Polynomial-Time Approximation Schemes for Independent Packing Problems on Fractionally Tree-Independence-Number-Fragile Graphs

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Abstract

We investigate a relaxation of the notion of treewidth-fragility, namely tree-independence-number-fragility. In particular, we obtain polynomial-time approximation schemes for independent packing problems on fractionally tree-independence-number-fragile graph classes. Our approach unifies and extends several known polynomial-time approximation schemes on seemingly unrelated graph classes, such as classes of intersection graphs of fat objects in a fixed dimension or proper minor-closed classes. We also study the related notion of layered tree-independence number, a relaxation of layered treewidth.

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1 Introduction

Many optimization problems involving collections of geometric objects in the d-dimensional space are known to admit a polynomial-time approximation scheme (PTAS). Arguably the earliest example of such behavior is the problem of finding the maximum number of pairwise non-intersecting disks or squares in a collection of unit disks or unit squares, respectively [38]. Such subcollection is usually called an *independent packing*. This result was later extended to collections of arbitrary disks and squares and, more generally, fat objects [11, 30]. The reason for the abundance of approximation schemes for geometric problems is that shifting and layering techniques can be used to reduce the problem to small subproblems that can be solved by dynamic programming. In fact, the same phenomenon occurs for graph problems, as evidenced by the seminal work of Baker [4] on approximation schemes for local problems, such as Independent Set, on planar graphs and its generalizations first to apex-minor-free graphs [29] and further to graphs embeddable on a surface of bounded genus with a bounded number of crossings per edge [37]. The notion of intersection graph allows to jump from the geometric world to the graph-theoretic one. Given a collection $\mathcal{O}$ of geometric objects in $\mathbb{R}^d$, we can consider its *intersection graph*, the graph whose vertices are the objects in $\mathcal{O}$ and where two vertices $O_i, O_j \in \mathcal{O}$ are adjacent if and only if $O_i \cap O_j \neq \varnothing$. An independent packing in $\mathcal{O}$ is then nothing but an independent set in the corresponding intersection graph. Notice that intersection graphs of unit disks or squares are not minor-closed, as they contain arbitrarily large cliques. Our motivating question is the following:

Is there any underlying graph-theoretical reason for the existence of the seemingly unrelated PTASes for Independent Set *mentioned above?*

We provide a positive answer to this question that also allows us to further generalize to a family of independent packing problems. The similar question of whether there is a general notion under which PTASes using Baker's technique can be obtained was asked in [37].

Baker's layering technique relies on a form of decomposition theorem for planar graphs that can be roughly summarized as follows. Given a planar graph G and $k \in \mathbb{N}$, the vertex set of G can be partitioned into $k + 1$ possibly empty sets in such a way that deleting any part induces a graph of treewidth at most $O(k)$ in G. Moreover, such a partition together with tree decompositions of width at most $O(k)$ of the respective graphs can be found in polynomial time. A statement of this form is typically referred to as a *Vertex Decomposition Theorem* (VDT) [48]. VDTs are known to exist in planar graphs [4], graphs of bounded-genus and apex-minor-free graphs [29], and H-minor-free graphs [17, 19]. However, their existence is in general something too strong to ask for, as is the case of intersection graphs of unit disks or squares and hence fat objects in general. There are then two natural ways in which one can try to relax the notion of VDT. First, we can consider an approximate partition of the vertex set, where a vertex can belong to some constant number of sets. Second, we can look for a width parameter less restrictive than treewidth.

Dvořák [24] pursued the first direction and introduced the notion of *efficient fractional treewidth-fragility.* We state here an equivalent formulation from [28]. A class of graphs $\mathcal{G}$ is *efficiently fractionally treewidth-fragile* if there exists a function $f\colon \mathbb{N} \to \mathbb{N}$ and an algorithm that, for every $k \in \mathbb{N}$ and $G \in \mathcal{G}$, returns in time $\mathsf{poly}(|V(G)|)$ a collection of subsets $X_1, X_2, \ldots, X_m \subseteq V(G)$ such that each vertex of G belongs to at most m/k of the subsets and moreover, for $i = 1, \ldots, m$, the algorithm also returns a tree decomposition of $G - X_i$ of width at most $f(k)$. Several graph classes are known to be efficiently fractionally treewidth-fragile. In fact, a hereditary class $\mathcal{G}$ is efficiently fractionally treewidth-fragile in each of the following cases (see, e.g., [28]): $\mathcal{G}$ has sublinear separators and bounded maximum degree, $\mathcal{G}$ is proper minor-closed, or $\mathcal{G}$ consists of intersection graphs of convex objects with bounded aspect ratio in $\mathbb{R}^d$ (for fixed d) and the graphs in $\mathcal{G}$ have bounded clique number. Dvořák [24] showed that Independent Set admits a PTAS on every efficiently fractionally treewidth-fragile graph class. This result was later extended [26, 28] to a framework of maximization problems including, for example, Max Weight Distance-d Independent Set, Max Weight Induced Forest and Max Weight Induced Matching. However, the notion of fractional treewidth-fragility falls short of capturing classes such as unit disk graphs, as it implies the existence of sublinear separators [24].

One can then try to pursue the second direction mentioned above and further relax the notion of efficient fractional fragility by considering width parameters *more powerful* than treewidth (i.e., bounded on a larger class of graphs) and algorithmically useful. A natural candidate is the recently introduced *tree-independence number* [15], a width parameter defined in terms of tree decompositions which is more powerful than treewidth (see Section 3). Several algorithmic applications of boundedness of tree-independence number have been provided, most notably polynomial-time solvability of Max Weight Independent Packing [15] (see Section 5 for the definition), a common generalization of Max Weight Independent Set and Max Weight Induced Matching, and of its distance-d version, for d even [45]. Investigating the notion of efficient fractional tree-independence-number-fragility (tree-α-fragility for short) was recently suggested in a talk by Dvořák [25], where it was stated that, using an argument from [27], it is possible to show that intersection graphs of balls and cubes in $\mathbb{R}^d$ are fractionally tree-α-fragile.

A successful notion related to fractional treewidth-fragility is the layered treewidth of a graph [21]. Despite currently lacking any direct algorithmic application, it proved useful especially in the context of coloring problems (we refer to [22] for additional references). We just mention that classes of bounded layered treewidth include planar graphs and, more generally, apex-minor-free graphs and graphs embeddable on a surface of bounded genus with a bounded number of crossings per edge, amongst others [20]. It can be shown that bounded layered treewidth implies fractional treewidth-fragility (see Section 4). Layered treewidth is also related to local treewidth, a notion first introduced by Eppstein [29], and in fact, on proper minor-closed classes, having bounded layered treewidth coincides with having bounded local treewidth (see, e.g., [20]).

1.1 Our results

In this paper, we investigate the notion of efficient fractional tree-α-fragility and show that it answers our motivating question in the positive, thus allowing to unify and extend several known results. Our main result can be summarized as follows and will be proved in Section 4 and Section 5.

▶ **Theorem 1.** *Max Weight Independent Packing admits a PTAS on every efficiently fractionally* tree-α*-fragile class. Moreover, the class of intersection graphs of fat objects in* $\mathbb{R}^d$*, for fixed d, is efficiently fractionally* tree-α*-fragile*[1]*.*

The message of Theorem 1 is that a doubly-relaxed version of a VDT suffices for algorithmic applications and is general enough to hold for several interesting graph classes. Theorem 1 cannot be improved to guarantee an EPTAS, unless $\mathsf{FPT} = \mathsf{W}[1]$. Indeed, Marx [42] showed that Independent Set remains $\mathsf{W}[1]$-complete on intersection graphs of unit disks and unit squares. The natural trade-off in extending the tractable families with respect to approximation is that fewer problems will admit a PTAS. In our case this is exemplified by the minimization problem Feedback Vertex Set, which admits no PTAS, unless $\mathsf{P} = \mathsf{NP}$, on unit ball graphs in $\mathbb{R}^3$ [32] but admits an EPTAS on disk graphs in $\mathbb{R}^2$ [41].

In Section 4, we also show that fractionally tree-α-fragile classes have bounded biclique number, where the *biclique number* of a graph G is the maximum $n \in \mathbb{N}$ such that the complete bipartite graph $K_{n,n}$ is an induced subgraph of G. This shows in particular that, unsurprisingly, the notion of fractional tree-α-fragility falls short of capturing intersection graphs of rectangles in the plane. Whether Independent Set admits a PTAS on these graphs remains one of the major open problems in the area (see, e.g., [34]). We also show that the absence of large bicliques is not sufficient for guaranteeing fractional tree-α-fragility: n-dimensional grids of width n are $K_{2,3}$-free but not fractionally tree-α-fragile.

We begin our study of fractional tree-α-fragility by introducing, in Section 3, a subclass of fractionally tree-α-fragile graphs, namely the class of graphs with bounded layered tree-independence number. We obtain the notion of *layered tree-independence number* by relaxing the successful notion of layered treewidth and show that, besides graphs of bounded layered treewidth, classes of intersection graphs of unit disks in $\mathbb{R}^2$ and of paths with bounded horizontal part[2] on a grid have bounded layered tree-independence number. Moreover, we observe that, for minor-closed classes, having bounded layered tree-independence number is equivalent to having bounded layered treewidth, thus extending a characterization of bounded layered treewidth from [20].

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[1] Here we use a definition of fatness slightly generalizing that of Chan [11] (see Section 4.1).
[2] The *horizontal part* of a path is the interval corresponding to the projection of the path onto the x-axis.

We then consider the behavior of layered tree-independence number with respect to graph powers. We show that odd powers of graphs of bounded layered tree-independence number have bounded layered tree-independence number and that this does not extend to even powers. Combined with Theorem 1, this gives the following result which applies, for example, to unit disk graphs and cannot be extended to odd $d \in \mathbb{N}$ (see Section 5.2).

▶ **Theorem 2.** *For a fixed positive even integer d, the distance-d version of* Max Weight Independent Packing *admits a PTAS on every class of bounded layered tree-independence number, provided that a tree decomposition and a layering witnessing small layered tree-independence number can be computed efficiently.*

Finally, we show that the approach to PTASes through tree-independence number is competitive in terms of running time for some classes of intersection graphs. Specifically, in Section 5.3, we obtain PTASes for Max Weight Independent Set for intersection graphs of families of unit disks, unit-height rectangles, and paths with bounded horizontal part on a grid, which improve or generalize results from [6, 12, 43] mentioned in the next section.

We believe that the notion of fractional tree-α-fragility can find further applications in the design of PTASes. In fact, it would be interesting to obtain an algorithmic meta-theorem similar to those for fractionally treewidth-fragile classes [28, 26] and classes of bounded tree-independence number [45]. Although our interest is in approximation schemes, we notice en passant that the observations from Section 3 lead to a subexponential-time algorithm for the distance-d version of Max Weight Independent Packing, for d even, on unit disk graphs. We finally remark that all our PTASes for intersection graphs of geometric objects are not robust i.e., they require a geometric realization to be part of the input.

1.2 Other related work

Disk graphs. Very recently, Lokshtanov et al. [41] established a framework for designing EPTASes for a broad class of minimization problems (specifically, vertex-deletion problems) on disk graphs including, among others, Feedback Vertex Set and d-Bounded Degree Vertex Deletion. Previous sporadic PTASes on this class were known only for Vertex Cover [30, 50], Dominating Set [35], Independent Set [11, 30] and Max Clique [8]. Theorem 1 adds several maximization problems to this list (see Section 5).

Unit disk graphs. Unit disk graphs are arguably one of the most well-studied graph classes in computational geometry, as they naturally model several real-world problems. Great attention has been devoted to approximation algorithms for Max Weight Independent Set on this class (see, e.g., [39, 46, 49]). To the best of our knowledge, the fastest known PTAS is a $(1-1/k)$-approximation algorithm with running time $O(kn^{4\lceil\frac{2(k-1)}{\sqrt{3}}\rceil})$ [43]. We also remark that a special type of Decomposition Theorem was recently shown to hold for the class of unit disk graphs. A Contraction Decomposition Theorem (CDT) is a statement of the following form: given a graph G, for any $p \in \mathbb{N}$, one can partition the edge set of G into $E_1, \ldots, E_p$ such that contracting the edges in each E_i in G yields a graph of treewidth at most $f(p)$, for some function $f\colon \mathbb{N} \to \mathbb{N}$. CDTs are useful in designing efficient approximation and parameterized algorithms and are known to hold for classes such as graphs of bounded genus [18] and unit disk graphs [5]. Since these classes are efficiently fractionally tree-α-fragile, our results can be seen as providing a different type of relaxed decomposition theorems for them.

Intersection graphs of unit-height rectangles. As observed by Agarwal et al. [1], this class of graphs arises naturally as a model for the problem of labeling maps with labels of the same font size. Improving on [38], they obtained a $(1-1/k)$-approximation algorithm for Max Weight Independent Set on this class with running time $O(n^{2k-1})$. Chan [12] provided a $(1-1/k)$-approximation algorithm with running time $O(n^k)$.

Intersection graphs of paths on a grid. Asinowski et al. [3] introduced the class of *Vertex intersection graphs of Paths on a Grid* (*VPG graphs* for short). A graph G is a *VPG graph* if there exists a collection $\mathcal{P}$ of paths on a grid $\mathcal{G}$ such that $\mathcal{P}$ is in one-to-one correspondence with $V(G)$ and two vertices are adjacent in G if and only if the corresponding paths intersect. It is not difficult to see that this class coincides with the well-known class of string graphs. If every path in $\mathcal{P}$ has at most k *bends* i.e., 90 degrees turns at a grid-point, the graph is a *B_k-VPG graph*. Golumbic et al. [36] introduced the class of *Edge intersection graphs of Paths on a Grid* (*EPG graphs* for short) which is defined similarly to VPG, except that two vertices are adjacent if and only if the corresponding paths share a grid-edge. It turns out that every graph is EPG [36] and B_k-EPG graphs have been defined similarly to B_k-VPG graphs. Approximation algorithms for Independent Set on VPG and EPG graphs have been deeply investigated, especially when the number of bends is a small constant (see, e.g., [7, 33, 40, 44]). It is an open problem whether Independent Set admits a PTAS on B_1-VPG graphs [44]. Concerning EPG graphs, Bessy et al. [6] showed that the problem admits no PTAS on B_1-EPG graphs, unless $\mathsf{P} = \mathsf{NP}$, even if each path has its vertical segment or its horizontal segment of length at most 1. On the other hand, they provided a PTAS for Independent Set on B_1-EPG graphs where the length of the horizontal part of each path is at most a constant c with running time $O^*(n^{\frac{3c}{\varepsilon}})$.

2 Preliminaries

We consider only finite simple graphs. If G' is a subgraph of G and G' contains all the edges of G with both endpoints in $V(G')$, then G' is an *induced subgraph* of G and we write $G' = G[V(G')]$. For a vertex $v \in V(G)$ and $r \in \mathbb{N}$, the *r-closed neighborhood* $N_G^r[v]$ is the set of vertices at distance at most r from v in G. The *degree* $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to v in G. The *maximum degree* $\Delta(G)$ of G is the quantity $\max\{d_G(v) : v \in V\}$. Given a graph $G = (V, E)$ and $V' \subseteq V$, the operation of *deleting the set of vertices* V' from G results in the graph $G - V' = G[V \setminus V']$. A graph is *$Z$-free* if it does not contain induced subgraphs isomorphic to graphs in a set Z. The complete bipartite graph with parts of sizes r and s is denoted by $K_{r,s}$. An *independent set* of a graph is a set of pairwise non-adjacent vertices. The maximum size of an independent set of G is denoted by $\alpha(G)$. A *clique* of a graph is a set of pairwise adjacent vertices. A *matching* of a graph is a set of pairwise non-incident edges. An *induced matching* in a graph is a matching M such that no two vertices belonging to different edges in M are adjacent in the graph.

Intersection graphs of unit disks and rectangles. We now explain how the geometric realizations of these intersection graphs are encoded. A collection of unit disks with a common radius $c \in \mathbb{R}$ is encoded by a collection of points in $\mathbb{R}^2$ representing the centers of the disks. Unless otherwise stated, when we refer to a rectangle we mean an axis-aligned closed rectangle in $\mathbb{R}^2$. As is typically done for intersection graphs of rectangles, we assume that the vertices of the rectangles are on an integer grid $\mathcal{G}$ and each rectangle is encoded by the coordinates of its vertices. Given an intersection graph G of a family $\mathcal{R}$ of rectangles, a *grid representation* of G is a pair $(\mathcal{G}, \mathcal{R})$ as above.

VPG and EPG graphs. Given a rectangular grid $\mathcal{G}$, its horizontal lines are referred to as *rows* and its vertical lines as *columns*. For a VPG (EPG) graph G, the pair $\mathcal{R} = (\mathcal{G}, \mathcal{P})$ is a *VPG representation* (*EPG representation*) of G. More generally, a *grid representation* of a graph G is a triple $\mathcal{R} = (\mathcal{G}, \mathcal{P}, x)$ where $x \in \{e, v\}$, such that $(\mathcal{G}, \mathcal{P})$ is an EPG representation of G if $x = e$, and $(\mathcal{G}, \mathcal{P})$ is a VPG representation of G if $x = v$. Note that, irrespective of whether $x = e$ (that is, G is an EPG graph) or $x = v$ (that is, G is a VPG graph), if two vertices $u, v \in V(G)$ are adjacent in G then P_u and P_v share at least one grid-point. A *bend-point* of a path $P \in \mathcal{P}$ is a grid-point corresponding to a bend of P and a *segment* of P is either a vertical or horizontal line segment in the polygonal curve constituting P. Paths in $\mathcal{P}$ are encoded as follows. For each $P \in \mathcal{P}$, we have one sequence $s(P)$ of points in $\mathbb{R}^2$: $s(P) = (x_1, y_1), (x_2, y_2), \ldots, (x_{\ell_P}, y_{\ell_P})$ consists of the endpoints (x_1, y_1) and (x_{ℓ_P}, y_{ℓ_P}) of P and all the bend-points of P in their order of appearance when traversing P from (x_1, y_1) to (x_{ℓ_P}, y_{ℓ_P}). If each path in $\mathcal{P}$ has a number of bends polynomial in $|V(G)|$, then the size of this data structure is polynomial in $|V(G)|$. Given $s(P)$, we can easily determine the horizontal part $h(P)$ of the path P. Note that our results for VPG and EPG graphs (Theorems 11 and 26), although stated for constant number of bends, still hold for polynomial (in $|V(G)|$) number of bends, with a worse polynomial running time.

PTAS. A PTAS for a maximization problem is an algorithm which takes an instance I of the problem and a parameter $\varepsilon > 0$ and produces a solution within a factor $1 - \varepsilon$ of the optimal in time $n^{O(f(1/\varepsilon))}$. A PTAS with running time $f(1/\varepsilon) \cdot n^{O(1)}$ is called an efficient PTAS (EPTAS for short).

3 Layered and local tree-independence number

The key definitions of this section are those of tree-independence number and layering, which we now recall. A *tree decomposition* of a graph G is a pair $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$, where T is a tree whose every node t is assigned a vertex subset $X_t \subseteq V(G)$, called a *bag*, such that the following conditions are satisfied:

(T1) Every vertex of G belongs to at least one bag;

(T2) For every $uv \in E(G)$, there exists a bag containing both u and v;

(T3) For every $u \in V(G)$, the subgraph T_u of T induced by $\{t \in V(T) : u \in X_t\}$ is connected.

The *width* of $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ is the maximum value of $|X_t| - 1$ over all $t \in V(T)$. The *treewidth* of a graph G, denoted $\mathrm{tw}(G)$, is the minimum width of a tree decomposition of G. The *independence number* of $\mathcal{T}$, denoted $\alpha(\mathcal{T})$, is the quantity $\max_{t \in V(T)} \alpha(G[X_t])$. The *tree-independence number* of a graph G, denoted tree-$\alpha(G)$, is the minimum independence number of a tree decomposition of G. Clearly, tree-$\alpha(G) \leq \mathrm{tw}(G) + 1$, for any G. On the other hand, tree-independence number is more powerful than treewidth, as there exist classes with bounded tree-independence number and unbounded treewidth (for example, chordal graphs have tree-independence number 1 [15]).

A *layering* of a graph G is a partition $(V_0, V_1, \ldots, V_t)$ of $V(G)$ such that, for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leq 1$. Each set V_i is a *layer*. The *layered width* of a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of a graph G is the minimum integer ℓ such that, for some layering $(V_0, V_1, \ldots)$ of G, and for each bag X_t and layer V_i, we have $|X_t \cap V_i| \leq \ell$. The *layered treewidth* of a graph G is the minimum layered width of a tree decomposition of G. Layerings with one layer show that the layered treewidth of G is at most $\mathrm{tw}(G) + 1$. We now introduce the analogue of layered treewidth for the width parameter tree-independence number.

▶ **Definition 3.** *The layered independence number of a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of a graph G is the minimum integer ℓ such that, for some layering $(V_0, V_1, \ldots)$ of G, and for each bag X_t and layer V_i, we have $\alpha(G[X_t \cap V_i]) \leq \ell$. The layered tree-independence number of a graph G is the minimum layered independence number of a tree decomposition of G.*

Layerings with one layer show that the layered tree-independence number of G is at most tree-$\alpha(G)$. Moreover, the layered tree-independence number of a graph is clearly at most its layered treewidth. The proof of [21, Lemma 10] shows, mutatis mutandis, that graphs of bounded layered tree-independence number have $O(\sqrt{n})$ tree-independence number:

▶ **Lemma 4.** *Every n-vertex graph with layered tree-independence number k has tree-independence number at most $2\sqrt{kn}$.*

Given a width parameter p, a graph class $\mathcal{G}$ has *bounded local p* if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every integer $r \in \mathbb{N}$, graph $G \in \mathcal{G}$, and vertex $v \in V(G)$, the subgraph $G[N^r[v]]$ has p-width at most $f(r)$. In [21], it is shown that if every graph in a class $\mathcal{G}$ has layered treewidth at most ℓ, then $\mathcal{G}$ has bounded local treewidth with $f(r) = \ell(2r+1) - 1$.

▶ **Lemma 5** ($\star$)**.** *If every graph in a class $\mathcal{G}$ has layered tree-independence number at most ℓ, then $\mathcal{G}$ has bounded local tree-independence number with $f(r) = \ell(2r+1)$.*

▶ **Corollary 6** ($\star$)**.** *The layered tree-independence number of $K_{n,n}$ is at least $n/5$.*

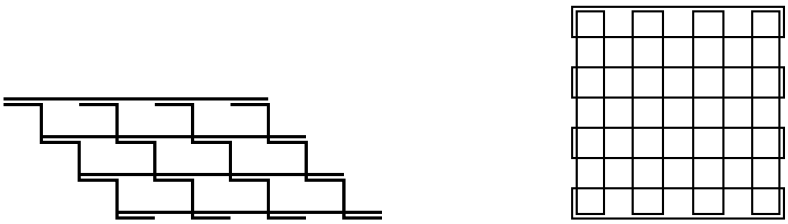

Figure 1 Examples showing that VPG/EPG graphs and intersection graphs of rectangles have unbounded layered tree-independence number: VPG/EPG representation (left) and representation by intersection of rectangles (right) of $K_{4,4}$.

▶ **Theorem 7** ($\star$)**.** *The following are equivalent for a minor-closed class $\mathcal{G}$:*
1. *Some apex[3] graph is not in $\mathcal{G}$;*
2. *$\mathcal{G}$ has bounded local tree-independence number;*
3. *$\mathcal{G}$ has linear local tree-independence number (i.e., $f(r)$ is linear in r);*
4. *$\mathcal{G}$ has bounded layered tree-independence number.*

For $p \in \mathbb{N}$, the p-th power of a graph G is the graph G^p with vertex set $V(G^p) = V(G)$, where $uv \in E(G^p)$ if and only if u and v are at distance at most p in G. Bonomo-Braberman and Gonzalez [9] showed that fixed powers of bounded treewidth and bounded degree graphs are of bounded treewidth: For any graph G and $p \geq 2$, $\operatorname{tw}(G^p) \leq (\operatorname{tw}(G)+1)(\Delta(G)+1)^{\lceil \frac{p}{2} \rceil} - 1$. It follows from [23] that powers of graphs of bounded layered treewidth and bounded maximum degree have bounded layered treewidth. The upper bound therein was later improved by Dujmović et al. [22], who showed that if G has layered treewidth k, then G^p has layered treewidth less than $2pk\Delta(G)^{\lfloor \frac{p}{2} \rfloor}$. Using a result from [45], we show that odd powers of bounded layered tree-independence number graphs have bounded layered tree-independence number and that this does not extend to even powers.

[3] An apex graph is a graph that can be made planar by deleting a single vertex.

▶ **Theorem 8** ($\star$). *Let G be a graph and let d be a positive integer. Given a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ of G and a layering $(V_1, \ldots, V_m)$ of G such that, for each bag X_t and layer V_i, $\alpha(G[X_t \cap V_i]) \leq k$, it is possible to compute in $O(|V(T)| \cdot (|V(G)| + |E(G)|))$ time a tree decomposition $\mathcal{T}' = (T, \{X'_t\}_{t\in V(T)})$ of G^{1+2d} and a layering $(V'_1, \ldots, V'_{\lceil \frac{m}{1+2d} \rceil})$ of G^{1+2d} such that, for each bag X'_t and layer V'_i, $\alpha(G^{1+2d}[X'_t \cap V'_i]) \leq (1 + 4d)k$. In particular, if G has layered tree-independence number k, then G^{1+2d} has layered tree-independence number at most $(1 + 4d)k$.*

▶ **Lemma 9** ($\star$). *Fix an even $k \in \mathbb{N}$. There exist graphs G with layered tree-independence number 1 and such that the layered tree-independence number of G^k is arbitrarily large.*

3.1 Intersection graphs with bounded layered tree-independence number

▶ **Theorem 10** ($\star$). *Let G be the intersection graph of a family $\mathcal{D}$ of n unit disks. It is possible to compute, in $O(n)$ time, a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)}\})$ and a layering $(V_1, V_2, \ldots)$ of G such that $|V(T)| = O(n)$ and, for each bag X_t and layer V_i, $\alpha(G[X_t \cap V_i]) \leq 8$. In particular, G has layered tree-independence number at most 8.*

▶ **Theorem 11** ($\star$). *Let G be a graph on n vertices together with a grid representation $\mathcal{R} = (\mathcal{G}, \mathcal{P}, x)$ such that each path in $\mathcal{P}$ has horizontal part of length at most $\ell - 1$, for some fixed $\ell \geq 1$, and number of bends constant. It is possible to compute, in $O(n^2)$ time, a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ and a layering $(V_1, V_2 \ldots)$ of G such that $|V(T)| = O(n^2)$ and, for each bag X_t and layer V_i, $\alpha(G[X_t \cap V_i]) \leq 4\ell - 1$. In particular, G has layered tree-independence number at most $4\ell - 1$.*

4 Fractional tree-α-fragility

Let p be a width parameter in $\{\text{tw}, \text{tree-}\alpha\}$. Fractional tw-fragility was first defined in [24]. We provide here an equivalent definition from [26], which was explicitly extended to the case $p = \text{tree-}\alpha$ in [25].

▶ **Definition 12.** *For $\beta \leq 1$, a β-general cover of a graph G is a multiset $\mathcal{C}$ of subsets of $V(G)$ such that each vertex belongs to at least $\beta|\mathcal{C}|$ elements of the cover. The p-width of the cover is $\max_{C\in\mathcal{C}} p(G[C])$.*

For a parameter p, a graph class $\mathcal{G}$ is fractionally p-fragile if there exists a function $f \colon \mathbb{N} \to \mathbb{N}$ such that, for every $r \in \mathbb{N}$, every $G \in \mathcal{G}$ has a $(1 - 1/r)$-general cover with p-width at most $f(r)$.

A fractionally p-fragile class $\mathcal{G}$ is efficiently fractionally p-fragile if there exists an algorithm that, for every $r \in \mathbb{N}$ and $G \in \mathcal{G}$, returns in $\mathsf{poly}(|V(G)|)$ time a $(1 - 1/r)$-general cover $\mathcal{C}$ of G and, for each $C \in \mathcal{C}$, a tree decomposition of $G[C]$ of width (if $p = \text{tw}$) or independence number (if $p = \text{tree-}\alpha$) at most $f(r)$, for some function $f \colon \mathbb{N} \to \mathbb{N}$.

Note that classes of bounded tree-independence number are efficiently fractionally tree-α-fragile thanks to [14]. Hence, the family of efficiently fractionally tree-α-fragile classes contains the two incomparable families of bounded tree-independence number classes and efficiently fractionally tw-fragile classes (to see that they are incomparable, consider chordal graphs and planar graphs). We now identify one more subfamily:

▶ **Lemma 13.** *Let $\ell \in \mathbb{N}$ and let G be a graph. For each $r \in \mathbb{N}$, given a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ of G and a layering $(V_0, V_1, \ldots)$ of G such that, for each bag X_t and layer V_i, $\alpha(G[X_t \cap V_i]) \leq \ell$, it is possible to compute in $O(|V(G)|)$ time a $(1 - 1/r)$-general* *cover $\mathcal{C}$ of G and, for each $C \in \mathcal{C}$, a tree decomposition of $G[C]$ with independence number at most $\ell(r-1)$. In particular, if every graph in a class $\mathcal{G}$ has layered tree-independence number at most ℓ, then $\mathcal{G}$ is fractionally* tree-α-*fragile with $f(r) = \ell(r-1)$.*

Proof. Fix $r \in \mathbb{N}$. Let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ and $(V_0, V_1, \ldots)$ be the given tree decomposition and layering of G, respectively. For each $m \in \{0, \ldots, r-1\}$, let $C_m = \bigcup_{i \not\equiv m \pmod r} V_i$. We claim that $\mathcal{C} = \{C_m : 0 \leq m \leq r-1\}$ is a $(1-1/r)$-general cover of G with tree-independence number at most $\ell(r-1)$. Observe first that each $v \in V(G)$ is not covered by exactly one element of $\mathcal{C}$ and so it belongs to $r-1 = (1-1/r)|\mathcal{C}|$ elements of $\mathcal{C}$. Let now $C \in \mathcal{C}$. Each component K of $G[C]$ is contained in at most $r-1$ (consecutive) layers and so, since $\alpha(G[X_t \cap V_i]) \leq \ell$ for each bag X_t and layer V_i, restricting the bags in $\mathcal{T}$ to $V(K)$, gives a tree decomposition of K with independence number at most $\ell(r-1)$. We then merge the tree decompositions of the components of $G[C]$ into a tree decomposition of $G[C]$ with independence number at most $\ell(r-1)$ in linear time. ◀

Note that the same argument of Lemma 13 shows that, if every graph in a class $\mathcal{G}$ has bounded layered treewidth, then $\mathcal{G}$ is fractionally tw-fragile. The following result implies that, if a class is fractionally tree-α-fragile, then it has bounded biclique number.

▶ **Theorem 14.** *For any function $f\colon \mathbb{N} \to \mathbb{N}$ and integer $r > 2$, there exists $n \in \mathbb{N}$ such that no $(1-1/r)$-general cover of $K_{n,n}$ has tree-independence number less than $f(r)$. Hence, the class $\{K_{n,n} : n \in \mathbb{N}\}$ is not fractionally* tree-α-*fragile.*

Proof. Fix arbitrary $f\colon \mathbb{N} \to \mathbb{N}$ and $r > 2$. Consider a copy G of $K_{n,n}$, with $n > f(r)/(1-2/r)$. Let $\mathcal{C}$ be a $(1-1/r)$-general cover of G. Then, every vertex of G belongs to at least $(1-1/r)|\mathcal{C}|$ elements of $\mathcal{C}$ and so there exists $C \in \mathcal{C}$ of size at least $2n(1-1/r)$. Let A and B be the two bipartition classes of G. Then, $|A \cap C| \geq |C| - |B| \geq 2n(1-1/r) - n = n(1-2/r) > f(r)$ and, similarly, $|B \cap C| > f(r)$. Therefore, $G[C]$ contains $K_{f(r),f(r)}$ as an induced subgraph and since tree-$\alpha(K_{f(r),f(r)}) = f(r)$ [15], tree-$\alpha(G[C]) \geq f(r)$. ◀

However, the following result shows that small biclique number does not guarantee fractional tree-α-fragility.

▶ **Theorem 15.** *The class of $K_{2,3}$-free graphs is not fractionally* tree-α-*fragile.*

Proof. Let G_n be the n-dimensional grid graph of width n, i.e., the graph with vertex set $V(G_n) = [n]^n = \{(a_1, \ldots, a_n) : 1 \leq a_1, \ldots, a_n \leq n\}$, where two vertices $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are adjacent if and only if $\sum_{1 \leq i \leq n} |a_i - b_i| = 1$. It is not difficult to see that G_n is $K_{2,3}$-free, for each $n \in \mathbb{N}$. We show that the class $\{G_n : n \in \mathbb{N}\}$ is not fractionally tree-α-fragile.

Fix arbitrary $f\colon \mathbb{N} \to \mathbb{N}$ and $r > 2$. For such a choice, fix $n \in \mathbb{N}$ such that $\frac{r-4}{2r}n + 1 \geq R(3, f(r))$, where $R(3, s)$ denotes the smallest integer m for which every graph on m vertices either contains a clique of size 3 or an independent set of size s. We now show that every $(1-1/r)$-general cover of G_n has tree-independence at least $f(r)$. Let $\mathcal{C}$ be a $(1-1/r)$-general cover of G_n. Then, every vertex of G_n belongs to at least $(1-1/r)|\mathcal{C}|$ elements of $\mathcal{C}$ and so there exists $C \in \mathcal{C}$ containing at least $(1-1/r)|V(G_n)| = (1-1/r)n^n$ vertices of G_n. Fix such a C and let G be the subgraph of G_n induced by C. We claim that tree-$\alpha(G) \geq f(r)$.

Observe first that, for each $v \in V(G_n)$, $n \leq d_{G_n}(v) \leq 2n$. Hence, $2|E(G_n)| = \sum_{v \in V(G_n)} d_{G_n}(v) \geq n \cdot n^n$. Consider now the graph G' obtained from G_n by deleting the vertex set C. Clearly, G' has at most n^n/r vertices. Since deleting a vertex from G_n decreases the number of edges of the resulting graph by at most $2n$, we have that $|E(G)| \geq$

$|E(G_n)| - 2n|V(G')|$, from which $\sum_{v\in V(G)} d_G(v) \geq n \cdot n^n - 2 \cdot 2n \cdot n^n/r = n \cdot n^n(1 - 4/r)$. Therefore, the average degree of G is at least $n(1 - 4/r)$ and so $\text{tw}(G) \geq \frac{r-4}{2r}n$, for example by [13, Corollary 1]. This implies that every tree decomposition of G has a bag of size at least $\frac{r-4}{2r}n + 1 \geq R(3, f(r))$ and, since G is triangle-free, it follows that tree-$\alpha(G) \geq f(r)$. ◀

4.1 Intersection graphs of fat objects

In this section we show that the class of intersection graphs of fat objects in $\mathbb{R}^d$ is efficiently fractionally tree-α-fragile. Let $d \geq 2$ be a fixed integer. A *box of size* r is an axis-aligned hypercube in $\mathbb{R}^d$ of side length r. The *size* of an object O in $\mathbb{R}^d$, denoted $s(O)$, is the side length of its smallest enclosing axis-aligned hypercube.

Chan [11] considered the following definition of fatness: A collection of objects in $\mathbb{R}^d$ is *fat* if, for any r and size-r box R, we can choose c points in $\mathbb{R}^d$ such that every object that intersects R and has size at least r contains at least one of the chosen points. Chan also stated that a collection of balls or boxes with bounded aspect ratios are fat (recall that the aspect ratio of a box is the ratio of its largest side length over its smallest side length). We slightly generalize this fatness definition as follows.

▶ **Definition 16.** *A collection of objects in $\mathbb{R}^d$ is c-fat if, for any r and any size-r closed box R, for every sub-collection $\mathcal{P}$ of pairwise non-intersecting objects which intersect R and are of size at least r, we can choose c points in $\mathbb{R}^d$ such that every object in $\mathcal{P}$ contains at least one of the chosen points.*

▶ Remark 17. When working with a c-fat collection of objects, we assume that some reasonable operations can be done in constant time: determining the center and size of an object, deciding if two objects intersect and constructing the geometric realization of the collection.

▶ **Theorem 18** ($\star$). *Let $\mathcal{O}$ be a c-fat collection of objects in $\mathbb{R}^d$ and let G be its intersection graph. For each $r_0 > 1$, let $f(r_0) = 2\left\lceil \frac{1}{1-\left(1-\frac{1}{r_0}\right)^{\frac{1}{d}}} \right\rceil$. Then, we can compute in linear time a $(1 - 1/r_0)$-general cover $\mathcal{C}$ of G of size at most $(f(r_0)/2 - 1)^d$. Moreover, for each $C \in \mathcal{C}$, we can compute in linear time a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t\in V(T)})$ of $G[C]$, with $|V(T)| \leq |V(G)| + 1$, such that $\alpha(\mathcal{T}) \leq cf(r_0)^{2d}$.*

▶ **Corollary 19** ($\star$). *There exist fractionally* tree-α*-fragile classes of unbounded local tree-independence number.*

5 PTASes

Let us begin by defining Max Weight Independent Packing. Given a graph G and a finite family $\mathcal{H} = \{H_j\}_{j\in J}$ of connected non-null subgraphs of G, an *independent $\mathcal{H}$-packing* in G is a subfamily $\mathcal{H}' = \{H_i\}_{i\in I}$ of subgraphs from $\mathcal{H}$ (that is, $I \subseteq J$) that are at pairwise distance at least 1, that is, they are vertex-disjoint and there is no edge between any two of them. If the subgraphs in $\mathcal{H}$ are equipped with a weight function $w\colon J \to \mathbb{Q}_+$ assigning weight w_j to each subgraph H_j, the *weight* of an independent $\mathcal{H}$-packing $\mathcal{H}' = \{H_i\}_{i\in I}$ in G is $\sum_{i\in I} w_i$. Given a graph G, a finite family $\mathcal{H} = \{H_j\}_{j\in J}$ of connected non-null subgraphs of G, and a weight function $w\colon J \to \mathbb{Q}_+$ on the subgraphs in $\mathcal{H}$, the problem Max Weight Independent Packing asks to find an independent $\mathcal{H}$-packing in G of maximum weight. In the special case when $\mathcal{F}$ is a *fixed* finite family of connected non-null graphs and $\mathcal{H}$ is the set of all subgraphs of G isomorphic to a member of $\mathcal{F}$, the problem is called Max Weight Independent $\mathcal{F}$-Packing and is a common generalization of several problems, among which: INDEPENDENT $\mathcal{F}$-PACKING [10], MAX WEIGHT INDEPENDENT SET ($\mathcal{F} = \{K_1\}$), MAX WEIGHT INDUCED MATCHING ($\mathcal{F} = \{K_2\}$), DISSOCIATION SET ($\mathcal{F} = \{K_1, K_2\}$ and the weight function assigns to each subgraph H_j the weight $|V(H_j)|$) [47, 51].

5.1 Packing subgraphs at distance at least 1 in efficiently fractionally tree-α-fragile classes

Our PTAS relies on the following result.

▶ **Theorem 20** (Dallard et al. [15]). *Let k and h be two positive integers. Given a graph G and a finite family $\mathcal{H} = \{H_j\}_{j \in J}$ of connected non-null subgraphs of G such that $|V(H_j)| \leq h$ for every $j \in J$, MAX WEIGHT INDEPENDENT PACKING can be solved in time $O(|V(G)|^{h(k+1)} \cdot |V(T)|)$ if G is given together with a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ with $\alpha(\mathcal{T}) \leq k$.*

▶ **Theorem 21.** *Let $h \in \mathbb{N}$ and let $f \colon \mathbb{N} \to \mathbb{N}$ be a function. There exists an algorithm that, given $r \in \mathbb{N}$, an n-vertex graph G equipped with a $(1 - 1/r)$-general cover $\mathcal{C} = \{C_1, C_2, \ldots\}$ and, for each i, a tree decomposition $\mathcal{T}_i = (T_i, \{X_t\}_{t \in V(T_i)})$ of $G[C_i]$ with $\alpha(\mathcal{T}_i) \leq f(r)$, a finite family $\mathcal{H} = \{H_j\}_{j \in J}$ of connected non-null subgraphs of G such that $|V(H_j)| \leq h$ for every $j \in J$, and a weight function $w \colon J \to \mathbb{Q}_+$ on the subgraphs in $\mathcal{H}$, returns in time $|\mathcal{C}| \cdot O(n^{h(f(r)+1)} \cdot t)$, where $t = \max_i |V(T_i)|$, an independent $\mathcal{H}$-packing in G of weight at least a factor $(1 - h/r)$ of the optimal.*

Proof. For each $i \geq 1$, we proceed as follows. Using the algorithm from Theorem 20, we simply compute a maximum-weight independent $\mathcal{H}$-packing $\mathcal{P}_i$ in $G[C_i]$ in time $O(n^{h(f(r)+1)} \cdot t)$. The total running time is then $|\mathcal{C}| \cdot O(n^{h(f(r)+1)} \cdot t)$. For a collection $\mathcal{A}$ of subgraphs of G, each isomorphic to a member of $\mathcal{H}$, and a subset $C \subseteq V(G)$, let $w(\mathcal{A}) = \sum_{A \in \mathcal{A}} w(A)$ and let $\mathcal{A} \cap C = \{A \in \mathcal{A} : A \subseteq C\}$. Observe that, given a subgraph H of G, each vertex $v \in V(H)$ is not contained in at most $|\mathcal{C}|/r$ elements of the $(1 - 1/r)$-general cover $\mathcal{C}$. Hence, $V(H)$ is contained in at least $(1 - |V(H)|/r)|\mathcal{C}|$ elements of $\mathcal{C}$. Let $\mathcal{P} = \{P_1, P_2, \ldots\}$ be an independent $\mathcal{H}$-packing in G of maximum weight. Then,

$$\begin{aligned}
\sum_{C_i \in \mathcal{C}} w(\mathcal{P} \cap C_i) &= \sum_{C_i \in \mathcal{C}} \sum_{P_j \in \mathcal{P}} w(P_j) \mathbb{1}_{\{P_j \subseteq C_i\}} \\
&= \sum_{P_j \in \mathcal{P}} w(P_j) \sum_{C_i \in \mathcal{C}} \mathbb{1}_{\{P_j \subseteq C_i\}} \\
&\geq \sum_{P_j \in \mathcal{P}} w(P_j)(1 - |V(P_j)|/r)|\mathcal{C}| \\
&\geq \sum_{P_j \in \mathcal{P}} w(P_j)(1 - h/r)|\mathcal{C}| \\
&= |\mathcal{C}|(1 - h/r)w(\mathcal{P}).
\end{aligned}$$

By the pigeonhole principle, there exists $C_i \in \mathcal{C}$ such that $w(\mathcal{P} \cap C_i) \geq (1 - h/r)w(\mathcal{P})$. We then return the maximum-weight independent $\mathcal{H}$-packing $\mathcal{P}_i$ in $G[C_i]$ computed above. Since $\mathcal{P} \cap C_i$ is an independent $\mathcal{H}$-packing in $G[C_i]$, we have that $w(\mathcal{P}_i) \geq w(\mathcal{P} \cap C_i) \geq (1 - h/r)w(\mathcal{P})$. ◀

Theorem 21 immediately implies that MAX WEIGHT INDEPENDENT PACKING admits a PTAS in any efficiently fractionally tree-α-fragile class. A special case is the following.

▶ **Corollary 22** ($\star$). *There exists an algorithm that, given $r \in \mathbb{N}$, a c-fat collection $\mathcal{O}$ of n objects in $\mathbb{R}^d$ and its intersection graph G, and a weight function $w\colon V(G) \to \mathbb{Q}_+$, returns in time $(f(r)/2-1)^d \cdot O(n^{(cf(r)^{2d}+2)})$, where $f(r) = 2\left\lceil \frac{1}{1-\left(1-\frac{1}{r}\right)^{\frac{1}{d}}} \right\rceil$, an independent set in G of weight at least a factor $(1-1/r)$ of the optimal.*

5.2 Packing subgraphs at distance at least d in graphs with bounded layered tree-independence number

Max Weight Independent Packing has a natural generalization. For a fixed positive integer d, given a graph G and a finite family $\mathcal{H} = \{H_j\}_{j \in J}$ of connected non-null subgraphs of G, a *distance-d $\mathcal{H}$-packing* in G is a subfamily $\mathcal{H}' = \{H_i\}_{i \in I}$ of subgraphs from $\mathcal{H}$ that are at pairwise distance at least d. If we are also given a weight function $w\colon J \to \mathbb{Q}_+$, Max Weight Distance-$d$ Packing is the problem of finding a distance-d $\mathcal{H}$-packing in G of maximum weight. The case $d = 2$ coincides with Max Weight Independent Packing.

▶ **Theorem 23** ($\star$). *Let $h, \ell \in \mathbb{N}$. Let d be an even positive integer. There exists an algorithm that, given $r \in \mathbb{N}$, an n-vertex graph G equipped with a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ and a layering $(V_1, V_2, \ldots)$ of G such that, for each bag X_t and layer V_i, $\alpha(G[X_t \cap V_i]) \leq \ell$, a finite family $\mathcal{H} = \{H_j\}_{j \in J}$ of connected non-null subgraphs of G such that $|V(H_j)| \leq h$ for every $j \in J$, and a weight function $w\colon J \to \mathbb{Q}_+$, returns in time $r \cdot |V(T)| \cdot n^{O(r)}$ a distance-d $\mathcal{H}$-packing in G within a factor $(1-h/r)$ of the optimal.*

Combining Theorem 23 with Theorem 10, we obtain the following:

▶ **Corollary 24.** *Let $d \in \mathbb{N}$ be even. Max Weight Distance-d Packing admits a* PTAS *for unit disk graphs.*

Observe that Theorem 23 cannot be extended to odd values of d, unless P = NP. Indeed, Eto et al. [31] showed that, for each $\varepsilon > 0$ and fixed odd $d \geq 3$, it is NP-hard to approximate Distance-d Independent Set to within a factor of $n^{1/2-\varepsilon}$ for chordal graphs.

Since unit disk graphs have $O(\sqrt{n})$ tree-independence number (Theorem 10 and Lemma 4) and since Max Weight Distance-d Packing is solvable in time $n^{O(k)}$, where k is the tree-independence number of the input graph [45], we immediately obtain a subexponential-time algorithm on unit disk graphs.

▶ **Lemma 25.** *For any fixed even $d \in \mathbb{N}$, Max Weight Distance-d Packing can be solved in $2^{O(\sqrt{n}\log n)}$ time on unit disk graphs.*

A subexponential-time algorithm for Independent Set on unit disk graphs was first given in [2] and later extended in [16] to intersection graphs of fat objects.

5.3 Packing independent unit disks, unit-width rectangles and paths with bounded horizontal part on a grid

The following PTASes are obtained by showing that the tree-independence number of graphs whose geometric realizations are contained in an axis-aligned rectangle with bounded width is bounded.

▶ **Theorem 26** ($\star$). *Max Weight Independent Set admits a PTAS when restricted to:*

- *Intersection graphs of a family of n unit disks of common radius $c \geq 1$. The running time is $O(c\lceil \frac{2}{\varepsilon} \rceil \cdot n^{2\lceil \frac{2}{\varepsilon} \rceil + 3})$.*
- *Intersection graphs of a family of n width-c rectangles together with a grid representation $(\mathcal{G}, \mathcal{R})$. The running time is $O(c\lceil \frac{1}{\varepsilon} \rceil \cdot n^{\lceil \frac{1}{\varepsilon} \rceil \cdot \frac{c}{2} + 4})$.*

- *Graphs on n vertices with a grid representation $\mathcal{R} = (\mathcal{G}, \mathcal{P}, x)$ such that each path in $\mathcal{P}$ has number of bends constant and the horizontal part of each path in $\mathcal{P}$ has length at most c, for some fixed $c \in \mathbb{N}$. If $x = v$, the running time is $O(c\lceil \frac{1}{\varepsilon} \rceil \cdot n^{\lceil \frac{1}{\varepsilon} \rceil c+4})$. If $x = e$, the running time is $O(c\lceil \frac{1}{\varepsilon} \rceil \cdot n^{3(\lceil \frac{1}{\varepsilon} \rceil c+1)})$.*

References

1 Pankaj K. Agarwal, Marc van Kreveld, and Subhash Suri. Label placement by maximum independent set in rectangles. *Computational Geometry*, 11(3):209–218, 1998.

2 Jochen Alber and Jiří Fiala. Geometric separation and exact solutions for the parameterized independent set problem on disk graphs. *Journal of Algorithms*, 52(2):134–151, 2004.

3 Andrei Asinowski, Elad Cohen, Martin Charles Golumbic, Vincent Limouzy, Marina Lipshteyn, and Michal Stern. Vertex intersection graphs of paths on a grid. *Journal of Graph Algorithms and Applications*, 16(2):129–150, 2012.

4 Brenda S. Baker. Approximation algorithms for NP-complete problems on planar graphs. *J. ACM*, 41(1):153–180, 1994.

5 Sayan Bandyapadhyay, William Lochet, Daniel Lokshtanov, Saket Saurabh, and Jie Xue. True contraction decomposition and almost ETH-tight bipartization for unit-disk graphs. In Xavier Goaoc and Michael Kerber, editors, *38th International Symposium on Computational Geometry (SoCG 2022)*, volume 224 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 11:1–11:16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022.

6 Stephane Bessy, Marin Bougeret, Steven Chaplick, Daniel Gonçalves, and Christophe Paul. On independent set in B_1-EPG graphs. *Discrete Applied Mathematics*, 278:62–72, 2020.

7 Therese Biedl and Martin Derka. Splitting B_2-VPG graphs into outer-string and co-comparability graphs. In Faith Ellen, Antonina Kolokolova, and Jörg-Rüdiger Sack, editors, *Algorithms and Data Structures - 15th International Symposium, (WADS 2017)*, volume 10389 of *Lecture Notes in Computer Science*, pages 157–168. Springer, 2017.

8 Marthe Bonamy, Édouard Bonnet, Nicolas Bousquet, Pierre Charbit, Panos Giannopoulos, Eun Jung Kim, Paweł Rzążewski, Florian Sikora, and Stéphan Thomassé. EPTAS and subexponential algorithm for maximum clique on disk and unit ball graphs. *J. ACM*, 68(2), 2021.

9 Flavia Bonomo-Braberman and Carolina Lucía Gonzalez. A new approach on locally checkable problems. *Discrete Applied Mathematics*, 314:53–80, June 2022.

10 Kathie Cameron and Pavol Hell. Independent packings in structured graphs. *Mathematical Programming*, 105(2-3):201–213, 2006.

11 Timothy M Chan. Polynomial-time approximation schemes for packing and piercing fat objects. *Journal of Algorithms*, 46(2):178–189, 2003.

12 Timothy M. Chan. A note on maximum independent sets in rectangle intersection graphs. *Information Processing Letters*, 89(1):19–23, 2004.

13 L.Sunil Chandran and C.R. Subramanian. Girth and treewidth. *Journal of Combinatorial Theory, Series B*, 93(1):23–32, 2005.

14 Clément Dallard, Fedor V. Fomin, Petr A. Golovach, Tuukka Korhonen, and Martin Milanič. Computing tree decompositions with small independence number. *CoRR*, abs/2207.09993, 2022. URL: `https://arxiv.org/abs/2207.09993`.

15 Clément Dallard, Martin Milanič, and Kenny Štorgel. Treewidth versus clique number. II. Tree-independence number. *CoRR*, abs/2111.04543, 2022. URL: `https://arxiv.org/abs/2111.04543`.

16 Mark de Berg, Hans L. Bodlaender, Sándor Kisfaludi-Bak, Dániel Marx, and Tom C. van der Zanden. A framework for exponential-time-hypothesis–tight algorithms and lower bounds in geometric intersection graphs. *SIAM Journal on Computing*, 49(6):1291–1331, 2020.

17 Erik D. Demaine, Mohammad Taghi Hajiaghayi, and Ken-ichi Kawarabayashi. Algorithmic graph minor theory: Decomposition, approximation, and coloring. In *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2005)*, pages 637–646. IEEE Computer Society, 2005.

18 Erik D. Demaine, MohammadTaghi Hajiaghayi, and Bojan Mohar. Approximation algorithms via contraction decomposition. *Combinatorica*, 30(5):533–552, 2010.

19 Matt DeVos, Guoli Ding, Bogdan Oporowski, Daniel P. Sanders, Bruce Reed, Paul Seymour, and Dirk Vertigan. Excluding any graph as a minor allows a low tree-width 2-coloring. *Journal of Combinatorial Theory, Series B*, 91(1):25–41, 2004.

20 Vida Dujmović, David Eppstein, and David R. Wood. Structure of graphs with locally restricted crossings. *SIAM Journal on Discrete Mathematics*, 31(2):805–824, 2017.

21 Vida Dujmović, Pat Morin, and David R. Wood. Layered separators in minor-closed graph classes with applications. *Journal of Combinatorial Theory, Series B*, 127:111–147, 2017.

22 Vida Dujmović, Louis Esperet, Pat Morin, Bartosz Walczak, and David R. Wood. Clustered 3-colouring graphs of bounded degree. *Combinatorics, Probability and Computing*, 31(1):123–135, 2022.

23 Vida Dujmović, Pat Morin, and David R. Wood. Graph product structure for non-minor-closed classes. *CoRR*, abs/1907.05168, 2022. URL: `https://arxiv.org/abs/1907.05168`.

24 Zdeněk Dvořák. Sublinear separators, fragility and subexponential expansion. *European Journal of Combinatorics*, 52:103–119, 2016.

25 Zdeněk Dvořák. Talk at GWP 2022 - satellite workshop of ICALP 2022, 2022. URL: `https://homepages.ecs.vuw.ac.nz/~bretteni/GWP2022/`.

26 Zdeněk Dvořák. Approximation Metatheorems for Classes with Bounded Expansion. In Artur Czumaj and Qin Xin, editors, *18th Scandinavian Symposium and Workshops on Algorithm Theory (SWAT 2022)*, volume 227 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 22:1–22:17. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022.

27 Zdeněk Dvořák, Daniel Gonçalves, Abhiruk Lahiri, Jane Tan, and Torsten Ueckerdt. On comparable box dimension. In Xavier Goaoc and Michael Kerber, editors, *38th International Symposium on Computational Geometry (SoCG 2022)*, volume 224 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 38:1–38:14. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2022.

28 Zdeněk Dvořák and Abhiruk Lahiri. Approximation schemes for bounded distance problems on fractionally treewidth-fragile graphs. In Petra Mutzel, Rasmus Pagh, and Grzegorz Herman, editors, *29th Annual European Symposium on Algorithms (ESA 2021)*, volume 204 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 40:1–40:10. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2021.

29 David Eppstein. Diameter and treewidth in minor-closed graph families. *Algorithmica*, 27(3):275–291, 2000.

30 Thomas Erlebach, Klaus Jansen, and Eike Seidel. Polynomial-time approximation schemes for geometric intersection graphs. *SIAM Journal on Computing*, 34(6):1302–1323, 2005.

31 Hiroshi Eto, Fengrui Guo, and Eiji Miyano. Distance-d independent set problems for bipartite and chordal graphs. *Journal of Combinatorial Optimization*, 27(1):88–99, 2014.

32 Fedor V. Fomin, Daniel Lokshtanov, and Saket Saurabh. Bidimensionality and geometric graphs. In Yuval Rabani, editor, *Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2012)*, pages 1563–1575. SIAM, 2012.

33 Jacob Fox and János Pach. Computing the independence number of intersection graphs. In Dana Randall, editor, *Twenty-Second Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2011)*, pages 1161–1165. SIAM, 2011.

34 Waldo Gálvez, Arindam Khan, Mathieu Mari, Tobias Mömke, Madhusudhan Reddy Pittu, and Andreas Wiese. A 3-approximation algorithm for Maximum Independent Set of rectangles. In Joseph (Seffi) Naor and Niv Buchbinder, editors, *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms (SODA 2022)*, pages 894–905. SIAM, 2022.

35 Matt Gibson and Imran A. Pirwani. Algorithms for dominating set in disk graphs: Breaking the $\log n$ barrier. In Mark de Berg and Ulrich Meyer, editors, *Algorithms – ESA 2010*, volume 6346 of *Lecture Notes in Computer Science*, pages 243–254. Springer Berlin Heidelberg, 2010.

36 Martin Charles Golumbic, Marina Lipshteyn, and Michal Stern. Edge intersection graphs of single bend paths on a grid. *Networks*, 54(3):130–138, 2009.

37 Alexander Grigoriev and Hans L. Bodlaender. Algorithms for graphs embeddable with few crossings per edge. *Algorithmica*, 49(1):1–11, 2007.

38 Dorit S. Hochbaum and Wolfgang Maass. Approximation schemes for covering and packing problems in image processing and VLSI. *J. ACM*, 32(1):130–136, 1985.

39 Harry B Hunt, Madhav V Marathe, Venkatesh Radhakrishnan, S.S Ravi, Daniel J Rosenkrantz, and Richard E Stearns. NC-approximation schemes for NP- and PSPACE-hard problems for geometric graphs. *Journal of Algorithms*, 26(2):238–274, 1998.

40 Abhiruk Lahiri, Joydeep Mukherjee, and C. R. Subramanian. Maximum Independent Set on B_1-VPG graphs. In Zaixin Lu, Donghyun Kim, Weili Wu, Wei Li, and Ding-Zhu Du, editors, *Combinatorial Optimization and Applications - 9th International Conference (COCOA 2015)*, volume 9486 of *Lecture Notes in Computer Science*, pages 633–646. Springer, 2015.

41 Daniel Lokshtanov, Fahad Panolan, Saket Saurabh, Jie Xue, and Meirav Zehavi. A framework for approximation schemes on disk graphs. In Nikhil Bansal and Viswanath Nagarajan, editors, *Proceedings of the 2023 ACM-SIAM Symposium on Discrete Algorithms (SODA 2023)*, pages 2228–2241. SIAM, 2023.

42 Dániel Marx. Efficient approximation schemes for geometric problems? In Gerth Stølting Brodal and Stefano Leonardi, editors, *Algorithms – ESA 2005*, volume 3669 of *Lecture Notes in Computer Science*, pages 448–459. Springer Berlin Heidelberg, 2005.

43 Tomomi Matsui. Approximation algorithms for maximum independent set problems and fractional coloring problems on unit disk graphs. In Jin Akiyama, Mikio Kano, and Masatsugu Urabe, editors, *Discrete and Computational Geometry*, pages 194–200. Springer Berlin Heidelberg, 1998.

44 Saeed Mehrabi. Approximating domination on intersection graphs of paths on a grid. In Roberto Solis-Oba and Rudolf Fleischer, editors, *Approximation and Online Algorithms - 15th International Workshop (WAOA 2017)*, volume 10787 of *Lecture Notes in Computer Science*, pages 76–89. Springer, 2017.

45 Martin Milanič and Paweł Rzążewski. Tree decompositions with bounded independence number: beyond independent sets. *CoRR*, abs/2209.12315, 2022. URL: `https://arxiv.org/abs/2209.12315`.

46 Tim Nieberg, Johann L. Hurink, and Walter Kern. A robust PTAS for maximum weight independent sets in unit disk graphs. In Juraj Hromkovic, Manfred Nagl, and Bernhard Westfechtel, editors, *Graph-Theoretic Concepts in Computer Science, 30th International Workshop (WG 2004)*, volume 3353 of *Lecture Notes in Computer Science*, pages 214–221. Springer, 2004.

47 Yury Orlovich, Alexandre Dolgui, Gerd Finke, Valery Gordon, and Frank Werner. The complexity of dissociation set problems in graphs. *Discrete Applied Mathematics*, 159(13):1352–1366, 2011.

48 Fahad Panolan, Saket Saurabh, and Meirav Zehavi. Contraction decomposition in unit disk graphs and algorithmic applications in parameterized complexity. In Timothy M. Chan, editor, *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2019)*, pages 1035–1054. SIAM, 2019.

49 Erik Jan van Leeuwen. Approximation algorithms for unit disk graphs. In Dieter Kratsch, editor, *Graph-Theoretic Concepts in Computer Science, 31st International Workshop (WG 2005)*, volume 3787 of *Lecture Notes in Computer Science*, pages 351–361. Springer, 2005.

50 Erik Jan van Leeuwen. Better approximation schemes for disk graphs. In Lars Arge and Rusins Freivalds, editors, *Algorithm Theory - SWAT 2006, 10th Scandinavian Workshop on Algorithm Theory*, volume 4059 of *Lecture Notes in Computer Science*, pages 316–327. Springer, 2006.

51 Mihalis Yannakakis. Node-deletion problems on bipartite graphs. *SIAM Journal on Computing*, 10(2):310–327, 1981.