



The Complexity of Translationally Invariant Problems Beyond Ground State Energies

James D. Watson  

University College London, UK

Johannes Bausch  

University of Cambridge, UK

Sevag Gharibian  

Universität Paderborn, Germany

Abstract

The physically motivated quantum generalisation of k -SAT, the k -Local Hamiltonian (k -LH) problem, is well-known to be QMA-complete (“quantum NP”-complete). What is surprising, however, is that while the former is easy on 1D Boolean formulae, the latter remains hard on 1D local Hamiltonians, *even if all constraints are identical* [Gottesman, Irani, FOCS 2009]. Such “translation-invariant” systems are much closer in structure to what one might see in Nature. Moving beyond k -LH, what is often more physically interesting is the computation of *properties of the ground space* (i.e. “solution space”) itself. In this work, we focus on two such recent problems: Simulating local measurements on the ground space (APX-SIM, analogous to computing properties of optimal solutions to MAX-SAT formulae) [Ambainis, CCC 2014], and deciding if the low energy space has an energy barrier (GSCON, analogous to classical reconfiguration problems) [Gharibian, Sikora, ICALP 2015]. These problems are known to be $P^{\text{QMA}[\log]}$ - and QCMA-complete, respectively, in the general case. Yet, to date, it is not known whether they remain hard in such simple 1D translationally invariant systems.

In this work, we show that the 1D translationally invariant versions of both APX-SIM and GSCON are intractable, namely are $P^{\text{QMA}_{\text{EXP}}}$ - and QCMA_{EXP} -complete (“quantum P^{NEXP} ” and “quantum NEXP”), respectively. Each of these results is attained by giving a respective generic “lifting theorem”. For APX-SIM we give a framework for lifting any abstract local circuit-to-Hamiltonian mapping H satisfying mild assumptions to hardness of APX-SIM on the family of Hamiltonians produced by H , while preserving the structural properties of H (e.g. translation invariance, geometry, locality, etc). Each result also leverages counterintuitive properties of our constructions: for APX-SIM, we compress the answers to polynomially many parallel queries to a QMA oracle into a single qubit. For GSCON, we show strong robustness, i.e. soundness even against adversaries acting on all but a single qudit in the system.

2012 ACM Subject Classification Theory of computation → Problems, reductions and completeness; Theory of computation → Oracles and decision trees; Theory of computation → Quantum complexity theory

Keywords and phrases Complexity, Quantum Computing, Physics, Constraint Satisfaction, Combinatorial Reconfiguration, Many-Body Physics

Digital Object Identifier 10.4230/LIPIcs.STACS.2023.54

Related Version *Full Version:* <https://arxiv.org/abs/2012.12717>

Funding *James D. Watson:* EPSRC Centre for Doctoral Training in Delivering Quantum Technologies (grant EP/L015242/1).

Johannes Bausch: Draper’s Research Fellowship at Pembroke College.

Sevag Gharibian: DFG grant 432788384



© James D. Watson, Johannes Bausch, and Sevag Gharibian; licensed under Creative Commons License CC-BY 4.0

40th International Symposium on Theoretical Aspects of Computer Science (STACS 2023).
Editors: Petra Berenbrink, Patricia Bouyer, Anuj Dawar, and Mamadou Moustapha Kanté;
Article No. 54; pp. 54:1–54:21



Leibniz International Proceedings in Informatics

LIPICs Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany



1 Introduction

The quantum generalisation of a Boolean constraint satisfaction problem such as a 3-SAT formula is known as a *k-local Hamiltonian*, $H = \sum_i H_i$. Here, a Hermitian operator H acts on N quantum systems of dimension $d \in O(1)$, so that H is a $d^N \times d^N$ complex matrix. Yet, H has a succinct description of $\text{poly}(N)$ bits, in that each H_i acts non-trivially¹ on only $k \in O(1)$ qudits, and hence requires $O(1)$ bits to specify. Any Hamiltonian H captures the static and dynamic properties of some many-body quantum system (via the Schrödinger equation), such as its ground state energy, spectral gap, and time-evolution.

For this reason, the complexity of computing properties of local Hamiltonians has seen intense interest in the last two decades (e.g., [32, 15] for surveys). The “benchmark” problem here has been the quantum generalisation of MAX- k -SAT – approximating the ground state energy of H (i.e. smallest eigenvalue $\lambda_{\min}(H)$ of H , which represents the energy level the system settles into when cooled to low temperature). This is the *Local Hamiltonian Problem (LH)*, known to be QMA-complete [26]. (Quantum-Merlin Arthur (QMA) is a quantum generalisation of Merlin-Arthur (MA), with a quantum proof and verifier.) LH remains hard on physically motivated setups, such as qubits on a 2D lattice [31] and 1D chains of local dimension 8 [2, 30, 22]. Amazingly, it is QMA_{EXP}-complete² even for *1D translationally invariant, nearest neighbour* systems [21], meaning on a line of N qudits, with each constraint $H_{i,i+1}$ being *identical* for $i \in [N - 1]$ and each constraint only acting between neighbouring pairs of qudits. This in stark contrast to 1D *classical* constraint satisfaction, which can be efficiently solved via divide-and-conquer in time polynomial in the length of the chain (even in non-translationally invariant systems).

Beyond ground state energies. From a physical perspective, what is often more interesting than ground state energies is computing *properties of the ground space* itself – a problem analogous to computing properties of optimal MAX- k -SAT assignments. In this direction, recent works have studied determining a system’s density of states [10, 34]; minimising interaction terms yielding frustrated ground spaces [16]; deciding if a ground space has an energy barrier [18, 20]; simulating local measurements on ground spaces [3, 19, 17]; estimating spectral gaps of local Hamiltonians [3, 12, 19]; and “universal” Hamiltonian models simulating other quantum many-body systems [8, 11, 33, 27]. Here, we focus on two of these problems: Simulating local measurements on ground spaces (APX-SIM) [3] and deciding if a ground space has an energy barrier (GSCON) [18].

APX-SIM. The first problem, *Approximate Simulation (APX-SIM)*, asks: How difficult is it to simulate a local measurement on a ground state of a local Hamiltonian? Given that much of condensed matter physics is devoted to determining the low-energy properties of materials, and that local measurements are the only tools available to experimentalists to examine these systems, this is an extremely important problem.

► **Definition 1** (APX-SIM(H, A, k, l, a, b, δ)) [3]). *Given k -local Hamiltonian $H = \sum_i H_i$ on N qubits, l -local observable A , and $a, b, \delta \in \mathbb{R}$ such that $b - a \geq N^{-c}$ and $\delta \geq N^{-c'}$, for $c, c' > 0$ constant, decide:*

YES. *If H has a ground state $|\psi\rangle$ satisfying $\langle \psi | A | \psi \rangle \leq a$.*

NO. *If for all $|\psi\rangle$ satisfying $\langle \psi | H | \psi \rangle \leq \lambda_{\min}(H) + \delta$, it holds that $\langle \psi | A | \psi \rangle \geq b$.*

¹ Formally, if H_i acts on a subset S_i of qudits, it is specified via $H_i \otimes I_{[N] \setminus S_i}$, for $[N] = \{1, \dots, N\}$.

² QMA_{EXP} (Definition 20) is a quantum analogue of NEXP, meaning an exp-length quantum proof and exp-time quantum verifier.

We will also be interested in the version of the problem with a translationally invariant Hamiltonian.

► **Definition 2** (TI-APX-SIM($N, H_i, A, k, l, a, b, \delta$) [3]). *Defined similarly to Definition 1 except the input is now the systems size N described in $n = O(\log(N))$ many bits, and a local interaction term H_i . The overall Hamiltonian is a translationally invariant Hamiltonian on N qudits $H = \sum_i H_i$, with parameters $b - a \geq N^{-c} = O(2^{-cn})$, $\delta \geq N^{-c'} = O(2^{-c'n})$, for $c, c' > 0$.*

APX-SIM was originally shown $P^{\text{QMA}[\log]}$ -complete for 5-local Hamiltonians with 1-local measurements [3, 19]. Here, $P^{\text{QMA}[\log]}$ is the quantum analogue of $P^{\text{NP}[\log]}$, meaning all languages decidable by a P machine making logarithmically many queries to a QMA oracle. It was subsequently shown that APX-SIM remains $P^{\text{QMA}[\log]}$ -complete on physically motivated 2D models, and on (non-translationally invariant) 1D chains [17]. While it is unlikely that $\text{QMA} = P^{\text{QMA}[\log]}$ (since $P^{\text{QMA}[\log]}$ trivially contains co-QMA), $P^{\text{QMA}[\log]}$ is “not too much harder” than QMA, in that $P^{\text{QMA}[\log]} \subseteq \text{PP}$ [19]. Finally, in this work we use $P^{\text{QMA}[\log]} = P^{\parallel\text{QMA}}$ [17], where $P^{\parallel\text{QMA}}$ allows polynomially many parallel queries to the QMA oracle.

GSCON. The second problem we study is *Ground State Connectivity (GSCON)* [18], which asks: given two ground states $|\psi\rangle$ and $|\phi\rangle$ of a local Hamiltonian, is there a low energy path connecting $|\psi\rangle$ to $|\phi\rangle$? Physically, this captures the question of determining if a ground space has an energy barrier.

► **Definition 3** (Ground State Connectivity (GSCON ($H, \eta_1, \eta_2, \eta_3, \eta_4, \delta, b, m, U_\psi, U_\phi, |\psi\rangle, |\phi\rangle$)) [18]). *Let $H = \sum_i H_i$ be a k -local Hamiltonian on N qubits. Consider parameters $\eta_1, \eta_2, \eta_3, \eta_4, \delta \in \mathbb{R}$, and $m \in \mathbb{Z}^+$, with $\eta_2 - \eta_1, \eta_4 - \eta_3 \geq \delta$. Let U_ψ and U_ϕ be poly(N)-size quantum circuits generating “start” and “target” states $|\psi\rangle = U_\psi |0 \dots 0\rangle$ and $|\phi\rangle = U_\phi |0 \dots 0\rangle$, respectively, satisfying $\langle \psi | H | \psi \rangle \leq \eta_1$ and $\langle \phi | H | \phi \rangle \leq \eta_1$. Output:*

YES: *If there exists a sequence of b -local unitaries $(U_i)_{i=1}^m$ such that:*

1. *(Intermediate states remain in low energy space) For all $i \in [m]$ and intermediate states $|\psi_i\rangle := U_i \dots U_2 U_1 |\psi\rangle$, one has $\langle \psi_i | H | \psi_i \rangle \leq \eta_1$, and*
2. *(Final state close to target state) $\|U_m \dots U_1 |\psi\rangle - |\phi\rangle\|_2 \leq \eta_3$.*

NO: *If for all b -local sequences of unitaries $(U_i)_{i=1}^m$, either:*

1. *(Intermediate state obtains high energy) There exists $i \in [m]$ and an intermediate state $|\psi_i\rangle := U_i \dots U_2 U_1 |\psi\rangle$, such that $\langle \psi_i | H | \psi_i \rangle \geq \eta_2$, or*
2. *(Final state far from target state) $\|U_m \dots U_1 |\psi\rangle - |\phi\rangle\|_2 \geq \eta_4$.*

The translationally invariant version of GSCON can be specified in a similar way, where instead the Hamiltonian is specified by a TI local interaction terms and the systems size.

GSCON is QCMA-complete for 5-local Hamiltonians [18], and remarkably (and in contrast to LH) remains hard even for *commuting* Hamiltonians (i.e. when H_i and H_j commute for all pairs) [20]. (Quantum-Classical Merlin Arthur (QCMA) is QMA, except with a *classical* witness.) GSCON is motivated via quantum memories and stabilizer codes. For example, a Hamiltonian H for a YES instance of GSCON has a short sequence of local unitaries mapping between low energy states $|\psi\rangle$ and $|\phi\rangle$ through the low energy space of H . In a quantum memory, $|\psi\rangle$ and $|\phi\rangle$ may encode logical states. As errors in physical systems are often local, this implies H might not be a good quantum memory – not only can a short adversarial circuit corrupt $|\psi\rangle$ to $|\phi\rangle$ (since there is no energy barrier “separating” $|\psi\rangle$ from $|\phi\rangle$), but this corrupting process takes place completely in the low energy space, meaning such errors are not easily detectable.

When are these problems relevant to physics? From a quantum complexity theoretic standpoint, a key goal is to show that even local Hamiltonian systems mirroring those in Nature can encode “hard” problems. *Translationally-invariant* Hamiltonians, in particular, not only possess symmetries seen in Nature, but are also believed simpler than general Hamiltonians. Intuitively, due to the spatial invariance of the system, the degrees of freedom available to encode complex behaviour appears limited. A second physically motivated restriction is *spatial structure*, such as one-dimensional systems, which can sometimes be more tractable [7, 29, 1, 36, 28, 14]. Combining these two properties, we arrive at one of the simplest systems imaginable – 1D, translationally-invariant (TI) systems. Are these “more tractable” than their higher-dimensional counterparts? As mentioned earlier, LH on such systems surprisingly turns out to be QMA_{EXP} -complete [21], and the question of existence of a spectral gap undecidable [5]. Yet for natural problems such as APX-SIM and GSCON, the verdict is still open.

1.1 Results

In this work, we give generic “lifting frameworks”, which we then apply to show (a) that APX-SIM and GSCON remain “hard” in the 1D TI setting and (b) the Local Hamiltonian problem determines the complexity of these problems. We now discuss these results in depth.

A lifting framework for APX-SIM

We begin with a generic framework for “lifting” hardness results about ground state energies (i.e. LH) to hardness results for APX-SIM. Formally achieved via our Lifting Lemma (Lemma 10) and applications thereof in Section 2.3, the general premise is informally:

► **Theorem 4** (LH to APX-SIM (informal)). *If a family of Hamiltonians \mathcal{F} admits a circuit-to-Hamiltonian mapping H_w such that approximating the ground state energy for \mathcal{F} is C -hard (for complexity class C), then APX-SIM for \mathcal{F} is $\text{P}^{C[\log]}$ - or P^C -hard for non-TI and TI Hamiltonians, respectively.*

The key point of the lifting map underlying Theorem 4 is that it automatically *preserves structural properties* of \mathcal{F} , such as locality, geometry, translational invariance, etc. This has two advantages: First, it obviates the need to reprove hardness for APX-SIM each time a new physically motivated circuit-to-Hamiltonian construction H_w is discovered (modulo mild assumptions on H_w as per Definition 9). Second, it reveals that LH itself fundamentally characterises the complexity of computing properties, such as simulating measurements on the low energy states of a family of Hamiltonians.

For clarity, throughout this work, 1D TI versions of computational problems assume the input size is n , whereas the length of the 1D chain is $N \in O(\exp(n))$. This is because in the TI setup, it is standard for the input to be given succinctly by (1) the length of the chain in binary and (2) a description of the single $H_{i,i+1}$ term to be repeated along the chain [21].

► **Theorem 5.** *APX-SIM is $\text{P}^{\text{QMA}_{\text{EXP}}}$ -complete for 1D TI, nearest neighbour, Hamiltonians on N qudits of local dimension 44, for $\delta = \Omega(1/\text{poly}(N))$, $b - a = \Omega(1/\text{poly}(N))$.*

We also obtain PSPACE-completeness for 1D TI APX-SIM when $\delta, b - a \in \Omega(1/\exp(N))$. We thus find the first known hardness result for APX SIM in the TI setting. Two points worth highlighting: First, counter-intuitively, our construction “stores” the answers to m QMA queries into a *single* qubit. A similar phenomenon is trivially impossible classically. This “compression” is what allows us to make our setup so generic. Second, one of the steps

to establishing Theorem 5 is to show $\text{EXP}^{\parallel\text{QMA}} = \text{P}^{\text{QMA}_{\text{EXP}}}$. In other words, an exp-time Turing machine making exponentially many parallel QMA queries is equivalent to a poly-time machine making poly-many adaptive queries to QMA_{EXP} .

Hardness of Ground State Connectivity (GSCON)

Via different techniques, we next give a Lifting Lemma for GSCON (Lemma 15), obtaining the first GSCON hardness result in a physically motivated setting:

► **Theorem 6.** *GSCON is QCMA_{EXP} -complete for 1D TI Hamiltonians on N qudits of constant local dimension, for $m \in \text{poly}(N)$, $\delta \in \Theta(1/\text{poly}(N))$, and any $b \in \{2, \dots, N - 1\}$.*

Here, QCMA_{EXP} (Definition 21) is to QCMA as QMA_{EXP} is to QMA , i.e. one has an exp-long classical proof and exp-time quantum verifier. Worth highlighting is that, perhaps surprisingly, Theorem 6 holds even for $b = N - 1$. In words, even if an adversary can act jointly on *all but a single qudit* per time step, our construction remains sound. This is significantly more robust than [18], and is tight, since an adversary acting on all N qudits can trivially cheat by mapping $|\psi\rangle$ to $|\phi\rangle$ via a single N -qudit unitary. We remark that the applicability of our lifting theorem for GSCON is not as wide as that for APX-SIM; details in Section 1.2.

1.2 Techniques

Circuit-to-Hamiltonian Mappings

We begin with a brief overview of circuit-to-Hamiltonian mappings. In the literature, a *circuit-to-Hamiltonian mapping* roughly means a map which takes as input any quantum circuit $U = U_T U_{T-1} \dots U_1$ (e.g. consisting of 2-qubit gates U_i), where U acts on some Hilbert space \mathcal{H}_q , and outputs a so-called “history state Hamiltonian” H , whose low energy space “encodes” U . The prototypical example is Kitaev’s construction [26], which outputs $H = H_{\text{in}} + H_{\text{prop}} + H_{\text{out}} + H_{\text{stab}}$, where H_{in} forces the input of U to be initialised correctly, H_{prop} that each gate of U follows correctly after all previous gates are applied, H_{out} checks that U accepts, and H_{stab} ensures the clock register is encoded correctly. (The Cook-Levin theorem has analogous Boolean formulae for $H_{\text{in}}, H_{\text{prop}}, H_{\text{out}}$, but does not require H_{stab} , as time is explicitly encoded via rows of the table.) Of these, the most relevant to our discussion is

$$H_{\text{prop}} = \sum_{t=0}^{T-1} h_t \quad \text{where} \quad h_t := \sum_{|e\rangle} (|t\rangle|e\rangle - |t+1\rangle U_t |e\rangle) (\langle t| \langle e| - \langle t+1| \langle e| U_t^\dagger),$$

where we sum over a basis $\{|e\rangle\}$ for \mathcal{H}_q . The “history states” are then any state $|\psi\rangle$ of the following form (which span the null space of H_{prop})

$$|\psi\rangle = \frac{1}{\sqrt{T}} \sum_{t=0}^T |t\rangle |\psi_t\rangle \quad \text{where} \quad |\psi_t\rangle := U_t U_{t-1} \dots U_1 |\psi_0\rangle \tag{1}$$

for any $|\psi_0\rangle \in \mathcal{H}_q$. For our purposes, we formulate precise definitions of such mappings (Definition 9 and Definition 26) in order to rigorously prove our lifting theorems with as broad generality as possible.

For APX-SIM

To make our lifting framework as general as possible, instead of focusing on $P^{\parallel\text{QMA}}$ (which equals $P^{\text{QMA}[\log]}$ [17]), we consider arbitrary classes $D^{\parallel\text{Q}}$. Briefly, D is any “deterministic class” (e.g. P , EXP ; Definition 22), Q any “existentially quantified quantum verification classes (QVClass)” Q (e.g. NP , QCMA , QMA ; Definition 23). We allow almost arbitrary “local circuit-to-Hamiltonian mappings” H_w (e.g. Kitaev [25]; Definition 9), the primary requirement of which is that a measurement in the standard basis on the first and second output qubits of any unitary U can be simulated by a measurement on the low energy space of Hamiltonian $H_w(U)$. Formal requirements with all minor assumptions in Definition 9.

With just this basic property of measuring two qubits of the ground state in hand, we give a Lifting Lemma (Lemma 7) which takes any $D^{\parallel\text{Q}}$ computation and embeds it in an APX-SIM instance, while automatically preserving structural properties of the circuit-to-Hamiltonian mapping H_w . At a high level, we begin with an idea similar to the 1D non-TI $P^{\text{QMA}[\log]}$ -hardness result of [17] by replacing all parallel oracle calls to Q with explicit *verification circuits* for Q . In contrast to [17], we then “count” the number of YES Q -queries via a single qubit – each time a Q -verifier outputs YES, we rotate a designated “flag qubit” by a small fixed amount. We then push this entire “bootstrapped” computation through the circuit-to-Hamiltonian mapping H_w , followed by use of the “primary requirement” above to simulate a penalty on the flag qubit. Remarkably, by carefully adjusting the weight on this one flag qubit, with high probability we can force *all* Q -queries to simultaneously be answered correctly. *A priori*, this is perhaps surprising; for example, Holevo’s theorem [23] says that n qubits cannot transmit more than n bits of information, and yet here we are cramming polynomially many query answers into a single flag qubit, while still meaningfully utilising the information therein.

This flag qubit construction now allows us to circumvent the 1D TI restrictions (since there is only a *single* flag qubit to keep track of, we are not worried about how it is arranged geometrically within the final system). Additionally, a key part of the soundness analysis is an exchange argument (Lemma 11), which may be of independent interest: given a joint entangled proof $|w_{1\dots m}\rangle$ to m Q -verifiers V_i , if the i^{th} local component of $|w_{1\dots m}\rangle$ is ϵ -suboptimal for verifier V_i , we give a rigorous lower bound on the deviation from the optimal “counted sum” on the flag qubit.

For GSCON

Using different techniques, we next give a generic Lifting Lemma for GSCON (Lemma 15), although less generic than what we are able to achieve for APX-SIM. Namely, we restrict attention to quantum verification classes such as QCMA or QCMA_{EXP} (since we require the ability to prepare low energy/history states efficiently in the YES case), and to general 1D TI circuit-to-Hamiltonian mappings (again, with mild restrictions; see Definition 26). We then apply Lemma 15 to the 1D TI Gottesman-Irani construction [21] to obtain QCMA_{EXP} -hardness of GSCON on 1D TI systems (Theorem 6).

At a high level, the starting setup for our lifting framework is similar to [18], which we briefly review: given a QCMA verification circuit V , apply Kitaev’s circuit-to-Hamiltonian construction to obtain local Hamiltonian $H = \sum_i H_i$, such that if V is a YES (NO) instance, $\lambda_{\min}(H)$ is small (large). Then, attach a “switch gadget” to H to obtain a new Hamiltonian H' , so that any polynomial-length traversal of the low energy space of H' , from start state $|\psi\rangle$ to target state $|\phi\rangle$, forcibly “switches on” all terms of H . In the NO case, switching on H incurs a large energy penalty, i.e. we hit the claimed energy barrier.

Extending this to the 1D TI setting presents various challenges. First, the construction of [18] is highly non-local geometrically, as each switch qubit is coupled to *all* local terms H_i of H . In order to maintain this level of coupling in 1D, we first use an idea reminiscent of the space-time circuit-to-Hamiltonian construction of Breuckmann and Terhal [9], and instead endow each qudit on the 1D chain with its *own* “local switch qudit”. We then wish to add “string constraints” on these “local switch qubits” to force a prover to “switch on” each term $H_{i,i+1}$ of the 1D Hamiltonian one at a time on the chain. But here we face an additional pair of challenges: first, any naïve implementation of string constraints allows a cheating prover to switch on only $N - b$ of the chain’s local terms $H_{i,i+1}$ – this is because the prover is allowed to apply arbitrary b -local unitaries in each step, allowing it to “shortcut” the last b switch qudits in one step. Second, we cannot satisfy the desired completeness properties for GSCON by simply switching on local terms $H_{i,i+1}$ iteratively from left to right. Rather, we must allow a *non-linear* order of activation.

It turns out that, not only can both the second and third challenges above be addressed in a unified black-box fashion, but the unified fix will also make the construction remarkably robust from a soundness perspective. Specifically, we first increase the local switch Hilbert space dimension to 7, which roughly will allow non-linear activation orders when switching on the local constrain terms. We then carefully construct our string constraints so that any ground space evolution satisfying said constraints becomes “trapped” in a low-dimensional joint switch subspace on all qudits. This low-dimensional space is precisely set up to achieve two things: (1) force *all* local terms of H to be simultaneously switched on, and (2) be “logically protected” from any switch subspace deviating from property (1) by a “string” of local unitaries of length $\Theta(N)$. The formal proof of correctness uses, among other tools, the Traversal Lemma (Lemma 14) of [18, 20]. Perhaps counterintuitively, soundness holds even if a cheating prover can apply $(N - 1)$ -local unitaries in each step, i.e. can act on *all but one* qudit of the chain per step.

1.3 Open questions

For APX-SIM, our lifting framework not only simplifies existing $\text{P}^{\text{QMA}[\log]}$ -hardness proofs of APX-SIM [3, 19, 17], but also yields new hardness results, notably for 1D TI systems: $\text{P}^{\text{QMA}_{\text{EXP}}}$ -completeness and PSPACE-completeness for inverse polynomial and inverse exponential precision (with respect to the length of the chain), respectively. Can our techniques, such as “compressing” multiple queries into a single qubit (Lemmas 7) and 11, find use elsewhere in studying quantum oracle classes? Can our APX-SIM results be generalised to yet more physical Hamiltonians (e.g. using Hamiltonian simulation techniques [17])? For GSCON, does our “logically protected” switch subspace design have applications beyond complexity theory, e.g. to robust quantum memories? Can our GSCON lifting framework be generalised to the broader class of “local circuit-to-Hamiltonian constructions” (Definition 9), as in APX-SIM? Most interestingly, do there exist non-trivial classes of Hamiltonians for which GSCON is easy?

Organisation

Section 2 and Section 3 give detailed proof sketches of our main results.

Notation

$\text{Herm}(\mathcal{X})$ and $\text{U}(\mathcal{X})$ denote the sets of linear and unitary operators acting on space \mathcal{X} , respectively. For $H \in \text{Herm}(\mathcal{X})$, $\lambda_{\min}(H)$ is its smallest eigenvalue, and $\Delta(H)$ its spectral gap (i.e. gap between two smallest distinct eigenvalues of A), i.e. $\Delta(A) := \lambda_1(A) - \lambda_{\min}(A)$. $\text{Null}(A)$ is the null-space/kernel of A .

2 Hardness via Lifting for APX-SIM

We now prove our lifting results for APX-SIM. Where appropriate, full proofs are deferred or can be found in [35].

2.1 Reducing $P^{\parallel\text{QMA}}$ to a single quantum verification circuit

While it is instructive to view $P^{\parallel\text{QMA}}$ as the “guiding example” for this section, our statements below apply more generally to classes of form $D^{\parallel\text{Q}}$. Namely, let D be a *Deterministic Decision Class* if the set of languages it contains can be deterministically decided with some time and space resources as a function of the input length (e.g. P , $PSPACE$, EXP , etc., Definition 22). Let Q be an *Existentially quantified quantum verification class (QVClass)* if it consists of promise problems verifiable by a uniform family of quantum verifiers given access to a quantum proof $|\psi\rangle$, and with some completeness/soundness parameters c and s (e.g. NP , QMA , QMA_{EXP} . Definition 23 makes no restrictions on c , s , uniformity resources, etc).

The first step of our construction is to map an arbitrary $D^{\parallel\text{Q}}$ computation to a single “verification circuit”.

► **Lemma 7.** *Let $x \in \{0, 1\}^n$ be an instance of a problem in $D^{\parallel\text{Q}}$, which is decided by $D^{\parallel\text{Q}}$ machine U . There exists an efficiently computable (in encoding sizes of x and U) quantum circuit V satisfying:*

1. V takes as input $m+2$ registers: input register A containing $x \in \{0, 1\}^n$, m proof registers B_i containing joint quantum proof $|w_{1\dots m}\rangle$, with register B_i to be verified by a Q -circuit V_i (Figure 1). Without loss of generality, each verifier V_i has the same completeness and soundness parameters c and s , respectively.
2. V has two designated output wires: q_{out} encodes the output of U , and q_{flag} , the state of which encodes the number of Q queries made by U which were YES instances. Let $|\psi\rangle$ denote the output state of V , given joint proof $|w_{1\dots m}\rangle$.

Let S_0 and S_1 partition $\{0, 1\}^m$ such that the D machine underlying U rejects (accepts) given a string of query responses $y \in S_0$ ($y \in S_1$). Define $p_{y,w} := \Pr\left(\bigwedge_{i=1}^m V_i \text{ outputs } y_i \mid |w_{1\dots m}\rangle\right)$. Then,

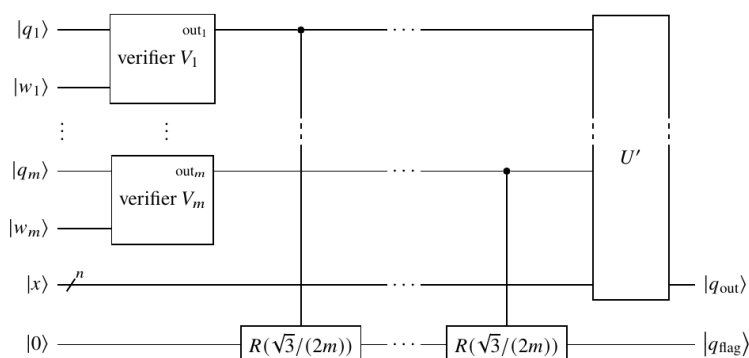
$$\text{Tr}\left(|\psi\rangle\langle\psi| \cdot |1\rangle\langle 1|_{q_{\text{out}}}\right) = \sum_{y \in S_1} p_{y,w} \quad (2)$$

$$\text{Tr}\left(|\psi\rangle\langle\psi| \cdot |1\rangle\langle 1|_{q_{\text{flag}}}\right) = \sum_{y \in \{0,1\}^m} p_{y,w} \cdot \sin^2\left(\frac{\sqrt{3}}{2m} \cdot \text{HW}(y)\right). \quad (3)$$

where $\text{HW}(y)$ is the Hamming weight of $y \in \{0, 1\}^m$.

Proof. As depicted in Figure 1, V is constructed by translating the D machine underlying U into a quantum circuit U' , and then “simulating” the m (parallel) oracle calls U makes as sub-routines by executing their Q -verification circuits V_i on the relevant subsets of $|w_{1\dots m}\rangle$. Note U' is diagonal in the standard basis, and U computes the inputs $|q_i\rangle$ to Q -verification circuits V_i on-the-fly given x . Gate $R(\theta)$ in Figure 1 denotes 2×2 the rotation matrix $R(\theta) = (\cos \theta, -\sin \theta; \sin \theta, \cos \theta)$.

Let X, Y, Z denote the input registers to U' holding input $x \in \{0, 1\}^n$, query response string $y = y_1 \cdots y_m$, and ancilla (initialised to all zeroes), respectively. Since U' is a classical circuit, without loss of generality it maps any $|x\rangle_X |y\rangle_Y |0 \cdots 0\rangle_Z \mapsto |x\rangle_X |y\rangle_Y |0 \cdots 0 f(y)\rangle_Z$, where $f(y)$ is the output of U' given query response string $y \in \{0, 1\}^m$. If F denotes the flag qubit register, the output $|\psi\rangle$ of V is given by



■ **Figure 1** The circuit V constructed in Lemma 7. The V_i are Q-verifiers, each taking input $|q_i\rangle$ and proof/witness $|w_i\rangle$. For simplicity, states $|w_i\rangle$ are illustrated in tensor product, but our proofs treat the general case of entangled joint proof $|w_{1\dots m}\rangle$. U' denotes the host postprocessing circuit in the original $D^{\parallel Q}$ circuit U , which takes the Q-query responses and outputs U 's final answer. We omit any preprocessing needed by U to compute inputs $|q_i\rangle$ and ancilla register C .

$$|\psi\rangle = \sum_{y \in \{0,1\}^m} \alpha_y |x\rangle_X |y\rangle_Y |0 \dots 0 f(y)\rangle_Z \left(\cos\left(\frac{\sqrt{3}}{2m} \cdot \text{HW}(y)\right) |0\rangle + \sin\left(\frac{\sqrt{3}}{2m} \cdot \text{HW}(y)\right) |1\rangle \right)_F, \quad (4)$$

where we omit registers such as those containing proof $|w_{1\dots m}\rangle$, $\text{HW}(y)$ is the Hamming weight of y , and $|\alpha_y|^2 = \Pr\left(\bigwedge_{i=1}^m V_i \text{ outputs } y_i \mid |w_{1\dots m}\rangle\right)$. This immediately yields Equations (2) and (3). ◀

► **Remark.** The flag qubit q_{flag} does not use binary to store the number of queries which output YES, but rather the number is stored in the angle the qubit is rotated by from its initial state.

► **Remark.** It is *not* true in Figure 1 that the optimal strategy of a dishonest prover is to send the *optimal* proof $|w_i^*\rangle$ for each verifier V_i . This is because intentionally sending a rejecting proof $|w_i\rangle$ to V_i (even if q_i is a YES instance) sets $y_i = 0$, which may cause U' to incorrectly output 1 (whereas setting $y_i = 1$ might cause U' to output 0). Indeed, if sending $|w_i^*\rangle$ was the malicious prover's optimal strategy, then Lemma 7 itself maps an arbitrary $P^{\parallel QMA}$ computation to a single QMA instance V , implying $P^{\parallel QMA} = QMA$.

2.2 Generic Hardness Constructions via a Lifting Lemma

We will in particular be interested in Hamiltonians which satisfy the following condition:

► **Definition 8 (Conformity).** Let H be a Hamiltonian with some well-defined structure S (such as k -local interactions, all constraints drawn from a fixed finite family, with a fixed geometry such as 1D, translational invariance, etc). We say a Hermitian operator P conforms to H if $H + P$ also has structure S .

For example, if H is a 1D translationally invariant Hamiltonian on qubits, then P conforms to H if $H + P$ is also 1D translationally invariant.

► **Definition 9 (Local Circuit-to-Hamiltonian Mapping).** Let $\mathcal{X} = (\mathbb{C}^2)^{\otimes m}$ and $\mathcal{Y} = (\mathbb{C}^2)^{\otimes n}$. A map $H_w : U(\mathcal{X}) \mapsto \text{Herm}(\mathcal{Y})$ is a local circuit-to-Hamiltonian mapping if, for any $T > 0$ and any sequence of 2-qubit unitary gates $U = U_T U_{T-1} \dots U_1$, the following hold:

1. (Overall structure) $H_w(U) \succeq 0$ has a non-trivial null space, i.e. $\text{Null}(H_w(U)) \neq \{0\}$. This null space is spanned by (some appropriate notion of) “correctly initialised computation history states”³, i.e. with ancillae qubits set “correctly” and gates in U “applied” sequentially.
2. (Local penalisation and measurement) Let q_1 and q_2 be the first two output wires of U (each a single qubit), respectively. Let $S_{\text{pre}} \subseteq \mathcal{X}$ and $S_{\text{post}} \subseteq \mathcal{Y}$ denote the sets of input states to U satisfying the structure enforced by $H_w(U)$ (e.g. ancillae initialised to zeroes), and null states of $H_w(U)$, respectively. Then, there exist projectors M_1 and P_T , projector M_2 conforming to $H_w(U)$, and a bijection $f : S_{\text{pre}} \mapsto S_{\text{post}}$, such that for all $i \in \{1, 2\}$ and $|\phi\rangle \in S_{\text{pre}}$, the state $|\psi\rangle = f(|\phi\rangle)$ satisfies

$$\text{Tr}(|0\rangle\langle 0|_i (U_T U_{T-1} \dots U_1) |\phi\rangle\langle \phi| (U_T U_{T-1} \dots U_1)^\dagger) = \text{Tr}(|\psi_T\rangle\langle \psi_T| M_i), \quad (5)$$

where $|\psi_T\rangle = P_T |\psi\rangle / \|P_T |\psi\rangle\|_2$ is $|\psi\rangle$ postselected on measurement outcome P_T (we require $P_T |\psi\rangle \neq 0$). Moreover, there exists a function $g : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}$ such that

$$\|P_T |\psi\rangle\|_2^2 = g(m, T) \text{ for all } |\psi\rangle \in \text{Null}(H_w(U)), \quad (6)$$

$$M_i = P_T M_i P_T. \quad (7)$$

The map H_w , and all operators/functions above (M_1, M_2, P_T, f, g) are computable given U .

We next embed the circuits V constructed in Lemma 7 into a “local circuit-to-Hamiltonian construction”, and carefully penalise the *flag* qubit – not the output qubit – to encourage the ground space of $H_w(V)$ to encode correct query answers made by the D machine to the Q oracle. The latter is necessary since the reduction of Lemma 7 is not *sound*, meaning a NO instance of P^{QMA} is not necessarily mapped to a “NO QMA circuit” V . For this, we formalise and use a broad notion of “local circuit-to-Hamiltonian mapping” H_w in Definition 9.

Coupling our single-qubit flag register setup from Lemma 7 with generic black-box usage of “local circuit-to-Hamiltonian mappings H_w ” allows us to give the main workhorse of this section, the Lifting Lemma for APX-SIM. Crucially, Definition 8 and Definition 9 ensure H below automatically maintains desirable structural properties of H_w (such as translational invariance, geometric constraints, etc).

► **Lemma 10** (Lifting Lemma for APX-SIM). *Let $x \in \{0, 1\}^n$ be an instance of an arbitrary D^{Q} problem, U a D^{Q} machine deciding x , and V the verification circuit output by Lemma 7. Fix a local circuit-to-Hamiltonian mapping H_w , and assume the notation in Definition 9. Fix any $\alpha : \mathbb{N} \mapsto \mathbb{N}$ such that $\alpha > \max\left(\frac{4\|M_2\|}{\Delta(H_w(V))}, \frac{\Delta(H_w(V))}{3\|M_2\|^2}, 1\right)$. Then, for any ϵ satisfying*

$0 \leq \epsilon \leq \frac{1}{\alpha} \left(\frac{1}{\alpha} + \frac{12\|M_2\|^2}{\Delta(H_w(V))}\right) \left(\frac{8m^2}{3g(m, T)}\right)$, the Hamiltonian $H := \alpha(n)H_w(V) + M_2$ satisfies:

- *If x is a YES instance, then for all $|\psi\rangle$ with $\langle \psi | H | \psi \rangle \leq \lambda_{\min}(H) + \frac{1}{\alpha^2}$, $\langle \psi | M_1 | \psi \rangle \leq g(m, T) \cdot m \cdot \max(1 - c + \epsilon, s) + \frac{12\|M_2\|}{\alpha\Delta}$.*
- *If x is a NO instance, then for all $|\psi\rangle$ with $\langle \psi | H | \psi \rangle \leq \lambda_{\min}(H) + \frac{1}{\alpha^2}$,*

$$\langle \psi | M_1 | \psi \rangle \geq g(m, T) (1 - m \cdot \max(1 - c + \epsilon, s)) - \frac{12\|M_2\|}{\alpha\Delta}.$$

³ We are intentionally being vague here, as the only formal requirement on the null space of $H_w(U)$ is that of the following bullet on “local penalisation and measurement”. Intuitively, this would appear to necessitate the null space of $H_w(U)$ to indeed encode “correct initialisations” and “correct gate applications”, as do all known circuit-to-Hamiltonian constructions.

■ **Table 1** Completeness of APX-SIM for families of many-body systems. Measurement precision is relative to system size N , not input size n . NN = 2-local nearest-neighbour interactions. TI = translationally-invariant.

circuit-to-Ham. construction	interaction topology	measurement precision	completeness
[31]	2D planar, NN, local dim 2	$\frac{1}{\text{poly}}$	$\mathsf{P}^{\parallel\text{QMA}}$
[2]	1D line, NN, local dim 12	$\frac{1}{\text{poly}}$	
[13]	3-local, local dim 2	1/exp	PSPACE
[4]	TI, 1D line, NN, local dim 44	$\frac{1}{\text{poly}}$	$\mathsf{EXP}^{\parallel\text{QMA}}$
[6]	TI, 3D fcc lattice, 4-local, local dim 4	$\frac{1}{\text{poly}}$	$\mathsf{EXP}^{\parallel\text{QMA}}$
[27]	TI, 1D line, NN, local dim 42	1/exp	PSPACE

Note the Lifting Lemma’s sole degree of freedom is the function α . All other quantities stem from the choices of H_w , D , and Q . M_1 plays the role of observable A from Definition 24.

Proof sketch. We proceed via three steps: (i) We first use the Extended Projection Lemma (Lemma 25) to show that for α as in the lemma claim, for any $|\psi\rangle$ such that $\langle\psi|H|\psi\rangle \leq \lambda_{\min}(H) + \delta$, there exists a uniform history state $|\phi\rangle \in \text{Null}(H_w(V))$ such that $\| |\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| \|_{\text{tr}} \leq \frac{12\|M_2\|}{\alpha\Delta}$ and where $|\phi\rangle$ has energy $\langle\phi|H|\phi\rangle \leq \lambda_{\min}(H) + \delta + \frac{12\|M_2\|^2}{\alpha\Delta}$. (ii) These history states correctly simulate V from Lemma 7 (as given in Equation (1)) on any claimed proof $|w_{1\dots m}\rangle$ to the parallel Q -verifiers. However, the M_2 term in H , which simulates penalising the flag qubit of V , enforces that any such low energy history state must in fact send the *locally optimal* proofs $|w_{1\dots m}\rangle = |w_1^*\rangle \otimes \dots \otimes |w_m^*\rangle$, ensuring all Q -queries are answered correctly. The main technical ingredient behind rigorously proving this is the following lemma.

► **Lemma 11.** *Assume the notation of Lemma 7, which showed $\text{Tr}(|\psi\rangle\langle\psi| \cdot |0\rangle\langle 0|_{q_{\text{flag}}}) = \sum_{y \in \{0,1\}^m} p_{y,w} \cdot \cos^2(\text{HW}(y)\sqrt{3}/(2m))$, where $|\psi\rangle$ denoted the output of V given joint proof $|w_{1\dots m}\rangle$, and $p_{y,w} = \Pr(\bigwedge_{i=1}^m V_i \text{ outputs } y_i \mid |w_{1\dots m}\rangle)$. Suppose there exists an $i \in \{1, \dots, m\}$ and $\epsilon > 0$ such that $|w_{1\dots m}\rangle$ is “ ϵ -suboptimal on proof i ”, meaning there exists a local proof $|w'_i\rangle$ such that $\Pr(V_i \text{ outputs } 1 \mid |w_{1\dots m}\rangle) = \Pr(V_i \text{ outputs } 1 \mid |w'_i\rangle) - \epsilon$. Then there exists a proof $|w'_{1\dots m}\rangle = |w'_1\rangle \otimes \dots \otimes |w'_m\rangle$ which causes V to output $|\psi'\rangle$ satisfying $\text{Tr}(|\psi\rangle\langle\psi| \cdot |0\rangle\langle 0|_{q_{\text{flag}}}) \geq \text{Tr}(|\psi'\rangle\langle\psi'| \cdot |0\rangle\langle 0|_{q_{\text{flag}}}) + \frac{3}{8m^2}\epsilon$.*

Lemma 11 ties the energy penalty on the flag qubit of V to the optimality of all Q -queries. It is proven via a careful exchange argument involving a pair of recursions – the delicacy lies in the fact that we require a rigorous deviation bound (i.e. $3\epsilon/8m^2$), and for this we must take conditional probabilities into account due to potential entanglement between proofs. A full proof is provided in the full version of the paper [35]. ◀

2.3 Applying the Lifting Lemma

With Lemma 10 in hand, we show the hardness results depicted in the rightmost column of Table 1. Technically, our construction gives a slightly stronger result, namely hardness for \forall -APX-SIM (Definition 24), for which the YES case reads “for all $|\psi\rangle$ satisfying $\langle\psi|H|\psi\rangle \leq \lambda_{\min}(H) + \delta$, $\langle\psi|A|\psi\rangle \leq a$ ”. This immediately implies hardness for APX-SIM as well, since \forall -APX-SIM trivially reduces to it.

► **Corollary 12.** \forall -APX-SIM is complete for the Hamiltonians and respective classes in Table 1. In particular, \forall -APX-SIM (and hence APX-SIM) is $\text{EXP}^{\|\text{QMA}}$ -hard for a 1D TI Hamiltonian on qudits of local dimension $4a$, 1-local observable A , $\delta = 1/\text{poly}(N)$, $b - a = \Omega(1/\text{poly}(N))$.

► **Remark.** (1) In TI settings, one need be slightly careful in how Lemma 10 is applied, as we cannot explicitly write out the full circuit of Figure 1. (2) We also obtain PSPACE-completeness for inverse exponential promise gap (relative to system size N). Previously, LH with such promise gap was shown PSPACE-complete [13]. (3) We also recover existing $\text{P}^{\|\text{QMA}}$ -hardness results of [3, 19, 17] except for $\text{P}^{\|\text{StoqMA}}$ -completeness [17] (Lemma 10 requires error reduction, not known to hold for StoqMA).

3 Hardness via Lifting for GSCON

We now show Theorem 6, which recall says GSCON is QCMA_{EXP} -complete for 1D TI Hamiltonians on N qudits. To begin, we require the following tools.

► **Definition 13** (*b*-orthogonal states and subspaces [18]). For $b \geq 1$, a pair of states $|v\rangle, |w\rangle \in (\mathbb{C}^d)^{\otimes N}$ is *b*-orthogonal if for all *b*-qudit unitaries U , we have $\langle w|U|v\rangle = 0$. We call subspaces $S, T \subseteq (\mathbb{C}^d)^{\otimes N}$ *b*-orthogonal if any pair of vectors $|v\rangle \in S$ and $|w\rangle \in T$ are *b*-orthogonal.

► **Lemma 14** (Traversal Lemma [18, 20]). Let $S, T \subseteq (\mathbb{C}^d)^{\otimes N}$ be *b*-orthogonal subspaces. Fix arbitrary states $|v\rangle \in S$ and $|w\rangle \in T$, and consider a sequence of *b*-qudit unitaries $(U_i)_{i=1}^m$ such that $\| |w\rangle - U_m \cdots U_1 |v\rangle \|_2 \leq \epsilon$ for some $0 \leq \epsilon < 1/2$. Define $|v_i\rangle := U_i \cdots U_1 |v\rangle$ and $P := I - \Pi_S - \Pi_T$. Then, there exists an $i \in [m]$ such that $\langle v_i|P|v_i\rangle \geq ((1 - \epsilon)/m)^2$.

3.1 Generic hardness constructions via a Lifting Lemma

We now give a black-box mapping for “lifting” 1D TI circuit-to-Hamiltonian constructions to QCMA_{EXP} -hardness results for GSCON. While the goal is similar to the Lifting Lemma for APX-SIM, here we restrict attention to a broad class of 1D TI circuit-to-Hamiltonian mappings we denote *TI-standard* (Definition 26), with two primary properties: (1) In the YES case, there exists a low energy state $|\psi_{\text{low}}\rangle$ preparable in time $\text{poly}(N)$, for N the length of the chain (hence our focus on QCMA_{EXP} , not QMA_{EXP}), and (2) the local H_i need not be positive, but the set of H_i with $\langle \psi_{\text{low}}|H_i|\psi_{\text{low}}\rangle < 0$ is computable in time $\text{poly}(N)$. Again, these assumptions are rather mild, and satisfied by most, if not all known circuit-to-Hamiltonian constructions.

► **Lemma 15** (Lifting Lemma for GSCON). Let V be a verifier for a QCMA_{EXP} promise problem. Fix any *TI-standard* circuit-to-Hamiltonian mapping which produces 1D TI Hamiltonians on qudits of local dimension d , and any $b \in \{2, \dots, N - 1\}$. Then, there exists a $\text{poly}(\log N)$ -time many-one reduction mapping any instance x for V to a 1D TI GSCON instance H on N qudits of local dimension $7d$ with *b*-local unitaries U_i , such that $m \in \text{poly}(N)$ and $\delta \in \Theta(1/\text{poly}(N))$.

3.1.1 Proof of Lemma 15

The construction is stated below – Definition 16 gives the Hamiltonian, and the remaining parameters are set subsequently. The intuition was outlined in Section 1, and is fleshed out in the full version in multiple steps. Here, we recount the most crucial points, and point the

	0	1	2	3	4	5	6	Phase	Switch register contents
0	✓	✓	✗	✗	✗	✗	✗	Warm up	0000 ↦ 1000 ↦ 1010
1	✓	✓	✓	✗	✗	✗	✗	Full blast	1010 ↦ 1110 ↦ 1111
2	✗	✗	✓	✗	✗	✗	✗	Left deke	1111 ↦ 1112 ↦ 1122 ↦ 1222 ↦ 2222
3	✗	✗	✓	✓	✓	✗	✗	Right deke	2222 ↦ 3222 ↦ 3322 ↦ 3332 ↦ 3333
4	✗	✗	✗	✗	✓	✗	✗	Left deke	3333 ↦ 3334 ↦ 3344 ↦ 3444 ↦ 4444
5	✗	✗	✗	✗	✓	✓	✓	Right deke	4444 ↦ 5444 ↦ 5544 ↦ 5554 ↦ 5555
6	✗	✗	✗	✗	✗	✓	✓	Cool down	5555 ↦ 5556 ↦ 5656
6	✗	✗	✗	✗	✗	✓	✓	Complete shutdown	5656 ↦ 5666 ↦ 6666

■ **Figure 2** *Left:* A ✓ (✗) at position (i, j) means symbol i can (cannot) appear immediately to the left of j in joint clock space \mathcal{B} . *Right:* An honest prover’s sequence of states in \mathcal{B} , organised by phase, and for $N = 4$. In this example, for concreteness we assume $F = \{1, 3\}$ (see Definition 26).

reader to the completeness analysis of Appendix C.2 for further insight: (1) Each qudit in our system is given its own “clock register” to permit a 1D construction, and (2) the Hamiltonian H , start/end states $|\psi\rangle$ and $|\phi\rangle$ are designed to force any low-energy space evolution to effectively “wind through” a pre-defined path in the “clock space”, along which lies a carefully placed “bottleneck” which “switches” on a simulation of the QCMA_{EXP} verifier V .

► **Definition 16** (Lifted Hamiltonian). *Let x be an instance of a QCMA_{EXP} promise problem, with verification circuit V such that for any YES instance x , V accepts some classical proof with probability at least $1 - \epsilon$, and for all NO instances x , V accepts all proofs with probability at most ϵ . Let $H' = \sum_{i=1}^{N-1} H'_{i,i+1}$, be the Hamiltonian generated by applying a TI-standard construction (Definition 26) to V . Define E as the 2-local projector onto the set of forbidden 2-local nearest-neighbour substrings in figure 2 (left), where the \mathcal{B}_i are the local clock registers. Define the lifted Hamiltonian as $H = \sum_{i=1}^{N-1} H_{i,i+1}$, where:*

$$H_{i,i+1} := (H'_{i,i+1})_{\mathcal{A}_i, \mathcal{A}_{i+1}} \otimes (|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4| + |5\rangle\langle 5|)_{\mathcal{B}_i} + \Delta E_{\mathcal{B}_i, \mathcal{B}_{i+1}} \quad (8)$$

for $\Delta \in \mathbb{R}$ to be chosen as needed. Note each $\mathcal{A}_i \otimes \mathcal{B}_i$ is viewed jointly as qudit i .

The start and final states are $|\psi\rangle = \bigotimes_{i=1}^N |0\rangle_{\mathcal{A}_i} |0\rangle_{\mathcal{B}_i}$ and $|\phi\rangle = \bigotimes_{i=1}^N |0\rangle_{\mathcal{A}_i} |6\rangle_{\mathcal{B}_i}$, respectively. Set $\eta_1 = \alpha$, $\eta_2 = \beta/(8m^2)$, $\eta_3 = 0$, $\eta_4 = 1/2$, and $m = 2L + 7N$, where $L \in \text{poly}(N)$ is the size of the circuit preparing the ground state of H' in the YES case. Set $\delta = (\eta_1 + \eta_2)/2$, which by definition of TI-standard is at least inverse polynomial in N (since β is at least inverse polynomial in N). Finally, set any $b \in \{2, \dots, N - 1\}$ (b the locality of each U_i); $b = 2$ suffices to show completeness, and soundness holds for all $b \in \{2, \dots, N - 1\}$.

Completeness

Suppose x is a YES instance. The following lemma shows there is a short path through the low energy subspace between states $|\psi\rangle$ and $|\phi\rangle$, as desired.

► **Lemma 17** (Completeness). *Let $\lambda_{\min}(H') \leq \alpha$. There exists a circuit $U = U_m \dots U_2 U_1$ of 2-local gates U_i such that $U|\psi\rangle = |\phi\rangle$, and all intermediate states $|\psi_i\rangle = U_i \dots U_2 U_1 |\psi\rangle$ satisfy $\langle \psi_i | H | \psi_i \rangle \leq \eta_1$.*

Proof sketch. The full analysis is given in Appendix C. The high-level idea is as follows: Since x is a YES instance, H' has a low energy history state $|\psi_{\text{low}}\rangle$, which by Definition 26 can be prepared in $\text{poly}(N)$ time. To traverse the low-energy space of H from $|\psi\rangle$ to $|\phi\rangle$,

prepare $|\psi_{\text{low}}\rangle$ in \mathcal{A} , map the clock registers from $|0\rangle^{\otimes N}$ to $|6\rangle^{\otimes N}$ according to the rules of Figure 2 (left); an $N = 4$ example is in Figure 2 (right). Roughly, the Warm up and Full blast (and by symmetry, Cool down, Complete shutdown) phases allow the honest prover to “switch on” local terms of H_i in an arbitrary order, required since we cannot assume $H'_i \succeq 0$ for all i in Definition 26 (necessary for [21]). The remaining four phases are for soundness (Lemma 18). ◀

Soundness. Suppose x is a NO instance. The following lemma shows that any short path from $|\psi\rangle$ to $|\phi\rangle$ must leave the low-energy subspace.

► **Lemma 18.** *Let $\lambda_{\min}(H') \geq \beta$, and fix any $b \in \{2, \dots, N - 1\}$. Consider any sequence $U = U_m \cdots U_1$ of b -local unitary operators acting on $\bigotimes_{i=1}^N \mathcal{A}_i \otimes \mathcal{B}_i$. Then, either there exists $i \in [m]$ such that intermediate state $|\psi_i\rangle := U_i \cdots U_2 U_1 |\psi\rangle$ satisfies $\langle \psi_i | H | \psi_i \rangle \geq \frac{1}{2} \left(\frac{\beta}{4m^2} \right) = \eta_2$, or $\| |\psi_m\rangle - |\phi\rangle \|_2 \geq 1/2 = \eta_4$.*

Proof sketch. The full analysis is given in Appendix C. At a high level, in the NO case, H' does not have a low-energy state $|\psi_{\text{low}}\rangle$ to prepare in \mathcal{A} . Thus, the aim is to force the cheating prover to switch on all terms H'_i , which inflicts energy penalty at least β on register \mathcal{A} . The catch is that the prover can apply b -local unitaries, potentially attempting to bypass the last b switches on the chain via a single unitary. Via a careful application of the Traversal Lemma, we show that there exists a time step i , such that intermediate state $|\psi_i\rangle$ has non-trivial overlap in $\mathcal{B}_{N-b} \otimes \cdots \otimes \mathcal{B}_N$ on regular expression $33^*(2^* \cup 4^*)$. By the rules of Figure 2 (left), we deduce that all switch qudits to the left of \mathcal{B}_{N-b} are also set to $|3\rangle$; but this guarantees all terms of H'_i are switched on, as desired. Note the Traversal Lemma alone cannot ensure that *all* H'_i are turned on; it is the delicate combination of the rules of Figure 2 (left) and the Traversal Lemma which make this possible. ◀

3.2 Proof of QCMA_{EXP}-completeness

With the Lifting Lemma for GSCON (Lemma 15) in hand, we obtain our QCMA_{EXP}-completeness result.

► **Theorem 19.** *GSCON is QCMA_{EXP}-complete for 1D, nearest neighbour, translationally invariant Hamiltonians on N qudits, for $m \in \text{poly}(N)$, $\delta \in \Theta(1/\text{poly}(N))$, and any $b \in \{2, \dots, N - 1\}$.*

Proof sketch. Containment in QCMA_{EXP} for $\delta \in \Omega(1/\text{poly}(N))$ is immediate since GSCON \in QCMA for any interaction graph [18]. QCMA_{EXP}-hardness of GSCON follows from plugging GI into Lemma 15. Note GI itself is not TI-standard, as it does not satisfy the requirement $\beta \geq 16(2L + 7N)\alpha \geq 0$, but this is easily addressed via energy shifts. ◀

References

- 1 Ian Affleck, Tom Kennedy, Elliott H. Lieb, and Hal Tasaki. Rigorous results on valence-bond ground states in antiferromagnets. *Physical Review Letters*, 59(7):799–802, August 1987. doi:10.1103/PhysRevLett.59.799.
- 2 Dorit Aharonov, Daniel Gottesman, Sandy Irani, and Julia Kempe. The power of quantum systems on a line. *Communications in Mathematical Physics*, 287(1):41–65, May 2009. doi:10.1007/s00220-008-0710-3.
- 3 Andris Ambainis. On physical problems that are slightly more difficult than QMA. In *29th IEEE Conference on Computational Complexity (CCC)*, pages 32–43, 2014.

- 4 Johannes Bausch, Toby Cubitt, and Maris Ozols. The Complexity of Translationally Invariant Spin Chains with Low Local Dimension. *Annales Henri Poincaré*, 18(11):3449–3513, November 2017. doi:10.1007/s00023-017-0609-7.
- 5 Johannes Bausch, Toby S. Cubitt, Angelo Lucia, and David Perez-Garcia. Undecidability of the Spectral Gap in One Dimension. *Physical Review X*, 10(3):031038, August 2020. doi:10.1103/PhysRevX.10.031038.
- 6 Johannes Bausch and Stephen Piddock. The complexity of translationally invariant low-dimensional spin lattices in 3D. *Journal of Mathematical Physics*, 58(11):111901, November 2017. doi:10.1063/1.5011338.
- 7 H Bethe. Zur Theorie der Metalle. *Zeitschrift für Physik*, 71(3–4):205–226, 1931.
- 8 S. Bravyi and M. Hastings. On complexity of the quantum Ising model. *Communications in Mathematical Physics*, 349(1):1–45, 2017. doi:10.1007/s00220-016-2787-4.
- 9 Nikolas P. Breuckmann and Barbara M. Terhal. Space-time circuit-to-Hamiltonian construction and its applications. *Journal of Physics A Mathematical General*, 47(19):195304, May 2014. doi:10.1088/1751-8113/47/19/195304.
- 10 Brielin Brown, Steven T. Flammia, and Norbert Schuch. Computational Difficulty of Computing the Density of States. *PRL*, 107(4):040501, July 2011. doi:10.1103/PhysRevLett.107.040501.
- 11 Toby S Cubitt, Ashley Montanaro, and Stephen Piddock. Universal quantum hamiltonians. *Proceedings of the National Academy of Sciences*, 115(38):9497–9502, 2018. arXiv:1701.05182.
- 12 Toby S. Cubitt, David Perez-Garcia, and Michael M. Wolf. Undecidability of the spectral gap. *Nature*, 528(7581):207–211, December 2015. doi:10.1038/nature16059.
- 13 Bill Fefferman and Cedric Yen-Yu Lin. A Complete Characterization of Unitary Quantum Space. In *Leibniz International Proceedings in Informatics (LIPIcs)*. 9th Innovations in Theoretical Computer Science Conference, April 2016. doi:10.4230/LIPIcs.ITCS.2018.4.
- 14 Fabio Franchini. *An Introduction to Integrable Techniques for One-Dimensional Quantum Systems*, volume 940 of *Lecture Notes in Physics*. Springer International Publishing, Cham, 2017. doi:10.1007/978-3-319-48487-7.
- 15 S. Gharibian, Y. Huang, Z. Landau, and S. W. Shin. Quantum Hamiltonian complexity. *Foundations and Trends® in Theoretical Computer Science*, 10(3):159–282, 2014. doi:10.1561/04000000066.
- 16 S. Gharibian and J. Kempe. Hardness of approximation for quantum problems. In *39th International Colloquium on Automata, Languages and Programming (ICALP)*, pages 387–398, 2012.
- 17 Sevag Gharibian, Stephen Piddock, and Justin Yirka. Oracle Complexity Classes and Local Measurements on Physical Hamiltonians. In *37th International Symposium on Theoretical Aspects of Computer Science (STACS 2020)*, volume 154, pages 20:1–20:37, 2020.
- 18 Sevag Gharibian and Jamie Sikora. Ground state connectivity of local Hamiltonians. In *42nd International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 617–628, 2015.
- 19 Sevag Gharibian and Justin Yirka. The complexity of simulating local measurements on quantum systems. *Quantum*, 3:189, 2019. doi:10.22331/q-2019-09-30-189.
- 20 David Gosset, Jenish C. Mehta, and Thomas Vidick. QCMA hardness of ground space connectivity for commuting Hamiltonians. *Quantum*, 1:16, July 2017. doi:10.22331/q-2017-07-14-16.
- 21 Daniel Gottesman and Sandy Irani. The Quantum and Classical Complexity of Translationally Invariant Tiling and Hamiltonian Problems. *Theory of Computing*, 9(1):31–116, May 2009. doi:10.4086/toc.2013.v009a002.
- 22 S. Hallgren, D. Nagaj, and S. Narazanaswami. The Local Hamiltonian problem on a line with eight states is QMA-complete. *Quantum Information & Computation*, 13(9&10):0721–0750, 2013.

- 23 A. S. Holevo. Bounds for the quantity of information transmitted by a quantum communication channel. *Problems of Information Transmission*, 9(3):177–183, 1973.
- 24 Julia Kempe, Alexei Yu. Kitaev, and Oded Regev. The Complexity of the Local Hamiltonian Problem. *SIAM Journal on Computing*, 35(5):1070–1097, January 2006. doi:10.1137/S0097539704445226.
- 25 A. Kitaev, A. Shen, and M. Vyalyi. *Classical and Quantum Computation*. American Mathematical Society, 2002.
- 26 Alexei Yu. Kitaev, Alexander Shen, and Mikhail N. Vyalyi. Classical and quantum computing. In *Quantum Information*, pages 203–217. Springer New York, New York, NY, 2002. doi:10.1007/978-0-387-36944-0_13.
- 27 Tamara Kohler, Stephen Piddock, Johannes Bausch, and Toby Cubitt. Translationally invariant universal quantum hamiltonians in 1d. *Annales Henri Poincaré*, 23(1):223–254, 2021. doi:10.1007/s00023-021-01111-7.
- 28 Zeph Landau, Umesh Vazirani, and Thomas Vidick. A polynomial time algorithm for the ground state of one-dimensional gapped local Hamiltonians. *Nature Physics*, 11(7):566–569, June 2015. doi:10.1038/nphys3345.
- 29 Elliott Lieb, Theodore Schultz, and Daniel Mattis. Two soluble models of an antiferromagnetic chain. *Annals of Physics*, 16(3):407–466, December 1961. doi:10.1016/0003-4916(61)90115-4.
- 30 Daniel Nagaj. *Local Hamiltonians in Quantum Computation*. PhD thesis, Massachusetts Institute of Technology, Boston, 2008. Available at arXiv.org quant-ph/0808.2117v1.
- 31 Roberto I. Oliveira and Barbara M. Terhal. The complexity of quantum spin systems on a two-dimensional square lattice. *Quantum Information and Computation*, 8(10):1–23, April 2005. arXiv:0504050.
- 32 T. J. Osborne. Hamiltonian complexity. *Reports on Progress in Physics*, 75(2):022001, 2012. URL: <http://stacks.iop.org/0034-4885/75/i=2/a=022001>.
- 33 Stephen Piddock and Johannes Bausch. Universal Translationally-Invariant Hamiltonians. *arXiv*, January 2020. arXiv:2001.08050.
- 34 Y. Shi and S. Zhang. Note on quantum counting classes. Available at <http://www.cse.cuhk.edu.hk/syzhang/papers/SharpBQP.pdf>.
- 35 James D. Watson, Johannes Bausch, and Sevag Gharibian. The Complexity of Translationally Invariant Problems beyond Ground State Energies. *arXiv e-prints*, page arXiv:2012.12717, December 2020. arXiv:2012.12717.
- 36 Steven R. White. Density matrix formulation for quantum renormalization groups. *Physical Review Letters*, 69(19):2863–2866, November 1992. doi:10.1103/PhysRevLett.69.2863.

A Additional Definitions

► **Definition 20** (QMA_{EXP} [21]). *A promise problem $\Pi = (A_{\text{yes}}, A_{\text{no}})$ is in QMA_{EXP} if and only if there exists a $k \in O(1)$ and a Quantum Turing Machine M such that for any input $x \in \{0, 1\}^n$, and any proof $|\psi\rangle \in (\mathbb{C}^2)^{\otimes 2^{n^k}}$, on input $(x, |\psi\rangle)$, M halts in 2^{n^k} steps. Furthermore,*

- (Completeness) *If $x \in A_{\text{yes}}$, $\exists |\psi\rangle \in (\mathbb{C}^2)^{\otimes 2^{n^k}}$ such that M accepts $(x, |\psi\rangle)$ with probability $\geq 2/3$.*
- (Soundness) *If $x \in A_{\text{no}}$, then $\forall |\psi\rangle \in (\mathbb{C}^2)^{\otimes 2^{n^k}}$, M accepts $(x, |\psi\rangle)$ with probability $\leq 1/3$.*

We take care to distinguish QMA_{EXP} from the class QMA_{exp} of [13] which is for an exponentially small promise gap in the input size, but polynomial length run time, also called PreciseQMA.

Next, QMA with a classical witness yields the complexity class QCMA.

► **Definition 21** (QCMA_{EXP}). A promise problem $\Pi = (A_{\text{yes}}, A_{\text{no}})$ is in QCMA_{EXP} if and only if there exists $k \in O(1)$ and an exponential-time uniform family of quantum circuits $\{Q_n\}$, where Q_n takes as input a string $x \in \Sigma^n$, a classical proof $y \in \{0, 1\}^{2^{n^k}}$, and 2^{n^k} ancilla qubits in state $|0\rangle^{\otimes 2^{n^k}}$, such that:

- (Completeness) If $x \in A_{\text{yes}}$, $\exists y \in \{0, 1\}^{2^{n^k}}$ such that Q_n accepts (x, y) with probability $\geq 2/3$.
- (Soundness) If $x \in A_{\text{no}}$, $\forall y \in \{0, 1\}^{2^{n^k}}$, Q_n accepts (x, y) with probability $\leq 1/3$.

► **Definition 22** (Deterministic Decision class). A set C of languages is a deterministic decision class if, for any language $L \in C$, there exists a deterministic Turing machine M which can decide L under the resource constraints specified by C . Formally, given any input $x \in \{0, 1\}^n$, M halts after using $R(n)$ resources (where R may specify bounds on time or space), and accepts if $x \in L$ or rejects if $x \notin L$.

Standard examples of deterministic classes include P, PSPACE, and EXP.

► **Definition 23** (Existentially quantified quantum verification class (QVClass)). A set C of promise problems is an existentially quantified quantum verification class if any promise problem $A = (A_{\text{yes}}, A_{\text{no}}, A_{\text{inv}})$ in C satisfies the following. There exist computable functions $f, g, h : \mathbb{N} \mapsto \mathbb{N}$, as well as a deterministic Turing machine M which, for any input $x \in \{0, 1\}^n$, uses $R(n)$ resources to produce a quantum verification circuit V (consisting of 1- and 2-qubit gates) and thresholds $c, s \in \mathbb{R}^+$ such that $c - s > 1/h(n)$. Here, $R(x)$ refers to resources such as time, space, etc, as required by C . The circuit V takes in a quantum proof $|\psi\rangle$ on $f(n)$ qubits, $g(n)$ ancilla qubits initialised to all zeroes, and has a designated output qubit, such that:

- (YES case) If $x \in A_{\text{yes}}$, there exists a quantum proof $|\psi\rangle$ on $f(n)$ qubits such that measuring the output qubit of $V|\psi\rangle|0 \cdots 0\rangle$ in the standard basis yields 1 with probability at least c .
- (NO case) If $x \in A_{\text{no}}$, for all quantum proofs $|\psi\rangle$ on $f(n)$ qubits, measuring the output qubit of $V|\psi\rangle|0 \cdots 0\rangle$ in the standard basis yields 1 with probability at most s .

Without loss of generality, we assume the output qubit of V is the first wire exiting V .

In this way, classes such as NP, NEXP, QCMA, QMA, and so forth are examples of a QVClass.

► **Definition 24** ($(\forall\text{-APX-SIM}(H, A, k, l, a, b, \delta)$ [17]). Given a k -local Hamiltonian $H = \sum_i H_i$ acting on N qubits, an l -local observable A , and real numbers a, b , and δ such that $b - a \geq N^{-c}$ and $\delta \geq N^{-c'}$, for $c, c' > 0$ constant, decide:

YES. If for all $|\psi\rangle$ satisfying $\langle \psi | H | \psi \rangle \leq \lambda_{\min}(H) + \delta$, it holds that $\langle \psi | A | \psi \rangle \leq a$.

NO. If for all $|\psi\rangle$ satisfying $\langle \psi | H | \psi \rangle \leq \lambda_{\min}(H) + \delta$, it holds that $\langle \psi | A | \psi \rangle \geq b$.

B Complexity of APX-SIM

B.1 Useful Lemmas

► **Lemma 25** (Extended Projection Lemma ([24, 19])). Let $H = H_1 + H_2$ be the sum of two Hamiltonians operating on some Hilbert space $\mathcal{H} = \mathcal{S} + \mathcal{S}^\perp$. The Hamiltonian H_1 is such that \mathcal{S} is a zero eigenspace and the eigenvectors in \mathcal{S}^\perp have eigenvalue at least $J > 2 \|H_2\|_\infty$. Let $K := \|H_2\|_\infty$. Then, for any $\delta \geq 0$ and $|\psi\rangle$ satisfying $\langle \psi | H | \psi \rangle \leq \lambda_{\min}(H) + \delta$, there exists a $|\psi'\rangle \in \mathcal{S}$ such that the ground state energy is bounded as $\lambda_{\min}(H_2|_{\mathcal{S}}) - \frac{K^2}{J-2K} \leq$

$\lambda_{\min}(H) \leq \lambda_{\min}(H_2|_{\mathcal{S}})$, where $\lambda_{\min}(H_2|_{\mathcal{S}})$ denotes the smallest eigenvalue of H_2 restricted to space \mathcal{S} . Furthermore, the ground state is perturbed as $|\langle \psi | \psi' \rangle|^2 \geq 1 - \left(\frac{K + \sqrt{K^2 + \delta(J-2K)}}{J-2K} \right)^2$, and satisfies $\langle \psi' | H | \psi' \rangle \leq \lambda_{\min}(H) + \delta + 2K \frac{K + \sqrt{K^2 + \delta(J-2K)}}{J-2K}$.

C Complexity of GSCON

C.1 Definitions for GSCON

► **Definition 26** (TI-standard). A circuit-to-Hamiltonian mapping from verification circuits V to 1D, nearest neighbor, translationally invariant Hamiltonians $H = \sum_{i=1}^{N-1} H_{i,i+1}$ is TI-standard if it satisfies the following conditions. Below, N denotes the number of qudits H acts on, and α and β the completeness/soundness (a.k.a. “low energy” and “high energy”) parameters for H , respectively:

1. $H \succeq 0$, although the local terms may satisfy $H_{i,i+1} \not\succeq 0$.
2. In the YES case, if the optimal proof to verifier V is a classical string $y \in \{0, 1\}^{\text{poly}(N)}$, then there exists a (potentially non-uniformly generated) quantum circuit of size $L \in \text{poly}(N)$ preparing a low energy state $|\psi_{\text{low}}\rangle$ for H , i.e. $\langle \psi_{\text{low}} | H | \psi_{\text{low}} \rangle \leq \alpha$.
3. In the YES case, the subset of indices $F \subseteq [N-1]$ for which $H_{i,i+1}$ contributes negative energy to the low-energy state, i.e. all i for which $H_{i,i+1}$ satisfies $\langle \psi_{\text{low}} | H_{i,i+1} | \psi_{\text{low}} \rangle < 0$, is computable in $\text{poly}(N)$ time⁴.
4. In the NO case, $\lambda_{\min}(H) \geq \beta$. Here, we require $|\alpha - \beta| \geq 1/\text{poly}(N)$ (which is standard in the literature) and $\beta \geq 16(2L + 7N)\alpha \geq 0$ (which is specific to our construction).

All of these assumptions are rather mild, as we now clarify.

Remarks regarding Definition 26.

- Assumptions 1 and 4 must be taken together (otherwise, $H \succeq 0$ can always be achieved by adding a multiple of the identity).
- The setting of $H \succeq 0$ but $H_i \not\succeq 0$ arises when applying our construction to the Gottesman-Irani 1D TI mapping (henceforth GI) [21] in Section 3.2. Specifically, GI is not TI-standard in its original form, since it violates the final requirement $\beta \geq 16(2L + 7N)\alpha$, which is crucial to our use of the Traversal Lemma.
- Assumption 3 is vacuously true when all $H_i \succeq 0$. When $H_i \not\succeq 0$ for some i , however, this is also generally a mild assumption, since it only cares about energies against $|\psi_{\text{low}}\rangle$, which is typically a history state of some form.

C.2 Proof of GSCON Lifting Lemma

We now prove the Lifting Lemma for GSCON, Lemma 15.

C.2.1 Completeness

Suppose x is a YES instance. The following lemma shows there is a short path through the low energy subspace between states $|\psi\rangle$ and $|\phi\rangle$, as desired.

⁴ One can replace the < 0 condition here with $< -1/p(N)$ for some sufficiently small polynomial p ; we omit this for simplicity.

► **Lemma 27.** *Using the notation of Definition 16, let $\lambda_{\min}(H') \leq \alpha$. There exists a circuit $U = U_m \dots U_2 U_1$ of 2-local gates U_i such that $U|\psi\rangle = |\phi\rangle$, and all intermediate states $|\psi_i\rangle = U_i \dots U_2 U_1 |\psi\rangle$ satisfy $\langle \psi_i | H | \psi_i \rangle \leq \eta_1$.*

Proof. By definition of TI-standard, since the optimal QCMA_{EXP} proof is a classical string of size $\text{poly}(N)$, there exists a $\text{poly}(N)$ -length sequence $U' = U_L \dots U_1$ of L 1- and 2-qudit unitaries (acting on $\bigotimes_{i=1}^N \mathcal{A}_i$) which prepares a low energy state $|\psi_{\text{low}}\rangle$ of H' . The circuit U of the claim now acts as follows (see Figure 2 for an explicit example when $N = 4$).

1. *Prepare low energy state.* Compute $U'_{\mathcal{A}} \otimes I_{\mathcal{B}} |\psi\rangle_{\mathcal{A}, \mathcal{B}}$, i.e. perform the mapping

$$|\psi\rangle_{\mathcal{A}, \mathcal{B}} \mapsto |\psi_{\text{low}}\rangle_{\mathcal{A}} |0 \dots 0\rangle_{\mathcal{B}}.$$

2. *“Warm up”.* Let $F \subseteq [N-1]$ denote the set of indices i for which $\langle \psi_{\text{low}} | H'_{i, i+1} | \psi_{\text{low}} \rangle < 0$, which is efficiently computable by definition of TI-standard. One at a time, map $|0\rangle_{\mathcal{B}_i} \mapsto |1\rangle_{\mathcal{B}_i}$ for each $i \in F$, in any order.
3. *“Full blast”.* One at a time, map $|0\rangle_{\mathcal{B}_i} \mapsto |1\rangle_{\mathcal{B}_i}$ for all $i \in [N] \setminus F$, in any order.
4. *“Left deke”.* Map $|1\rangle_{\mathcal{B}_i} \mapsto |2\rangle_{\mathcal{B}_i}$ for all i in sequence $(N, \dots, 1)$ (i.e. right to left).
5. *“Right deke”.* Map $|2\rangle_{\mathcal{B}_i} \mapsto |3\rangle_{\mathcal{B}_i}$ for all i in sequence $(1, \dots, N)$ (i.e. left to right).
6. *“Left deke”.* Map $|3\rangle_{\mathcal{B}_i} \mapsto |4\rangle_{\mathcal{B}_i}$ for all i in sequence $(N, \dots, 1)$ (i.e. right to left).
7. *“Right deke”.* Map $|4\rangle_{\mathcal{B}_i} \mapsto |5\rangle_{\mathcal{B}_i}$ for all i in sequence $(1, \dots, N)$ (i.e. left to right).
8. *“Cool down”.* One at a time, map $|5\rangle_{\mathcal{B}_i} \mapsto |6\rangle_{\mathcal{B}_i}$ for all $i \in [N] \setminus F$, in any order.
9. *“Complete shut down”.* One at a time, map $|5\rangle_{\mathcal{B}_i} \mapsto |6\rangle_{\mathcal{B}_i}$ for each $i \in F$, in any order.
10. *Uncompute low energy state.* Apply $(U')_{\mathcal{A}}^{\dagger} \otimes I_{\mathcal{B}}$ to our state.

Analysing each step above shows that for each step satisfy $\langle \psi_i | H | \psi_i \rangle \leq \eta_1$. ◀

C.2.2 Soundness

Suppose x is a NO instance. The following lemma shows that any short path from $|\psi\rangle$ to $|\phi\rangle$ must leave the low-energy subspace, as desired.

► **Lemma 28.** *Using the notation of Definition 16, let $\lambda_{\min}(H') \geq \beta$, and fix any $b \in \{2, \dots, N-1\}$. Consider any sequence $U = U_m \dots U_1$ of b -local unitary operators acting on $\bigotimes_{i=1}^N \mathcal{A}_i \otimes \mathcal{B}_i$. Then, either there exists $i \in [m]$ such that intermediate state $|\psi_i\rangle := U_i \dots U_2 U_1 |\psi\rangle$ satisfies $\langle \psi_i | H | \psi_i \rangle \geq \frac{1}{2} \left(\frac{\beta}{4m^2} \right) = \eta_2$, or $\| |\psi_m\rangle - |\phi\rangle \|_2 \geq 1/2 = \eta_4$.*

Proof. Assume, for sake of contradiction, that $\| |\psi_m\rangle - |\phi\rangle \|_2 < 1/2$, and that $\langle \psi_i | H | \psi_i \rangle < \eta_2$ for all $i \in [m]$. Define b -orthogonal subspaces

$$\begin{aligned} S_{012} &= I_{\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_{N-b-1}} \otimes \text{Span} \left(\left\{ |s\rangle_{\mathcal{B}_{N-b, N}} \mid s \in \{0, 1, 2\}^{b+1} \right\} \right) \\ S_{456} &= I_{\mathcal{A}, \mathcal{B}_1, \dots, \mathcal{B}_{N-b-1}} \otimes \text{Span} \left(\left\{ |s\rangle_{\mathcal{B}_{N-b, N}} \mid s \in \{4, 5, 6\}^{b+1} \right\} \right), \end{aligned}$$

and recall that we set

$$H_{i, i+1} := H'_{i, i+1} \otimes (|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4| + |5\rangle\langle 5|)_{\mathcal{B}_i} + \Delta E_{\mathcal{B}_i, \mathcal{B}_{i+1}} \quad (9)$$

for E the projector forbidding the 2-local substrings depicted in Figure 2 (left). We first show that, for sufficiently large Δ , $|\psi_i\rangle$ has almost all its amplitude on a state in the null space of $H_E := \Delta \sum_{i=1}^N E_{\mathcal{B}_i, \mathcal{B}_{i+1}}$.

► **Lemma 29.** *Assume $\langle \psi_i | H | \psi_i \rangle < \eta_2$ for all $i \in [m]$. Then, there exists $i \in [m]$ such that $|\psi_i\rangle$ can be written as $|\psi_i\rangle = \gamma_1 |\gamma_1\rangle + \gamma_2 |\gamma_2\rangle$, with $\langle \gamma_1 | \gamma_2 \rangle = 0$, $|\gamma_1\rangle \in \text{Null}(H_E)$, and*

$$\| |\psi_i\rangle \langle \psi_i| - |\gamma_1\rangle \langle \gamma_1| \|_{\text{tr}} < 2\sqrt{\frac{\eta_2}{\Delta}}, \quad (10)$$

$$\langle \gamma_1 | I - \Pi_{S_{012}} - \Pi_{S_{456}} | \gamma_1 \rangle > \frac{1}{4m^2} - 2\sqrt{\frac{\eta_2}{\Delta}}. \quad (11)$$

The proof of Lemma 29 follows from Lemma 14. We draw the following conclusions:

1. Recalling that S_{012} (S_{456}) projects onto $\{0, 1, 2\}^*$ ($\{4, 5, 6\}^*$) in $\mathcal{B}_{N-b, \dots, N}$, respectively, Equation (11) implies $|\gamma_1\rangle = \chi_1 |\chi_1\rangle + \chi_2 |\chi_2\rangle$ for orthonormal vectors $\{|\chi_1\rangle, |\chi_2\rangle\}$, $|\chi_1|^2 > (4m^{-2}) - 2\sqrt{\eta_2/\Delta}$, such that $|\chi_1\rangle$ has registers $\mathcal{B}_{N-b, \dots, N}$ supported solely on the intersection of two sets:

- All strings in the null space of H_E (since $|\gamma_1\rangle \in \text{Null}(H_E)$ implies $|\chi_1\rangle \in \text{Null}(H_E)$), and
- all strings in the null space of $I - \Pi_{S_{012}} - \Pi_{S_{456}}$ (by Equation (11)).

But the intersection of these two sets has precisely the regular expression

$$33^*(2^* \cup 4^*), \quad (12)$$

where the first 3 is located in \mathcal{B}_{N-b} . This follows since by Figure 2 (left), a 3 can only have a 2, 3, or 4 to its right, and once we put down a 2 to the right (resp. 4), we can only put down more 2's (resp. 4's).

Similarly, $|\chi_2\rangle$ is supported in registers $\mathcal{B}_{N-b, \dots, N}$ solely on the span of strings from set $\{0, 1, 2\}^{b+1} \cup \{4, 5, 6\}^{b+1}$ (note Figure 2 (left) disallows a digit from set $\{0, 1, 2\}$ to be neighbours with a digit from $\{4, 5, 6\}$). Thus, $|\chi_1\rangle$ and $|\chi_2\rangle$ are orthogonal on the last $b+1$ switch qudits (since the former must have a $|3\rangle$ on these qudits, but the latter cannot).

2. Again since $|\chi_1\rangle \in \text{Null}(H_E)$, combining Equation (12) with Figure 2 (left) now implies, in fact, that *all* switch qudits “to the left” of \mathcal{B}_{N-b} are also set to $|3\rangle$, i.e.

$$|\chi_1\rangle = |3 \cdots 3\rangle_{\mathcal{B}_{1, \dots, N-b}} \otimes |\chi'_1\rangle_{\mathcal{A}, \mathcal{B}_{N-b+1, \dots, N}},$$

for some unit vector $|\chi'_1\rangle$. Together with Equation (12), this implies the entire register \mathcal{B} of $|\chi_1\rangle$ is supported only on symbols from $\{2, 3, 4\}$.

3. Since all of \mathcal{B} is now supported on symbols from $\{2, 3, 4\}$, it follows from Equation (9) that *all* terms of H' are switched on (thus resolving Obstacle 2). Hence,

$$\begin{aligned} \langle \gamma_1 | H | \gamma_1 \rangle &= \langle \gamma_1 | \sum_{i=1}^{N-1} H'_{i, i+1} \otimes (|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4| + |5\rangle\langle 5|)_{\mathcal{B}_i} | \gamma_1 \rangle \\ &> \left(\frac{1}{4m^2} - 2\sqrt{\frac{\eta_2}{\Delta}} \right) \langle \chi_1 | H | \chi_1 \rangle \\ &= \left(\frac{1}{4m^2} - 2\sqrt{\frac{\eta_2}{\Delta}} \right) \text{Tr}(H' \text{Tr}_{\mathcal{B}}(|\chi_1\rangle\langle \chi_1|)) \\ &\geq \left(\frac{1}{4m^2} - 2\sqrt{\frac{\eta_2}{\Delta}} \right) \beta, \end{aligned} \quad (13)$$

where the first statement follows since $|\gamma_1\rangle \in \text{Null}(H_E)$, the second since (1) $H' \succeq 0$ and (2) since

$$\langle \chi_1 | H'_{i, i+1} \otimes (|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4| + |5\rangle\langle 5|)_{\mathcal{B}_i} | \chi_2 \rangle = 0,$$

since $|\chi_1\rangle$ and $|\chi_2\rangle$ are orthogonal on the last $b+1$ switch qudits (even when projected down onto $\text{Span}(|1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle)$), the third statement since all of register \mathcal{B} is supported on symbols $\{2, 3, 4\}$, and the last statement since $\lambda_{\min}(H') \geq \beta$ by assumption.

We conclude that $|\gamma_1\rangle$ is high energy against H . We now show a similar result for $|\psi_i\rangle$, giving the desired contradiction. To do so, we follow the proof of the Projection Lemma of [24]. For brevity, define $H_1 := \sum_{i=1}^{N-1} H'_{i,i+1} \otimes (|1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + |4\rangle\langle 4| + |5\rangle\langle 5|)_{\mathcal{B}_i}$ so that $H = H_1 + H_E$. Then, for $\Delta > 2 \|H'\|_\infty = 2 \|H_1\|_\infty$, recalling that $H_E |\gamma_1\rangle = 0$,

$$\begin{aligned} \langle \psi_i | H | \psi_i \rangle &\geq [(1 - |\gamma_2|^2) \langle \gamma_1 | H_1 | \gamma_1 \rangle + 2\text{Re}(\gamma_1 \gamma_2 \langle \gamma_1 | H_1 | \gamma_2 \rangle) + |\gamma_2|^2 \langle \gamma_2 | H_1 | \gamma_2 \rangle] + \Delta |\gamma_2|^2 \\ &\geq \langle \gamma_1 | H_1 | \gamma_1 \rangle + (\Delta - 2 \|H_1\|_\infty) |\gamma_2|^2 - 2 \|H_1\|_\infty |\gamma_2| \\ &> \langle \gamma_1 | H | \gamma_1 \rangle - 2 \|H_1\|_\infty \sqrt{\frac{\eta_2}{\Delta}} \\ &> \frac{1}{4m^2} \beta - 2 \sqrt{\frac{\eta_2}{\Delta}} (\beta + \|H'\|_\infty), \end{aligned}$$

where the first statement follows since $|\gamma_1|^2 + |\gamma_2|^2 = 1$ and $H_E |\gamma_1\rangle = 0$, the second since $|\gamma_1| \leq 1$, the third when $\Delta > 2 \|H_1\|_\infty$ and since $|\gamma_2|^2 < \eta_2/\Delta$, and the last by Equation (13) and since $\|H_1\|_\infty = \|H'\|_\infty$. Crucially, note that H' is independent of Δ (recall H' is the TI-standard Hamiltonian we have plugged in as a black-box). Thus, we may set Δ to a sufficiently large fixed polynomial in N so that $\langle \psi_i | H | \psi_i \rangle > \frac{1}{2} \left(\frac{\beta}{4m^2} \right) = \eta_2$. This yields the desired contradiction. \blacktriangleleft