A Lambda Calculus Satellite

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Abstract

We shortly summarize the contents of the book “A Lambda Calculus Satellite”, presented at the 8th International Conference on Formal Structures for Computation and Deduction (FSCD 2023).

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Introduction

The λ-calculus was introduced by Alonzo Church around 1930 as the kernel of a more general investigation on the foundations of mathematics and logic [18, 19], and played a prominent role in theoretical computer science for more than fifty years.[1] Under the influence of the pioneering work of Corrado Böhm, who established his famous separation theorem [11], and Dana Scott, who constructed the first model of λ-calculus [56], the research on λ-calculus has flourished in the seventies.[2] In that period a wealth of results were established by applying techniques arising from several areas of computer science and mathematics, namely recursion theory, algebra, topology and category theory. These approaches revealed that λ-calculus can be studied from different, although interconnected, perspectives:

- *The λ-calculus as a rewriting system.* The set Λ of *λ-terms* can be endowed with *notions of reductions*, thus it deserves the status of a higher-order term rewriting system:

  $$(\lambda x.M)N \to_\beta M[x:=N], \qquad\qquad \lambda x.Mx \to_\eta M, \text{ if } x \notin \mathrm{FV}(M).$$

  This approach has its roots in the Church-Rosser Theorem [20] establishing the confluence of $\beta(\eta)$-reduction, and therefore the unicity of normal forms, and in the Standardization Theorem [23] from which it follows that the leftmost-outermost reduction strategy is normalizing. Subsequently, researchers studied the reduction sequences originating from a λ-term by performing a fine-grained analysis of the creation of redexes and by tracking the residuals of sets of redexes. This analysis culminated in a proof of the Finite Developments Theorem, in its various formulations [25, 34, 32].
- *The analysis of the cost of normalization.* The idea of considering the λ-calculus an actual programming language was taken very seriously by Böhm and Gross, who designed in the sixties the CUCH machines as an implementational model [13]. Therefore, the

[1] We consider here the research on untyped λ-calculus – we could argue that variations of λ-calculus are still omnipresent in the literature concerning logical systems as well as idealized programming languages.
[2] For a more detailed survey of the early history of λ-calculus, we refer the reader to [17].

problem of determining the computational complexity of normalization via β-reduction arose naturally. This led Jean-Jacques Lévy to develop in his *Thèse de doctorat d'État* an interesting notion of *optimal reduction* [45].

- *The λ-calculus as a model of computation.* A well-established result shows that any partial recursive numerical function can be represented by a λ-term operating on Church numerals. However, observing the behavior of λ-terms exclusively on numerals represents a very narrow point of view. In general, a λ-term F can be seen as a total function

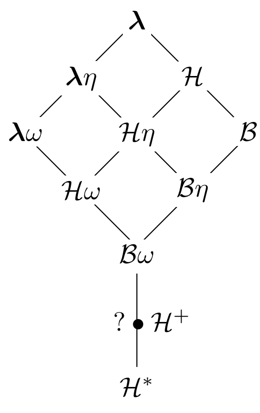
  $$F : \Lambda \to \Lambda$$

  and one can study its $\mathsf{Range}(F) = \{FN \mid N \in \Lambda\}$ or the set of its fixed points $\mathsf{Fix}(F) = \{N \mid FN =_\beta N\}$ and wonder if they are finite or infinite modulo β-conversion [6, 37]. Moreover, λ-terms can be classified into solvable and unsolvable, depending on their capability of interaction with the environment: a *solvable* term can be transformed into the identity I when plugged in a suitable context $C[\,]$, while *unsolvables* are unable to interact with any context. Equivalently, solvable terms must provide at least a stable portion of their output (their head nf) while unsolvables correspond to looping programs. This classification led Barendregt to define the *Böhm tree* $\mathrm{BT}(M)$ of a λ-term M, a possibly infinite tree constructed by coinductively collecting all stable pieces of information coming out of its computations, and eventually representing the complete evaluation of M [4].
- *The lattice of λ-theories.* The equational theories of λ-calculus are called λ*-theories* and become the main subject of study when one is more interested in program equivalence than in the process of reduction. Some λ-theories are particularly interesting for computer scientists because they capture operational properties of λ-terms, e.g., *extensional theories* equating all extensionally equivalent λ-terms, or *sensible theories* collapsing all unsolvables, or theories equating all λ-terms having the same evaluation trees. Morris introduced a class of *observational theories* specifying when two λ-terms are observationally equivalent [49]. This means that two λ-terms M and N are considered equivalent whenever one can plug either M or N into any context $C[\,]$ without noticing any difference in the global behavior. Observational equivalences thus depend on the kind of behavior one is interested in observing – a parametricity that can be represented by a set $\mathcal{O} \subseteq \Lambda$ of *observables*:

  $$\mathcal{T}_{\mathcal{O}} = \{M = N \mid \forall C[\,]\,.\,[\; C[M] \in \mathcal{O} \iff C[N] \in \mathcal{O}\;]\}.$$

  Therefore, even if the set of λ-theories constitutes a complete lattice of cardinality $2^{\aleph_0}$, researchers have mostly focussed on the following λ-theories that form a kite-shaped diagram where $\mathcal{T}_1$ is depicted above $\mathcal{T}_2$ whenever $\mathcal{T}_1 \subsetneq \mathcal{T}_2$ holds:

  $$\begin{array}{lcl}
  \boldsymbol{\lambda} & = & \{M = N \mid M =_\beta N\}.\\
  \boldsymbol{\lambda}\eta & = & \{M = N \mid M =_{\beta\eta} N\}.\\
  \mathcal{H} & = & \{M = N \mid M, N \text{ are unsolvable}\}.\\
  \mathcal{H}^+ & = & \mathcal{T}_{\mathrm{NF}},\ \text{where } \mathrm{NF} = \{M \in \Lambda \mid M \text{ has a } \beta\text{-nf}\}.\\
  \mathcal{H}^* & = & \mathcal{T}_{\mathrm{SOL}},\ \text{where } \mathrm{SOL} = \{M \in \Lambda \mid M \text{ is solvable}\}.\\
  \mathcal{B} & = & \{M = N \mid \mathrm{BT}(M) = \mathrm{BT}(N)\}.\\
  \mathcal{T}\eta & = & \text{the closure of a } \lambda\text{-theory } \mathcal{T} \text{ under the } \eta\text{-rule}.\\
  \mathcal{T}\omega & = & \text{the closure of a } \lambda\text{-theory } \mathcal{T} \text{ under the } \omega\text{-rule, which is}\\
  & & \dfrac{FZ = GZ \text{ for all } Z \in \Lambda^o}{F = G}\ \omega\text{-rule}
  \end{array}$$



The ω-*rule* defined above states if two λ-terms F, G are equal whenever they are applied to the same closed argument Z (i.e. $Z \in \Lambda^o$), then they are equal. This form of extensionality is inspired from set-theoretical definition of "function equality" and encompasses the η-rule. Proving that the inclusion $\boldsymbol{\lambda}\eta \subsetneq \boldsymbol{\lambda}\omega$ is strict requires some clever construction [53].

- *The model theoretic approach.* The model theory of λ-calculus has been developed along three axes. The first one is algebraic and proposes as models (a subclass of) the variety of *combinatory algebras*, based on the combinators $\mathbf{K}, \mathbf{S}$ from Combinatory Logic [23]. The second one, based on syntactic λ-models, is closely related to the algebraic definition, but more set-theoretical [31]. The third one is category-theoretic and focuses on the notion of reflexive object living in Cartesian closed categories [2]. The relationship among these notions has been explored by Koymans [42]: in general, categorical models correspond to λ-*algebras* (describing, e.g., closed term models), that are moreover λ-*models* when the underlying category is "well-pointed".
  Concerning the construction of individual models, the most famous ones are Scott's $\mathcal{D}_\infty$ whose theory is the observational theory $\mathcal{H}^*$, as proved by Hyland [33] and Wadsworth [67] (independently), and Plotkin-Scott's $\mathcal{P}_\omega$ that induces the theory $\mathcal{B}$ of Böhm trees [52, 57].

At the end of the seventies, Barendregt decided to collect all these results (and others) in the monograph "The Lambda Calculus. Its syntax and semantics" [5], presenting the state of the art of research in λ-calculus at that time. Subsequently translated in Russian and Chinese, this book is nowadays omnipresent in academic libraries of computer science all around the world. Several open problems concerning semantical and syntactic aspects of λ-calculus were proposed in the book, often in the form of conjectures:

1. Do invertible λ-terms correspond to bijective λ-terms, modulo $=_{\beta\eta}$? [5, Exercise 21.4.9].
2. Does the Perpendicular Lines Property hold, modulo β-equality? [5, Chapter 14].
3. The λ-theory $\mathcal{H}$ satisfies the range property. [5, Conjecture 20.2.8].
4. The position of $\mathcal{H}^+$ in the kite of λ-theories is $\mathcal{B}\omega \subsetneq \mathcal{H}^+ \subsetneq \mathcal{H}^*$. Conjecture by P. Sallé reported in the proof of [5, Theorem 17.4.16].
5. The λ-theory $\mathcal{H}\omega$ is Π^1_1-complete. [5, Conjecture 17.4.15].

Most of these problems have been solved in the subsequent 35 years, but the results are scattered throughout the literature and difficult to piece together. Some of these solutions occupied an entire PhD thesis, e.g. Folkerts [29] (1995), or part of a thesis, e.g. Polonsky [54] (2011), with the complexity (but not the method) of a proof using priority for results on the degrees of undecidability. In 2017, Hyland suggested to Barendregt and the present author that the time had come to write another book, intended as a "satellite" of [5], in which all these solutions could be presented in a more clear and uniform way, improving and simplifying the proofs (when possible). We accepted his challenge, and profited from the occasion to include the solution of problems that weren't explicitly stated in [5], and other related research that was conducted in the last decades.

The resulting monograph, entitled "A Lambda Calculus Satellite", has been published in 2022 by College Publications [9] and is structured in six parts:

1. Preliminaries
2. Reduction
3. Conversion
4. Theories
5. Models
6. Open Problems

In the reminder of the paper, we briefly describe the contents of these parts – each composed by several chapters – and discuss some pedagogical choices that we made.

1 Preliminaries

In order to make the satellite book [9] as self-contained as possible[3], we start with a preliminary part surveying all the notions and results about λ-calculus[4] that are needed in the rest of the book. This part is divided into three chapters: 1) about syntactic properties; 2) Böhm trees and their variations; 3) models and theories of λ-calculus.

Besides presenting the syntax and operational semantics of λ-calculus in a nutshell, Chapter 1 contains more developed sections discussing properties of reduction, like finite developments and standardization, the Reduction under Substitution (RuS) technique by Diederik van Daalen [65], and its consequences [26]. Chapter 2 is devoted to present three kinds of coinductively defined trees representing the operational behavior of λ-terms, namely Böhm trees, Berarducci trees and Lévy-Longo trees. We profit from the occasion to mention that coinduction is treated in this book using the modern approach suggested in [43]. Since coinduction has been around for decades, and it is nowadays better understood in the scientific community, we adopt a more informal style of coinductive reasoning that greatly improves the readability and benefits the reader. Intuitively, we can do this without compromising the soundness of our proofs because the definition of Böhm trees is inherently *productive*. At the end of the chapter, we introduce two distinct notions of extensional Böhm trees corresponding to η-Böhm trees and Nakajima trees (respectively). The reason why these notions are incompatible with Berarducci and Lévy-Longo trees becomes clear later. In Chapter 3 we recall the main results concerning the lattice of λ-theories [46], and describe the relationships existing among the different definitions of a model of λ-calculus. In particular, we revisit Koymans' construction of a combinatory algebra starting from a categorical model – giving a λ-model only when the underlying category is *well pointed* – in favor of the more general construction described in [16] that works in every Cartesian closed category.

2 Reduction

In this part we consider properties of reductions in several systems: the regular λ-calculus, its infinitary version, and the **S**-fragment of Combinatory Logic.

Chapter 4. Leaving a β-reduction plane. Any λ-term M belongs to some *β-reduction plane*, which is defined as follows. Given $P, Q \in \Lambda$, write $P \circlearrowleft_\beta Q$ if $P \twoheadrightarrow_\beta Q \twoheadrightarrow_\beta P$. Then, a *$\beta$-reduction plane* is any $\circlearrowleft_\beta$-equivalence class. Clearly, M belongs to its own plane $[M]_{\circlearrowleft_\beta}$. It is possible to *leave a plane $\mathcal{P}$ at point $M \in \mathcal{P}$* if there is an N such that $M \to_\beta N \notin \mathcal{P}$. In 1980, Jan Willem Klop conjectured that if one can leave a plane at one of its points, then such a plane can be left at any of its points [41]. A few years later, Hans Mulder [50] and Sekimoto and Hirokawa [58] have refuted this conjecture (independently). In this chapter, we present Mulder's counterexample because he constructs a λ-term M containing a free variable x, but β-reducing to a closed term: $M \twoheadrightarrow_\beta P \in \Lambda^o$. Therefore, in order to conclude that $P \notin [M]_{\circlearrowleft_\beta}$ it is sufficient to invoke the fact that a λ-term cannot create free variables along β-reduction. The counterexample in [58] is a closed term, whence proving that it actually leaves its plane requires a more subtle reasoning. Both counterexamples are minimal, in the sense that they contain a minimal amount of redexes (i.e. 3) and rely on the fact that λ-calculus satisfies the ξ-rule.

[3] In our intent, it should be readable without having the previous book at hand.
[4] Auxiliary definitions and results about category theory, domain theory and universal algebra are presented in a technical appendix.

Chapter 5. Optimal lambda reduction. A consequence of the Standardization Theorem is that, if a λ-term is normalizing, then its normal form can be reached by repeatedly contracting the leftmost-outermost redex. However, due to the duplication of redexes, the leftmost-outermost reduction strategy might not be optimal, in the sense that the normal form could be reached in a number of steps which is not minimal. In [45], Jean-Jacques Lévy introduced the notion of *redex family* to capture an intuitive idea of *optimal sharing* between "copies" of the same redex. By studying the causal history of redexes using a suitable labeled extension of λ-calculus it is possible to define an optimal reduction method for λ-calculus. Compared to the most recent presentation of this material [1], our discussion enters deeply into the technical details of the so-called *extraction method*, furnishes more examples, and discusses some implementations.

Chapter 6. Infinitary lambda calculus. In λ-calculus there are terms, like Turing's fixed point combinator, generating an infinite reduction sequence. Pushing this reduction to infinity, one generates the infinite term $\lambda f.f(f(f(\cdots)))$, which is an in-line depiction of its Böhm tree. Inspired by this phenomenon, Kennaway, Klop, Sleep, and de Vries introduced the *infinitary λ-calculus* [40], whose terms and reductions can possibly be infinite. The resulting infinitary term rewriting system is well defined and enjoys the unicity of its normal forms, but many properties fail like normalization and – most importantly – confluence. Berarducci showed that collapsing meaningless terms allows to restore confluence, a property recently proved in `Coq` by Czajka used coinductive methods [24], and induces a new model of λ-calculus based on Berarducci trees [10]. By modifying the notion of meaningless terms, one also retrieves Böhm trees and Lévy-Longo trees as infinitary normal forms of λ-terms. Adding η-reduction to Berarducci trees or Lévy-Longo trees breaks confluence again, while this notion of reduction is compatible with Böhm trees.

Chapter 7. Starlings. In this birdwatching chapter our binoculars are focused on the *starling* **S**, living in Smullyan's enchanted forest of combinatory terms [59] and exhibiting the following behavior:

$$\mathbf{S}xyz \to_w xz(yz)$$

We consider the **S**-fragment of Combinatory Logic, namely combinatory terms built up from application and **S** exclusively. Many of these have a weak normal form, like **SSSSSSS**. Others do not, like **S**(**SS**)**SSSS** and **SSS**(**SSS**)(**SSS**). Many properties that are undecidable in the context of λ-calculus and Combinatory Logic, become decidable in this setting. For instance, the question of whether the normalization of **S** is decidable was answered positively by Johannes Waldmann in his PhD thesis [68]. We also present original results by Vincent Padovani on the **S**-fragment of Combinatory Logic, including:

- the fact that termination of head reduction is decidable;
- the existence of two non-interconvertible terms having the same Berarducci tree.

Whether the inter-convertibility $P =_w Q$ is decidable is still an open problem.

3 Conversion

In this part we consider properties of $\beta(\eta)$-conversion, although the classification here is rather subjective. Indeed, these properties could equivalently be formulated using the λ-theories $\boldsymbol{\lambda}, \boldsymbol{\lambda}\eta$, or even the associated term models. In general, given a λ-theory $\mathcal{T}$, it is equivalent to work with the associated equality $=_{\mathcal{T}}$ or in the term model $\mathcal{M}(\mathcal{T})$, while working in the closed term model $\mathcal{M}^o(\mathcal{T})$ is equivalent to consider *closed* λ-terms modulo $=_{\mathcal{T}}$.

Chapter 8. Perpendicular Lines Property. In the Cartesian plane $\mathbb{R}^2$ the two lines $\{(x,2) \mid x \in \mathbb{R}\}$ and $\{(3,y) \mid y \in \mathbb{R}\}$ are perpendicular. Translated to λ-terms, one says that $\{(X,\mathsf{I}) \mid X \in \Lambda\}$ and $\{(\mathsf{K},Y) \mid Y \in \Lambda\}$ are "perpendicular", and similarly in higher dimensions. The *perpendicular lines property* (PLP) states that if a λ-definable function F of k arguments is constant on k perpendicular lines, then F is constant everywhere. As usual, the validity of this property depends on the notion of equality which is considered, and on whether one focuses on closed terms, or open terms are also allowed. The validity of PLP has been shown for $\mathcal{M}(\mathcal{B})$ by using Berry's stability [5], for $\mathcal{M}^o(\mathcal{B})$ in the satellite book by using coinductive methods, and for $\mathcal{M}(\boldsymbol{\lambda})$ by Endrullis and de Vrijer using Reduction under Substitution [26]. Statman and Barendregt proved that $\mathcal{M}^o(\boldsymbol{\lambda}) \not\models$ PLP using variants of Plotkin terms [64].

Chapter 9. Bijectivity and invertibility in $\lambda\eta$. In set theory a function $f : X \to X$ is bijective if and only if it is invertible. More precisely, f is surjective whenever it is right-invertible, and injective whenever it is left-invertible. Now, given a λ-theory $\mathcal{T}$, every closed λ-term $F \in \Lambda^o$ can be considered as a function $F\colon \mathcal{M}^o(\mathcal{T}) \to \mathcal{M}^o(\mathcal{T})$, so it makes sense to wonder whether this correspondence still holds:

Assuming that F is a bijection, can one conclude that F is $\mathcal{T}$-invertible?

The problem is more difficult because the inverses of F are required to be λ-definable. In other words, is there a λ-term $G \in \Lambda^o$ such that $F \circ G =_{\mathcal{T}} G \circ F =_{\mathcal{T}} \mathsf{I}$? For $\mathcal{T} = \boldsymbol{\lambda}$ the answer is positive because the only β-invertible closed λ-term is the identity I. This was shown in [12]. The invertibility problem for $\mathcal{T} = \boldsymbol{\lambda}\eta$ was first raised in [5, Exercise 21.4.9]. More than 10 years later, Enno Folkerts showed that this correspondence does hold [29]. As a consequence of this, and of combined results by Dezani and Bergstra-Klop, a closed λ-term is $\boldsymbol{\lambda}\eta$-invertible if and only if it is a finite hereditary permutator.

4 Theories

In this part we study sensible λ-theories, that is, theories that equate all unsolvable λ-terms. Such terms indeed correspond to "looping" programs deprived of any computational content.

Chapter 10. Sensible theories. In this chapter we discuss two celebrated problems in λ-calculus, known as the range property and the fixed point property.

- The *range property* for a λ-theory $\mathcal{T}$ states that a combinator F, seen as a total function $F : \mathcal{M}^o(\mathcal{T}) \to \mathcal{M}^o(\mathcal{T})$ has either an infinite range or a singleton range (in other words, it is a constant function). For $\mathcal{T} = \boldsymbol{\lambda}$, this property has been conjectured by Böhm in [11] and proved by Barendregt in [5, Theorem 17.1.16]. The proof constitutes a striking example of the power of the hyperconnectedness property enjoyed by the Visser topology [66]. It is easy to check that the λ-theory $\mathcal{B}$ generated by equating all λ-terms having the same Böhm tree satisfies the range property, and the same reasoning generalizes to all λ-theories $\mathcal{T}$ in the interval $\mathcal{B} \subseteq \mathcal{T} \subseteq \mathcal{H}^*$. Barendregt conjectured that $\mathcal{H}$ satisfies the range property in [5, Conjecture 20.2.8], and this problem remained open for 30 years. This conjecture was refuted by Polonsky in his PhD thesis [54], where a λ-term having range 2 modulo $=_{\mathcal{H}}$ is constructed.
- The *fixed point property* for a λ-theory $\mathcal{T}$ states that every combinator has either one or infinitely many pairwise $\mathcal{T}$-distinct closed fixed points. The question whether this property holds for $\mathcal{T} = \boldsymbol{\lambda}$ was first raised by [37] and appears as Problem 25 in the TLCA list of open problems [36]. In this chapter, we present a λ-term violating the fixed point property for every sensible λ-theory, a result first appeared in [47].

Chapter 11. The kite. As mentioned in the introduction, researchers are mostly interested in those λ-theories constituting the kite-shape diagram depicted on Page 2. In the seventies, Patrick Sallé conjectured that the observational theory $\mathcal{H}^+$ should be placed in the diagram between $\mathcal{B}\omega$ and $\mathcal{H}^*$, that is, $\mathcal{B}\omega \subsetneq \mathcal{H}^+ \subsetneq \mathcal{H}^*$. The second inclusion actually follows from the work of Lévy [44] and Hyland [33] proving that $\mathcal{H}^+$ and $\mathcal{H}^*$ respectively capture the equalities induced by η-Böhm trees and Nakajima trees. The former inclusion turned out to be false: in an FSCD article [38], Intrigila *et al.* proved that $\mathcal{B}\omega = \mathcal{H}^+$, thus refuting Sallé's conjecture. We describe the main ingredients used in the proof:
$\underline{\mathcal{B}\omega \subseteq \mathcal{H}^+}$. This inclusion follows from the fact that $\mathcal{H}^+$ satisfies the ω-rule. This is a consequence of a weak separation theorem, first proved in [14], and satisfied by $\mathcal{H}^*$-equivalent terms, that are however different in $\mathcal{H}^+$.
$\underline{\mathcal{H}^+ \subseteq \mathcal{B}\omega}$. This inclusion is the difficult one. The proof exploits Lévy's characterization of $\mathcal{H}^+$ in terms of η-Böhm trees, the property that the "η-supremum" of two λ-terms (if any) is always λ-definable, the capability of λ-terms to work on the "codes" of other λ-terms via Gödelization, and the lemma stating that every closed λ-term becomes unsolvable when fed enough copies of Ω. To understand how these ingredients can be mixed together to obtain a proof of this theorem, the reader will need to actually study the chapter.

5 Models

In this part we present some advances on the denotational semantics of λ-calculus. Chapter 12 contains some preliminaries that are needed to understand the subsequent chapters.

Chapter 12. Ordered models and theories. In general, a model of λ-calculus only induces an equational theory (λ-theory) through the kernel of its interpretation function. In practice, most of the models individually introduced in the literature live in some cpo-enriched Cartesian closed category, whence they also induce an inequational theory. In this chapter, we introduce the inequational theories of λ-calculus independently from denotational considerations, and show that they inherit notions like extensionality and sensibility from λ-theories. We focus on observational inequational theories:

$$M \sqsubseteq_{\mathcal{O}} N \iff \forall C[\,]\,.\,[\; C[M] \in \mathcal{O} \quad \Rightarrow \quad C[N] \in \mathcal{O}\;],$$

still depending on a set $\mathcal{O}$ of observables. E.g., $\mathcal{H}^+ \vdash M = N \Leftrightarrow M \sqsubseteq_{\mathrm{NF}} N \;\&\; N \sqsubseteq_{\mathrm{NF}} M$. We conclude the chapter by recalling the (in)equational theories of the most known denotational models, like Scott's $\mathcal{D}_\infty$, Engeler's graph model $\mathcal{E}$, Plotkin's $\mathcal{P}_\omega$ and the like.

Chapter 13. Filter models. Intersection type assignment systems, introduced by [21], allow to give a logical description of several operational properties of λ-terms, like solvability and various forms of normalization. Moreover, thanks to the celebrated Stone's duality, they correspond to a class of denotational models of λ-calculus, called *filter models*. This nomenclature derives from the fact that the denotation of a λ-term is given by the filter of its types. We show that classical lattice models, whose construction mimics the one of Scott's $\mathcal{D}_\infty$, can be presented as filter models. Others examples are the original filter model $\mathcal{F}_{\mathrm{BCD}}$ defined by Barendregt, Coppo and Dezani [8] and the model $\mathcal{F}_{\mathrm{CDZ}}$ by Coppo, Dezani and Zacchi [22]. We mainly focus on the latter since it has been largely overlooked in the literature, with the notable exception of [55]. Exploiting Girard/Tait's reducibility candidates, we show that $\mathcal{F}_{\mathrm{CDZ}}$ satisfies an Approximation Theorem. Finally, using Lévy's extensional approximants, we prove that it is (in)equationally fully abstract for $\mathcal{H}^+$, a result first established in [22].

Chapter 14. Relational models. In the eighties Jean-Yves Girard realized that the category of sets and relations constitutes a simple quantitative semantics of Linear Logic [30], where the promotion $!A$ is given by the set of all finite multisets over A. The relational semantics of λ-calculus, obtained by applying the coKleisli construction, has been largely studied in the last decades because of its peculiar properties. First, its quantitative features allow to expose semantically intensional properties of λ-terms, like the amount of head-reduction steps needed to reach their head normal form. This also allows to endow fundamental results, like Approximation Theorems, with easy inductive proofs bypassing the usual techniques based on Tait's computability. Second, relational models can be expressed through tensor type[5] assignment systems whose inhabitation problem is decidable. Finally, the fact that it is a non-well-pointed category contributes to justify, together with categories of games, the interest in non well-pointed categorical models.

Chapter 15. Church algebras for λ-calculus. Combinatory algebras are considered algebraically pathological because they are never commutative, associative, finite or recursive. In fact, at first sight, they seem to have little in common with the mathematical structures that are usually considered in universal algebra. Salibra viewed these topics from a wider perspective and introduced the variety of *Church algebras* [48], namely algebras possessing two distinguished nullary terms representing the truth values, and a ternary term representing the if-then-else conditional construct, which is ubiquitous in programming languages. Beyond combinatory algebras, this class includes all Boolean algebras, Heyting algebras and rings with unity. Manzonetto and Salibra also proved that combinatory algebras satisfy a Representation Theorem stating that every combinatory algebra can be decomposed as a direct product of directly indecomposable combinatory algebras. It is therefore natural to study the *indecomposable semantics* of λ-calculus, namely the class of λ-models that are indecomposable in this sense. It turns out that this class is large enough to include all the main semantics of λ-calculus, but also largely incomplete: there is a wealth of λ-theories whose models must be decomposable. This furnishes a uniform algebraic proof of incompleteness for the main semantics.

6 Open Problems

In the last part of the book, we present some open problems in the hope that the next generation of scientists will solve them (and possibly write a book, forty years from now). Some are longstanding open problems or conjectures already present in the TLCA or RTA lists, we simply wish to draw the attention on them. Others arose during our discussions. We present here a few problems that should be of interest for the FSCD community.

6.1 Are there hyper-recurrent λ-terms?

In the article [63], Statman presents several notions of combinators that are not supposed to exist in λ-calculus, but whose existence is difficult to disprove. This list includes hyper-recurrent terms, uniform universal generators, and double fixed point combinators.

A closed λ-term M is called: *recurrent* if $M \twoheadrightarrow_\beta N$ entails $N \twoheadrightarrow_\beta M$; *hyper-recurrent* if, for every $N \in \Lambda^o$, $N =_\beta M$ implies that N is recurrent.

▶ **Problem 1** (Statman 1993). *Do hyper-recurrent λ-terms exist?*

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[5] Intuitively, relevant intersection type systems where $\wedge$ is a non-idempotent operator, i.e. $\sigma \wedge \sigma \neq \sigma$.

This problem appears as Problem 52 in the RTA list of open problems [62]. Statman proved that hyper-recurrent combinators do not exist in Combinatory Logic [60], however the two calculi are known to satisfy different properties as term rewriting systems. For instance, in Combinatory Logic based on $\{\mathsf{S}, \mathsf{K}, \mathsf{I}\}$ there are no pure cycles, as shown by [28].

6.2 Is there a double fixed point combinator?

Consider the λ-term $\boldsymbol{\delta} = \lambda yx.x(yx)$. It was remarked by Böhm and van der Mey that Y is a fixed point combinator if and only if $\boldsymbol{\delta}Y =_\beta Y$ holds. It follows that, if Y is a fixed point combinator, then both $\boldsymbol{\delta}Y$ and $Y\boldsymbol{\delta}$ are fixed point generators. A *double fixed point combinator* is a λ-term Y satisfying

$$\boldsymbol{\delta}Y =_\beta Y =_\beta Y\boldsymbol{\delta}.$$

▶ **Problem 2** (Statman 1993)**.** *Does there exist a double fixed point combinator?*

This problem appears as Problem 52 in the RTA list of open problems [61], and it is marked as "solved" since the appearance of [35]. However, in 2011, Endrullis has discovered a gap in a crucial case of the argument and the problem should therefore be considered as open. Klop considers this problem one of the most interesting problems in term rewriting, and Endrullis *et al.* [27] have developed a clocked mechanism in the hope of distinguishing every Y from $Y\boldsymbol{\delta}$, but their attempts were unsuccessful. For more information about this problem and other suggestions for a proof strategy, we refer to [47].

6.3 Does Combinatory Logic satisfy the Plane Property?

We consider here the Combinatory Logic with the basis of combinators $\{\mathbf{K}, \mathbf{S}\}$.

▶ **Problem 3** (Jan Willem Klop 1980)**.** *If a combinatory term can leave a plane at one of its points, can such a plane be left at any of its points?*

This problem was first raised in [41], and a positive answer was conjectured. We mentioned before that the λ-calculus does not satisfy this property, but also that the counterexamples rely on the fact that λ-calculus satisfies the ξ-rule:

$$\frac{M = N}{\lambda x.M = \lambda x.N}\ (\xi)$$

Since Combinatory Logic does not satisfy this rule, the constructions in [50, 58] do not generalize to this setting.

6.4 Is the word problem for S decidable?

The following is another longstanding open problem concerning the $\mathbf{S}$-fragment of Combinatory Logic. Recall that the problems of determining whether an $\mathbf{S}$-term is strongly normalizing or head normalizing are both decidable.

▶ **Problem 4** (The word problem for $\mathbf{S}$)**.** *Given* $\mathbf{S}$*-terms* P, Q*, is the problem of determining whether* $P =_w Q$ *decidable?*

First raised in [3]. Notice that, because of Padovani's counterexample [9, Theorem 7.49], the w-conversion $=_w$ does not coincide with the equality induced by Berarducci trees.

6.5 Are there denotational models of other observational theories?

Among the inequational theories defined in Chapter 12, the following have never been seriously studied in the literature:

$$M \sqsubseteq_{m1} N \iff \forall C[\,]\,.\,[\;C[M] \text{ has a } \beta\text{-nf} \quad \Rightarrow \quad C[N] \text{ has the same } \beta\text{-nf}\;]$$
$$M \sqsubseteq_{[\mathsf{I}]_\beta} N \iff \forall C[\,]\,.\,[\;C[M] =_\beta \mathsf{I} \quad \Rightarrow \quad C[N] =_\beta \mathsf{I}\;]$$

As a warm-up exercise, consider $X = \lambda zx.xzx$ and $Y = \lambda zx.xz(\lambda y.xy)$ and convince yourself that the following inequalities hold: $X\Omega \sqsubseteq_{m1} Y\Omega$ and $Y\Omega \sqsubseteq_{m1} X\Omega$.

▶ **Problem 5** (Manzonetto 2022). *Are there (in)equationally fully abstract denotational models for $\sqsubseteq_{m1}$ and $\sqsubseteq_{[\mathsf{I}]_\beta}$?*

Preliminary investigations made by Manzonetto in collaboration with Barbarossa, Breuvart and Kerinec suggest that fully abstract models for these theories might exist in the strongly stable semantics defined by Bucciarelli and Ehrhard [15].

6.6 Is the λ-theory $\mathcal{H}\omega$ Π^1_1-complete?

In [7], Barendregt *et al.* proved that for any Π^1_1-predicate P there exist closed λ-terms $\mathsf{B}^n_0, \mathsf{B}^n_1$ such that

$$\mathrm{P}(n) \quad \Rightarrow \quad \mathcal{H}\omega \vdash \mathsf{B}^n_0 = \mathsf{B}^n_1,$$

in which the fact that $\mathrm{P}(n)$ holds seems to have been properly used (which is not the case if one takes e.g. $\mathsf{B}^n_0 = \mathsf{I} = \mathsf{B}^n_1$). See also [5, Theorem 17.4.14]. This result motivated the conjecture that the λ-theory $\mathcal{H}\omega$ is Π^1_1-complete. This conjecture first appeared in [7] and was subsequently stated in [5, Conjecture 17.4.15]. A confirmation of Barendregt's conjecture was proposed by Intrigila and Statman in [39], but their proof contains a few issues that we explain in detail in [9, §6.3]. The problem should be then considered open.

▶ **Problem 6** (Barendregt 1978). *Is the λ-theory $\mathcal{H}\omega$ Π^1_1-complete?*

This is the last open problem from Barendregt's book [5] which is still standing.

Conclusions

I want to conclude by sharing some personal impressions about this experience. Writing a book on my research domain has been an incredibly enriching journey from a scientific perspective. Not only it gave me the opportunity to pick from Henk's mathematical wisdom, but it allowed me to study the state of the art of research in the field from a broader viewpoint. This brought to my attention many results that I wasn't aware of, and forced me to enter into the technicalities of proofs that I previously studied only superficially, eventually becoming familiar with the associated proof-techniques.

I believe that too often researchers get caught, not by their fault, in a "publish or perish" situation in which finding the next result is more profitable in terms of their academic career than studying existing results. I believe it is important, when a research domain reaches a stable point of maturity, to take the time to organize the material in a clear and well-structured text, so that the whole community can benefit from the effort.

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