** is defined in the usual manner. We furthermore define another variant of intuitionistic affine logic by adding the following *right-weakening* rule (W') to **ILZW**:

$$\frac{\Gamma \vdash}{\Gamma \vdash A}\ (\mathrm{W'})$$

The resulting system is denoted by $\mathbf{ILZW'}$.

Let $\mathcal{K}$ be a non-empty subset of $\mathcal{L}$. An *intuitionistic $\mathcal{K}$-formula* is an intuitionistic formula containing only logical connectives from $\mathcal{K}$. An *intuitionistic $\mathcal{K}$-sequent* is an intuitionistic sequent consisting only of intuitionistic $\mathcal{K}$-formulas. The *$\mathcal{K}$-fragment* of **ILZW** (resp. $\mathbf{ILZW'}$) is the sequent calculus obtained from **ILZW** (resp. $\mathbf{ILZW'}$) by dropping all the inference rules concerning connectives not in $\mathcal{K}$.

Each of the logical systems within the scope of this paper is a fragment of **ILZW** or $\mathbf{ILZW'}$. We list them below:

- *Full Lambek calculus with exchange and left-weakening* ($\mathbf{FL_{ei}}$) is the $\{\otimes, \multimap, \&, \oplus, \mathbf{1}, \bot\}$-fragment of **ILZW**.
- *Full Lambek calculus with exchange and weakening* ($\mathbf{FL_{ew}}$) is the $\{\otimes, \multimap, \&, \oplus, \mathbf{1}, \bot\}$-fragment of $\mathbf{ILZW'}$.
- The *positive fragment* ($\mathbf{FL^+_{ei}}$) of $\mathbf{FL_{ei}}$ is the $\{\otimes, \multimap, \&, \oplus, \mathbf{1}\}$-fragment of **ILZW**.
- *BCK logic* (**BCK**) is the implicational fragment (i.e., $\{\multimap\}$-fragment) of **ILZW**.

Unfortunately, our notation is considerably different from the notation widely employed in the substructural logic community; we refer the reader to [14, Table 2.1] for the notational correspondence between linear logic and substructural logic.

$$\frac{}{\vdash A, A^{\perp}}\ (\mathrm{Init}) \qquad \frac{}{\vdash \mathbf{1}}\ (\mathbf{1}) \qquad \frac{}{\vdash \Gamma, \top}\ (\top) \qquad \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}\ (\mathrm{Cut})$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ (\perp) \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ (\otimes) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ (\parr)$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ (\&) \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ (\oplus 1) \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ (\oplus 2)$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ (?) \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ (!) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, A}\ (\mathrm{W}) \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ (?\mathrm{C})$$

Figure 2 Inference rules of **LLW**; metavariables A, B range over classical $\mathcal{L}_C$-formulas, and Γ, Δ over finite multisets of $\mathcal{L}_C$-formulas.

We next formulate right-hand sided sequent calculi for *classical affine logic* (**LLW**), cf. [16, 24]. For our purpose here, we again employ the countable set V of propositional variables, and introduce their duals $V^{\perp} = \{p^{\perp}, q^{\perp}, r^{\perp}, \ldots\}$. Elements in $V \cup V^{\perp}$ are often referred to as *literals*. The language $\mathcal{L}_C$ consists of binary operation symbols $\&, \oplus, \otimes, \parr$, and constants $\mathbf{1}, \top, \perp, \mathbf{0}$, and unary operation symbols $!, ?$. Given a sublanguage $\mathcal{K}$ of $\mathcal{L}_C$, *Classical $\mathcal{K}$-formulas* are built up from literals using operation symbols in $\mathcal{K}$. For each classical $\mathcal{L}_C$-formula A, we inductively define the formula $A^{\perp}$ by the de Morgan duality; $(p^{\perp})^{\perp} = p, (A \,\&\, B)^{\perp} = A^{\perp} \oplus B^{\perp}, (A \oplus B)^{\perp} = A^{\perp} \,\&\, B^{\perp}, (A \otimes B)^{\perp} = A^{\perp} \parr B^{\perp}, (A \parr B)^{\perp} = A^{\perp} \otimes B^{\perp}, (!A)^{\perp} = ?A^{\perp}, (?A)^{\perp} = !A^{\perp}, \mathbf{1}^{\perp} = \perp, \perp^{\perp} = \mathbf{1}, \top^{\perp} = \mathbf{0}, \mathbf{0}^{\perp} = \top$. It is easy to see that $A = A^{\perp\perp}$ for any classical $\mathcal{L}_C$-formula A. A *classical $\mathcal{K}$-sequent* is an expression of the form $\vdash \Gamma$, where Γ is a finite multiset of classical $\mathcal{K}$-formulas. The inference rules of **LLW** are presented in Figure 2.

As with the intuitionistic sequent systems discussed earlier, various fragments of **LLW** can be defined in the usual manner. Among such fragments, of importance to us is the $\{\otimes, \parr, \&, \oplus, \mathbf{1}, \perp\}$-fragment of **LLW**, called *involutive full Lambek calculus with exchange and weakening* ($\mathbf{InFL_{ew}}$).

In [17] Girard proposed a logic which captures elementary recursive computation, called *elementary linear logic* (**ELL**). We review here some affine variants of **ELL**, following [2, 8, 10]. *Intuitionistic elementary affine logic with bottom* (**IEZW**) is obtained from **ILZW** by dropping the rules of (!D) and (!P) and by adding the following functorial promotion rule:

$$\frac{\Gamma \vdash A}{!\Gamma \vdash !A}\ (!\mathrm{F})$$

Similarly, *classical elementary affine logic* (**ELLW**) is obtained from **LLW** by deleting the rules of (?) and (!) and by adding the following rule:

$$\frac{\vdash \Gamma, A}{\vdash ?\Gamma, !A}\ (\mathrm{F})$$

It is easy to see that **IEZW** (resp. **ELLW**) is a subsystem of **ILZW** (resp. **LLW**), i.e., every sequent that is provable in **IEZW** (resp. **ELLW**) is provable in **ILZW** (resp. **LLW**). Henceforce, we write $\mathcal{L}^{+}$ for the language $\mathcal{L} \setminus \{\perp\}$. The notation **ILLW** (resp. **IELW**) is used to denote the $\mathcal{L}^{+}$-fragment of **ILZW** (resp. **IEZW**). Every sequent in such $\perp$-free logical systems is of the form $\Gamma \vdash A$.

Let **L** be one of the sequent calculi described so far and Φ a set of formulas in **L**. We write $\mathbf{L}[\Phi]$ for the sequent calculus obtained from **L** by adding $\vdash B$ as an initial sequent for every $B \in \Phi$. In this paper, we consider the following two types of decision problems for **L**.

▶ **Problem** (Provability in $\mathbf{L}$).
Instance: *A formula F in $\mathbf{L}$.*
Question: *Is the sequent $\vdash F$ provable in $\mathbf{L}$?*

▶ **Problem** (Deducibility in $\mathbf{L}$).
Instance: *A finite set $\Phi \cup \{F\}$ of formulas in $\mathbf{L}$.*
Question: *Is the sequent $\vdash F$ provable in $\mathbf{L}[\Phi]$?*

Our argument depends heavily on the following cut-elimination theorem, as we will see in the remaining sections:

▶ **Theorem 1** (cf. [10, 24, 30]). *The sequent calculi for* **ILLW**, **ILZW**, **ILZW′**, **LLW**, **IELW**, **IEZW**, *and* **ELLW** *all enjoy cut-elimination.*

As far as we know, for instance, the cut-elimination for **ILZW′** has not been settled. The reader can however show this without great difficulty, using a proof-theoretic or algebraic manner; see also [28, 29, 40] for technical details on cut-elimination in linear logic. We thus omit the proof in this paper.

Of course, the cut-elimination theorem also holds for various fragments of the systems stated in Theorem 1, e.g., **BCK**, $\mathbf{FL_{ei}}$, the $\{\multimap, !\}$-fragment of **ILZW**.

2.2 Translation from classical affine logic to intuitionistic affine logic

In this subsection, we present an efficient (i.e., polynomial-time) translation from **LLW** (resp. **ELLW**) into **ILLW** (resp. **IELW**). It is a modification of Laurent's *parametric negative translation* from classical linear logic to intuitionistic linear logic; see [25, Definition 2.2].

Let us fix an intuitionistic $\mathcal{L}$-formula F. Given a classical $\mathcal{L}_C$-formula A, we inductively define the intuitionistic $\mathcal{L}$-formula $A^{[F]}$ as follows:

$$\begin{aligned}
p^{[F]} &:= \neg_F p & (p^\perp)^{[F]} &:= p \\
\mathbf{1}^{[F]} &:= \neg_F \mathbf{1} & \perp^{[F]} &:= \mathbf{1} \\
\top^{[F]} &:= \mathbf{0} & \mathbf{0}^{[F]} &:= \neg_F \mathbf{0} \\
(B \otimes C)^{[F]} &:= \neg_F B^{[F]} \multimap C^{[F]} & (B \parr C)^{[F]} &:= \neg_F (B^{[F]} \multimap \neg_F C^{[F]}) \\
(B \,\&\, C)^{[F]} &:= B^{[F]} \oplus C^{[F]} & (B \oplus C)^{[F]} &:= \neg_F(\neg_F B^{[F]} \oplus \neg_F C^{[F]}) \\
(!B)^{[F]} &:= \neg_F ! \neg_F B^{[F]} & (?B)^{[F]} &:= !B^{[F]}
\end{aligned}$$

where $\neg_F A$ is an abbreviation for $A \multimap F$. We have the following theorem. The proof can be found in the full version.

▶ **Theorem 2.** *Let $\vdash \Gamma$ be a classical $\mathcal{L}_C$-sequent and x a fresh propositional variable not occurring in Γ.*
(1) $\vdash \Gamma$ *is provable in* **LLW** *if and only if* $\Gamma^{[x]} \vdash x$ *is provable in* **ILLW**.
(2) $\vdash \Gamma$ *is provable in* **ELLW** *if and only if* $\Gamma^{[x]} \vdash x$ *is provable in* **IELW**.

This translation is convenient to show the complexity of the contraction-free logics that we deal with, e.g., the NP-completeness of the provability problem for **BCK**.

▶ **Corollary 3.** *The provability problem for* **BCK** *is NP-complete.*

Proof. Membership in NP is an immediate consequence of cut elimination for **BCK**. The proof is based on that of [28, Lemma 5.3]. In any cut-free proof in the system, the only applicable rules are (Init), ($\multimap$L), ($\multimap$R) and (W). Thus each subformula occurring in the endsequent is analyzed at most once in such a proof-tree. This means that, the size of a cut-free proof in the system is polynomially bounded in the size of the endsequent. Hence the problem is in NP.[1]

For the hardness, we construct a polynomial-time reduction from provability in the $\{\otimes, \parr\}$-fragment of **LLW** (i.e., the constant-free fragment of multiplicative classical affine logic) into provability in **BCK**. The NP-completeness of the former is shown by Lincoln-Mitchell-Scedrov-Shankar [28], and Kanovich [23]. Let A be a classical $\{\otimes, \parr\}$-formula and x a fresh variable not in A. Our goal is to show that $\vdash A$ is provable in the $\{\otimes, \parr\}$-fragment of **LLW** if and only if $\vdash \neg_x A^{[x]}$ is provable in **BCK**. As a consequence of cut elimination for **LLW**, we can easily show that **LLW** is conservative over its $\{\otimes, \parr\}$-fragment. That is, $\vdash A$ is provable in the $\{\otimes, \parr\}$-fragment of **LLW** if and only if $\vdash A$ is provable in **LLW**. By Theorem 2, $\vdash A$ is provable in **LLW** if and only if $A^{[x]} \vdash x$ is provable in **ILLW**. Here $A^{[x]} \vdash x$ is an intuitionistic $\{\multimap\}$-sequent. Again, by the cut elimination theorem for **ILLW**, we know that **ILLW** is conservative over **BCK**; hence $A^{[x]} \vdash x$ is provable in **ILLW** if and only if $A^{[x]} \vdash x$ is provable in **BCK**. By the invertibility of ($\multimap$R), $A^{[x]} \vdash x$ is provable in **BCK** if and only if $\vdash \neg_x A^{[x]}$ is provable in **BCK**; thus we conclude that $\vdash A$ is provable in the $\{\otimes, \parr\}$-fragment of **LLW** if and only if $\vdash \neg_x A^{[x]}$ is provable in **BCK**. ◀

In particular, the translation $(\_)^{[\bot]}$ is a sort of standard negative translation from **LLW** (resp. **ELLW**) to **ILZW** (resp. **IEZW**). One can also show the following:

▶ **Theorem 4.** *Let $\vdash \Gamma$ be a classical $\mathcal{L}_C$-sequent. $\vdash \Gamma$ is provable in* **LLW** *(resp.* **ELLW***) if and only if $\Gamma^{[\bot]} \vdash$ is provable in* **ILZW** *(resp.* **IEZW***).*

3 Alternating branching VASS

The whole content of this section is taken from [26, Section 3]. Let d be in $\mathbb{N}$. The symbols $\bar{\mathsf{v}}_1, \bar{\mathsf{v}}_2, \ldots$ are used to denote d-dimensional vectors. In particular, we write $\bar{\mathsf{e}}_i$ for the *i-th unit vector* in $\mathbb{N}^d$ (i.e., the vector with a one in the i-th coordinate and zeros elsewhere), and $\bar{0}$ for the vector whose every coordinate is zero.

An *alternating branching vector addition system with states and full zero tests* ($\text{ABVASS}_{\bar{0}}$, for short) is a structure of the form $\mathcal{A} = \langle Q, d, T_u, T_s, T_f, T_z \rangle$ where:

- Q is a finite set,
- d is in $\mathbb{N}$,
- T_u is a finite subset of $Q \times \mathbb{Z}^d \times Q$,
- T_s and T_f are subsets of Q^3, and
- T_z is a subset of Q^2.

We call Q a *state space*, d a *dimension*, and T_u (resp. T_s, T_f, T_z) the set of *unary* (resp. *split*, *fork*, *full zero test*) rules of $\mathcal{A}$. For readability, we always write $q \xrightarrow{\bar{\mathsf{u}}} q'$ for $(q, \bar{\mathsf{u}}, q') \in T_u$, $q \to q_1 \wedge q_2$ for $(q, q_1, q_2) \in T_f$, $q \to q_1 + q_2$ for $(q, q_1, q_2) \in T_s$, and $q \xrightarrow{\stackrel{?}{=}\bar{0}} q'$ for $(q, q') \in T_z$. A *configuration* of $\mathcal{A}$ is an element of $Q \times \mathbb{N}^d$.

Given an $\text{ABVASS}_{\bar{0}}$ $\mathcal{A} = \langle Q, d, T_u, T_s, T_f, T_z \rangle$, the operational semantics for $\mathcal{A}$ is given by a deduction system over configurations in $Q \times \mathbb{N}^d$. It consists of the deduction rules depicted in Figure 3. Given a subset Q_ℓ of Q, a *$Q_\ell \times \{\bar{0}\}$-leaf-covering deduction tree* in $\mathcal{A}$ is a finite tree labeled by configurations, where leaves are all in $Q_\ell \times \{\bar{0}\}$ and each other node is obtained

[1] This was already pointed out in [20, p. 71].

$$\frac{q, \bar{\mathsf{v}}}{q', \bar{\mathsf{v}} + \bar{\mathsf{u}}}\ (q \xrightarrow{\bar{\mathsf{u}}} q') \qquad \frac{q, \bar{0}}{q', \bar{0}}\ (q \xrightarrow{\stackrel{?}{=}\bar{0}} q')$$

$$\frac{q, \bar{\mathsf{v}}}{q_1, \bar{\mathsf{v}} \qquad q_2, \bar{\mathsf{v}}}\ (q \to q_1 \wedge q_2) \qquad \frac{q, \bar{\mathsf{v}}_1 + \bar{\mathsf{v}}_2}{q_1, \bar{\mathsf{v}}_1 \qquad q_2, \bar{\mathsf{v}}_2}\ (q \to q_1 + q_2)$$

Figure 3 The deduction rules of an $\mathrm{ABVASS}_{\bar{0}}$ $\mathcal{A} = \langle Q, d, T_u, T_s, T_f, T_z \rangle$, where $q \xrightarrow{\stackrel{?}{=}\bar{0}} q' \in T_z$, $q \xrightarrow{\bar{\mathsf{u}}} q' \in T_u$, $q \to q_1 + q_2 \in T_s$, and $q \to q_1 \wedge q_2 \in T_f$. The symbol $+$ stands for componentwise addition and $\bar{\mathsf{v}} + \bar{\mathsf{u}}$ must be in $\mathbb{N}^d$.

from its children by applying one of the deduction rules derived from $T_u \cup T_s \cup T_f \cup T_z$. A deduction tree $\mathcal{T}$ whose root configuration is $q, \bar{\mathsf{v}}$ is denoted by the following figure:

$$\begin{array}{c} q, \bar{\mathsf{v}} \\ \mathcal{T} \end{array}$$

We write $\mathcal{A}, Q_\ell \rhd q, \bar{\mathsf{v}}$ if there exists a $Q_\ell \times \{\bar{0}\}$-leaf-covering deduction tree whose root configuration is $q, \bar{\mathsf{v}}$.

In addition, we also give an account of another semantics for $\mathrm{ABVASS}_{\bar{0}}$s. Given an $\mathrm{ABVASS}_{\bar{0}}$ $\mathcal{A} = \langle Q, d, T_u, T_s, T_f, T_z \rangle$, the *lossy semantics* for $\mathcal{A}$ is given by the aforementioned deduction system of $\mathcal{A}$ augmented with the following additional deduction rules:

$$\frac{q, \bar{\mathsf{v}} + \bar{\mathsf{e}}_i}{q, \bar{\mathsf{v}}}\ (\mathsf{loss})$$

for every $q \in Q$ and every $i \in \{1, \ldots, d\}$. In the natural way, we define the notion of $Q_\ell \times \{\bar{0}\}$*-leaf-covering lossy deduction tree* in $\mathcal{A}$. We write $\mathcal{A}, Q_\ell \rhd_\ell q, \bar{\mathsf{v}}$ if there exists a $Q_\ell \times \{\bar{0}\}$-leaf-covering lossy deduction tree with root $q, \bar{\mathsf{v}}$.

4 Some Tower-complete problems

Following the terminology of [26, 35], we define

$$\textsc{Tower} := \bigcup_{f \in \mathsf{FELEMENTARY}} \textsc{DTime}(2^{\cdot^{\cdot^{\cdot^{2}}}}\}^{f(n) \text{ times}})$$

where $\mathsf{FELEMENTARY}$ denotes the set of elementary functions. This class of problems is closed under elementary many-one reductions (and elementary Turing reductions), i.e., for any two languages X and Y, if there is an elementary reduction from X to Y and Y is in Tower, then X is in Tower. The notion of *Tower-completeness* is defined with respect to elementary reductions in the usual manner. For an elaborate discussion on the fast-growing complexity hierarchy, we refer the reader to [35].

For later use, we summarize here some Tower-complete problems. Given an $\mathrm{ABVASS}_{\bar{0}}$ $\mathcal{A} = \langle Q, d, T_u, T_s, T_f, T_z \rangle$, $Q_\ell \subseteq Q$, and $q_r \in Q$, the *reachability problem* (resp. *lossy reachability problem*) asks whether it holds that $\mathcal{A}, Q_\ell \rhd q_r, \bar{0}$ (resp. $\mathcal{A}, Q_\ell \rhd_\ell q_r, \bar{0}$).

▶ **Theorem 5** (Lazić and Schmitz [26], Theorem 3.6). *The lossy reachability problem for $\mathrm{ABVASS}_{\bar{0}}$s is Tower-complete.*

In contrast, the reachability problem is undecidable for $\mathrm{ABVASS}_{\bar{0}}$s. In fact, the same holds even for alternating VASSs, which are $\mathrm{ABVASS}_{\bar{0}}$s with only unary rules and fork rules; see [28, Section 3.4] and [26, Section 3.3.1] for details.

Intriguingly, the lossy reachability problem is complete for TOWER even when a restricted version of ABVASS$_{\bar{0}}$s is considered. An *ordinary ABVASS*$_{\bar{0}}$ is an ABVASS$_{\bar{0}}$ $\mathcal{A} = \langle Q, d, T_u, T_s, T_f, T_z \rangle$ where for every $q \xrightarrow{\bar{\mathsf{u}}} q' \in T_u$ it holds that either $\bar{\mathsf{u}} = \bar{\mathsf{e}}_i$ or $\bar{\mathsf{u}} = -\bar{\mathsf{e}}_i$ for some $1 \le i \le d$. A *branching vector addition system with states* (BVASS, for short) is an ABVASS$_{\bar{0}}$ $\mathcal{A} = \langle Q, d, T_u, T_s, T_f, T_z \rangle$ where T_f and T_z are both empty. The main results in the remaining sections rely crucially on the following two complexity results:

▶ **Theorem 6** (Lazić and Schmitz [26], Lemma 3.5 and Theorem 3.6)**.** *The lossy reachability problem for ordinary BVASSs is* TOWER*-complete.*

▶ **Theorem 7** (Lazić and Schmitz [26]; Fact 4.2, Corollary 5.4 and Corollary 6.3)**.** *The provability problems for* **ILZW** *and* **LLW** *are* TOWER*-complete.*

5 Membership in Tower of contraction-free logics

This section consists of two parts. We first show that provability in elementary affine logic is in TOWER (Section 5.1). Secondly, we also show the TOWER-membership of deducibility in $\mathbf{FL_{ew}}$ and related logical systems (Section 5.2).

5.1 Tower upper bound for provability in elementary affine logic

We show that there exists an elementary reduction from the provability problem for **IEZW** to the lossy reachablity problem for ABVASS$_{\bar{0}}$s. Our reduction is a slightly modified version of the (polynomial-space) reduction, given in [26, Section 4.1.2], from provability in **ILZW** to lossy reachability in ABVASS$_{\bar{0}}$.

Let F be an intuitionistic $\mathcal{L}$-formula. Let S be the set of subformulas of F, $S_!$ the set of formulas in S of the form $!B$, and $\bullet$ a fresh symbol not in S. For $\Pi \in S \cup \{\varepsilon\}$, we define $\Pi^\dagger = A$ if $\Pi = A$, and $\Pi^\dagger = \bullet$ otherwise. A multiset over S is just a map from S to $\mathbb{N}$, i.e., an element of $\mathbb{N}^S$. Given a multiset m over S, $m(B)$ denotes the multiplicity of a formula B in m.

Fix an enumeration $F_1, \ldots, F_d$ of all the formulas in S. A multiset m over S can be expressed as $F_1^{m(F_1)}, \ldots, F_d^{m(F_d)}$. For each multiset m over S, we write $\bar{\mathsf{v}}_m$ for the vector $\langle m(F_1), \ldots, m(F_d) \rangle$ in $\mathbb{N}^d$. In particular, we write $\bar{\mathsf{e}}_B$ for the vector $\bar{\mathsf{v}}_B$ corresponding to a formula B in S. Note that $\bar{\mathsf{v}}_m = \bar{0}$ if m is the empty multiset. We write $\sigma(m)$ for the *support* of a multiset m, i.e., $\{B \in S \mid m(B) > 0\}$. We then construct an ABVASS$_{\bar{0}}$ $\mathcal{A}_F^E$ as below:

- The dimension of $\mathcal{A}_F^E$ is d $(= |S|)$.
- The state space of $\mathcal{A}_F^E$ contains $\mathcal{P}(S_!) \times (S \cup \{\bullet\})$, a distinguished leaf state q_ℓ, and several intermediate states which are needed for defining the rules for ($\multimap$L), (&L), ($\oplus$L), ($\otimes$L), (0L), ($\top$R) and (func) in $\mathcal{A}_F^E$ (cf. Figure 4).
- The rules and intermediate states of $\mathcal{A}_F^E$ are defined as in Figure 4.

The construction of $\mathcal{A}_F^E$ is quite similar to that of the ABVASS$_{\bar{0}}$ $\mathcal{A}_F^I$ defined in [26, Section 4.1.2]. We stress that $\mathcal{A}_F^E$ has the rules for (func) instead of the rules for (!D) and (!P) (see Figure 5 in Section 5.2), whereas $\mathcal{A}_F^I$ has the rules for (!D) and (!P) instead of the rules for (func). Notice that $\mathcal{A}_F^E$ does not have the rules corresponding to the inference rule of (W) in **ILZW**. The left-weakening rule is implemented by loss rules derived from the lossy semantics for $\mathcal{A}_F^E$.

Let $\Theta, \Gamma \vdash \Pi$ be an intuitionistic $\mathcal{L}$-sequent such that Θ is a multiset of formulas in $S_!$, Γ is a multiset of formulas in $S \setminus S_!$, and Π is in $S \cup \{\varepsilon\}$. It is translated as the configuration $\sigma(\Theta), \Pi^\dagger, \bar{\mathsf{v}}_\Gamma$ in $\mathcal{P}(S_!) \times (S \cup \{\bullet\}) \times \mathbb{N}^d$. We have the key theorem of this subsection; see the full version for a detailed proof.

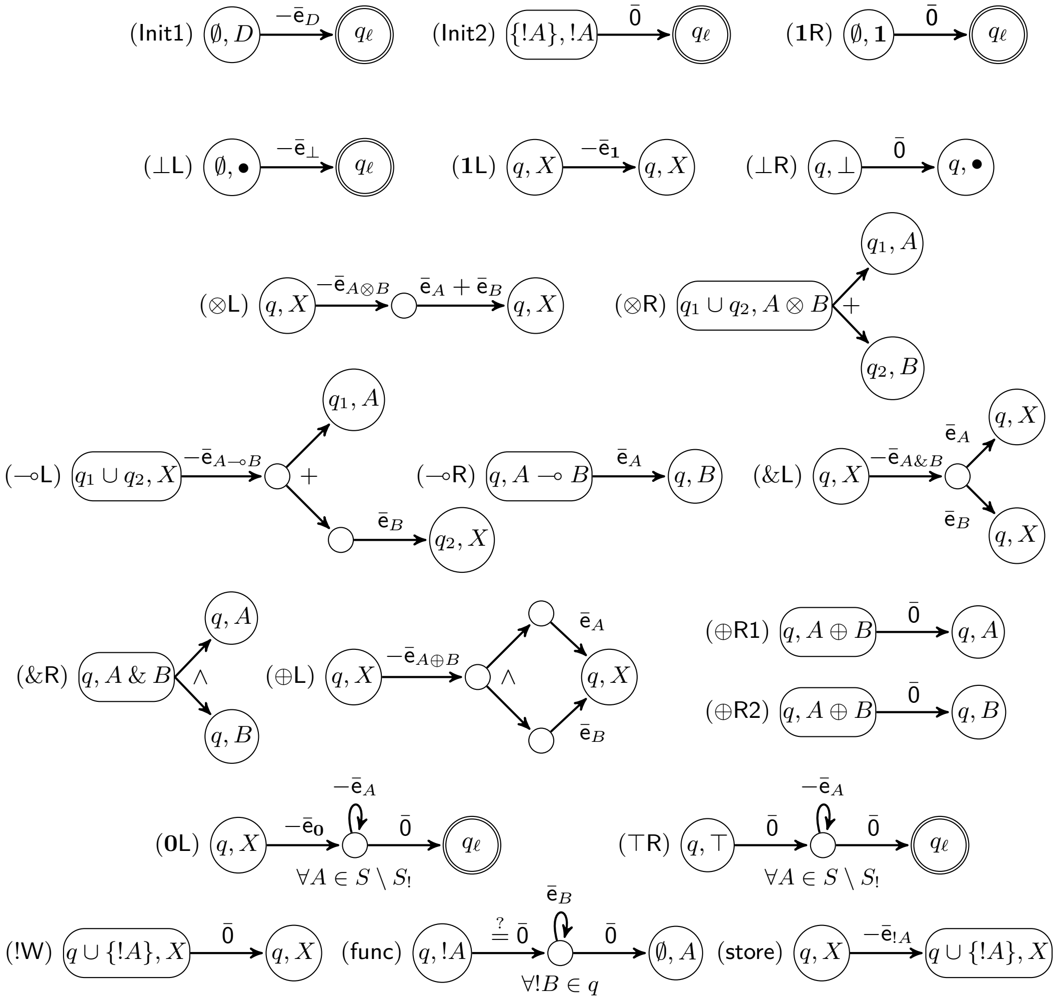

Figure 4 Rules and intermediate states of $\mathcal{A}^E_F$. All formulas range over S, q, q_1, q_2 range over $\mathcal{P}(S_!)$, D ranges over $S \setminus S_!$, and X ranges over $S \cup \{\bullet\}$. Small circles stand for intermediate states.

▶ **Theorem 8.** *Let Π be in $S \cup \{\varepsilon\}$, Θ a multiset of formulas in $S_!$, Γ a multiset of formulas in $S \setminus S_!$. $\Theta, \Gamma \vdash \Pi$ is provable in* **IEZW** *if and only if $\mathcal{A}^E_F, \{q_\ell\} \triangleright_\ell \sigma(\Theta), \Pi^\dagger, \bar{\mathsf{v}}_\Gamma$.*

In particular, Theorem 8 guarantees that for any intuitionistic $\mathcal{L}$-formula F, F is provable in **IEZW** if and only if $\mathcal{A}^E_F, \{q_\ell\} \triangleright_\ell \emptyset, F, \bar{0}$. By Theorems 5 and 4:

▶ **Corollary 9.** *The provability problems for* **IEZW** *and* **ELLW** *are in* Tower.

We stress that the Tower upper bound also holds for fragments of **IEZW** and **ELLW**, e.g., the $\{\multimap, !\}$-fragment of **IEZW**.

5.2 Tower upper bound for deducibility in FLew and related systems

We first describe the notion of !-*prenex sequent* which originates in [38, Section 2]. Let $\mathcal{K}$ be a language such that $\bot \in \mathcal{K} \subseteq \mathcal{L}$. A !-*prenex $\mathcal{K}$-sequent* is an intuitionistic sequent of the form $!\Gamma, \Delta \vdash \Pi$, where Γ and Δ are finite multisets of intuitionistic $\mathcal{K}$-formulas, and Π is an intuitionistic $\mathcal{K}$-formula or the empty multiset. Similarly, let $\mathcal{K}$ be a sublanguage of $\mathcal{L}_C$. A

?*-prenex* $\mathcal{K}$*-sequent* is a right-hand sided sequent of the form $\vdash ?\Gamma, \Delta$, where Γ and Δ are finite multisets of classical $\mathcal{K}$-formulas. We write Γ^n for the multiset sum of n copies of Γ for each $n \geq 0$.

▶ **Lemma 10.** *Let* $!\Gamma, \Delta \vdash \Pi$ *be a* !*-prenex* $\{\otimes, \multimap, \&, \oplus, \mathbf{1}, \bot\}$*-sequent.*
(1) *If* $!\Gamma, \Delta \vdash \Pi$ *is provable in* **ILZW***, then* $\Gamma^n, \Delta \vdash \Pi$ *is provable in* $\mathbf{FL_{ei}}$ *for some* $n \geq 0$*.*
(2) *If* $!\Gamma, \Delta \vdash \Pi$ *is provable in* **ILZW′***, then* $\Gamma^n, \Delta \vdash \Pi$ *is provable in* $\mathbf{FL_{ew}}$ *for some* $n \geq 0$*.*

Proof. The proof of Statement (1) proceeds by induction on the size of the cut-free proof of $!\Gamma, \Delta \vdash \Pi$ in **ILZW**. We perform a case analysis, depending on which inference rule is applied last.

We consider only the case of ($\multimap$L). If $!\Gamma, !\Sigma, A \multimap B, \Delta, \Xi \vdash \Pi$ is obtained from $!\Gamma, \Delta \vdash A$ and $!\Sigma, B, \Xi \vdash \Pi$ by an application of ($\multimap$L), then by the induction hypothesis, $\Gamma^{n'}, \Delta \vdash A$ is provable in $\mathbf{FL_{ei}}$ for some n', and $\Sigma^{n''}, B, \Xi \vdash \Pi$ is provable in $\mathbf{FL_{ei}}$ for some n''. Applying ($\multimap$L) we obtain a proof of $\Gamma^{n'}, \Sigma^{n''}, A \multimap B, \Delta, \Xi \vdash \Pi$ in $\mathbf{FL_{ei}}$. Note that n' is not always equal to n''. However, we may unify them using the rule of (W); thus $(\Gamma, \Sigma)^{n'+n''}, A \multimap B, \Delta, \Xi \vdash \Pi$ is provable in $\mathbf{FL_{ei}}$. The remaining cases are similar.

One can show Statement (2) similarly. ◀

Similarly to the above theorem, one can also show the following:

▶ **Lemma 11.** *Let* $\vdash ?\Gamma, \Delta$ *be a* ?*-prenex* $\{\otimes, ⅋, \&, \oplus, \mathbf{1}, \bot\}$*-sequent. If* $\vdash ?\Gamma, \Delta$ *is provable in* **LLW***, then* $\vdash \Gamma^n, \Delta$ *is provable in* $\mathbf{InFL_{ew}}$ *for some* $n \geq 0$*.*

Lemmas 10 and 11 are affine analogues of [38, Proposition 2.6]. Interestingly, a similar idea is found in the proof of the *local deduction theorem* for $\mathbf{FL_{ew}}$; see [15, Corollary 2.15]. The following lemma, which is inspired by [28, Lemmas 3.2 and 3.3], provides a very simple reduction from $\mathbf{FL_{ei}}$ deducibility (resp. $\mathbf{FL_{ew}}$ deducibility) into **ILZW** provability (resp. **ILZW′** provability).

▶ **Lemma 12.** *Let* Φ *be a finite set of intuitionistic* $\{\otimes, \multimap, \&, \oplus, \mathbf{1}, \bot\}$*-formulas and* $\Gamma \vdash \Pi$ *an intuitionistic* $\{\otimes, \multimap, \&, \oplus, \mathbf{1}, \bot\}$*-sequent.*
(1) $\Gamma \vdash \Pi$ *is provable in* $\mathbf{FL_{ei}}[\Phi]$ *if and only if* $!\Phi, \Gamma \vdash \Pi$ *is provable in* **ILZW***.*
(2) $\Gamma \vdash \Pi$ *is provable in* $\mathbf{FL_{ew}}[\Phi]$ *if and only if* $!\Phi, \Gamma \vdash \Pi$ *is provable in* **ILZW′***.*

Proof. We only prove the first statement, the proof of the second one being similar. For the *if* part, let us suppose that $!\Phi, \Gamma \vdash \Pi$ is provable in **ILZW**. Since $!\Phi, \Gamma \vdash \Pi$ is a !-prenex $\{\otimes, \multimap, \&, \oplus, \mathbf{1}, \bot\}$-sequent, $\Phi^n, \Gamma \vdash \Pi$ is provable in $\mathbf{FL_{ei}}$ for some n by Lemma 10. Since $\vdash B$ is an initial sequent of $\mathbf{FL_{ei}}[\Phi]$ for each $B \in \Phi$, we can construct a proof of $\Gamma \vdash \Pi$ in $\mathbf{FL_{ei}}[\Phi]$ by several applications of (Cut). The *only-if* part follows by induction on the height of the proof of $\Gamma \vdash \Pi$ in $\mathbf{FL_{ei}}[\Phi]$. ◀

In the same way as before, the deducibility problems for $\mathbf{FL_{ei}^{+}}$ and **BCK** are also reduced to the provability problem for **ILZW**. Furthermore, one can show that there is also a straightforward reduction from $\mathbf{InFL_{ew}}$ deducibility to **LLW** provability, using Lemma 11:

▶ **Lemma 13.** *Let* Φ *be a finite set of classical* $\{\otimes, ⅋, \&, \oplus, \mathbf{1}, \bot\}$*-formulas and* $\vdash \Gamma$ *a classical* $\{\otimes, ⅋, \&, \oplus, \mathbf{1}, \bot\}$*-sequent.* $\vdash \Gamma$ *is provable in* $\mathbf{InFL_{ew}}[\Phi]$ *if and only if* $\vdash ?\Phi^{\bot}, \Gamma$ *is provable in* **LLW***.*

Recall that the provability problems for **ILZW** and **LLW** are both in Tower by Theorem 7. We obtain:

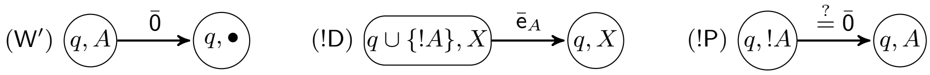

Figure 5 The ABVASS$_{\bar{0}}$ rules for (W′), (!D) and (!P).

▶ **Corollary 14.** *The following decision problems are in* Tower*.*
- *the deducibility problem for* **BCK***,*
- *the deducibility problem for* $\mathbf{FL}^{+}_{\mathbf{ei}}$*,*
- *the deducibility problem for* $\mathbf{FL}_{\mathbf{ei}}$*,*
- *the deducibility problem for* $\mathbf{InFL}_{\mathbf{ew}}$*.*

At the end of this section, we show the membership in Tower of the deducibility problem for $\mathbf{FL}_{\mathbf{ew}}$. It suffices by Lemma 12 to show that the provability problem for $\mathbf{ILZW}'$ is in Tower.

Let F be an instance of the provability problem for $\mathbf{ILZW}'$. As before, S denotes the set of subformulas of F, $S_!$ the set of formulas in S of the form $!B$, and $\bullet$ a distinguished symbol. We then construct an ABVASS$_{\bar{0}}$ $\mathcal{A}^{I'}_F$ of dimension $|S|$, by modifying the construction of $\mathcal{A}^{E}_F$ given in the previous subsection. The state space of $\mathcal{A}^{I'}_F$ contains $\mathcal{P}(S_!) \times (S \cup \{\bullet\})$, a distinguished leaf state q_ℓ, and intermediate states that are needed for defining the rules for ($\multimap$L), (&L), ($\oplus$L), ($\otimes$L), ($\mathbf{0}$L), and ($\top$R), cf. Figure 4. The rules of $\mathcal{A}^{I'}_F$ are (Init1), (Init2), (store), ($\mathbf{1}$L), ($\mathbf{1}$R), ($\perp$L), ($\perp$R), ($\multimap$L), ($\multimap$R), ($\otimes$L), ($\otimes$R), (&L), (&R), ($\oplus$L), ($\oplus$R1), ($\oplus$R2), ($\mathbf{0}$L), ($\top$R), (!W), (W′), (!D) and (!P), all of which except for (W′), (!D) and (!P), are depicted in Figure 4. The rules for (W′), (!D) and (!P) are defined as in Figure 5. Clearly, these three types of ABVASS$_{\bar{0}}$ rules correspond to the inference rules of (W’), (!D) and (!P) in $\mathbf{ILZW}'$, respectively. Note that $\mathcal{A}^{I'}_F$ is not equipped with the rule for (func). We can show that $\mathcal{A}^{I'}_F$ simulates the proof search of F in $\mathbf{ILZW}'$ with the lossy semantics; see the full version for a proof.

▶ **Theorem 15.** *Let* Π *be in* $S \cup \{\varepsilon\}$*,* Θ *a multiset of formulas in* $S_!$*,* Γ *a multiset of formulas in* $S \setminus S_!$*.* $\Theta, \Gamma \vdash \Pi$ *is provable in* $\mathbf{ILZW}'$ *if and only if* $\mathcal{A}^{I'}_F, \{q_\ell\} \triangleright_\ell \sigma(\Theta), \Pi^\dagger, \bar{v}_\Gamma$*.*

Specifically, $\mathcal{A}^{I'}_F, \{q_\ell\} \triangleright_\ell \emptyset, F, \bar{0}$ if and only if $\vdash F$ is provable in $\mathbf{ILZW}'$; thus Lemma 12 and Theorem 5 provide:

▶ **Corollary 16.** *The provability problem for* $\mathbf{ILZW}'$ *is in* Tower*.*

▶ **Corollary 17.** *The deducibility problem for* $\mathbf{FL}_{\mathbf{ew}}$ *is in* Tower*.*

6 Tower-hardness of contraction-free logics

For our purposes, we recall from [26, Section 4.2] a log-space reduction from the lossy reachability problem for ordinary BVASSs to the provability problem of ?-prenex $\{\otimes, \parr\}$-sequents in **LLW**.

Let $(\mathcal{B}, Q_\ell, q_r)$ be an instance of the lossy reachability problem for ordinary BVASSs, where $\mathcal{B} = \langle Q, d, T_u, T_s, \emptyset, \emptyset \rangle$. We fix a set $Q \cup \{e_i \mid 1 \leq i \leq d\}$ of propositional variables. Given $(q, \bar{v}) \in Q \times \mathbb{N}^d$, we define $\theta(q, \bar{v}) = q^\perp, (e_1^\perp)^{\bar{v}(1)}, \ldots, (e_d^\perp)^{\bar{v}(d)}$, where $\bar{v}(i)$ stands for the i-th coordinate of $\bar{v}$. We write T for the set of the three types of non-logical axioms, each of which is derived from $T_u \cup T_s$ as follows:
- $\vdash q^\perp, q_1 \otimes e_i$ for $q \xrightarrow{\bar{e}_i} q_1 \in T_u$,
- $\vdash q^\perp, e_i^\perp, q_1$ for $q \xrightarrow{-\bar{e}_i} q_1 \in T_u$,

- $\vdash q^\perp, q_1 \parr q_2$ for $q \to q_1 + q_2 \in T_s$.

Each sequent in T is of the form $\vdash q_1^\perp, \ldots, q_n^\perp, C$, where $q_1^\perp, \ldots, q_n^\perp$ are negative literals and C is a classical $\{\otimes, \parr\}$-formula. For any sequent $t = \vdash q_1^\perp, \ldots, q_n^\perp, C$ in T, we define $\ulcorner t \urcorner = q_1 \otimes \cdots \otimes q_n \otimes C^\perp$. Given a finite set $T = \{t_1, \ldots, t_n\}$ of sequents, $\ulcorner T \urcorner$ denotes the multiset $\ulcorner t_1 \urcorner, \ldots, \ulcorner t_n \urcorner$. It then holds that, for any $(q, \bar{\mathrm{v}}) \in Q \times \mathbb{N}^d$, $\mathcal{B}, Q_\ell \triangleright_\ell q, \bar{\mathrm{v}}$ if and only if $\vdash ?\ulcorner T \urcorner, ?Q_\ell, \theta(q, \bar{\mathrm{v}})$ is provable in **LLW**. In particular, the following holds:

▶ **Theorem 18** (Lazić and Schmitz [26], Section 4.2.3). *$\mathcal{B}, Q_\ell \triangleright_\ell q_r, \bar{0}$ if and only if $\vdash ?\ulcorner T \urcorner, ?Q_\ell, q_r^\perp$ is provable in* **LLW**.

The key observation here is that $\vdash ?\ulcorner T \urcorner, ?Q_\ell, q_r^\perp$ forms a ?-prenex $\{\otimes, \parr\}$-sequent. Thus in conjunction with Theorem 6, we obtain:

▶ **Corollary 19** (Lazić and Schmitz [26], Section 4.2.3). *The problem of determining if a given ?-prenex $\{\otimes, \parr\}$-sequent is provable in* **LLW** *is* Tower*-hard.*

6.1 Tower-hardness of deducibility in FLew and related systems

A *!-prenex $\{\multimap\}$-sequent* is an intuitionistic $\{\multimap, !\}$-sequent of the form $!\Gamma, \Delta \vdash C$ where the only connective occurring in Γ, Δ, C is $\multimap$. We prove:

▶ **Theorem 20.** *The problem of determining if a given !-prenex $\{\multimap\}$-sequent is provable in the $\{\multimap, !\}$-fragment of* **ILLW** *is* Tower*-hard.*

Proof. In view of Corollary 19, we reduce from the problem of whether a given ?-prenex $\{\otimes, \parr\}$-sequent is provable in **LLW**. Let $\vdash ?\Gamma, \Delta$ be a ?-prenex $\{\otimes, \parr\}$-sequent and x a new propositional variable not occurring in $?\Gamma, \Delta$. Theorem 2 guarantees that $\vdash ?\Gamma, \Delta$ is provable in **LLW** if and only if $(?\Gamma, \Delta)^{[x]} \vdash x$ is provable in **ILLW**. Clearly, the latter sequent forms a !-prenex $\{\multimap\}$-sequent. Due to the fact that **ILLW** admits cut-elimination, **ILLW** is a conservative extension of its $\{\multimap, !\}$-fragment; hence $(?\Gamma, \Delta)^{[x]} \vdash x$ is provable in **ILLW** if and only if it is provable in the $\{\multimap, !\}$-fragment of **ILLW**. We conclude that $\vdash ?\Gamma, \Delta$ is provable in **LLW** if and only if $(?\Gamma, \Delta)^{[x]} \vdash x$ is provable in the $\{\multimap, !\}$-fragment of **ILLW**. Hence the problem is hard for Tower. ◀

We furthermore show the following lemma:

▶ **Lemma 21.** *Let $!\Gamma, \Delta \vdash A$ be a !-prenex $\{\multimap\}$-sequent. The following statements are mutually equivalent:*
(1) *$!\Gamma, \Delta \vdash A$ is provable in the $\{\multimap, !\}$-fragment of* **ILLW**,
(2) *$\Delta \vdash A$ is provable in* $\mathbf{BCK}[\sigma(\Gamma)]$,
(3) *$\Delta \vdash A$ is provable in* $\mathbf{FL}^+_{\mathbf{ei}}[\sigma(\Gamma)]$,
(4) *$\Delta \vdash A$ is provable in* $\mathbf{FL}_{\mathbf{ei}}[\sigma(\Gamma)]$,
(5) *$\Delta \vdash A$ is provable in* $\mathbf{FL}_{\mathbf{ew}}[\sigma(\Gamma)]$.
Here $\sigma(\Gamma)$ stands for the support of Γ, i.e., the set of formulas that are contained in Γ at least once.

Proof. For starters, observe that the following claim holds:
(a) Let $!\Xi, \Sigma \vdash C$ be a !-prenex $\{\multimap\}$-sequent. If $!\Xi, \Sigma \vdash C$ is provable in the $\{\multimap, !\}$-fragment of **ILLW**, then $\Xi^n, \Sigma \vdash C$ is provable in **BCK** for some n.

Similarly to Lemma 10, this is shown by induction on the size of cut-free proofs in the $\{\multimap, !\}$-fragment of **ILLW**. To show that Statement (1) implies Statement (2), suppose that $!\Gamma, \Delta \vdash A$ is provable in the $\{\multimap, !\}$-fragment of **ILLW**. By Claim (a), $\Gamma^n, \Delta \vdash A$ is provable in **BCK** for some n. For each formula B in Γ, B is also in $\sigma(\Gamma)$. Hence we obtain a proof of $\Delta \vdash A$ in $\mathbf{BCK}[\sigma(\Gamma)]$, applying (Cut) several times. The implications $(2 \Rightarrow 3)$, $(3 \Rightarrow 4)$ and $(4 \Rightarrow 5)$ trivially hold.

We now embark on the proof of the remaining implication $(5 \Rightarrow 1)$. By straightforward induction on the length of derivations, we can show:

(b) Let Ξ be a finite multiset of formulas in $\mathbf{FL_{ew}}$ and $\Sigma \vdash \Pi$ a sequent of $\mathbf{FL_{ew}}$. If $\Sigma \vdash \Pi$ is provable in $\mathbf{FL_{ew}}[\sigma(\Xi)]$, then $!\Xi, \Sigma \vdash \Pi$ is provable in $\mathbf{ILZW}'$.

Let us assume that $\Delta \vdash A$ is provable in $\mathbf{FL_{ew}}[\sigma(\Gamma)]$. By Claim (b), $!\Gamma, \Delta \vdash A$ is provable in $\mathbf{ILZW}'$. By the cut elimination theorem for $\mathbf{ILZW}'$, we can easily check that $\mathbf{ILZW}'$ is conservative over the $\{\multimap, !\}$-fragment of **ILLW**. Hence $!\Gamma, \Delta \vdash A$ is provable in the $\{\multimap, !\}$-fragment of **ILLW**. ◀

The above lemma provides a polynomial time reduction from the provability problem of !-prenex $\{\multimap\}$-sequents in the $\{\multimap, !\}$-fragment of **ILLW** to deducibility in **BCK**, $\mathbf{FL^+_{ei}}$, $\mathbf{FL_{ei}}$, and $\mathbf{FL_{ew}}$; hence by Corollary 20 we obtain the TOWER-hardness of deducibility in these systems. In the same way as before, we can show:

▶ **Lemma 22.** *Let* $\vdash ?\Gamma, \Delta$ *be a ?-prenex* $\{\otimes, \parr, \&, \oplus, \mathbf{1}, \bot\}$*-sequent.* $\vdash ?\Gamma, \Delta$ *is provable in* **LLW** *if and only if* $\vdash \Delta$ *is provable in* $\mathbf{InFL_{ew}}[\sigma(\Gamma^\bot)]$.

By Corollary 19 and Lemma 22, the deducibility problem for $\mathbf{InFL_{ew}}$ is also hard for TOWER. The deducibility problems for **BCK**, $\mathbf{FL^+_{ei}}$, $\mathbf{FL_{ei}}$, $\mathbf{FL_{ew}}$, and $\mathbf{InFL_{ew}}$ are all in TOWER by Corollaries 14 and 17. We thus conclude:

▶ **Corollary 23.** *Each of the following decision problems is complete for TOWER:*

- *the deducibility problem for* **BCK***,*
- *the deducibility problem for* $\mathbf{FL^+_{ei}}$*,*
- *the deducibility problem for* $\mathbf{FL_{ei}}$*,*
- *the deducibility problem for* $\mathbf{FL_{ew}}$*,*
- *the deducibility problem for* $\mathbf{InFL_{ew}}$*.*

6.2 Tower-hardness of provability in elementary affine logic

▶ **Lemma 24.** *Let* Γ *be a finite multiset of classical* $\{\otimes, \parr, \&, \oplus, \mathbf{1}, \bot\}$*-formulas and* A *a classical* $\{\otimes, \parr, \&, \oplus, \mathbf{1}, \bot\}$*-formula.* $\vdash ?\Gamma, A$ *is provable in* **LLW** *if and only if* $\vdash ?\Gamma, !A$ *is provable in* **ELLW***.*

Proof. Suppose that $\vdash ?\Gamma, A$ is provable in **LLW**. By Lemma 11, $\vdash \Gamma^n, A$ is provable in $\mathbf{InFL_{ew}}$ for some n. Obviously, $\vdash \Gamma^n, A$ is provable in **ELLW** for some n. Using the rule of (F) and structural rules, we obtain a proof of $\vdash ?\Gamma, !A$ in **ELLW**. For the other direction, assume that $\vdash ?\Gamma, !A$ is provable in **ELLW**. It is provable in **LLW** since every sequent that is provable in **ELLW** is provable in **LLW**. Recall that $\vdash ?A^\bot, A$ is provable in **LLW** whereas it is not provable in **ELLW**. We therefore obtain a proof of $\vdash ?\Gamma, A$ in **LLW** by a single application of (Cut). ◀

It follows from Theorem 18 and Lemma 24 that $\mathcal{B}, Q_\ell \rhd_\ell q_r, \bar{0}$ if and only if $\vdash ?\ulcorner T \urcorner, ?Q_\ell, !q_r^\bot$ is provable in **ELLW**. Here the sequent $\vdash ?\ulcorner T \urcorner, ?Q_\ell, !q_r^\bot$ is built from literals using only connectives in $\{\otimes, \parr, !, ?\}$. We therefore obtain:

▶ **Corollary 25.** *The problem of determining if a given classical* $\{\otimes, \parr, !, ?\}$*-sequent is provable in* **ELLW** *is TOWER-hard, and hence so is the provability problem for* **ELLW***.*

Combining this with Corollary 9, we obtain:

▶ **Corollary 26.** *The provability problem for* **ELLW** *is* TOWER*-complete.*

Clearly, provability in the multiplicative-exponential fragment of **ELLW** is also complete for TOWER. The same holds for a very small fragment of **IELW**:

▶ **Theorem 27.** *The provability problem for the* $\{\multimap, !\}$*-fragment of* **IELW** *is* TOWER*-complete.*

Proof. Similar to the proof of Theorem 20. The problem is clearly in TOWER due to Corollary 9. Let $\vdash \Gamma$ be a classical $\{\otimes, \parr, !, ?\}$-sequent and x a fresh variable not occurring in this sequent. By Theorem 2, $\vdash \Gamma$ is provable in **ELLW** if and only if $\Gamma^{[x]} \vdash x$ is provable in **IELW**. By cut elimination for **IELW**, we know that **IELW** is a conservative extension of its $\{\multimap, !\}$-fragment. Thus $\Gamma^{[x]} \vdash x$ is provable in **IELW** if and only if it is provable in the $\{\multimap, !\}$-fragment of **IELW**. Consequently, $\vdash \Gamma$ is provable in **ELLW** if and only if $\Gamma^{[x]} \vdash x$ is provable in the $\{\multimap, !\}$-fragment of **IELW**. Hence there is a polynomial time reduction from the problem of whether a given classical $\{\otimes, \parr, !, ?\}$-sequent is provable in **ELLW** to provability in the $\{\multimap, !\}$-fragment of **IELW**. By Corollary 25, provability in the $\{\multimap, !\}$-fragment of **IELW** is hard for TOWER. ◀

7 Concluding remarks

We have shown the TOWER-completeness of deducibility in some contraction-free substructural logics without modal operators. We hope that our work sheds new light on computational aspects of fuzzy logic. This is because $\mathbf{FL_{ew}}$ forms a theoretical basis for a wide range of fuzzy logics. In fact, one can construct from $\mathbf{FL_{ew}}$ various fuzzy logics (such as monoidal t-norm based logic, basic logic, weak nilpotent minimum logic, Łukasiewicz logic and product logic) by adding some new axioms, cf. [13, 18]. Remarkably, the latter four of the above examples are shown to be coNP-complete with respect to both provability and deducibility; see [19] for a detailed survey. Together with those facts, our result suggests that there is a critical difference between $\mathbf{FL_{ew}}$ and its fuzzy extensions with respect to computational complexity.

We summarize the known complexity results in Tables 1 and 2. We adopt the notation of combinatory logic in Table 1:

- **BCI** = the implicational fragment of intuitionistic linear logic,
- **BCIW** = the extension of **BCI** by the rule of contraction,
- **BCK** = the extension of **BCI** by the rule of weakening.

BCIW is usually denoted by $\mathbf{R}_{\to}$ in the relevance logic community. $\mathbf{IL}_{\to}$ stands for the extension of **BCI** by weakening and contraction. It is nothing but the implicational fragment of intuitionistic propositional logic (**IL**). In Table 2, $\mathbf{FL_{e}}$ (resp. $\mathbf{FL_{ec}}$) denotes the extension of **BCI** (resp. **BCIW**) by the connectives $\otimes$, &, $\oplus$, **1**, and $\bot$.

Below we comment on some results not covered in the main sections.

Ackermannian complexity and deducibility in BCI. In [35] Schmitz also defined a non-primitive recursive complexity class by:

$$\text{ACKERMANN} := \bigcup_{f \in \mathsf{FPR}} \text{DTIME}(Ack(f(n)))$$

Table 1 Complexity results of extensions of **BCI**.

	Provability	Deducibility
BCI	NP-complete [7]	open (Ackermann-hard, cf. Corollary 31)
BCIW	2-ExpTime-complete [36]	decidable (2-ExpTime-hard, cf. [36])
BCK	NP-complete (Corollary 3)	Tower-complete (Corollary 23)
$\mathbf{IL}_{\to}$	Pspace-complete [37]	Pspace-complete [37]

Table 2 Complexity results of extensions of $\mathbf{FL_e}$.

	Provability	Deducibility
$\mathbf{FL_e}$	Pspace-complete [28]	undecidable [28]
$\mathbf{FL_{ec}}$	Ackermann-complete, cf. [26, 41]	Ackermann-complete, cf. [26, 41]
$\mathbf{FL_{ei}}$	Pspace-complete, cf. [21]	Tower-complete (Corollary 23)
$\mathbf{FL_{ew}}$	Pspace-complete, cf. [21]	Tower-complete (Corollary 23)
$\mathbf{FL_{ewc}}$ (= **IL**)	Pspace-complete [37]	Pspace-complete [37]

where *Ack* is the Ackermannian function and FPR denotes the set of primitive recursive functions. The class contains the class PR of primitive recursive problems, and is closed under primitive recursive reductions. As in [26, 35], we define the notion of *Ackermann-hardness* using primitive recursive reductions.

The decidability of deducibility in **BCI** is a long-standing open problem. However, one reviewer pointed out that the Ackermann-hardness of deducibility in **BCI** follows from the recent breakthrough by Czerwiński and Orlikowski [9], and Leroux [27], who independently proved that the reachability problem for VASSs is Ackermann-hard.

Let us sketch the proof of the Ackermannian lower bound for deducibility in **BCI**. Clearly, by the aforementioned result in [9, 27], reachability in BVASSs is also hard for Ackermann. As is the case of lossy reachability for ABVASS$_{\bar{0}}$s, the BVASS-reachability problem is reduced to reachability in ordinary BVASSs (cf. Section 4); see [26, Lemma 3.5] for details. Furthermore, reachability in ordinary BVASSs can be efficiently encoded to the problem of whether classical linear logic (**LL**), i.e., **LLW** without the rule of (W), proves a given ?-prenex $\{\otimes, ⅋\}$-sequent; see [26, Section 4.2.2]. To sum up, the following holds:

▶ **Theorem 28** ([9, 26, 27])**.** *The problem of determining if a given ?-prenex $\{\otimes, ⅋\}$-sequent is provable in* **LL** *is Ackermann-hard.*

The translation in Section 2.2 also holds between **LL** and **ILL** (i.e., **ILLW** without the unrestricted left-weakening); see the full version for a sketch of a proof.

▶ **Theorem 29.** *Let $\vdash \Gamma$ be a classical $\mathcal{L}_C$-sequent and x a fresh propositional variable not occurring in Γ. $\vdash \Gamma$ is provable in* **LL** *if and only if $\Gamma^{[x]} \vdash x$ is provable in* **ILL***.*

Using this, we show the intuitionistic version of Theorem 28.[2]

▶ **Corollary 30** ([9, 12, 27])**.** *The problem of determining if a given !-prenex $\{\multimap\}$-sequent is provable in* **ILL** *is Ackermann-hard.*

[2] In [12] de Groote, Guillaume and Salvati provided a reduction from BVASS-reachability to provability of !-prenex $\{\multimap\}$-sequents in **ILL**. Thus one can also show Corollaries 30 and 31, using the results in [9, 12, 27].

Proof. By Theorem 28, it suffices to show that there is a reduction from the provability of ?-prenex $\{\otimes, \parr\}$-sequents in **LL** to the provability of !-prenex $\{\multimap\}$-sequents in **ILL**. The proof is essentially the same as that of Theorem 20, but we use Theorem 29 instead of Theorem 2. Let $\vdash ?\Gamma, \Delta$ be a ?-prenex $\{\otimes, \parr\}$-sequent and x a fresh propositional variable not in the sequent. By Theorem 29, $\vdash ?\Gamma, \Delta$ is provable in **LL** if and only if $(?\Gamma, \Delta)^{[x]} \vdash x$ is provable in **ILL**. Obviously, the latter sequent is a !-prenex implicational sequent. ◀

▶ **Corollary 31.** *The deducibility problem for* **BCI** *is* Ackermann*-hard.*

Proof. In view of Corollary 30, our goal is to show that: for any !-prenex $\{\multimap\}$-sequent $!\Gamma, \Delta \vdash C$, $\Delta \vdash C$ is provable in $\mathbf{BCI}[\sigma(\Gamma)]$ if and only if $!\Gamma, \Delta \vdash C$ is provable in **ILL**.

The proof of the *if* part goes by induction on the length of the cut-free proof of $!\Gamma, \Delta \vdash C$ in **ILL**. We only analyze the case where the rule of (!D) is applied last. If $!\Gamma, !A, \Delta \vdash C$ is obtained from $!\Gamma, A, \Delta \vdash C$ by an application of (!D), then by the induction hypothesis, $A, \Delta \vdash C$ is provable in $\mathbf{ILL}[\sigma(\Gamma)]$; thus it is also provable in $\mathbf{ILL}[\sigma(\Gamma, A)]$. Since $\vdash A$ is a non-logical axiom of $\mathbf{ILL}[\sigma(\Gamma, A)]$, we may construct a derivation of $\Delta \vdash C$ in $\mathbf{ILL}[\sigma(\Gamma, A)]$ by using (Cut). The remaining cases are similar.

The converse direction is shown by induction on the length of the derivation for $\Delta \vdash C$ in $\mathbf{BCI}[\sigma(\Gamma)]$. All cases are straightforward. ◀

Tower-hardness of light affine logic. We have also shown the Tower-completeness of provability in elementary affine logic. On the other hand, one can define another subsystem of affine logic, called *intuitionistic light affine logic* (**ILAL**) [1, 38], which characterizes polynomial time functions. We obtain it from the $\{\multimap, !\}$-fragment of **ILLW** by dropping the rules for (!D) and (!P), and by adding a new unary connective "§" and the following inference rules:

$$\frac{E \vdash A}{!E \vdash !A}\ (!) \qquad\qquad \frac{\Gamma, \Delta \vdash A}{!\Gamma, \S\Delta \vdash \S A}\ (\S)$$

where E is a formula or the empty multiset. It is known that provability in **ILAL** is decidable because Terui proved that it has the finite model property; see [38, Corollary 7.45]. We show that there is no elementary recursive algorithm for solving the provability problem for **ILAL**:

▶ **Lemma 32.** *Let* $!\Gamma, \Delta \vdash A$ *be a !-prenex* $\{\multimap\}$*-sequent.* $!\Gamma, \Delta \vdash A$ *is provable in the* $\{\multimap, !\}$*-fragment of* **ILLW** *if and only if* $!\Gamma, \S\Delta \vdash \S A$ *is provable in* **ILAL**.

Proof. Our proof is inspired by the proof of the undecidability of provability in light linear logic by Terui; see [38, Section 2.3.2]. For any intuitionistic $\{\multimap, !, \S\}$-formula A, we inductively define the intuitionistic $\{\multimap, !\}$-formula A^- by $p^- = p$, $(B \multimap C)^- = B^- \multimap C^-$, $(!B)^- = !B^-$, and $(\S B)^- = B^-$. It is easy to see that if $\Sigma \vdash C$ is provable in **ILAL** then $\Sigma^- \vdash C^-$ is provable in the $\{\multimap, !\}$-fragment of **ILLW**, for any sequent $\Sigma \vdash C$ of **ILAL**. This is checked by induction on the size of proofs. Thus the *if* direction holds. The *only-if* direction follows from Claim (a) used in the proof of Lemma 21, the rule of (§), and the structural rules. ◀

By Theorem 20, the provability problem for **ILAL** is hard for Tower. We strongly believe that this problem is in Tower.

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