Ephemeral Persistence Features and the Stability of Filtered Chain Complexes

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Abstract

We strengthen the usual stability theorem for Vietoris-Rips (VR) persistent homology of finite metric spaces by building upon constructions due to Usher and Zhang in the context of filtered chain complexes. The information present at the level of filtered chain complexes includes ephemeral points, i.e. points with zero persistence, which provide additional information to that present at homology level. The resulting invariant, called verbose barcode, which has a stronger discriminating power than the usual barcode, is proved to be stable under certain metrics which are sensitive to these ephemeral points. In some situations, we provide ways to compute such metrics between verbose barcodes. We also exhibit several examples of finite metric spaces with identical (standard) VR barcodes yet with different verbose VR barcodes thus confirming that these ephemeral points strengthen the discriminating power of the standard VR barcode.

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1 Introduction

In topological data analysis, *persistent homology* is one of the main tools used for extracting and analyzing multiscale geometric and topological information from metric spaces.

Typically, the *persistent homology pipeline* (as induced by the Vietoris-Rips filtration) is explained via the diagram:

$$\text{Metric Spaces} \to \text{Simplicial Filtrations} \to \text{Persistence Modules}$$

where, from left to right, the second map is homology with field coefficients. Throughout the paper, we fix a base field $\mathbb{F}$.

Pairs of birth and death times of topological features (such as connected components, loops, voids and so on) give rise to the *barcode*, or also called the *persistence diagram*, of a given metric space [13, 4]. The so-called *bottleneck distance* d_B between the persistent homology barcodes arising from the Vietoris-Rips filtration of metric spaces provides a polynomial time computable lower bound for the *Gromov-Hausdorff distance* d_{GH} between the underlying metric spaces [8, 10]. However, this bound is not tight, in general (cf. [17, Example 6.6]). A restricted version of this theorem states:

▶ **Theorem 1** (Stability Theorem for d_{B}). *Let X and Y be two finite metric spaces. Let $\mathcal{B}_k(X)$ (resp. $\mathcal{B}_k(Y)$) denote the barcode of the persistence module $\mathrm{H}_k(\mathrm{VR}_\bullet(X))$ (resp. $\mathrm{H}_k(\mathrm{VR}_\bullet(Y))$). Then, we have*

$$\sup_{k\in\mathbb{Z}_{\geq 0}} d_{\mathrm{B}}(\mathcal{B}_k(X),\mathcal{B}_k(Y)) \leq 2\cdot d_{\mathrm{GH}}(X,Y).$$

In this paper, with the goal of refining the standard stability result alluded to above, we concentrate on the usually implicit but conceptually important intermediate step which assigns a *filtered chain complex* (FCC) to a given simplicial filtration:

$$\text{Metric spaces} \to \text{Simplicial Filtrations} \to \boxed{\text{FCCs}} \to \text{Persistence Modules}.$$

Related work on FCCs

An FCC is an ascending sequence of chain complexes connected by monomorphisms. For instance, an FCC induced by a simplicial filtration $\{X_t\}_{t\in\mathbb{R}}$ can be represented by the following commutative diagram: for any $t\leq t'$,

$$\begin{array}{ccccccccc}
\mathrm{C}_*(X_t): & & \cdots & \xrightarrow{\partial_{k+2}} & \mathrm{C}_{k+1}(X_t) & \xrightarrow{\partial_{k+1}} & \mathrm{C}_k(X_t) & \xrightarrow{\partial_k} & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & \\
\mathrm{C}_*(X_{t'}): & & \cdots & \xrightarrow{\partial_{k+2}} & \mathrm{C}_{k+1}(X_{t'}) & \xrightarrow{\partial_{k+1}} & \mathrm{C}_k(X_{t'}) & \xrightarrow{\partial_k} & \cdots
\end{array},$$

where each X_t is a simplical complex and $\mathrm{C}_*(X_t)$ denotes the simplical chain complex of X_t.

Studies of the decomposition of FCCs in several different settings can be found in [18, 12, 15, 5, 6]. We follow the convention of Usher and Zhang [18], where they study a notion of Floer-type complexes as a generalization of FCCs and prove a stability result for the usual bottleneck distance of concise barcodes of Floer-type complexes. In particular, they studied FCCs in detail and considered the notion of *verbose barcode* $\mathcal{B}_{\mathrm{Ver},k}$ of FCCs, which consists of the standard barcode (which the authors call *concise barcode* and denote as $\mathcal{B}_{\mathrm{Con},k} := \mathcal{B}_k$) together with additional *ephemeral* bars, i.e. bars of length 0.

They also proved that every FCC decomposes into the direct sum of indecomposables $\mathcal{E}(a,a+L,k)$, which they called *elementary FCCs*, of the following form (see [18, Definition 7.2]): if $L\in[0,\infty)$ and $a\in\mathbb{R}$, then $\mathcal{E}(a,a+L,k)$ is given by

$$\begin{array}{rccccccccc}
t<a: & \cdots\to 0 & \xrightarrow{\partial_{k+2}=0} & 0 & \xrightarrow{\partial_{k+1}=0} & 0 & \xrightarrow{\partial_k=0} & 0\to\cdots \\
 & & & \downarrow= & & \downarrow & & \\
t\in[a,a+L): & \cdots\to 0 & \xrightarrow{\partial_{k+2}=0} & 0 & \xrightarrow{\partial_{k+1}=0} & \mathbb{F}x & \xrightarrow{\partial_k=0} & 0\to\cdots \\
 & & & \downarrow & & \downarrow= & & \\
t\in[a+L,\infty): & \cdots\to 0 & \xrightarrow{\partial_{k+2}=0} & \mathbb{F}y & \xrightarrow{\partial_{k+1}:\,y\mapsto x} & \mathbb{F}x & \xrightarrow{\partial_{k-1}} & 0\to\cdots
\end{array}$$

If $L=\infty$, then $\mathcal{E}(a,\infty,k)$ (with the convention that $a+\infty=\infty$) is given by

$$\begin{array}{rccccccc}
t<a: & \cdots\to 0 & \xrightarrow{\partial_{k+2}=0} & 0 & \xrightarrow{\partial_{k+1}=0} & 0 & \xrightarrow{\partial_k=0} & 0\to\cdots \\
 & & & \downarrow= & & \downarrow & & \\
t\in[a,\infty): & \cdots\to 0 & \xrightarrow{\partial_{k+2}=0} & 0 & \xrightarrow{\partial_{k+1}=0} & \mathbb{F}x & \xrightarrow{\partial_k=0} & 0\to\cdots
\end{array}$$

The degree-l verbose barcode of the elementary FCC $\mathcal{E}(a,a+L,k)$ is $\{(a,a+L)\}$ for $l=k$ and is empty for $l\neq k$.

The concise barcode of an FCC is defined as the collection of non-ephemeral bars, i.e. bars corresponding to elementary FCCs with $L\neq 0$ in its decomposition, which agrees with the standard barcode. Indeed, the k-th persistent homology of the elementary FCC $\mathcal{E}(a, a+L, k)$ is the interval persistence module associated to the interval $[a, a+L)$, for $L \in [0, \infty]$. In particular, $\mathrm{H}_k(\mathcal{E}(a, a, k))$ is the trivial persistence module.

In real calculations, barcodes are often computed for simplexwise filtrations first (i.e., simplices are assumed to enter the filtration one at a time), in which case all elementary FCCs corresponds to intervals with positive length. This implies that, although not outputted, verbose barcodes are computed in many persistence algorithms. For VR FCCs, we made a small modification of the software Ripser introduced by Bauer (see [1]) to extract verbose barcodes of finite metric spaces.

In this paper, we focus on the *ephemeral* bars in the barcode, or equivalently, on the diagonal points in the persistence diagram.

Overview of our results

One drawback of the bottleneck stability result described in Theorem 1 is that one asks for optimal matchings between the concise (i.e. standard) barcodes $\mathcal{B}_{\mathrm{Con},k}(X)$ and $\mathcal{B}_{\mathrm{Con},k}(Y)$ for each individual degree k *independently.*

With the goal of finding a *coherent* or *simultaneous* matching of barcodes across all degrees at once, we study the interleaving distance d_{I} between FCCs and establish an isometry theorem between d_{I} and the *matching distance* d_{M} between the verbose barcodes (see Definition 18):

▶ **Theorem 2** (Isometry theorem). *For any two FCCs $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$, let $\mathcal{B}^C_{\mathrm{Ver},k}$ and $\mathcal{B}^D_{\mathrm{Ver},k}$ denote their degree-k verbose barcodes, respectively, and let $d_{\mathrm{M}}\left(\mathcal{B}^C_{\mathrm{Ver}}, \mathcal{B}^D_{\mathrm{Ver}}\right) := \sup_{k\in\mathbb{Z}_{\geq 0}} d_{\mathrm{M}}\left(\mathcal{B}^C_{\mathrm{Ver},k}, \mathcal{B}^D_{\mathrm{Ver},k}\right)$. Then,*

$$d_{\mathrm{M}}\left(\mathcal{B}^C_{\mathrm{Ver}}, \mathcal{B}^D_{\mathrm{Ver}}\right) = d_{\mathrm{I}}\left((C_*, \partial_C, \ell_C), (D_*, \partial_D, \ell_D)\right).$$

To prove that $d_{\mathrm{M}} \leq d_{\mathrm{I}}$ (see §3.3.1), we adapted ideas implicit in [18, Proposition 9.3] which the authors used to establish the stability of *Floer*-type complexes (on the same underlying chain complex). For the other direction, $d_{\mathrm{M}} \geq d_{\mathrm{I}}$ (see §3.3.2), we use an idea similar to the one used for proving that the bottleneck distance d_{B} between concise barcodes is upper bounded by d_{I} between persistent modules, cf. [14, Theorem 3.4].

Unlike d_{B} between concise barcodes, d_{M} between verbose barcodes of VR RCCs is not stable under the Gromov-Hausdorff distance between metric spaces. Indeed, d_{M} is only finite if the two underlying metric spaces have the same cardinality. We remedy this issue in §4.2 by incorporating the notion of tripods as in [16].

For a surjection $\phi_X : Z \twoheadrightarrow X$, we equip Z with the pullback $\phi_X^* d_X$ of the distance function d_X and call the pair $(Z, \phi_X^* d_X)$ the *pullback (pseudo) metric space* (induced by ϕ_X). We call the degree-k verbose barcode of $(Z, \phi_X^* d_X)$ a *degree-k pullback barcode* of X. We define the *pullback bottleneck distance* between verbose barcodes of two finite metric spaces X and Y to be the infimum of the matching distance between the verbose barcodes of the VR FCCs induced by the respective pullbacks $(Z, \phi_X^* d_X)$ and $(Z, \phi_Y^* d_Y)$, where the infimum is taken over *tripods* $R : X \xtwoheadleftarrow{\phi_X} Z \xtwoheadrightarrow{\phi_Y} Y$. We denote the result by $\hat{d}_{\mathrm{B}}$; see Definition 24. Similarly, we define the *pullback interleaving distance* between two VR FCCs, and denote it by $\hat{d}_{\mathrm{I}}$ (see Definition 23).

▶ Remark 3 (Terminology). We point out the following regarding the use of the term "distance" when referring to $\hat{d}_{\mathrm{B}}$ and $\hat{d}_{\mathrm{I}}$:

(1) $\hat{d}_{\mathrm{B}}$ between degree-0 verbose barcodes satisfies the triangle inequality [17, Corollary 6.7].
(2) The question whether $\hat{d}_{\mathrm{B}}$ between positive-degree verbose barcodes satisfies the triangle inequality is still open.
(3) $\hat{d}_{\mathrm{I}}$ does not satisfy the triangle inequality; see [17, Remark 6.8] for details.

Due to Items (2) and (3), the term "distance" is being abused through the use of the terminology "pullback bottleneck distance" and "pullback interleaving distance". We do so for consistency with Item (1) and due to the fact that in [17, Remark 6.8] we provide a way to modify $\hat{d}_{\mathrm{I}}$ and $\hat{d}_{\mathrm{B}}$ so that they do satisfy the triangle inequality (while still being Gromov-Hausdorff stable).

It is important to note that in general, the pullback bottleneck distance $\hat{d}_{\mathrm{B}}$ (or the pullback interleaving distance $\hat{d}_{\mathrm{I}}$) depends on the underlying metric spaces, rather than solely on the verbose barcodes (or FCCs). Nonetheless, we use the current terminology to emphasize the roles of verbose barcodes and FCCs in our discussion.

It follows from Theorem 2 and the definitions of $\hat{d}_{\mathrm{B}}$ and $\hat{d}_{\mathrm{I}}$ that we have the following:

▶ **Corollary 4.** *Let (X, d_X) and (Y, d_Y) be two finite metric spaces. Then,*

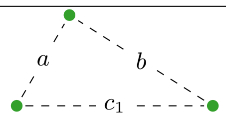
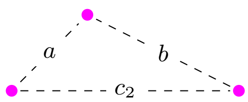
$$\sup_{k\in\mathbb{Z}_{\geq 0}} \hat{d}_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Ver},k}(X), \mathcal{B}_{\mathrm{Ver},k}(Y)\right) \leq \hat{d}_{\mathrm{I}}\left(\left(\mathrm{C}_*(\mathrm{VR}(X)), \partial^X, \ell^X\right), \left(\mathrm{C}_*(\mathrm{VR}(Y)), \partial^Y, \ell^Y\right)\right).$$

In the theorem below, we show that the pullback bottleneck distance $\hat{d}_{\mathrm{B}}$ is stable under the Gromov-Hausdorff distance d_{GH}, and that the bottleneck distance d_{B} between concise barcodes is not larger than $\hat{d}_{\mathrm{B}}$ between verbose barcodes. We show in several examples below and in §4.3 that $\hat{d}_{\mathrm{B}}$ between verbose barcodes can be strictly larger than d_{B} between concise barcodes. Thus, the stability of $\hat{d}_{\mathrm{B}}$ improves the stability of the standard bottleneck distance d_{B} (cf. Theorem 1). See §4.2 for the proof of Theorem 5.

▶ **Theorem 5** (Pullback stability theorem)**.** *Let (X, d_X) and (Y, d_Y) be two finite metric spaces. Then, for any $k \in \mathbb{Z}_{\geq 0}$,*

$$d_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Con},k}(X), \mathcal{B}_{\mathrm{Con},k}(Y)\right) \leq \hat{d}_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Ver},k}(X), \mathcal{B}_{\mathrm{Ver},k}(Y)\right) \leq 2 \cdot d_{\mathrm{GH}}(X, Y). \tag{1}$$

See Figure 1 for a pair of 3-point metric spaces which d_{B} between concise barcodes fails to distinguish, but the $\hat{d}_{\mathrm{B}}$ between verbose barcodes succeeds at telling apart.

	 X_1	 X_2
$\mathcal{B}_{\mathrm{Ver},0}$	$(0,a),(0,b),(0,\infty)$	$(0,a),(0,b),(0,\infty)$
$\mathcal{B}_{\mathrm{Ver},1}$	(c_1,c_1)	(c_2,c_2)

$d_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Con}}(X_1), \mathcal{B}_{\mathrm{Con}}(X_2)\right)$	$\hat{d}_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Ver}}(X_1), \mathcal{B}_{\mathrm{Ver}}(X_2)\right)$	$2 \cdot d_{\mathrm{GH}}(X_1, X_2)$
0	$\lvert c_1 - c_2\rvert$	$\lvert c_1 - c_2\rvert$

Figure 1 *First table:* three-point metric spaces X_1 and X_2 together with their verbose barcodes. Here $a \leq b \leq c_i$ for $i = 1, 2$. *Second table:* the bottleneck distance between concise barcodes, the pullback bottleneck distance between verbose barcodes and twice of the Gromov-Hausdorff distance between X_1 and X_2. See Example 28.

In order to have a more concrete understanding of the pullback bottleneck distance and in order to explore the possibility of computing it, we study the relation between the verbose barcode of a pullback metric space $(Z, \phi_X^* d_X)$ with the verbose barcode of the original space X. We conclude that the verbose barcodes of Z and X only differ on some distinguished diagonal points; see Proposition 6 below.

We now set up some notation about multisets[1]. For a non-negative integer m, by $\{x\}^m$ we will denote the multiset containing exactly m copies of x. For any multiset A and any $l \geq 1$, we let $P_l(A)$ be the multiset consisting of sub-multisets of A each with cardinality l.

▶ **Proposition 6** (Pullback barcodes). *Let $k \geq 0$, $m \geq 1$ and $Z = X \sqcup \{x_{j_1}, \ldots, x_{j_m}\}$ for some $j_1 \leq \cdots \leq j_m$. Then, for $k \geq 0$,*

$$\mathcal{B}_{\mathrm{Ver},k}(Z) = \mathcal{B}_{\mathrm{Ver},k}(X) \sqcup \bigsqcup_{i=0}^{m-1} \left\{ \mathrm{diam}([x_{j_{i+1}}, \beta_i]) \cdot (1,1) : \beta_i \in P_k\left((X \setminus \{x_{j_{i+1}}\}) \sqcup \{x_{j_1}, \ldots, x_{j_i}\} \right) \right\}. \tag{2}$$

In particular, $\mathcal{B}_{\mathrm{Ver},0}(Z) = \mathcal{B}_{\mathrm{Ver},0}(X) \sqcup \{(0,0)\}^m$.

Because concise barcodes can be obtained from verbose barcodes by excluding all diagonal points, the above proposition interestingly implies that $\mathcal{B}_{\mathrm{Con},k}(Z) = \mathcal{B}_{\mathrm{Con},k}(X)$ for any k.

To better understand Equation (2) in the case when $k \geq 1$, we give a graphical explanation in Figure 2. Let (X, d_X) be a finite metric space with $X = \{x_1, \ldots, x_n\}$. Each finite pullback metric space $(Z, \phi_X^* d_X)$ of X can be written as a multiset $Z = X \sqcup \{x_{j_1}, \ldots, x_{j_m}\}$ equipped with the pullback pseudo-metric $\phi_X^* d_X$ induced from d_X, for some $m \geq 0$ and $1 \leq j_1 \leq \cdots \leq j_m \leq n$. In other words, the extra points in Z are "repeats" of the points in X. We will call each point in X the *parent* of its repeated copies: to be more precise, for each $z \in Z$, the point $\phi_X(z) \in X$ will be called the parent of z. Write

$$Z = X \sqcup \Big\{ \underbrace{x_1, \ldots, x_1}_{m_1}, \ldots, \underbrace{x_n, \ldots, x_n}_{m_n} \Big\},$$

where each $m_j \geq 0$ is the multiplicity of the extra copies of x_j in Z and $m_1 + \cdots + m_n = m$.

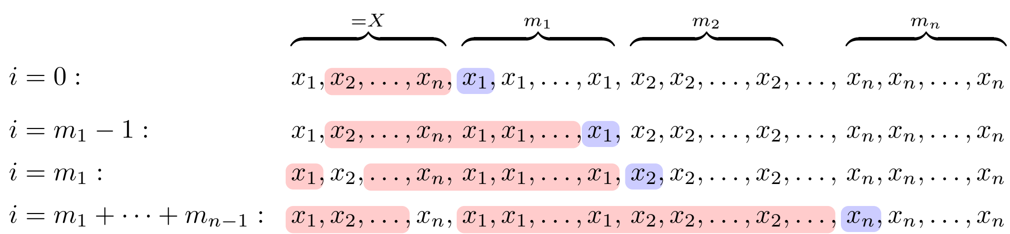


Figure 2 With the same notation as in Equation (2), for each i (i.e. for each row), the point $x_{j_{i+1}}$ is colored in blue. For each i the multiset β_i in Equation (2) ranges over all k element sub-multisets of the red-colored multiset. Notice that each red-colored multiset consists of every point before $x_{j_{i+1}}$ (from left to right) excluding the parent of $x_{j_{i+1}}$.

We examine the relationship between $\hat{d}_{\mathrm{B}}$ and d_{B}, and obtain an interpretation of $\hat{d}_{\mathrm{B}}$ in terms of matchings of points in the barcodes. To compute d_{B}, one looks for an optimal matching where points from a barcode can be matched to any points on the diagonal. However,

[1] We use the notation $\{\cdot\}$ for multisets as well when its meaning is clear from the content.

	X_1 : •- - a - -•	X_2 : •
$\mathcal{B}_{\mathrm{Ver},0} = \mathcal{B}_{\mathrm{Con},0}$	$\{(0,a),(0,\infty)\}$	$\{(0,\infty)\}$

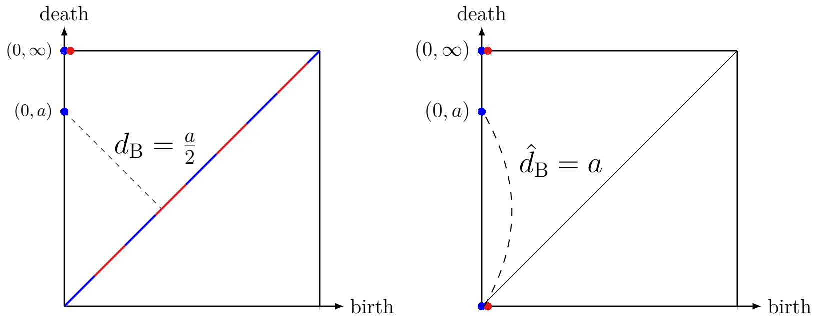


Figure 3 *Top*: X_1 a two-point space, X_2 the one-point space, and their 0-th verbose (or concise) barcode. *Bottom*: visualization of d_{B} and $\hat{d}_{\mathrm{B}}$, where in both figures the point $(0,\infty)$ is matched with $(0,\infty)$ and the ℓ_∞-metric is used to compute the distances.

in the computation of $\hat{d}_{\mathrm{B}}$, points are only allowed to be matched to verbose barcodes and a *particular sub-multiset* of the diagonal points, where the choice of these diagonal points depends on the metric structure of the two underlying metric spaces.

For degree-0, since the verbose barcode of any pullback (pseudo-)metric space Z of X only differs from the verbose barcode of X in multiple copies of the point $(0,0)$, the distance $\hat{d}_{\mathrm{B}}$ is indeed computing an optimal matching between concise barcodes which only allows bars to be matched to other bars or to the origin $(0,0)$ (see Figure 3). Combined with the fact that degree-0 bars are all born at 0, we obtain the following explicit formula for computing the distance $\hat{d}_{\mathrm{B}}$ for degree-0 (see [17, §6.2.1] for the proof):

▶ **Proposition 7** (Pullback bottleneck distance in degree 0)**.** *Let X and Y be two finite metric spaces such that* $\mathrm{card}(X) = n \leq n' = \mathrm{card}(Y)$. *Suppose the death time of finite-length degree-0 bars of X and Y are given by the sequences $a_1 \geq \cdots \geq a_{n-1}$ and $b_1 \geq \cdots \geq b_{n'-1}$, respectively. Then*

$$\hat{d}_{\mathrm{B}}(\mathcal{B}_{\mathrm{Ver},0}(X), \mathcal{B}_{\mathrm{Ver},0}(Y)) = \max\left\{ \max_{1\leq i\leq n-1} |a_i - b_i|, \max_{n\leq i\leq n'-1} b_i \right\}.$$

For higher degrees, the situation becomes more complicated because in addition to the point $(0,0)$, other choices of diagonal points need to be considered, as evidenced by the formula for pullback barcodes in Proposition 6. We leave this as our future work.

2 Filtered chain complexes (FCCs)

We recall the notion of FCCs and provide some properties and examples for VR FCCs.

Usher and Zhang express FCCs as the triples $(C_*, \partial_C, \ell_C)$, where (C_*, ∂_C) denotes a chain complex and $\ell_C : C_* \to \mathbb{R} \sqcup \{-\infty\}$ is a *filtration* function such that (1) $\ell_C \circ \partial_C \leq \ell_C$, and (2) $\ell_C(x) = -\infty$ iff $x = 0$, $\ell_C(\lambda x) = \ell_C(x)$ for $\lambda \neq 0$, and $\ell_C(x+y) \leq \max\{\ell_C(x), \ell_C(y)\}, \forall x, y \in C$. A *morphism of FCCs* from $(C_*, \partial_C, \ell_C)$ to $(D_*, \partial_D, \ell_D)$ is a chain map $\Phi_* : \mathrm{C}_* \to D_*$ that is *filtration preserving*, i.e. $\ell_D \circ \Phi_* \leq \ell_C$. Let **FCC** denote the category of FCCs. We refer readers to [18] or [17, §3] for more details about general FCCs.

VR FCCs. A *pseudo-metric* d_X on X is a function $d_X : X \times X \to [0, +\infty)$ satisfying the axioms for a metric, except that different points are allowed to have distance 0.

Given a finite pseudo-metric space (X, d_X) and $\epsilon \geq 0$, the ϵ*-Vietoris–Rips complex* $\mathrm{VR}_\epsilon(X)$ is the simplicial complex with vertex set X, where

$$\text{a finite subset } \sigma \subset X \text{ is a simplex of } \mathrm{VR}_\epsilon(X) \iff \mathrm{diam}(\sigma) \leq \epsilon.$$

Let $\mathrm{diam}(X)$ be the diameter of X. Let $\mathrm{VR}(X) := \mathrm{VR}_{\mathrm{diam}(X)}(X)$, which is the *full complex* on X. For each $k \in \mathbb{Z}_{\geq 0}$, we denote by $\mathrm{C}_k(\mathrm{VR}\,(X))$ the free $\mathbb{F}$-vector space generated by k-simplices in $\mathrm{VR}(X)$, and let $\mathrm{C}_*(\mathrm{VR}(X))$ be the free simplicial chain complex induced by $\mathrm{VR}(X)$ over coefficients in $\mathbb{F}$, with the standard simplicial boundary operator ∂^X. Notice that up to homotopy equivalence the simplicial complex $\mathrm{VR}(X)$ only depends on the cardinality of X, so does the chain complex $(\mathrm{C}_*(\mathrm{VR}(X)), \partial^X)$.

Define the filtration function $\ell^X : \mathrm{C}_*(\mathrm{VR}(X)) \to \mathbb{R} \sqcup \{-\infty\}$ by

$$\ell^X\left(\sum_{i=1}^{r} \lambda_i \sigma_i\right) := \max_{\lambda_i \neq 0} \{\mathrm{diam}(\sigma_i)\},$$

where the σ_i are simplices, and $\ell^X(0) := -\infty$. Then $\left(\mathrm{C}_*(\mathrm{VR}(X)), \partial^X, \ell^X\right)$ is an FCC.

2.1 Verbose and concise barcodes

For a vector space equipped with a filtration function ℓ, a finite collection $(x_1, \dots, x_r)$ of elements C is said to be *(ℓ-)orthogonal* if, for all $\lambda_1, \dots, \lambda_r \in \mathbb{F}$,

$$\ell\left(\sum_{i=1}^{r} \lambda_i x_i\right) = \max_{\lambda \neq 0} \ell(x_i).$$

Let $A : C \to D$ be a linear map with rank r. A *(unsorted) singular value decomposition of* A is a choice of orthogonal ordered bases $(y_1, \dots, y_n)$ for C and $(x_1, \dots, x_m)$ for D such that (see [18, Definition 3.1]):

- $(y_{r+1}, \dots, y_n)$ is an orthogonal ordered basis for $\mathrm{Ker}\, A$;
- $(x_1, \dots, x_r)$ is an orthogonal ordered basis for $\mathrm{Im}\, A$;
- $Ay_i = x_i$ for $i = 1, \dots, r$.

The existence of a singular value decomposition for linear maps between finite-dimensional orthogonalizable $\mathbb{F}$-spaces is guaranteed by [18, Theorem 3.4].

▶ **Definition 8** (Verbose barcode and concise barcode, [18, Definition 6.3])**.** *Let $(C_*, \partial_C, \ell_C)$ be an FCC over $\mathbb{F}$ and for each $k \in \mathbb{Z}$ write $\partial_k = \partial_C|_{\mathrm{C}_k}$. Given any $k \in \mathbb{Z}$ choose a singular value decomposition $((y_1, \dots, y_n), (x_1, \dots, x_m))$ for the $\mathbb{F}$-linear map $\partial_{k+1} : \mathrm{C}_{k+1} \to \mathrm{Ker}\, \partial_k$ and let r denote the rank of ∂_{k+1}. Then the degree-k* verbose barcode *of $(C_*, \partial_C, \ell_C)$ is the multiset $\mathcal{B}_{\mathrm{Ver},k}^{(C_*,\partial_C,\ell_C)}$ (or $\mathcal{B}_{\mathrm{Ver},k}^{C}$ for simplicity) of elements of $\mathbb{R} \times [0, \infty]$ consisting of*

- *a pair $(\ell(x_i), \ell(y_i))$ for each $i = 1, \dots, r = \mathrm{rank}(\partial_{k+1})$; and*
- *a pair $(\ell(x_i), \infty)$ for each $i = r + 1, \dots, m = \dim(\mathrm{Ker}\, \partial_k)$.*

The concise barcode *of $(C_*, \partial_C, \ell_C)$ is the submultiset of the verbose barcode consisting of those elements where $\ell(y_i) - \ell(x_i) \neq 0$.*

▶ Remark 9. Let X be a finite *metric* space. The degree-0 verbose barcode $\mathcal{B}_{\mathrm{Ver},0}$ and the degree-0 concise barcode $\mathcal{B}_{\mathrm{Con},0}$ of the VR FCC $\left(\mathrm{C}_*(\mathrm{VR}(X)), \partial^X, \ell^X\right)$ are the same. Notice that this is not necessarily true for pseudo-metric spaces, in which case verbose barcode may contain several copies of $(0, 0)$.

▶ **Example 10** (Verbose barcodes of VR FCCs)**.** Let $n := \mathrm{card}(X)$. The number of k-verbose barcodes (with multiplicity) of the VR FCC of a finite pseudo-metric space X is

$$\mathrm{card}(\mathcal{B}_{\mathrm{Ver},k}(X)) = \dim(\mathrm{Ker}(\partial_k)) = \begin{cases} n, & k = 0, \\ \binom{n-1}{k+1}, & \text{for } 1 \leq k \leq n-2, \\ 0, & \text{for } k \geq n-1. \end{cases}$$

2.2 Decomposition of FCCs

We recall from [18] that the collection of verbose barcodes is a *complete* invariant of FCCs, because every FCC decomposes uniquely up to isomorphism into the following form:

$$(C_*, \partial_C, \ell_C) \cong \bigoplus_{k\in\mathbb{Z}} \bigoplus_{(a,a+L)\in\mathcal{B}_{\mathrm{Ver},k}} \mathcal{E}(a, a+L, k).$$

Also, the collection of concise barcodes is an invariant up to the so-called filtered homotopy equivalence. In addition, for the case of VR FCCs, we show that isometry implies filtered chain isomorphism while the inverse is not true.

For the purpose of this paper, we use the theorem below as definitions of *filtered chain isomorphism (f.c.i.)* and *filtered homotopy equivalence (f.h.e.)* between FCCs, and refer readers to [18] for the original definitions of these two concepts.

▶ **Theorem 11** (Theorem A & B, [18])**.** *Two FCCs* $(C_*, \partial_C, \ell_C)$ *and* $(D_*, \partial_D, \ell_D)$ *are*

1. *filtered chain isomorphic iff they have identical verbose barcodes in all degrees;*
2. *filtered homotopy equivalent iff they have identical concise barcodes in all degrees.*

▶ **Example 12** (f.h.e. but not f.c.i.)**.** Let X and Y be (ultra-)metric spaces of 4 points given in Figure 4. The FCCs $\left(\mathrm{C}_*(\mathrm{VR}(X)), \partial^X, \ell^X\right)$ and $\left(\mathrm{C}_*(\mathrm{VR}(Y)), \partial^Y, \ell^Y\right)$ arising from Vietoris-Rips complexes have the same concise barcodes but different verbose barcodes.

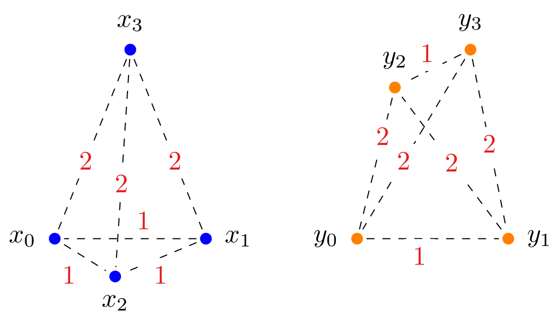


Figure 4 Four-point metric spaces X (left) and Y (right).

We compute from Definition 8 that the verbose barcodes of X and Y are

$$\mathcal{B}_{\mathrm{Ver},k}(X) = \begin{cases} \{(0,1),(0,1),(0,2),(0,\infty)\}, & k=0 \\ \{(1,1),(2,2),(2,2)\}, & k=1 \\ \{(2,2)\}, & k=2 \\ \emptyset, & \text{otherwise.} \end{cases}$$

and

$$\mathcal{B}_{\mathrm{Ver},k}(Y) = \begin{cases} \{(0,1),(0,1),(0,2),(0,\infty)\}\,, & k=0 \\ \{(2,2),(2,2),(2,2)\}\,, & k=1 \\ \{(2,2)\}\,, & k=2 \\ \emptyset, & \text{otherwise,} \end{cases}$$

respectively. The concise barcodes of X and Y are

$$\mathcal{B}_{\mathrm{Con},k}(X) = \mathcal{B}_{\mathrm{Con},k}(Y) = \begin{cases} \{(0,1),(0,1),(0,2),(0,\infty)\}\,, & k=0 \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let (X, d_X) and (Y, d_Y) be two finite pseudo-metric spaces with $|X| = |Y|$. Then, any bijection $f : X \to Y$ induces a chain isomorphism $f_* : \mathrm{C}_*(\mathrm{VR}(X)) \xrightarrow{\cong} \mathrm{C}_*(\mathrm{VR}(Y))$. It is not difficult to check that the respective VR FCCs of two isometric pseudo-metric spaces are filtered chain isomorphic.

▶ **Proposition 13** (Isometry implies f.c.i.). *Let (X, d_X) and (Y, d_Y) be two finite pseudo-metric spaces. If (X, d_X) and (Y, d_Y) are isometric, then the FCCs $\left(\mathrm{C}_*(\mathrm{VR}(X)), \partial^X, \ell^X\right)$ and $\left(\mathrm{C}_*(\mathrm{VR}(Y)), \partial^Y, \ell^Y\right)$ are filtered chain isomorphic.*

However, the converse of Proposition 13 is not true.

▶ **Example 14** (f.c.i. but not isometric). Let X and Y be (ultra-)metric spaces of 5 points given in Figure 5. The distance matrices for X and Y are, respectively:

$$\begin{pmatrix} 0 & 0.5 & 2 & 2 & 2 \\ 0.5 & 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & 1 & 1 \\ 2 & 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0.5 & 1 & 2 & 2 \\ 0.5 & 0 & 1 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 \\ 2 & 2 & 2 & 0 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{pmatrix}.$$

Notice that X and Y are not isometric. Indeed, in Y every vertex belongs to an edge of length 1, but the top point in X only belongs to edges of length 0.5 and 2.

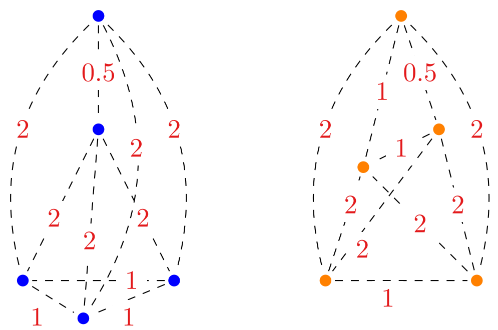


Figure 5 Five-point metric spaces X (left) and Y (right).

However, the VR FCCs of X and Y have the same verbose barcodes:

$$\mathcal{B}_{\mathrm{Ver},k}(X) = \mathcal{B}_{\mathrm{Ver},k}(Y) = \begin{cases} \{(0,0.5),(0,1),(0,1),(0,2),(0,\infty)\}, & k=0 \\ \{(1,1),(2,2),(2,2),(2,2),(2,2),(2,2)\}, & k=1 \\ \{(2,2),(2,2),(2,2),(2,2)\}, & k=2 \\ \{(2,2)\}, & k=3 \\ \emptyset, & \text{otherwise.} \end{cases}$$

3 Isometry theorem ($d_{\mathrm{I}} = d_{\mathrm{M}}$)

In TDA, it is well-known that, under mild conditions (e.g. q-tameness, see [9]), an isometry theorem holds: the interleaving distance between persistence modules is equal to the bottleneck distance between their *concise* barcodes (cf. [11, 7, 11]). In our notation, this means that for any degree k and any two FCCs $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$,

$$d_{\mathrm{B}}\left(\mathcal{B}^C_{\mathrm{Con},k}, \mathcal{B}^D_{\mathrm{Con},k}\right) = d_{\mathrm{I}}\left(\mathrm{H}_k \circ (C_*, \partial_C, \ell_C), \mathrm{H}_k \circ (D_*, \partial_D, \ell_D)\right).$$

We prove an analogous isometry theorem for the *verbose* barcode, i.e., Theorem 2.

3.1 Interleaving distance d_{I} between FCCs

For detailed proofs of results in this subsection, see [17, §4.1]. Let d_{I} be the categorical interleaving distance in the category of filtered chain complexes given by [2, Definition 3.2].

▶ **Proposition 15.** *Let $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$ be two FCCs. Then*

$$d_{\mathrm{I}}\left((C_*, \partial_C, \ell_C), (D_*, \partial_D, \ell_D)\right) < \infty \iff (C_*, \partial_C) \cong (D_*, \partial_D).$$

Because of Proposition 15, the interleaving distance between FCCs is only interesting when we consider the case when two FCCs have the *same* underlying chain complexes. Let (C_*, ∂_C) be a finite-dimensional non-zero chain complex over $\mathbb{F}$, and let $\mathrm{Iso}((C_*, \partial_C))$ be the set of chain isomorphisms on (C_*, ∂_C).

▶ **Theorem 16.** *Let (C_*, ∂_C) be a non-zero chain complex and let $\ell_1, \ell_2 : \mathrm{C}_* \to \mathbb{R} \sqcup \{-\infty\}$ be two filtration functions such that both $(C_*, \partial_C, \ell_1)$ and $(C_*, \partial_C, \ell_2)$ are FCCs. Then*

$$d_{\mathrm{I}}\left((C_*, \partial_C, \ell_1), (C_*, \partial_C, \ell_2)\right) = \inf_{\Phi_* \in \mathrm{Iso}(C_*, \partial_C)} \|\ell_1 - \ell_2 \circ \Phi_*\|_\infty.$$

Here we follow the convention $(-\infty) - (-\infty) = 0$ when computing $\|\ell_1 - \ell_2\|_\infty$. When ℓ_1 is the trivial filtration function, we have $d_{\mathrm{I}}\left((C_*, \partial_C, \ell_1), (C_*, \partial_C, \ell_2)\right) = \|\ell_2\|_\infty$.

▶ **Example 17** (d_{I} **between Elementary FCCs).** For $L_a, L_b < \infty$, the interleaving distance between elementary FCCs $\mathcal{E}(a, a+L_a, k)$ and $\mathcal{E}(b, b+L_b, l)$ is finite iff $k = l$. And

$$d_{\mathrm{I}}\left(\mathcal{E}(a, a+L_a, k), \mathcal{E}(b, b+L_b, k)\right) = \max\{|a-b|, |(a+L_a)-(b+L_b)|\}.$$

3.2 Matching distance d_{M} between verbose barcodes

Let $\mathcal{H} := \{(p,q) : 0 \le p < q \le \infty\}$, and let $\Delta := \{(r,r) : r \in \mathbb{R}_{\ge 0} \sqcup \{+\infty\}\}$. We denote $\overline{\mathcal{H}} := \mathcal{H} \sqcup \Delta$ the extended real upper plane. Let d_∞ be the metric on $\overline{\mathcal{H}}$ inherited from the l_∞-metric, where for $p, q, p', q' \in \mathbb{R}_{\ge 0} \sqcup \{\infty\}$,

$$d_\infty((p,q),(p',q')) = \begin{cases} \max\{|p-p'|, |q-q'|\}, & q, q' < \infty, \\ |p-p'|, & q = q' = \infty, \\ \infty, & \text{otherwise.} \end{cases}$$

Denote by Δ^∞ (or $\mathcal{H}^\infty$ and $\overline{\mathcal{H}}^\infty$, respectively) the multiset consisting of each point on Δ (or $\mathcal{H}$ and $\overline{\mathcal{H}}$, respectively), taken with (countably) infinite multiplicity. Let $\overline{\mathcal{H}}^\infty$ be equipped with the metric d_∞ inherited from $\overline{\mathcal{H}}$.

▶ **Definition 18** (The Matching Distance d_{M}). *Let A and B be two non-empty sub-multisets of $\overline{\mathcal{H}}^\infty$. The* matching distance *between A and B is*

$$d_{\mathrm{M}}(A,B) := \min\left\{\max_{a\in A} d(a,\phi(a)) : A \xrightarrow{\phi} B \text{ a bijection}\right\},$$

where $d_{\mathrm{M}}(A,B)=\infty$ if $\operatorname{card}(A)\neq\operatorname{card}(B)$.

▶ **Definition 19** (The Bottleneck Distance d_{B}). *Let A and B be two finite non-empty sub-multisets of $\mathcal{H}^\infty$. The* bottleneck distance *between A and B is*

$$d_{\mathrm{B}}(A,B) := d_{\mathrm{M}}(A\sqcup\Delta^\infty, B\sqcup\Delta^\infty).$$

Unlike the bottleneck distance d_{B}, the diagonal points that can be matched in d_{M} between verbose barcodes are limited (see also [17, Proposition 4.11]). Thus,

▶ **Proposition 20.** *Given two FCCs (C_*,∂_C,ℓ_C) and (D_*,∂_D,ℓ_D) and any degree k, we have*

$$d_{\mathrm{B}}\left(\mathcal{B}^C_{\mathrm{Con},k},\mathcal{B}^D_{\mathrm{Con},k}\right) \le d_{\mathrm{M}}\left(\mathcal{B}^C_{\mathrm{Ver},k},\mathcal{B}^D_{\mathrm{Ver},k}\right).$$

Given (C_*,∂_C,ℓ_C) and a chain isomorphism Φ_* on (C_*,∂_C), because (C_*,∂_C,ℓ) and $(C_*,\partial_C,\ell\circ\Phi_*)$ are filtered chain isomorphic, they have the same verbose barcode (see [17, Proposition 4.14]). By checking that Φ_* maps a singular value decomposition of (C_*,∂_C,ℓ) to a singular value decomposition of $(C_*,\partial_C,\ell\circ\Phi_*)$, we see that chain isomorphisms induce permutations of verbose barcodes. For more details, see [17, §4.2].

3.3 Proof of the isometry theorem

We now prove Theorem 2. If two FCCs have non-isomorphic underlying chain complexes, then d_{I} between the two FCCs is ∞, and so is d_{M} between their verbose barcodes. Thus, it remains to consider the case when two FCCs have the same (or isomorphic) underlying chain complexes.

3.3.1 The inequality $d_{\mathrm{M}} \le d_{\mathrm{I}}$

Although [18, Proposition 9.3] states a weaker result than the lemma below, their proof indeed implies the following (see [17, §4.3.1] for more details):

▶ **Lemma 21.** *Let (C_*,∂_C) be a finite-dimensional non-zero chain complex over $\mathbb{F}$ and let $\ell_1,\ell_2 : \mathrm{C}_* \to \mathbb{R}\sqcup\{-\infty\}$ be two filtration functions. Denote by $\mathcal{B}^1_{\mathrm{Ver}}$ and $\mathcal{B}^2_{\mathrm{Ver}}$ the verbose barcodes of (C_*,∂_C,ℓ_1) and (C_*,∂_C,ℓ_2), respectively. Then, we have*

$$d_{\mathrm{M}}\left(\mathcal{B}^1_{\mathrm{Ver}},\mathcal{B}^2_{\mathrm{Ver}}\right) = \sup_{k\in\mathbb{Z}_{\ge 0}} d_{\mathrm{M}}\left(\mathcal{B}^1_{\mathrm{Ver},k},\mathcal{B}^2_{\mathrm{Ver},k}\right) \le \|\ell_1-\ell_2\|_\infty.$$

▶ **Proposition 22.** *With the same notation as in Lemma 21, we have*

$$d_{\mathrm{M}}\left(\mathcal{B}^1_{\mathrm{Ver}},\mathcal{B}^2_{\mathrm{Ver}}\right) \le d_{\mathrm{I}}\left((C_*,\partial_C,\ell_1),(C_*,\partial_C,\ell_2)\right).$$

Proof. Given any $\Phi_* \in \mathrm{Iso}(C_*, \partial_C)$, [17, Proposition 4.14] implies that $\mathcal{B}^2_{\mathrm{Ver}} = \mathcal{B}^{(C_*,\partial_C,\ell_2\circ\Phi_*)}_{\mathrm{Ver}}$ agrees with the verbose barcodes of $(C_*, \partial_C, \ell_2 \circ \Phi_*)$. Combined with Lemma 21, we have

$$d_{\mathrm{M}}\left(\mathcal{B}^1_{\mathrm{Ver}}, \mathcal{B}^2_{\mathrm{Ver}}\right) = d_{\mathrm{M}}\left(\mathcal{B}^1_{\mathrm{Ver}}, \mathcal{B}^{(C_*,\partial_C,\ell_2\circ\Phi_*)}_{\mathrm{Ver}}\right) \le \|\ell_1 - \ell_2 \circ \Phi_*\|_\infty,$$

for any $\Phi_* \in \mathrm{Iso}(C_*, \partial_C)$. Therefore,

$$d_{\mathrm{M}}\left(\mathcal{B}^1_{\mathrm{Ver}}, \mathcal{B}^2_{\mathrm{Ver}}\right) \le \min_{\Phi_*\in\mathrm{Iso}(C_*,\partial_C)} \|\ell_1 - \ell_2 \circ \Phi_*\|_\infty = d_{\mathrm{I}}\left((C_*, \partial_C, \ell_1), (C_*, \partial_C, \ell_2)\right),$$

where the equality follows from Theorem 16. ◀

3.3.2 The inequality $d_{\mathrm{M}} \ge d_{\mathrm{I}}$

We prove $d_{\mathrm{M}} \ge d_{\mathrm{I}}$ via an idea similar to the one used for proving that d_{B} of concise barcodes is no larger than d_{I} between persistent modules, cf. [14, Theorem 3.4].

Proof of Theorem 2 "$d_{\mathrm{M}} \ge d_{\mathrm{I}}$". The proof is trivial if $(C_*, \partial_C, \ell_C)$ and $(D_*, \partial_D, \ell_D)$ have non-isomorphic underlying chain complexes. We now consider the case when the chain complexes (C_*, ∂_C) and (D_*, ∂_D) are isomorphic, and we assume without loss of generality that $(D_*, \partial_D) = (C_*, \partial_C)$ and write $\ell_1 := \ell_C, \ell_2 := \ell_D$.

Take any number $\delta \ge d_{\mathrm{M}}\left(\mathcal{B}^1_{\mathrm{Ver}}, \mathcal{B}^2_{\mathrm{Ver}}\right)$. Then for any $k \in \mathbb{Z}_{\ge 0}$, there is a bijection $f_k : \mathcal{B}^1_{\mathrm{Ver},k} \to \mathcal{B}^2_{\mathrm{Ver},k}$ such that

$$\max_{a\in\mathcal{B}^1_{\mathrm{Ver},k}} d_\infty(a, f_k(a)) \le \delta. \tag{3}$$

For $a \in \mathcal{B}^1_{\mathrm{Ver},k} \subset \overline{\mathcal{H}}^\infty$, assume that $a = (a_1, a_2)$. Also, write $b = f_k(a)$ and assume that $b = (b_1, b_2)$. Next we construct an isomorphism between the following elementary FCCs:

$$h_k : \mathcal{E}(a_1, a_2, k) \to \mathcal{E}(b_1, b_2, k).$$

Notice that a_2 and b_2 are either both finite or both infinite, otherwise the left hand side of Equation (3) is equal to ∞, which contradicts with $\delta < \infty$.

Case (1): $a_2 = b_2 = \infty$, so $\mathcal{E}(a_1, a_2, k)$ and $\mathcal{E}(b_1, b_2, k)$ have the same underlying chain complex:

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{F}x_k \xrightarrow{\partial_k=0} 0 \longrightarrow \ldots,$$

and the filtration functions are given by $\ell_1(x_k) = a_1$ and $\ell_2(x_k) = b_1$, respectively. We define the chain isomorphism to be

$$h_k : \mathcal{E}(a_1, \infty, k) \to \mathcal{E}(b_1, \infty, k) \text{ with } x_k \mapsto x_k.$$

Case (2): $a_2, b_2 < \infty$, so $\mathcal{E}(a_1, a_2, k)$ and $\mathcal{E}(b_1, b_2, k)$ have the same underlying chain complex:

$$\ldots \longrightarrow 0 \longrightarrow \mathbb{F}y_{k+1} \xrightarrow{\partial_{k+1}:y_{k+1}\mapsto x_k} \mathbb{F}x_k \xrightarrow{\partial_k=0} 0 \longrightarrow \ldots,$$

and the filtration functions are given by $\ell_1(x_k) = a_1$, $\ell_1(y_{k+1}) = a_2$ and $\ell_2(x_k) = b_1$, $\ell_2(y_{k+1}) = b_2$, respectively. We define the chain isomorphism to be

$$h_k : \mathcal{E}(a_1, a_2, k) \to \mathcal{E}(b_1, b_2, k) \text{ with } x_k \mapsto x_k, y_{k+1} \mapsto y_{k+1}.$$

In either case, it is straightforward to check that h_k satisfies the following condition

$$\|\ell_1 - \ell_2 \circ h_k\|_\infty \leq \max\{|a_1 - a_2, b_1 - b_2|\} = d_\infty(a, f(a)) \leq \delta.$$

We write $h_{k,a}$ whenever it is needed to emphasize that h_k depends on a.

By [18, Proposition 7.4] we have the following decomposition of FCCs

$$(C_*, \partial_C, \ell_1) \cong \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \bigoplus_{a\in\mathcal{B}^1_{\mathrm{Ver},k}} \mathcal{E}(a_1, a_2, k) \text{ and } (C_*, \partial_C, \ell_2) \cong \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \bigoplus_{b\in\mathcal{B}^2_{\mathrm{Ver},k}} \mathcal{E}(b_1, b_2, k).$$

Let $h := \bigoplus_{k\in\mathbb{Z}_{\geq 0}} \bigoplus_{a\in\mathcal{B}^1_{\mathrm{Ver},k}} h_{k,a} : (C_*, \partial_C, \ell_1) \to (C_*, \partial_C, \ell_2)$, which is then a chain isomorphism such that

$$\|\ell_1 - \ell_2 \circ h\|_\infty = \max_{k\in\mathbb{Z}_{\geq 0}} \max_{a\in\mathcal{B}^1_{\mathrm{Ver},k}} \|\ell_1 - \ell_2 \circ h_{k,a}\|_\infty \leq \delta.$$

It then follows from Theorem 16 that

$$d_{\mathrm{I}}\left((C_*, \partial_C, \ell_1), (C_*, \partial_C, \ell_2)\right) = \min_{\Phi_*\in\mathrm{Iso}(C_*,\partial_C)} \|\ell_1 - \ell_2 \circ \Phi_*\|_\infty \leq \|\ell_1 - \ell_2 \circ h\|_\infty \leq \delta.$$

Letting $\delta \searrow d_{\mathrm{M}}\left(\mathcal{B}^1_{\mathrm{Ver}}, \mathcal{B}^2_{\mathrm{Ver}}\right)$, we obtain the desired inequality $d_{\mathrm{I}} \leq d_{\mathrm{M}}$. ◀

4 Improved stability result for VR FCCs

In this section, we overcome the problem that the matching distance between verbose barcodes are not stable under the Gromov-Hausdorff distance, by incorporating the notion of tripods (see [16]). A *tripod* between two sets X and Y is a pair of surjections from another set Z to X and Y respectively, cf. [16]. We will express this by the diagram

$$R : X \xtwoheadleftarrow{\phi_X} Z \xtwoheadrightarrow{\phi_Y} Y.$$

Define $\mathrm{dis}(R) := \sup_{z,z'\in Z} |d_X(\phi_X(z), \phi_X(z')) - d_Y(\phi_Y(z), \phi_Y(z'))|$.

In §4.1, we define the *pullback bottleneck distance* between verbose barcodes of VR FCCs of two finite metric spaces X and Y and the *pullback interleaving distance* between VR FCCs. We prove the pullback stability theorem (Theorem 5) in §4.2, and provide examples in §4.3 to show that verbose barcodes improve the stability of concise barcodes in many cases.

For notational simplicity, we will omit the differential map ∂^X for VR FCC of X.

4.1 Pullback interleaving distance and pullback bottleneck distance

Using the notion of tripod, we construct a new distance between filtered chain complexes:

▶ **Definition 23** (Pullback interleaving distance). *For two finite metric spaces X and Y, we define the* pullback interleaving distance *between the VR FCCs of X and Y to be*

$$\hat{d}_{\mathrm{I}}\left(\left(\mathrm{C}_*(\mathrm{VR}(X)), \ell^X\right), \left(\mathrm{C}_*(\mathrm{VR}(Y)), \ell^Y\right)\right) :=$$
$$\inf\left\{d_{\mathrm{I}}\left(\left(\mathrm{C}_*(\mathrm{VR}(Z)), \ell^{Z_X}\right), \left(\mathrm{C}_*(\mathrm{VR}(Z)), \ell^{Z_Y}\right)\right) \mid X \xtwoheadleftarrow{\phi_X} Z \xtwoheadrightarrow{\phi_Y} Y \text{ a tripod}\right\},$$

where $Z_X := (Z, \phi_X^* d_X)$ and $Z_Y := (Z, \phi_Y^* d_Y)$.

With a similar idea and again using tripods, we refine the standard bottleneck distance and introduce a new notion of distance between verbose barcodes:

▶ **Definition 24** (Pullback bottleneck distance). *Let $k \in \mathbb{Z}_{\geq 0}$. For two finite metric spaces X and Y, the* pullback bottleneck distance *between $\mathcal{B}_{\mathrm{Ver},k}(X)$ and $\mathcal{B}_{\mathrm{Ver},k}(Y)$ is defined to be*

$$\hat{d}_{\mathrm{B}}\left((\mathcal{B}_{\mathrm{Ver},k}(X), \mathcal{B}_{\mathrm{Ver},k}(Y))\right) := \inf\left\{ d_{\mathrm{M}}\left(\mathcal{B}_{\mathrm{Ver},k}(Z_X), \mathcal{B}_{\mathrm{Ver},k}(Z_Y)\right) \mid X \xleftarrow{\phi_X} Z \xrightarrow{\phi_Y} Y \text{ a tripod} \right\},$$

where $Z_X := (Z, \phi_X^* d_X)$ and $Z_Y := (Z, \phi_Y^* d_Y)$. In addition, we define

$$\hat{d}_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Ver}}(X), \mathcal{B}_{\mathrm{Ver}}(Y)\right) := \sup_{k \in \mathbb{Z}_{\geq 0}} \hat{d}_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Ver},k}(X), \mathcal{B}_{\mathrm{Ver},k}(Y)\right).$$

We refer readers to Remark 3 for clarification regarding the usage of terminology, especially the term "distance", when referring to $\hat{d}_{\mathrm{B}}$ and $\hat{d}_{\mathrm{I}}$.

▶ Remark 25. For two finite metric spaces X and Y with the same cardinality, we have

$$\hat{d}_{\mathrm{B}}(\mathcal{B}_{\mathrm{Ver}}(X), \mathcal{B}_{\mathrm{Ver}}(Y)) \leq d_{\mathrm{M}}(\mathcal{B}_{\mathrm{Ver}}(X), \mathcal{B}_{\mathrm{Ver}}(Y)).$$

The above inequality can be strict. For instance, consider the four-point metric spaces X and Y given in Example 12, for which we have (see [17, Remark 5.4])

$$\hat{d}_{\mathrm{B}}(\mathcal{B}_{\mathrm{Ver}}(X), \mathcal{B}_{\mathrm{Ver}}(Y)) = 0 < 1 = d_{\mathrm{M}}(\mathcal{B}_{\mathrm{Ver}}(X), \mathcal{B}_{\mathrm{Ver}}(Y)).$$

4.2 Pullback stability theorem

In this section, we prove that the pullback interleaving distance $\hat{d}_{\mathrm{I}}$ and the pullback bottleneck distance $\hat{d}_{\mathrm{B}}$ are stable under the Gromov-Hausdorff distance d_{GH} (cf. Theorem 5) and see that it improves that stability of the standard bottleneck distance d_{B} (cf. Theorem 1).

We first show that $\hat{d}_{\mathrm{I}}$ is stable.

▶ **Proposition 26** (Stability of Pullback Interleaving Distance). *Let (X, d_X) and (Y, d_Y) be two finite metric spaces. Then,*

$$\hat{d}_{\mathrm{I}}\left(\left(\mathrm{C}_*(\mathrm{VR}(X)), \ell^X\right), \left(\mathrm{C}_*(\mathrm{VR}(Y)), \ell^Y\right)\right) \leq 2 \cdot d_{\mathrm{GH}}(X, Y).$$

Corollary 4 and Proposition 26 together yield the stability of $\hat{d}_{\mathrm{B}}$. In addition, we prove that $\hat{d}_{\mathrm{B}}$ is an improvement of d_B, as lower bounds of d_{GH} between metric spaces:

Proof of Theorem 5. It remains to prove $d_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Con},k}(X), \mathcal{B}_{\mathrm{Con},k}(Y)\right) \leq \hat{d}_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Ver},k}(X), \mathcal{B}_{\mathrm{Ver},k}(Y)\right)$. For any tripod $X \xleftarrow{\phi_X} Z \xrightarrow{\phi_Y} Y$, let $Z_X := (Z, \phi_X^* d_X)$ and $Z_Y := (Z, \phi_Y^* d_Y)$. By Proposition 6 and the fact that concise barcode is the corresponding verbose barcode excluding the diagonal points, we have that $\mathcal{B}_{\mathrm{Con},k}(X) = \mathcal{B}_{\mathrm{Con},k}(Z_X)$ and $\mathcal{B}_{\mathrm{Con},k}(Y) = \mathcal{B}_{\mathrm{Con},k}(Z_Y)$. Combined with Proposition 20, we have

$$d_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Con},k}(X), \mathcal{B}_{\mathrm{Con},k}(Y)\right) = d_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Con},k}(Z_X), \mathcal{B}_{\mathrm{Con},k}(Z_Y)\right) \leq d_{\mathrm{M}}\left(\mathcal{B}_{\mathrm{Ver},k}(Z_X), \mathcal{B}_{\mathrm{Ver},k}(Z_Y)\right).$$

◀

To prove Proposition 26, we first establish the stability of the interleaving distance between VR FCCs by showing that it is stable under the ℓ_∞ metric between two metrics over the same underlying set.

▶ **Proposition 27.** *Let X be a finite set. Let d_1 and d_2 be two distance functions on X, and let ℓ^1 and ℓ^2 be the filtration functions induced by d_1 and d_2 respectively. Then,*

$$|\|d_1\|_\infty - \|d_2\|_\infty| \leq d_{\mathrm{I}}\left((\mathrm{C}_*(\mathrm{VR}(X)),\ell^1),(\mathrm{C}_*(\mathrm{VR}(X)),\ell^2)\right) \leq \|d_1 - d_2\|_\infty.$$

Proof of Proposition 26. Suppose $R : X \xleftarrow{\phi_X} Z \xrightarrow{\phi_Y} Y$ is a tripod between X and Y with distortion $\mathrm{dis}(R) \leq \delta$. By Proposition 27, we obtain

$$d_{\mathrm{I}}\left(\left(\mathrm{C}_*(\mathrm{VR}\left(Z\right)),\ell^{Z_X}\right),\left(\mathrm{C}_*(\mathrm{VR}\left(Z\right)),\ell^{Z_Y}\right)\right) \leq \|\phi_X^* d_X - \phi_Y^* d_Y\|_\infty = \mathrm{dis}(R).$$

We finish the proof, by taking infimum over all tripods R on the above inequality and using the fact that $2 \cdot d_{\mathrm{GH}}(X,Y) = \inf_R \mathrm{dis}(R)$ (see [3, §7.3.3]). ◀

See [17, §5.3] for the remaining proofs and examples for results in this subsection.

4.3 Tightness and strictness of the pullback stability theorem

We show through examples that both inequalities in Theorem 5 are tight and can be strict.

▶ **Example 28.** Recall the 3-point metric spaces X_1 and X_2 from Figure 1, assuming $a \leq b \leq c_i$ for $i = 1,2$. Computing each of the distance given in Theorem 5, we obtain:

$d_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Con}}(X_1),\mathcal{B}_{\mathrm{Con}}(X_2)\right)$	$\hat{d}_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Ver}}(X_1),\mathcal{B}_{\mathrm{Ver}}(X_2)\right)$	$2 \cdot d_{\mathrm{GH}}(X_1,X_2)$
0	$\lvert c_1 - c_2\rvert$	$\lvert c_1 - c_2\rvert$

The first and third column in the above table are straightforward calculations. For the second column, notice that for any tripod $X_1 \xleftarrow{\phi_1} Z \xrightarrow{\phi_2} X_2$, we have

$$\mathcal{B}_{\mathrm{Ver},\mathrm{card}(Z)-2}(Z_1) = \{(c_1,c_1)\} \text{ and } \mathcal{B}_{\mathrm{Ver},\mathrm{card}(Z)-2}(Z_2) = \{(c_2,c_2)\},$$

where $Z_1 := (Z, \phi_1^* d_{X_1})$ and $Z_2 := (Z, \phi_2^* d_{X_2})$. Thus,

$$\hat{d}_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Ver}}(X_1),\mathcal{B}_{\mathrm{Ver}}(X_2)\right) \geq d_{\mathrm{M}}\left(\mathcal{B}_{\mathrm{Ver},\mathrm{card}(Z)-2}(Z_1),\mathcal{B}_{\mathrm{Ver},\mathrm{card}(Z)-2}(Z_2)\right) = |c_1 - c_2|.$$

This example shows that $\hat{d}_{\mathrm{B}}$ between verbose barcodes gives a better bound for the Gromov-Hausdorff distance d_{GH}, compared with d_{B} between concise barcodes.

▶ **Example 29.** Let X and Y be metric spaces of 4 points given in Figure 4. Let Z be the complete graph on 4 vertices with edge length 1, and W be the cycle graph on 4 vertices with edge length 1. See Figure 6 for the illustration of all 4 spaces and their verbose barcodes.

From Figure 7, we notice that the pair of metric spaces (X,Y) is such that

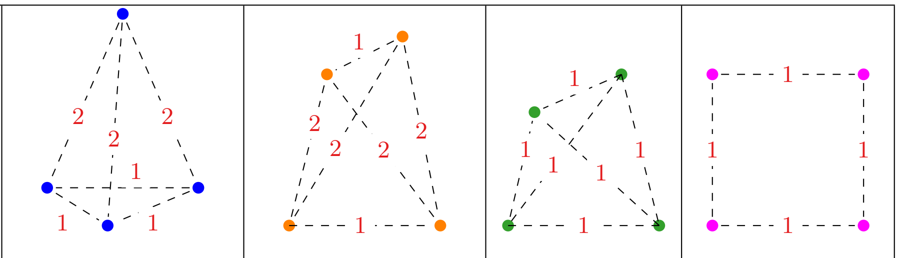
$$d_{\mathrm{B}}(\mathcal{B}_{\mathrm{Con}}(X),\mathcal{B}_{\mathrm{Con}}(Y)) = \hat{d}_{\mathrm{B}}(\mathcal{B}_{\mathrm{Ver}}(X),\mathcal{B}_{\mathrm{Ver}}(Y)) = 0 < 1 = 2 \cdot d_{\mathrm{GH}}(X,Y),$$

which tells us the fact that $\hat{d}_{\mathrm{B}}$ between distinct verbose barcodes can be zero. To see $\hat{d}_{\mathrm{B}}(\mathcal{B}_{\mathrm{Ver},1}(X),\mathcal{B}_{\mathrm{Ver},1}(Y)) = 0$, consider that pullback metric space Z_X that repeats the top point in X and Z_Y that repeats any one point in Y, and see that $\mathcal{B}_{\mathrm{Ver},1}(Z_X) = \mathcal{B}_{\mathrm{Ver},1}(Z_Y) = \{(1,1),(2,2)^5\}$. The pair (X,Y) shows the tightness of $d_{\mathrm{B}} \leq \hat{d}_{\mathrm{B}}$.

The pair (Z,W) is such that

$$d_{\mathrm{B}}(\mathcal{B}_{\mathrm{Con}}(Z),\mathcal{B}_{\mathrm{Con}}(W)) = \frac{1}{2} < 1 = \hat{d}_{\mathrm{B}}(\mathcal{B}_{\mathrm{Ver}}(Z),\mathcal{B}_{\mathrm{Ver}}(W)) = 2 \cdot d_{\mathrm{GH}}(Z,W),$$

which is another example of $\hat{d}_{\mathrm{B}}$ and $\hat{d}_{\mathrm{I}}$ providing better bounds of d_{GH} compared to the standard bottleneck distance d_{B}, as well as an example for the tightness of $\hat{d}_{\mathrm{B}} \leq 2 \cdot d_{\mathrm{GH}}$.



	X	Y	Z	W
$\mathcal{B}_{\mathrm{Ver},0}$	$(0,1)^2,(0,2),(0,\infty)$	$(0,1)^2,(0,2),(0,\infty)$	$(0,1)^3,(0,\infty)$	$(0,1)^3,(0,\infty)$
$\mathcal{B}_{\mathrm{Ver},1}$	$(1,1),(2,2)^2$	$(2,2)^3$	$(1,1)^3$	$(1,2),(2,2)^2$
$\mathcal{B}_{\mathrm{Ver},2}$	$(2,2)$	$(2,2)$	$(1,1)$	$(2,2)$

Figure 6 The 4-point metric spaces X, Y, Z and W; and their verbose barcodes.

One more example that the stability of $\hat{d}_{\mathrm{B}}$ improves that of d_{B} (see [17, Example 5.11]):

▶ **Proposition 30.** *Let X be the one-point space, and Y be any finite metric space. Then,*

$$d_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Con}}(X),\mathcal{B}_{\mathrm{Con}}(Y)\right)=\frac{\mathrm{diam}(Y)}{2}<\mathrm{diam}(Y)=\hat{d}_{\mathrm{B}}\left(\mathcal{B}_{\mathrm{Ver}}(X),\mathcal{B}_{\mathrm{Ver}}(Y)\right)=2\cdot d_{\mathrm{GH}}(X,Y).$$

5 About computing the pullback bottleneck distance

To have a more concrete understanding of the pullback bottleneck distance, we study verbose barcodes under pullbacks.

Let (X,d_X) be a finite metric space with $X=\{x_1,\ldots,x_n\}$. For any surjection $\phi:Z\twoheadrightarrow X$, the *pullback (pseudo) metric space* (induced by ϕ) is defined as the pair (Z,ϕ^*d_X), where ϕ^*d_X is the pullback of the distance function d_X. In other words, for any $z_1,z_2\in Z$,

$$(\phi^*d_X)(z_1,z_2):=d_X\left(\phi_X(z_1),\phi_X(z_2)\right).$$

For each $z\in Z$, the point $\phi_X(z)\in X$ is called the *parent* of z.

▶ **Proposition 31.** *Assume $X=\{x_1,\ldots,x_n\}$ is a pseudo-metric space and $Z=X\sqcup\{z\}$. Suppose $\phi:Z\to X$ is such that $z\mapsto x_j$ for some $j=1,\ldots,n$. Then*

$$\mathcal{B}_{\mathrm{Ver},0}(Z)=\mathcal{B}_{\mathrm{Ver},0}(X)\sqcup\{(0,0)\},$$

and for $k\geq 1$,

$$\begin{aligned}\mathcal{B}_{\mathrm{Ver},k}(Z)&=\mathcal{B}_{\mathrm{Ver},k}(X)\sqcup\{\mathrm{diam}([x_j,x_j,x_{i_1},\ldots,x_{i_k}])\cdot(1,1):x_{i_l}\in X-\{x_j\},\forall l=1,\ldots,k\}\\&=\mathcal{B}_{\mathrm{Ver},k}(X)\sqcup\{\mathrm{diam}([x_j,x_j,\beta])\cdot(1,1):\beta\in P_k(X\setminus\{x_j\})\}.\end{aligned}$$

Each finite pullback metric space Z of X can be written as a multiset $Z=X\sqcup\{x_{j_1},\ldots,x_{j_m}\}$ equipped with the inherited metric from X for some $m\geq 0$ and $j_1\leq\cdots\leq j_m$.

We apply the Proposition 31 to prove Proposition 6.

Proof of Proposition 6. We prove by induction on m. When $m=1$, the statement follows immediately from Proposition 31. Suppose $m\geq 2$ and that the statement holds for $Z':=X\sqcup\{x_{j_1},\ldots,x_{j_{m-1}}\}$. Recall that $P_k(A)$ denotes the multiset consisting of sub-multisets of A each with cardinality k. By Proposition 31 and the induction hypothesis, we obtain:

$d_{\mathrm{B}}(\mathcal{B}_{\mathrm{Con},0}(\cdot),\mathcal{B}_{\mathrm{Con},0}(\cdot))$	X	Y	Z	W
X	0	0	1	1
Y		0	1	1
Z			0	0
W				0

$\hat{d}_{\mathrm{B}}(\mathcal{B}_{\mathrm{Ver},0}(\cdot),\mathcal{B}_{\mathrm{Ver},0}(\cdot))$	X	Y	Z	W
X	0	0	1	1
Y		0	1	1
Z			0	0
W				0

$d_{\mathrm{B}}(\mathcal{B}_{\mathrm{Con},1}(\cdot),\mathcal{B}_{\mathrm{Con},1}(\cdot))$	X	Y	Z	W
X	0	0	0	$\frac{1}{2}$
Y		0	0	$\frac{1}{2}$
Z			0	$\frac{1}{2}$
W				0

$\hat{d}_{\mathrm{B}}(\mathcal{B}_{\mathrm{Ver},1}(\cdot),\mathcal{B}_{\mathrm{Ver},1}(\cdot))$	X	Y	Z	W
X	0	0	1	1
Y		0	1	1
Z			0	1
W				0

$2\cdot d_{\mathrm{GH}}(\cdot,\cdot)$	X	Y	Z	W
X	0	1	1	1
Y		0	1	1
Z			0	1
W				0

Figure 7 The bottleneck distance d_{B} between concise barcodes, the pullback bottleneck distance $\hat{d}_{\mathrm{B}}$ between verbose barcodes, and the Gromov-Hausdorff distance between spaces.

$$\begin{aligned}\mathcal{B}_{\mathrm{Ver},k}(Z) &= \mathcal{B}_{\mathrm{Ver},k}(Z') \sqcup \left\{\mathrm{diam}([x_{j_m},\beta])\cdot(1,1) : \beta\in P_k\left((X\setminus\{x_{j_m}\})\sqcup\left\{x_{j_1},\dots,x_{j_{m-1}}\right\}\right)\right\}\\
&= \mathcal{B}_{\mathrm{Ver},k}(X) \sqcup \bigsqcup_{i=0}^{m-2}\left\{\mathrm{diam}([x_{j_{i+1}},\beta_i])\cdot(1,1) : \beta_i\in P_k\left((X\setminus\{x_{j_{i+1}}\})\sqcup\{x_{j_1},\dots,x_{j_i}\}\right)\right\}\\
&\qquad \sqcup\left\{\mathrm{diam}([x_{j_m},\beta_{m-1}])\cdot(1,1) : \beta_{m-1}\in P_k\left((X\setminus\{x_{j_m}\})\sqcup\left\{x_{j_1},\dots,x_{j_{m-1}}\right\}\right)\right\}\\
&= \mathcal{B}_{\mathrm{Ver},k}(X) \sqcup \bigsqcup_{i=0}^{m-1}\left\{\mathrm{diam}([x_{j_{i+1}},\beta_i])\cdot(1,1) : \beta_i\in P_k\left((X\setminus\{x_{j_{i+1}}\})\sqcup\{x_{j_1},\dots,x_{j_i}\}\right)\right\}.\end{aligned}$$

◀

When considering degree 0, Proposition 6 implies Proposition 7, which imposes the strategy of matching bars in concise barcodes only to other bars or to the origin $(0,0)$ unlike in the case of d_{B} when bars are allowed to be matched to any point on the diagonal.

See [17, §6] for proofs and further details of this section.

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