Homotopy Type Theory as Internal Languages of Diagrams of ∞-Logoses

Taichi Uemura
Stockholm University, Sweden

Abstract

We show that certain diagrams of ∞-logoses are reconstructed in internal languages of their oplax limits via lex, accessible modalities, which enables us to use plain homotopy type theory to reason about not only a single ∞-logos but also a diagram of ∞-logoses. This also provides a higher dimensional version of Sterling's synthetic Tait computability – a type theory for higher dimensional logical relations.

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1 Introduction

Homotopy type theory [31] is a type theory where one can do homotopy theory. It extends Martin-Löf's type theory [19] by the *univalence axiom* and *higher inductive types*. The former forces types to behave like spaces rather than sets, and the latter allow us to build types representing spaces such as spheres and tori.

An ∞-*logos*, also known as an ∞-*topos* [16, 2][1], is another place to do homotopy theory, among other aspects of it. An ∞-logos is an $(\infty, 1)$-category that looks like the $(\infty, 1)$-category of spaces just as an ordinary logos is a category that looks like the category of sets.

Homotopy type theory and ∞-logoses are closely related. Shulman [26] has shown that any ∞-logos is presented by a structure that admits an interpretation of homotopy type theory. In other words, homotopy type theory is an *internal language* of an ∞-logos. Any theorem proved in homotopy type theory can be translated in an arbitrary ∞-logos. For example, the proof of the Blakers-Massey connectivity theorem in homotopy type theory [10] has led to a new generalized Blakers-Massey theorem that holds in an arbitrary ∞-logos [1].

An ∞-logos, however, does not live alone. ∞-logoses are often connected by functors which are also connected by natural transformations. Plain homotopy type theory is, at first sight, not sufficient to reason about a diagram of ∞-logoses, because the actions of

[1] The term ∞-logos is Anel and Joyal's terminology [2] for ∞-topos considered as an algebraic structure rather than a geometric object. A morphism of ∞-logoses is always considered in the direction of the inverse image functor. We use this terminology to clarify the direction of morphisms when speaking about (co)limits of ∞-logoses.

the functors and natural transformations are not internalized to type theory. Even worse, it is impossible to naively internalize some diagrams: some internal adjunction leads a contradiction [14]; there are only trivial internal idempotent comonads [25].

While there is no chance of naive internalization of such interesting but problematic diagrams to plain homotopy type theory, some other diagrams can be internalized in a clever way pointed out by Shulman[2]. A minimal non-trivial example is a diagram consisting of two ∞-logoses and a lex, accessible functor between them in one direction. The two ∞-logoses are *lex, accessible localizations* of another ∞-logos obtained by the *Artin gluing* for the functor, and the functor is reconstructed by composing the inclusion from one localization and the reflector to the other. Moreover, this reconstruction is *internal* to the glued ∞-logos, because lex, accessible localizations of an ∞-logos correspond to *lex, accessible modalities* in its internal language. Hence, plain homotopy type theory as an internal language of the glued ∞-logos is sufficient to reason about the original diagram.

In this paper, we propose a class of shapes of diagrams of ∞-logoses for which the internal reconstruction technique explained in the previous paragraph works. We call shapes in the proposed class *mode sketches*. Our main result is summarized as follows. Let $\mathfrak{M}$ be a mode sketch. $\mathfrak{M}$ is regarded as a presentation of an $(\infty, 2)$-category. Then:

1. We associate to $\mathfrak{M}$ certain axioms in type theory, one of which is to postulate some lex, accessible modalities from which one can construct a diagram of ∞-logoses internally to type theory (Sections 3.1 and 3.2).
2. For any diagram $\mathcal{L}$ indexed over $\mathfrak{M}$ consisting of ∞-logoses and lex, accessible functors, the oplax limit of $\mathcal{L}$ is an ∞-logos that satisfies the axioms associated to $\mathfrak{M}$, and the diagram obtained in the internal language of the oplax limit corresponds to the original diagram $\mathcal{L}$. Conversely, any ∞-logos that satisfies the axioms associated to $\mathfrak{M}$ is obtained by this oplax limit construction (Theorem 48).

Recall that the *oplax limit* of a diagram of $(\infty, 1)$-categories is a generalization of the Artin gluing [35, 24].

1.1 Synthetic Tait computability

This work is closely related to Sterling's *synthetic Tait computability* [29, 27]. It is a technique of constructing *logical relations* using an internal language for the Artin gluing. A logos obtained by the Artin gluing is always equipped with a distinguished proposition in its internal language. The two lex, accessible modalities associated to the glued logos are the *open* and *closed* modalities associated to the proposition. The *fracture and gluing theorem* asserts that every type in the internal language is canonically fractured into an open type and a closed (unary, proof-relevant) relation on it which are glued back together. The internal language for the Artin gluing is thus a type theory with an indeterminate proposition in which *types are relations* and provides a synthetic method of constructing logical relations used in the study of type theories and programming languages. Applications include normalization for complex type theories [28, 8].

We relate synthetic Tait computability and mode sketches. The core axiom for synthetic Tait computability is to postulate some indeterminate propositions. Note that, although the original synthetic Tait computability is based on extensional type theory, postulating propositions makes sense also in homotopy type theory. We show that part of the axioms associated to a mode sketch is equivalent to postulating a lattice of propositions (Theorem 37).

[2] `https://golem.ph.utexas.edu/category/2011/11/internalizing_the_external_or.html`

Mode sketches thus provide an alternative synthetic method of constructing logical relations. This is also natural from Shulman's point of view [24] that interpretations of type theory in oplax limits are generalized logical relations. Since we work in homotopy type theory, what we get is actually *higher-dimensional logical relations*, and our primary application of mode sketches in upcoming paper(s) [34] will be normalization for ∞*-type theories* introduced by Nguyen and Uemura [20] as a higher-dimensional generalization of type theories.

1.2 Contributions

Our main result is Theorem 48: for every mode sketch $\mathfrak{M}$, the models of the axioms associated to $\mathfrak{M}$ are precisely the oplax limits of diagrams of ∞-logoses indexed over $\mathfrak{M}$. This allows us to reason about a diagram of ∞-logoses in plain homotopy type theory. We also relate mode sketches to synthetic Tait computability (Theorem 37). Mode sketches provide a higher-dimensional version of synthetic Tait computability.

A minor result is an improvement of the *fracture and gluing theorem* of Rijke, Shulman, and Spitters [22, Theorem 3.50]. It gives a construction of a canonical join of two strongly disjoint lex modalities. We show that this construction preserves accessibility as well (Proposition 14).

1.3 Organization

In Section 2, we review the theory of modalities in homotopy type theory [22]. Our focus is on the poset of lex, accessible modalities and on the open and closed modalities associated to propositions.

Sections 3 and 4 are the core of the paper. We introduce the notion of a *mode sketch* (Definition 25). For every mode sketch, we introduce two equivalent sets of axioms to encode a certain diagram of universes. One postulates some lex, accessible modalities while the other postulates a lattice of propositions. The open and closed modalities give a construction of the former from the latter which we show is an equivalence (Theorem 37). The latter is a higher dimensional analogue of Sterling's synthetic Tait computability [27].

In Section 5, we give a sketch of proof of our main result (Theorem 48): for any mode sketch, the space of ∞-logoses satisfying the axioms associated to the mode sketch is equivalent to the space of diagrams of ∞-logoses and lex, accessible functors indexed over the mode sketch. For reasons of space, details are not presented in this version. See [33] for full details.

1.4 Preliminaries

We assume that the reader is familiar with homotopy type theory [31]. By *homotopy type theory* we mean dependent type theory with (dependent) function types, (dependent) pair types, a unit type, identity types, univalent universes $\mathcal{U} : \Uparrow\mathcal{U} : \Uparrow^2\mathcal{U} : \dots$, and all higher inductive types we need. The universe $\Uparrow^n\mathcal{U}$ is usually written as $\mathcal{U}_n$, but the latter conflicts with the notation for the subuniverse of modal types. The notation $\Uparrow^n\mathcal{U}$ also indicates that large types are interpreted in universe enlargements of an ∞-logos; see Section 5. We mainly follow the HoTT Book [31] for terminologies and notations in homotopy type theory.

1.5 Related work

An earlier version of *cohesive homotopy type theory* [23] uses modalities in plain homotopy type theory to internalize a series of adjunctions that arises in Lawvere's axiomatic cohesion [13]. However, because naive internalization of adjunctions do not work well [14, 25], the axiomatization is tricky and not ideal to work with. The newer version of cohesive homotopy type theory [25] instead extends homotopy type theory by another layer of context and new modal operators. The resulting type theory works well for axiomatic cohesion but is complicated compared to plain homotopy type theory. It is also too optimized for axiomatic cohesion.

A more general framework for internal diagrams is *multimodal dependent type theory* [9]. It is roughly a family of type theories related to each other via modal operators and interpreted in a diagram of presheaf categories among a more general notion of model. The shape of diagram is specified directly by an arbitrary 2-category which is called a *mode theory* in this context. Our terminology "mode sketch" is chosen to mean a sketch of a mode theory. Multimodal dependent type theory is potentially an internal language for diagrams of ∞-logoses, but for this one would have to rectify not only ∞-logoses but also functors and natural transformations between them.

Our work brings back the ideas of earlier cohesive homotopy type theory [23]. Although it might not be the best type theory, it has a lot of advantages: modalities are internal to plain homotopy type theory, and thus all results are ready to formalize in existing proof assistants; keeping type theory simple is also important in informal use of type theory in which the correctness of application of inference rules is not checked by computer; the semantics is clear, since the ∞-logos semantics of homotopy type theory is well-established [4, 3, 24, 26]; it also opens the door to internalization of more general diagrams in a uniform way, which is the motivation for the current work.

2 Modalities in homotopy type theory

We review the theory of *modalities* in homotopy type theory [22]. In this section, we work in homotopy type theory. A modality is in short a reflective subuniverse closed under pair types.

▶ **Definition 1.** *A* subuniverse $\mathfrak{m}$ *is a a function* $\mathsf{In}_{\mathfrak{m}} : \mathcal{U} \to \Uparrow\mathcal{U}$ *such that* $\mathsf{In}_{\mathfrak{m}}(A)$ *is a proposition for all* $A : \mathcal{U}$*. A type* A *satisfying* $\mathsf{In}_{\mathfrak{m}}(A)$ *is called* $\mathfrak{m}$-modal*. We define a subtype* $\mathcal{U}_{\mathfrak{m}} \subset \mathcal{U}$ *to be* $\{A : \mathcal{U} \mid \mathsf{In}_{\mathfrak{m}}(A)\}$*.*

▶ **Definition 2.** *A subuniverse* $\mathfrak{m}$ *is* reflective *if it is equipped with functions* $\bigcirc_{\mathfrak{m}} : \mathcal{U} \to \mathcal{U}_{\mathfrak{m}}$ *and* $\eta_{\mathfrak{m}} : \prod_{A:\mathcal{U}} A \to \bigcirc_{\mathfrak{m}} A$ *such that that the precomposition* $\lambda f.f \circ \eta_{\mathfrak{m}}(A) : (\bigcirc_{\mathfrak{m}} A \to B) \to (A \to B)$ *is an equivalence for any* $B : \mathcal{U}_{\mathfrak{m}}$*. Note that such a pair* $(\bigcirc_{\mathfrak{m}}, \eta_{\mathfrak{m}})$ *is unique.*

▶ **Definition 3.** *A reflective subuniverse* $\mathfrak{m}$ *is a* modality *if* $\mathsf{In}_{\mathfrak{m}}$ *is closed under pair types, that is, for* $A : \mathcal{U}$ *and* $B : A \to \mathcal{U}$*, if* $\mathsf{In}_{\mathfrak{m}}(A)$ *and* $\prod_{a:A} \mathsf{In}_{\mathfrak{m}}(B(a))$*, then* $\mathsf{In}_{\mathfrak{m}}(\sum_{a:A} B(a))$*.*

An important class of modalities is *accessible* modalities which are roughly modalities "presented by small data".

▶ **Definition 4.** *For types* $A, B : \mathcal{U}$*, we define*

$$(A \perp B) \equiv \mathsf{IsEquiv}(\lambda(b : B).\lambda(\_ : A).b).$$

Note that $\lambda(b : B).\lambda(\_ : A).b$ *is a function of type* $B \to (A \to B)$*. For a subuniverse* $\mathfrak{m}$*, we define subuniverses* $\mathfrak{m}^{\perp}$ *and* ${}^{\perp}\mathfrak{m}$ *by*

$$\mathsf{In}_{\mathfrak{m}^{\perp}}(B) \equiv \prod_{A:\mathcal{U}_{\mathfrak{m}}} A \perp B$$
$$\mathsf{In}_{{}^{\perp}\mathfrak{m}}(A) \equiv \prod_{B:\mathcal{U}_{\mathfrak{m}}} A \perp B.$$

▶ **Definition 5.** *A* null generator μ *consists of* $I_\mu : \mathcal{U}$ *and* $Z_\mu : I_\mu \to \mathcal{U}$. *We write* $\mathsf{NullGen}$ *for the type of null generators. Given a null generator* μ*, we define a subuniverse* $\mathfrak{Null}(\mu)$ *by* $\mathsf{In}_{\mathfrak{Null}(\mu)}(A) \equiv \prod_{i:I_\mu} Z_\mu(i) \perp A$*. It is shown that* $\mathfrak{Null}(\mu)$ *is a modality using a higher inductive type [22, Theorem 2.19]. A modality* $\mathfrak{m}$ *is* accessible *if it is in the image of* $\mathfrak{Null}$*, that is,* $\| \sum_{\mu:\mathsf{NullGen}} \mathfrak{m} = \mathfrak{Null}(\mu) \|$.

Another important class of modalities is *lex* modalities.

▶ **Definition 6.** *For a modality* $\mathfrak{m}$*, a type* $A : \mathcal{U}$ *is* $\mathfrak{m}$-connected *if* $\bigcirc_{\mathfrak{m}} A$ *is contractible. This is equivalent to* $\mathsf{In}_{\perp \mathfrak{m}}(A)$ *by [22, Corollary 1.37].*

▶ **Definition 7.** *A modality* $\mathfrak{m}$ *is* lex *if for any* $\mathfrak{m}$*-connected type* $A : \mathcal{U}$*, the identity type* $a_1 = a_2$ *is* $\mathfrak{m}$*-connected for any* $a_1, a_2 : A$.

Modalities that are both lex and accessible are of particular importance because they correspond to subtoposes of an ∞-topos under the interpretation of types as sheaves on the ∞-topos. From now on, we are mostly interested lex, accessible modalities, so we give them a short name.

▶ **Terminology 8.** LAM is an acronym for lex, accessible modality.

Fundamental examples of LAMs are *open* and *closed* modalities which correspond to open and closed, respectively, subtoposes.

▶ **Construction 9.** Let P be a proposition. We define the *open modality* $\mathfrak{Op}(P)$ by $\bigcirc_{\mathfrak{Op}(P)} A \equiv (P \to A)$ and $\eta_{\mathfrak{Op}(P)}(A, a) \equiv \lambda\_.a$. It is lex and accessible by [22, Example 2.24 and Example 3.10]. We also define the *closed modality* $\mathfrak{Cl}(P)$ by $\mathsf{In}_{\mathfrak{Cl}(P)}(A) \equiv (P \to \mathsf{IsContr}(A))$. It is lex and accessible by [22, Example 2.25 and Example 3.14]. Note that $\mathfrak{Cl}(P) = {}^{\perp}\mathfrak{Op}(P)$ [22, Example 1.31].

2.1 The poset of lex, accessible modalities

We have the posets

$$\mathsf{SU} \supset \mathsf{RSU} \supset \mathsf{Mdl} \supset \mathsf{AccMdl} \supset \mathsf{LAM}$$

of subuniverses, reflective subuniverses, modalities, accessible modalities, and lex, accessible modalities, respectively, where all the inclusions are full. We also have the full subposet $\mathsf{Lex} \subset \mathsf{Mdl}$ of lex modalities. By definition, $\mathsf{LAM} = \mathsf{Lex} \cap \mathsf{AccMdl}$. We study the poset LAM in more detail.

▶ **Definition 10** ([22, Theorem 3.25])**.** *Let* $I : \mathcal{U}$ *and* $\mathfrak{m} : I \to \mathsf{LAM}$*. A* canonical meet $\bigwedge_{i:I} \mathfrak{m}(i)$ *is a LAM that is the meet of* $\mathfrak{m}(i)$*'s in* SU*. A* canonical join $\bigvee_{i:I} \mathfrak{m}(i)$ *is a LAM satisfying that a type* $A : \mathcal{U}$ *is* $(\bigvee_{i:I} \mathfrak{m}(i))$*-connected if and only if it is* $\mathfrak{m}(i)$*-connected for all* $i : I$*. Note that a canonical join is the join in* Mdl*.*

▶ **Example 11.** The *top modality* $\mathfrak{Top}$, for which all the types are modal, is the canonical meet of the empty family. The *bottom modality* $\mathfrak{Bot}$, for which only the contractible types are modal, is the canonical join of the empty family.

The canonical meet of an arbitrary family of LAMs exists [22, Theorem 3.29 and Remark 3.23]. Canonical joins are less understood than canonical meets. One important case when canonical joins exist and can be computed is the following.

▶ **Definition 12.** *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be LAMs.* $\mathfrak{n}$ *is* strongly disjoint from $\mathfrak{m}$ *if any* $\mathfrak{m}$*-modal type is* $\mathfrak{n}$*-connected or equivalently if* $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$ *in* SU*.*

▶ **Proposition 13** (Fracture and gluing theorem)**.** *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be LAMs such that* $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$*.*

1. *The canonical join* $\mathfrak{m} \vee \mathfrak{n}$ *exists.*
2. *A type* A *is* $(\mathfrak{m} \vee \mathfrak{n})$*-modal if and only if the function* $\eta_{\mathfrak{n}}(A) : A \to \bigcirc_{\mathfrak{n}} A$ *has* $\mathfrak{m}$*-modal fibers.*
3. $\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}} \simeq \sum_{A:\mathcal{U}_{\mathfrak{m}}} \sum_{B:\mathcal{U}_{\mathfrak{n}}} A \to \bigcirc_{\mathfrak{m}} B$*.*

In the special case when $\mathfrak{m} = {}^{\perp}\mathfrak{n}$*, we have* $\mathfrak{m} \vee \mathfrak{n} = \mathfrak{Top}$*.*

Proof. All but the accessibility of $\mathfrak{m} \vee \mathfrak{n}$ are proved by Rijke, Shulman, and Spitters [22, Theorem 3.50]. We will prove the accessibility of $\mathfrak{m} \vee \mathfrak{n}$ in Proposition 14 using an open modality. ◀

2.2 Accessibility of the canonical join

Let us fill the gap in the proof of Proposition 13. This subsection is devoted to proving the following.

▶ **Proposition 14.** *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be LAMs such that* $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$*. Then the canonical join* $\mathfrak{m} \vee \mathfrak{n}$ *(in* Lex*) is accessible.*

We have to find a null generator for $\mathfrak{m} \vee \mathfrak{n}$. A natural guess is the following.

▶ **Construction 15.** Let μ and ν be null generators. We define a null generator $\mu \star \nu$ by $I_{\mu\star\nu} \equiv I_\mu \times I_\nu$ and $Z_{\mu\star\nu}(i,j) \equiv Z_\mu(i) \star Z_\nu(j) \equiv Z_\mu(i) +_{Z_\mu(i)\times Z_\nu(j)} Z_\nu(j)$.

▶ **Lemma 16.** *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be LAMs, and let* μ *and* ν *be null generators for* $\mathfrak{m}$ *and* $\mathfrak{n}$*, respectively. Then* $Z_{\mu\star\nu}(i,j)$ *is both* $\mathfrak{m}$*-connected and* $\mathfrak{n}$*-connected for all* $i : I_\mu$ *and* $j : I_\nu$*.*

Proof. Recall that a function is $\mathfrak{m}$-*connected* if its fibers are $\mathfrak{m}$-connected and that the class of $\mathfrak{m}$-connected functions is the left class of a (stable) orthogonal factorization system [22, Theorem 1.34]. Then the claim follows by the pushout stability and the right cancellability of connected functions. ◀

Lemma 16 shows $\mathfrak{m} \vee \mathfrak{n} \leq \mathfrak{Null}(\mu \star \nu)$ for arbitrary accessible modalities $\mathfrak{m}$ and $\mathfrak{n}$ and for arbitrary choices of μ and ν. We know neither if the other direction holds in general for some choices of μ and ν nor if $\mathfrak{Null}(\mu \star \nu)$ is independent of μ and ν. Note that Finster [7] observed that $\mathfrak{Null}(\mu \star \nu)$ is lex whenever $\mathfrak{Null}(\mu)$ and $\mathfrak{Null}(\nu)$ are lex. In the special case when $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$, the idea of the proof of $\mathfrak{m} \vee \mathfrak{n} = \mathfrak{Null}(\mu \star \nu)$ is to show that $\mathfrak{n}$ is an *open modality* within the subuniverse of $\mathfrak{Null}(\mu \star \nu)$-modal types.

▶ **Lemma 17.** *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be LAMs such that* $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$*. Then* $\mathfrak{n} \leq \mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})$*.*

Proof. This is because $\bigcirc_{\mathfrak{m}} \mathbf{0}$ is $\mathfrak{n}$-connected by assumption. ◀

▶ **Lemma 18.** *Let* $\mathfrak{m}$ *and* $\mathfrak{n}$ *be LAMs such that* $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$*. Suppose that* μ *and* ν *are null generators for* $\mathfrak{m}$ *and* $\mathfrak{n}$*, respectively, and that* μ *admits a function* $f : \bigcirc_{\mathfrak{m}} \mathbf{0} \to I_\mu$ *such that* $\mathbf{0} \simeq Z_\mu(f(i))$ *for all* $i : \bigcirc_{\mathfrak{m}} \mathbf{0}$*. Then* $\bigcirc_{\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})} A$ *is* $\mathfrak{n}$*-modal for any* $\mathfrak{Null}(\mu \star \nu)$*-modal type* A*. Consequently, the canonical function* $\bigcirc_{\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})} A \to \bigcirc_{\mathfrak{n}} A$ *induced by Lemma 17 is an equivalence for any* $\mathfrak{Null}(\mu \star \nu)$*-modal type* A*.*

Proof. We show that $\bigcirc_{\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})} A \equiv (\bigcirc_{\mathfrak{m}} \mathbf{0} \to A)$ is $\mathfrak{n}$-modal. Since ν is a null generator for $\mathfrak{n}$, it suffices to show that $Z_\nu(j) \perp (\bigcirc_{\mathfrak{m}} \mathbf{0} \to A)$ for all $j : I_\nu$. This is equivalent to that $Z_\nu(j) \perp A$ under an assumption $i : \bigcirc_{\mathfrak{m}} \mathbf{0}$. This holds since $Z_\nu(j) \simeq \mathbf{0} \star Z_\nu(j) \simeq Z_\mu(f(i)) \star Z_\nu(j)$ and since A is $\mathfrak{Null}(\mu \star \nu)$-modal. ◀

▶ **Lemma 19.** *Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs such that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$. Suppose that μ and ν are null generators for $\mathfrak{m}$ and $\mathfrak{n}$, respectively, and that ν has an element $j : I_\nu$ such that $Z_\nu(j) \simeq \bigcirc_{\mathfrak{m}} \mathbf{0}$. Then, if a type A is $\mathfrak{Null}(\mu \star \nu)$-modal and $\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})$-connected, then it is $\mathfrak{m}$-modal.*

Proof. We show that $Z_\mu(i) \perp A$ for all $i : I_\mu$. By the definition of $\star$, we have the following pullback square.

$$\begin{array}{ccc}
(Z_\mu(i) \star \bigcirc_{\mathfrak{m}} \mathbf{0} \to A) & \xrightarrow{\simeq} & (Z_\mu(i) \to A) \\
\downarrow & \lrcorner & \downarrow \\
(\bigcirc_{\mathfrak{m}} \mathbf{0} \to A) & \xrightarrow[\simeq]{} & (Z_\mu(i) \to \bigcirc_{\mathfrak{m}} \mathbf{0} \to A)
\end{array}$$

Since A is $\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})$-connected, the domain and codomain of the bottom function are contractible, and thus the bottom function is an equivalence. It then follows that the top function is also an equivalence. Since A is $\mathfrak{Null}(\mu \star \nu)$-modal and since $Z_\nu(j) \simeq \bigcirc_{\mathfrak{m}} \mathbf{0}$, we have $A \simeq (Z_\mu(i) \star \bigcirc_{\mathfrak{m}} \mathbf{0} \to A) \simeq (Z_\mu(i) \to A)$, and thus $Z_\mu(i) \perp A$. ◀

Proof of Proposition 14. Let μ and ν be null generators for $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Note that adjoining a family of connected types to a null generator does not change the modality presented by the null generator. Under an assumption $i : \bigcirc_{\mathfrak{m}} \mathbf{0}$, the empty type $\mathbf{0}$ becomes $\mathfrak{m}$-connected, and thus we may assume that μ includes the type family $\lambda(\_ : \bigcirc_{\mathfrak{m}} \mathbf{0}).\mathbf{0}$. Since $\bigcirc_{\mathfrak{m}} \mathbf{0}$ is $\mathfrak{n}$-connected by assumption, we may assume that ν includes the type family $\lambda(\_ : \mathbf{1}).\bigcirc_{\mathfrak{m}} \mathbf{0}$.

We show that $\mathfrak{Null}(\mu \star \nu) = \mathfrak{m} \vee \mathfrak{n}$. By Lemma 16, $\mathfrak{m} \vee \mathfrak{n} \leq \mathfrak{Null}(\mu \star \nu)$. For the other direction, suppose that A is a $\mathfrak{Null}(\mu \star \nu)$-modal type. By [22, Theorem 3.50], it suffices to show that $\eta_{\mathfrak{n}}(A) : A \to \bigcirc_{\mathfrak{n}} A$ has $\mathfrak{m}$-modal fibers. By Lemma 18, $\bigcirc_{\mathfrak{n}} A \simeq \bigcirc_{\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})} A$. Then the fibers of $\eta_{\mathfrak{n}}(A)$ are $\mathfrak{Op}(\bigcirc_{\mathfrak{m}} \mathbf{0})$-connected. Since both A and $\bigcirc_{\mathfrak{n}} A$ are $\mathfrak{Null}(\mu \star \nu)$-modal, the fibers of $\eta_{\mathfrak{n}}(A)$ are also $\mathfrak{Null}(\mu \star \nu)$-modal. Thus, by Lemma 19, $\eta_{\mathfrak{n}}(A)$ has $\mathfrak{m}$-modal fibers. ◀

As a by-product, we have the following.

▶ **Corollary 20.** *Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs such that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$. If $\mathfrak{m} \vee \mathfrak{n} = \mathfrak{Top}$, then $\mathfrak{m}$ and $\mathfrak{n}$ are the closed and open, respectively, modalities associated to the proposition $\bigcirc_{\mathfrak{m}} \mathbf{0}$.* ◀

3 Mode sketches

We introduce *mode sketches* as shapes of diagrams of subuniverses definable internally to type theory. We work in homotopy type theory through the section.

3.1 Internal diagrams induced by modalities

We consider postulating some LAMs to encode some diagram of subuniverses. The fundamental observation is that a pair of LAMs induces a canonical functor between them.

▶ **Construction 21.** Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs. We define a function $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}} : \mathcal{U}_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{m}}$ to be the restriction of $\bigcirc_{\mathfrak{m}}$ to $\mathcal{U}_{\mathfrak{n}} \subset \mathcal{U}$.

▶ Remark 22. We can say that $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}}$ is a functor *externally*: we can construct a function $\prod_{A,B:\mathcal{U}_{\mathfrak{n}}}(A \to B) \to (\bigcirc^{\mathfrak{n}}_{\mathfrak{m}} A \to \bigcirc^{\mathfrak{n}}_{\mathfrak{m}} B)$ and every instance of the coherence laws. However, it is not known how to state that $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}}$ is a functor internally to type theory, because defining the type of $(\infty, 1)$-categories in plain homotopy type theory is still an open problem.

We have two functors $\bigcirc^{\mathfrak{m}}_{\mathfrak{n}} : \mathcal{U}_{\mathfrak{m}} \to \mathcal{U}_{\mathfrak{n}}$ and $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}} : \mathcal{U}_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{m}}$ for every pair of LAMs $\mathfrak{m}$ and $\mathfrak{n}$, but we are often interested in only one direction. It is thus useful to cut off one direction by postulating that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$: by the definition of connectedness, $\bigcirc^{\mathfrak{m}}_{\mathfrak{n}}$ becomes constant at the unit type. The other direction $\bigcirc^{\mathfrak{n}}_{\mathfrak{m}} : \mathcal{U}_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{m}}$ remains non-trivial. Therefore, a pair $(\mathfrak{m}, \mathfrak{n})$ of LAMs such that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$ encodes a functor $\mathcal{U}_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{m}}$. When $\mathfrak{n} \leq {}^{\perp}\mathfrak{m}$ is also assumed, $\mathcal{U}_{\mathfrak{m}}$ and $\mathcal{U}_{\mathfrak{n}}$ are considered unrelated.

Given more than two LAMs, we have canonical natural transformations between the canonical functors.

▶ **Construction 23.** Let $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2$ be LAMs. We define

$$\eta^{\mathfrak{m}_0;\mathfrak{m}_2}_{\mathfrak{m}_1} : \prod_{A:\mathcal{U}_{\mathfrak{m}_2}} \bigcirc^{\mathfrak{m}_2}_{\mathfrak{m}_0} A \to \bigcirc^{\mathfrak{m}_1}_{\mathfrak{m}_0} \bigcirc^{\mathfrak{m}_2}_{\mathfrak{m}_1} A$$

by $\eta^{\mathfrak{m}_0;\mathfrak{m}_2}_{\mathfrak{m}_1}(A) \equiv \bigcirc_{\mathfrak{m}_0} \eta_{\mathfrak{m}_1}(A)$. This family of functions is natural in the sense that for any $A, B : \mathcal{U}_{\mathfrak{m}_2}$ and $f : A \to B$, we have a homotopy filling the following square.

$$\begin{array}{ccc}
\bigcirc^{\mathfrak{m}_2}_{\mathfrak{m}_0} A & \xrightarrow{\eta^{\mathfrak{m}_0;\mathfrak{m}_2}_{\mathfrak{m}_1}(A)} & \bigcirc^{\mathfrak{m}_1}_{\mathfrak{m}_0} \bigcirc^{\mathfrak{m}_2}_{\mathfrak{m}_1} A \\
{\scriptstyle \bigcirc^{\mathfrak{m}_2}_{\mathfrak{m}_0} f} \downarrow & & \downarrow {\scriptstyle \bigcirc^{\mathfrak{m}_1}_{\mathfrak{m}_0} \bigcirc^{\mathfrak{m}_2}_{\mathfrak{m}_1} f} \\
\bigcirc^{\mathfrak{m}_2}_{\mathfrak{m}_0} B & \xrightarrow[\eta^{\mathfrak{m}_0;\mathfrak{m}_2}_{\mathfrak{m}_1}(B)]{} & \bigcirc^{\mathfrak{m}_1}_{\mathfrak{m}_0} \bigcirc^{\mathfrak{m}_2}_{\mathfrak{m}_1} B
\end{array}$$

Let $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ be LAMs. By naturality, the following diagram commutes.

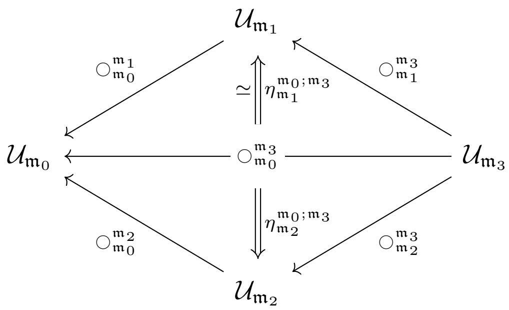
$$\begin{array}{ccc}
\bigcirc^{\mathfrak{m}_3}_{\mathfrak{m}_0} & \xrightarrow{\eta^{\mathfrak{m}_0;\mathfrak{m}_3}_{\mathfrak{m}_1}} & \bigcirc^{\mathfrak{m}_1}_{\mathfrak{m}_0} \bigcirc^{\mathfrak{m}_3}_{\mathfrak{m}_1} \\
{\scriptstyle \eta^{\mathfrak{m}_0;\mathfrak{m}_3}_{\mathfrak{m}_2}} \downarrow & & \downarrow {\scriptstyle \bigcirc^{\mathfrak{m}_1}_{\mathfrak{m}_0} \eta^{\mathfrak{m}_1;\mathfrak{m}_3}_{\mathfrak{m}_2}} \\
\bigcirc^{\mathfrak{m}_2}_{\mathfrak{m}_0} \bigcirc^{\mathfrak{m}_3}_{\mathfrak{m}_2} & \xrightarrow[\eta^{\mathfrak{m}_0;\mathfrak{m}_2}_{\mathfrak{m}_1} \bigcirc^{\mathfrak{m}_3}_{\mathfrak{m}_2}]{} & \bigcirc^{\mathfrak{m}_1}_{\mathfrak{m}_0} \bigcirc^{\mathfrak{m}_2}_{\mathfrak{m}_1} \bigcirc^{\mathfrak{m}_3}_{\mathfrak{m}_2}
\end{array}$$

For more than four LAMs, higher coherence laws are also satisfied. Hence, a tuple $(\mathfrak{m}_0, \ldots, \mathfrak{m}_n)$ of LAMs such that $\mathfrak{m}_i \leq {}^{\perp}\mathfrak{m}_j$ for all $i < j$ encodes an n-simplex with vertices $\mathcal{U}_{\mathfrak{m}_i}$, edges $\bigcirc^{\mathfrak{m}_j}_{\mathfrak{m}_i} : \mathcal{U}_{\mathfrak{m}_j} \to \mathcal{U}_{\mathfrak{m}_i}$ for $i < j$, triangles

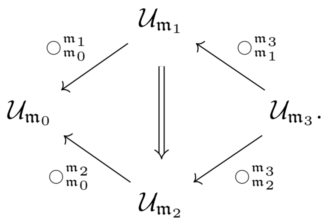
$$\begin{array}{ccc}
\mathcal{U}_{\mathfrak{m}_i} & \xleftarrow{\bigcirc^{\mathfrak{m}_k}_{\mathfrak{m}_i}} & \mathcal{U}_{\mathfrak{m}_k} \\
{\scriptstyle \bigcirc^{\mathfrak{m}_j}_{\mathfrak{m}_i}} \nwarrow & \Downarrow {\scriptstyle \eta^{\mathfrak{m}_i;\mathfrak{m}_k}_{\mathfrak{m}_j}} & \swarrow {\scriptstyle \bigcirc^{\mathfrak{m}_k}_{\mathfrak{m}_j}} \\
& \mathcal{U}_{\mathfrak{m}_j} &
\end{array}$$

for $i < j < k$, and higher homotopies.

Shapes other than simplices are expressed by postulating invertibility of some of $\eta_{\mathfrak{m}_j}^{\mathfrak{m}_i;\mathfrak{m}_k}$'s. For example, let $\mathfrak{m}_0, \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ be LAMs and suppose that $\mathfrak{m}_i \leq {}^{\perp}\mathfrak{m}_j$ for all $i < j$, that $\mathfrak{m}_2 \leq {}^{\perp}\mathfrak{m}_1$, and that $\eta_{\mathfrak{m}_1}^{\mathfrak{m}_0;\mathfrak{m}_3}$ is invertible. We have a diagram



which is equivalent to a diagram of the form



We cannot, however, naively postulate some properties of the functors $\bigcirc_{\mathfrak{m}}^{\mathfrak{n}}$'s such as conservativity, fullness, faithfulness, adjointness, and invertibility. This is because the internal statements of these conditions are too strong due to stability under substitution, and indeed some "no-go" theorems on internalizing properties of functors are known [14, Theorem 5.1][25, Theorem 4.1].

▶ Remark 24. It is *possible* to postulate arbitrary properties of $\bigcirc_{\mathfrak{m}_i}^{\mathfrak{m}_j}$'s in the following way. We first postulate a "base" LAM $\mathfrak{Base}$ and assume $\mathfrak{Base} \leq {}^{\perp}\mathfrak{m}_i$ for all i. The universe $\mathcal{U}_{\mathfrak{Base}}$ is intended to be interpreted as the $(\infty, 1)$-category of spaces, so statements in $\mathcal{U}_{\mathfrak{Base}}$ will correspond to external statements. Since $\bigcirc_{\mathfrak{Base}} : \mathcal{U} \to \mathcal{U}_{\mathfrak{Base}}$ preserves finite limits, it takes $(\infty, 1)$-categories to $(\infty, 1)$-categories and functors to functors. We can then postulate any property on the induced functor $\bigcirc_{\mathfrak{Base}}\mathcal{U}_{\mathfrak{m}_j} \to \bigcirc_{\mathfrak{Base}}\mathcal{U}_{\mathfrak{m}_i}$. In fact, cohesive homotopy type theory [23] was first formulated in a similar fashion where the $\sharp$ modality plays the role of $\mathfrak{Base}$. However, since we only know that $\bigcirc_{\mathfrak{Base}}\mathcal{U}_{\mathfrak{m}_i}$ is an $(\infty, 1)$-category *externally*, this approach is not so convenient to work with especially for formalization in proof assistants. For this and some other reasons, the newer version of cohesive homotopy type theory [25] is a proper extension of homotopy type theory. Nevertheless, this adding-base approach is attractive since it keeps type theory simple and works for any kind of diagram.

3.2 Mode sketches

We introduce *mode sketches* as shapes of diagrams definable by the methodology explained in Section 3.1.

▶ **Definition 25.** *A* mode sketch $\mathfrak{M}$ *consists of the following data:*

- *a decidable finite poset* $I_{\mathfrak{M}}$*;*
- *a subset* $T_{\mathfrak{M}}$ *of triangles in* $I_{\mathfrak{M}}$*.*

Here, by a decidable *poset we mean a poset whose ordering relation $\leq$ is decidable. A type is* finite *if it is merely equivalent to the coproduct of n copies of* **1** *for some $n : \mathbb{N}$ [21, Definition 16.3.1]. The identity type on a finite type is decidable [21, Remark 16.3.2]. The strict ordering relation $i < j$ defined as $(i \leq j) \wedge (i \neq j)$ is also decidable. By a* triangle *in $I_{\mathfrak{M}}$ we mean an ordered triple $(i_0 < i_1 < i_2)$ of elements of $I_{\mathfrak{M}}$. A triangle in $T_{\mathfrak{M}}$ is called* thin.

▶ Remark 26. The definition of mode sketches also makes sense in the metatheory. Every mode sketch $\mathfrak{M}$ in the metatheory can be encoded in type theory since it is finite.

Let $\mathfrak{M}$ be a mode sketch and $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ a function. We consider the following axioms.

▶ **Axiom A.** $\mathfrak{m}(i) \leq {}^{\perp}\mathfrak{m}(j)$ for any $j \not\leq i$ in $\mathfrak{M}$.

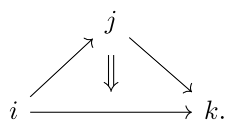
▶ **Axiom B.** For any triangle $(i_0 < i_1 < i_2) : T_{\mathfrak{M}}$, the natural transformation $\eta^{\mathfrak{m}(i_0);\mathfrak{m}(i_2)}_{\mathfrak{m}(i_1)} : \bigcirc^{\mathfrak{m}(i_2)}_{\mathfrak{m}(i_0)} \Rightarrow \bigcirc^{\mathfrak{m}(i_1)}_{\mathfrak{m}(i_0)} \bigcirc^{\mathfrak{m}(i_2)}_{\mathfrak{m}(i_1)}$ is invertible.

▶ **Axiom C.** The top modality is the canonical join $\bigvee_{\mathfrak{M}} \mathfrak{m}$.

▶ Remark 27. Assuming Axiom A, if $i < j$, then $\mathfrak{m}(i) \leq {}^{\perp}\mathfrak{m}(j)$.

Axioms A and B are motivated by the observation made in Section 3.1. That is, when $j \not\leq i$, the functor in the direction $\mathcal{U}_{\mathfrak{m}(i)} \to \mathcal{U}_{\mathfrak{m}(j)}$ is cut off. Our intended models constructed in Section 5 additionally satisfy Axiom C. This axiom is not so important in practical use, since our primary aim is to draw a diagram of ∞-logoses inside homotopy type theory, but Axiom C does nothing for this purpose. It is even better to work without Axiom C, because Axioms A and B are stable under restriction along a full inclusion $\mathfrak{M}' \subset \mathfrak{M}$ while Axiom C is not. Axiom C is meant to exclude models other than intended models.

▶ Remark 28. A mode sketch $\mathfrak{M}$ is regarded as a presentation of an $(\infty, 2)$-category. The strict ordering relation generates 1-cells $(i < j) : i \to j$, and the triangles $(i < j < k)$ generate 2-cells in the direction



When the triangle is thin, the corresponding 2-cell is made invertible. Longer chains $(i_0 < i_1 < \cdots < i_n)$ present coherence. Formally, we regard $\mathfrak{M}$ as a *scaled simplicial set* [17], one of models for $(\infty, 2)$-categories, by taking the nerve and marking thin triangles as thin 2-simplices, and then reverse 2-cells. A function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfying Axioms A–C is then considered as a diagram of subuniverses indexed over $\mathfrak{M}^{\mathrm{op}(1,2)}$, the $(\infty, 2)$-category obtained from $\mathfrak{M}$ by reversing the directions of 1-cells and 2-cells.

▶ **Example 29.** Every decidable finite poset is a mode sketch where no triangle is thin. The $(\infty, 2)$-category presented by it is obtained from the left adjoint of the Duskin nerve [6] by reversing 2-cells.

▶ **Example 30.** The *mode sketch for functors* is drawn as

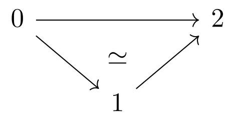
$$0 \longrightarrow 1.$$

Axiom A asserts $\mathfrak{m}(0) \leq {}^{\perp}\mathfrak{m}(1)$. Axiom B is empty since there is no triangle. Thus, we get the following diagram.

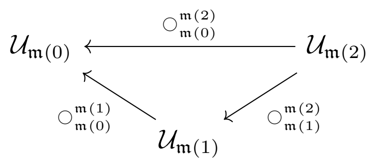
$$\mathcal{U}_{\mathfrak{m}(0)} \xleftarrow{\bigcirc^{\mathfrak{m}(1)}_{\mathfrak{m}(0)}} \mathcal{U}_{\mathfrak{m}(1)}$$

Axiom C asserts $\mathfrak{m}(0) \vee \mathfrak{m}(1) = \mathfrak{Top}$.

▶ **Example 31.** The *mode sketch for triangles* is drawn as



where "≃" indicates that the triangle is thin. Axiom A asserts $\mathfrak{m}(0) \leq {}^{\perp}\mathfrak{m}(1)$, $\mathfrak{m}(0) \leq {}^{\perp}\mathfrak{m}(2)$, and $\mathfrak{m}(1) \leq {}^{\perp}\mathfrak{m}(2)$. Axiom B asserts that $\eta^{\mathfrak{m}(0);\mathfrak{m}(2)}_{\mathfrak{m}(1)}$ is invertible. Thus, we have the following commutative triangle.



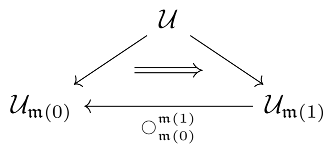
Axiom C asserts $\mathfrak{m}(0) \vee \mathfrak{m}(1) \vee \mathfrak{m}(2) = \mathfrak{Top}$. Notice that theorems for the mode sketch for functors proved without Axiom C also apply to the three edges in the above diagram. To keep this reusability, we should not assume Axiom C in practical use.

3.3 Intended models, internally

Let $\mathfrak{M}$ be a mode sketch. We can internally see what kind of an ∞-logos is a model of $\mathfrak{M}$. Here, by a model of $\mathfrak{M}$ we mean an ∞-logos that admits an interpretation of a postulated function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfying Axioms A–C.

▶ **Example 32.** Consider the case when $\mathfrak{M}$ is the mode sketch for functors (Example 30). Proposition 13 implies that $\mathcal{U} \simeq \mathcal{U}_{\mathfrak{m}(0)\vee\mathfrak{m}(1)}$ is the *Artin gluing* for the functor $\bigcirc^{\mathfrak{m}(1)}_{\mathfrak{m}(0)} : \mathcal{U}_{\mathfrak{m}(1)} \to \mathcal{U}_{\mathfrak{m}(0)}$. Therefore, our intended models of $\mathfrak{M}$ are ∞-logoses obtained by the Artin gluing.

A generalization of the Artin gluing is *oplax limits*. In the setting of Example 32, $\mathcal{U}$ fits into the following *universal oplax cone* over the diagram $\mathcal{U}_{\mathfrak{m}(0)} \xleftarrow{\bigcirc^{\mathfrak{m}(1)}_{\mathfrak{m}(0)}} \mathcal{U}_{\mathfrak{m}(1)}$.



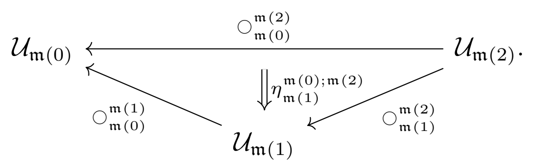
(1)

An oplax cone over a diagram is a kind of cone but every triangle formed by two projections and a functor in the diagram is only filled by a not necessarily invertible natural transformation in the direction of Diagram (1). The universal oplax cone or oplax limit is the terminal object in the $(\infty, 1)$-category of oplax cones.

▶ **Example 33.** Consider the case when $\mathfrak{M}$ is the mode sketch $\{0 \to 1 \to 2\}$ with no thin triangle. Iterating Proposition 13, we see that every type $A : \mathcal{U}$ is fractured into $A_0 : \mathcal{U}_{\mathtt{m}(0)}$, $A_1 : \mathcal{U}_{\mathtt{m}(1)}$, $A_2 : \mathcal{U}_{\mathtt{m}(2)}$, $f_{01} : A_0 \to \bigcirc^{\mathtt{m}(1)}_{\mathtt{m}(0)} A_1$, $f_{02} : A_0 \to \bigcirc^{\mathtt{m}(2)}_{\mathtt{m}(0)} A_2$, $f_{12} : A_1 \to \bigcirc^{\mathtt{m}(2)}_{\mathtt{m}(1)} A_2$, and $p_{012} : \bigcirc^{\mathtt{m}(1)}_{\mathtt{m}(0)} f_{12} \circ f_{01} = \eta^{\mathtt{m}(0);\mathtt{m}(2)}_{\mathtt{m}(1)}(A_2) \circ f_{02}$. Indeed, we have

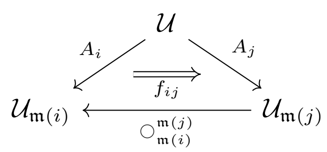
$$
\begin{aligned}
& \mathcal{U}_{\mathtt{m}(0) \vee \mathtt{m}(1) \vee \mathtt{m}(2)} \\
\simeq \quad & \{\text{Proposition 13 for } \mathtt{m}(0) \text{ and } \mathtt{m}(1) \vee \mathtt{m}(2)\} \\
& \textstyle\sum_{A_0 : \mathcal{U}_{\mathtt{m}(0)}} \sum_{A_{12} : \mathcal{U}_{\mathtt{m}(1) \vee \mathtt{m}(2)}} A_0 \to \bigcirc_{\mathtt{m}_0} A_{12} \\
\simeq \quad & \{\text{Proposition 13 for } \mathtt{m}(1) \text{ and } \mathtt{m}(2)\} \\
& \textstyle\sum_{A_0 : \mathcal{U}_{\mathtt{m}(0)}} \sum_{A_1 : \mathcal{U}_{\mathtt{m}(1)}} \sum_{A_2 : \mathcal{U}_{\mathtt{m}(2)}} \sum_{f_{12} : A_1 \to \bigcirc_{\mathtt{m}(1)} A_2} A_0 \to \bigcirc_{\mathtt{m}(0)}(A_1 \times_{\bigcirc_{\mathtt{m}(1)} A_2} A_2)
\end{aligned}
$$

where the pullback is taken for $f_{12} : A_1 \to \bigcirc_{\mathtt{m}(1)} A_2$ and $\eta_{\mathtt{m}(1)}(A_2) : A_2 \to \bigcirc_{\mathtt{m}(1)} A_2$. Since $\bigcirc_{\mathtt{m}(0)}$ preserves pullbacks, the component $A_0 \to \bigcirc_{\mathtt{m}(0)}(A_1 \times_{\bigcirc_{\mathtt{m}(1)} A_2} A_2)$ corresponds to the components f_{01}, f_{02}, and p_{012}. Then $\mathcal{U}$ is the oplax limit of the diagram

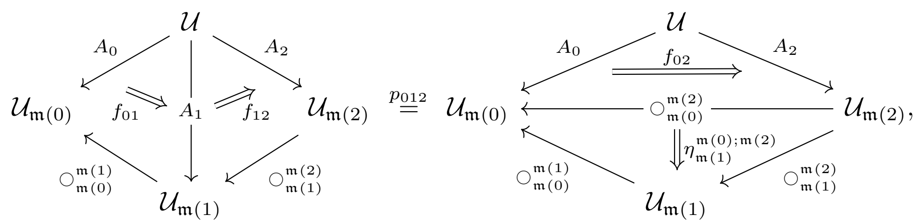


(2)

This means that we have projections $A_i : \mathcal{U} \to \mathcal{U}_{\mathtt{m}(i)}$ for all i, natural transformations



for all $i < j$, and a homotopy



and these data form a universal oplax cone over Diagram (2).

Let us make the triangle $(0 < 1 < 2)$ thin so that the natural transformation $\eta^{\mathtt{m}(0);\mathtt{m}(2)}_{\mathtt{m}(1)}$ becomes invertible. In this setting, $\mathcal{U}$ is still the oplax limit of Diagram (2), but the presentation can be simplified since the type of data (f_{02}, p_{012}) is contractible.

For a general mode sketch $\mathfrak{M}$, we apply Proposition 13 for a minimal element $\mathtt{m}(i_0)$ and the rest $\bigvee_{i : \mathfrak{M} \setminus i_0} \mathtt{m}(i)$ and repeat this for $\mathfrak{M} \setminus i_0$ to fracture types into modal types. Examples 32 and 33 suggest that $\mathcal{U}$ is the oplax limit of the diagram formed by $\mathcal{U}_{\mathtt{m}(i)}$'s explained in Remark 28. Thus, our intended models of $\mathfrak{M}$ are oplax limits of ∞-logoses indexed over the $(\infty, 2)$-category presented by $\mathfrak{M}$. The formal account of this is sketched in Section 5 and fully described in the extended version [33].

4 Mode sketches and synthetic Tait computability

We give an alternative set of axioms for mode sketches and exhibit a connection between mode sketches and *synthetic Tait computability* of Sterling [27]. The core axiom of synthetic Tait computability is to postulate a proposition. The proposition induces the open and closed modalities, and then every type is fractured into an open type equipped with a closed type family and behaves like a *logical relation*. In this story, the open and closed modalities seem more essential than the postulated proposition, so we aim to formulate synthetic Tait computability purely in terms of modalities. We work in homotopy type theory.

4.1 Alternative mode sketch axioms

The ∞-logoses obtained by the Artin gluing can be characterized as ∞-logoses equipped with a subterminal object; see [11, A4.5.6] for the 1-categorical case. We generalize this from the Artin gluing to oplax limits indexed by mode sketches, internally to type theory: the type of functions $\mathfrak{M} \to \mathsf{LAM}$ satisfying Axioms A and C is equivalent to the type of morphisms from the lattice of cosieves on $\mathfrak{M}$ to the lattice Prop (Theorem 37).

▶ **Definition 34.** *A* cosieve *on a decidable poset I is an upward-closed decidable subset of it. Let $\mathsf{coSieve}(I)$ denote the poset of cosieves on I ordered by inclusion. Note that cosieves are closed under finite meets and joins, so $\mathsf{coSieve}(I)$ is a lattice.*

▶ **Notation 35.** For $i : \mathfrak{M}$, let $(i \downarrow \mathfrak{M})$ denote the cosieve $\{j : \mathfrak{M} \mid i \leq j\}$ and $\partial(i \downarrow \mathfrak{M})$ the cosieve $(i \downarrow \mathfrak{M}) \setminus \{i\}$.

▶ **Construction 36.** Let $P : \mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}$ be a function. We define a function $\mathfrak{a}_P : \mathfrak{M} \to \mathsf{LAM}$ by $\mathfrak{a}_P(i) \equiv \mathfrak{Op}(P(i \downarrow \mathfrak{M})) \wedge \mathfrak{Cl}(P(\partial(i \downarrow \mathfrak{M})))$.

▶ **Theorem 37.** *Construction 36 is restricted to an equivalence between the following types:*
1. *the type of lattice morphisms $P : \mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}$;*
2. *the type of functions $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfying Axioms A and C.*

Before giving a proof of Theorem 37, let us relate Theorem 37 to *synthetic Tait computability* [29, 27, 30]. The core axiom of synthetic Tait computability is to postulate some propositions. One can work with those propositions directly but also with the induced open and closed modalities. Theorem 37 says that synthetic Tait computability can, in fact, be formulated completely in terms of modalities. The simplest version of synthetic Tait computability postulates a single proposition. The corresponding mode sketch is $\{0 \to 1\}$ as follows.

▶ **Example 38.** Let $\mathfrak{M}$ be the mode sketch for functors (Example 30). Then $\mathsf{coSieve}(\mathfrak{M}) = \{\{\}, \{1\}, \{0, 1\}\}$ is the free lattice generated by the single element $\{1\}$. We thus have $\{\text{lattice morphisms } \mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}\} \simeq \mathsf{Prop}$.

The rest of this subsection is devoted to the proof of Theorem 37. Because of space constraints, technical details are omitted and found in the extended version [33]. Here we focus on how to give an inverse construction to Construction 36. The key observation is that canonical joins of $\mathfrak{m}(i)$'s exist and are well-behaved under Axiom A.

▶ **Proposition 39.** *If a function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfies Axiom A, then the canonical join $\bigvee_S \mathfrak{m}$ exists for any decidable subset $S \subset \mathfrak{M}$.*

Proof. By induction on the size of S. If S is empty, then $\bigvee_{\emptyset} \mathfrak{m}$ is the bottom modality. Suppose that S is non-empty. Since $\mathfrak{M}$ is finite, there is an element i_0 minimal in S. Then $S \setminus \{i_0\}$ admits a canonical join by the induction hypothesis. Since i_0 is minimal, $\mathfrak{m}(i_0) \leq {}^{\perp}\mathfrak{m}(i)$ for any $i : S \setminus \{i_0\}$ by Axiom A, and thus $\mathfrak{m}(i_0) \leq {}^{\perp}(\bigvee_{S\setminus\{i_0\}} \mathfrak{m})$. Then we have the canonical join $\bigvee_S \mathfrak{m} \equiv \mathfrak{m}(i_0) \vee (\bigvee_{(S\setminus\{i_0\})} \mathfrak{m})$ by Proposition 13. ◀

▶ **Lemma 40.** *Let $\mathfrak{m}_0$, $\mathfrak{m}_1$, and $\mathfrak{m}_2$ be LAMs such that $\mathfrak{m}_i \leq {}^{\perp}\mathfrak{m}_j$ for any $i < j$. Then $\mathfrak{m}_0 \vee \mathfrak{m}_1 \leq {}^{\perp}\mathfrak{m}_2$.*

Proof. Let A be a $(\mathfrak{m}_0 \vee \mathfrak{m}_1)$-modal type. By Proposition 13, $\eta_{\mathfrak{m}_1}(A) : A \to \bigcirc_{\mathfrak{m}_1} A$ has $\mathfrak{m}_0$-modal fibers. Then, by assumption, $\bigcirc_{\mathfrak{m}_1} A$ and the fibers of $\eta_{\mathfrak{m}_1}(A)$ are made contractible by $\bigcirc_{\mathfrak{m}_2}$. Thus, $\bigcirc_{\mathfrak{m}_2} A$ is contractible. ◀

▶ **Proposition 41.** *If a function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfies Axiom A, then $\bigvee_{\mathfrak{M}\setminus S} \mathfrak{m} \leq {}^{\perp}(\bigvee_S \mathfrak{m})$ for any cosieve $S \subset \mathfrak{M}$.*

Proof. Since S is upward-closed, $j \not\leq i$ for any $i : \mathfrak{M} \setminus S$ and $j : S$. Thus, by Axiom A, $\mathfrak{m}(i) \leq {}^{\perp}\mathfrak{m}(j)$ for any $i : \mathfrak{M} \setminus S$ and $j : S$. The claim follows from Lemma 40 and the construction of the canonical join in Proposition 39. ◀

Sketch of proof of Theorem 37. Let $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ be a function satisfying Axioms A and C. We define a function $\varphi_{\mathfrak{m}} : \mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}$ by $\varphi_{\mathfrak{m}}(S) \equiv \bigcirc_{\bigvee_{\mathfrak{M}\setminus S} \mathfrak{m}} \mathbf{0}$ which exists by Proposition 39. By Corollary 20 and by Proposition 41, $\varphi_{\mathfrak{m}}(S)$ is the unique proposition such that $\mathfrak{Op}(\varphi_{\mathfrak{m}}(S)) = \bigvee_S \mathfrak{m}$. On the other hand, we have $\bigvee_S \mathfrak{a}_P = \mathfrak{Op}(P(S))$ by construction, from which one can derive that the constructions $P \mapsto \mathfrak{a}_P$ and $\mathfrak{m} \mapsto \varphi_{\mathfrak{m}}$ are mutually inverses. We again note that technical details are omitted and found in the extended version [33]. Certain amount of calculation is needed to prove that $\mathfrak{a}_P : \mathfrak{M} \to \mathsf{LAM}$ satisfies Axioms A and C and that $\varphi_{\mathfrak{m}}$ is a lattice morphism. ◀

4.2 Logical relations as types

We have seen in Section 4.1 that synthetic Tait computability is reformulated in terms of LAMs. The slogan of synthetic Tait computability is "logical relations as types" [29]. This is also formulated purely in terms of LAMs.

▶ **Fact 42** ([22, Theorem 3.11])**.** *For any LAM $\mathfrak{m}$, the universe of $\mathfrak{m}$-modal types $\mathcal{U}_{\mathfrak{m}} \equiv \{A : \mathcal{U} \mid \mathsf{In}_{\mathfrak{m}}(A)\}$ is $\mathfrak{m}$-modal.*

▶ **Proposition 43** (Fracture and gluing)**.** *Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs such that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$. Then we have an equivalence*

$$\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}} \simeq \textstyle\sum_{B:\mathcal{U}_{\mathfrak{n}}} B \to \mathcal{U}_{\mathfrak{m}}$$

whose right-to-left function sends a (B, A) to $\sum_{\mathtt{x}:B} A(\mathtt{x})$.

Proof. For any $B : \mathcal{U}_{\mathfrak{n}}$, we have

$$\begin{aligned} & \textstyle\sum_{A:\mathcal{U}_{\mathfrak{m}}} A \to \bigcirc_{\mathfrak{m}} B \\ \simeq \quad & \{\text{equivalence between fibrations and type families}\} \\ & \bigcirc_{\mathfrak{m}} B \to \mathcal{U}_{\mathfrak{m}} \\ \simeq \quad & \{\text{Fact 42}\} \\ & B \to \mathcal{U}_{\mathfrak{m}}. \end{aligned}$$

Then apply Proposition 13. ◀

Proposition 43 asserts that a type in $\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ is a $\mathfrak{n}$-modal type equipped with a $\mathfrak{m}$-modal unary (proof-relevant) relation on it, so *types (in $\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$) are relations*. More generally, for a mode sketch $\mathfrak{M}$ and a function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfying Axiom A, types in $\mathcal{U}_{\bigvee_{\mathfrak{M}} \mathfrak{m}}$ are fractured into a sort of generalized relations by iterated applications of Proposition 43. Intuitively, the ordering on $\mathfrak{M}$ is understood as "dependency": every type $A : \mathcal{U}_{\bigvee_{\mathfrak{M}} \mathfrak{m}}$ is fractured into a family of type families $\{A_{\mathfrak{m}(i)}\}_{i:\mathfrak{M}}$ such that $A_{\mathfrak{m}(i)}$ depends on $A_{\mathfrak{m}(j)}$ for all $j > i$. One may also regard the underlying finite poset of $\mathfrak{M}$ as a FOLDS signature [18].

▶ **Example 44.** When $\mathfrak{M}$ is the mode sketch $\{0 \leftarrow 01 \rightarrow 1\}$, we have an equivalence

$$\mathcal{U}_{\mathfrak{m}(01)\vee\mathfrak{m}(1)\vee\mathfrak{m}(0)} \simeq \sum_{A_0:\mathcal{U}_{\mathfrak{m}(0)}} \sum_{A_1:\mathcal{U}_{\mathfrak{m}(1)}} A_0 \to A_1 \to \mathcal{U}_{\mathfrak{m}(01)}.$$

▶ **Example 45.** When $\mathfrak{M}$ is the mode sketch $\{0 \rightarrow 1 \rightarrow 2\}$, we have an equivalence

$$\mathcal{U}_{\mathfrak{m}(0)\vee\mathfrak{m}(1)\vee\mathfrak{m}(2)} \simeq \sum_{A_2:\mathcal{U}_{\mathfrak{m}(2)}} \sum_{A_1:A_2\to\mathcal{U}_{\mathfrak{m}(1)}} \prod_{\mathtt{x}_2} A_1(\mathtt{x}_2) \to \mathcal{U}_{\mathfrak{m}(0)}.$$

The equivalence in Proposition 43 nicely interacts with type constructors, and we derive the *logical relation translation* (also called the parametricity translation) of dependent type theory [5, 24, 32, 12] as a *theorem* in type theory. Let $\mathfrak{m}$ and $\mathfrak{n}$ be LAMs such that $\mathfrak{m} \leq {}^{\perp}\mathfrak{n}$. Type constructors in $\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ behave in the same way as the definition of the logical relation translation of type constructors [12, Section 3] as follows.

- $\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}} : \Uparrow\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds (via Proposition 43) to the pair $(\mathcal{U}_{\mathfrak{n}}, \lambda B.B \to \mathcal{U}_{\mathfrak{m}})$.
- $\mathbf{1} : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to the pair $(\mathbf{1}, \lambda\_.\mathbf{1})$.
- Suppose that $A : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to a pair $(A_{\mathfrak{n}}, A_{\mathfrak{m}})$. Then $(A \to \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}) : \Uparrow\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to the pair

  $$(A_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{n}}, \lambda B. \prod_{\mathtt{x}:A_{\mathfrak{n}}} A_{\mathfrak{m}}(\mathtt{x}) \to B(\mathtt{x}) \to \mathcal{U}_{\mathfrak{m}}).$$

  Indeed,

  $$\begin{aligned} & A \to \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}} \\ \simeq \quad & \{\text{fracture and gluing}\} \\ & (\textstyle\sum_{\mathtt{x}:A_{\mathfrak{n}}} A_{\mathfrak{m}}(\mathtt{x})) \to (\textstyle\sum_{B:\mathcal{U}_{\mathfrak{n}}} B \to \mathcal{U}_{\mathfrak{m}}) \\ \simeq \quad & \{\textstyle\prod \text{ distributes over } \textstyle\sum\} \\ & \textstyle\sum_{B:\ \prod_{\mathtt{x}:A_{\mathfrak{n}}} A_{\mathfrak{m}}(\mathtt{x}) \to \mathcal{U}_{\mathfrak{n}}} \prod_{\mathtt{x}} \prod_{\mathtt{y}} B(\mathtt{x},\mathtt{y}) \to \mathcal{U}_{\mathfrak{m}} \\ \simeq \quad & \{\mathcal{U}_{\mathfrak{n}} \simeq (A_{\mathfrak{m}}(\mathtt{x}) \to \mathcal{U}_{\mathfrak{n}}) \text{ since } \mathfrak{m} \leq {}^{\perp}\mathfrak{n}\} \\ & \textstyle\sum_{B:A_{\mathfrak{n}}\to\mathcal{U}_{\mathfrak{n}}} \prod_{\mathtt{x}} A_{\mathfrak{m}}(\mathtt{x}) \to B(\mathtt{x}) \to \mathcal{U}_{\mathfrak{m}}. \end{aligned}$$

- Suppose that $A : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to a pair $(A_{\mathfrak{n}}, A_{\mathfrak{m}})$ and that $B : A \to \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to a pair $(B_{\mathfrak{n}}, B_{\mathfrak{m}})$. Then $\prod_{\mathtt{x}:A} B(\mathtt{x}) : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to the pair

  $$(\prod_{\mathtt{x}_{\mathfrak{n}}:A_{\mathfrak{n}}} B_{\mathfrak{n}}(\mathtt{x}_{\mathfrak{n}}), \lambda f. \prod_{\mathtt{x}_{\mathfrak{n}}} \prod_{\mathtt{x}_{\mathfrak{m}}:A_{\mathfrak{m}}(\mathtt{x}_{\mathfrak{n}})} B_{\mathfrak{m}}(\mathtt{x}_{\mathfrak{n}}, \mathtt{x}_{\mathfrak{m}}, f(\mathtt{x}_{\mathfrak{n}})))$$

  by a similar calculation to the previous clause. $\sum_{\mathtt{x}:A} B(\mathtt{x}) : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to the pair

  $$(\sum_{\mathtt{x}_{\mathfrak{n}}:A_{\mathfrak{n}}} B_{\mathfrak{n}}(\mathtt{x}_{\mathfrak{n}}), \lambda(a_{\mathfrak{n}}, b_{\mathfrak{n}}). \sum_{\mathtt{x}_{\mathfrak{m}}:A_{\mathfrak{m}}(a_{\mathfrak{n}})} B_{\mathfrak{m}}(a_{\mathfrak{n}}, \mathtt{x}_{\mathfrak{m}}, b_{\mathfrak{n}})).$$

- Suppose that $A : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to a pair $(A_{\mathfrak{n}}, A_{\mathfrak{m}})$, that $a : A$ corresponds to a pair $(a_{\mathfrak{n}}, a_{\mathfrak{m}})$, and that $a' : A$ corresponds to a pair $(a'_{\mathfrak{n}}, a'_{\mathfrak{m}})$. Then $a = a' : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ corresponds to the pair

  $$(a_{\mathfrak{n}} = a'_{\mathfrak{n}}, \lambda p. a_{\mathfrak{m}} =^{A_{\mathfrak{m}}}_{p} a'_{\mathfrak{m}}).$$

Thus, any type $A : \mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ constructed using these type constructors is fractured into a type $A_{\mathfrak{n}} : \mathcal{U}_{\mathfrak{n}}$ and a type family $A_{\mathfrak{m}} : A_{\mathfrak{n}} \to \mathcal{U}_{\mathfrak{m}}$, and $A_{\mathfrak{m}}$ is equivalent to the logical relation translation of $A_{\mathfrak{n}}$. In this sense, *types in $\mathcal{U}_{\mathfrak{m}\vee\mathfrak{n}}$ are logical relations.* The interaction of the equivalences in Examples 44 and 45 and type constructors is similarly calculated. We thus conclude that types in $\mathcal{U}_{\bigvee_{\mathfrak{M}} \mathfrak{m}}$ are generalized logical relations.

5 Semantics of mode sketches

We give an overview of the semantics of mode sketches in diagrams of ∞-logoses. Many details are omitted and found in the extended version [33].

We assume that we are given Grothendieck universes $\mathfrak{U} \in \Uparrow \mathfrak{U} \in \Uparrow^2 \mathfrak{U} \in \dots$. An ∞-*logos* (over $\mathfrak{U}$) is informally an $(\infty, 1)$-category of $\mathfrak{U}$-small sheaves over a “space”. Any ∞-logos $\mathcal{L}$ is embedded into its *universe enlargement* $\Uparrow^n \mathcal{L}$, the $(\infty, 1)$-category of $(\Uparrow^n \mathfrak{U})$-small sheaves. Homotopy type theory is interpreted in any ∞-logos $\mathcal{L}$: types in $\Uparrow^n \mathcal{U}$ are interpreted as objects in $\Uparrow^n \mathcal{L}$; terms are interpreted as morphisms. Note that, instead of choosing universes in $\mathcal{L}$, we enlarge $\mathcal{L}$ with respect to the fixed Grothendieck universes to interpret large types, so there is no ambiguity in the interpretation of universes. Coherence issues in this interpretation are solved by presenting an ∞-logos by a model category [26] and then by the local universe method [15]. The *internal language* of $\mathcal{L}$ is the type theory obtained from homotopy type theory by adjoining objects and morphisms in $\Uparrow^n \mathcal{L}$ as types and terms, respectively.

Let $\mathfrak{M}$ be a mode sketch. A *model of* $\mathfrak{M}$ is an ∞-logos $\mathcal{L}$ equipped with a function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ in its internal language satisfying Axioms A–C. We write $|\mathfrak{M}|$ for the $(\infty, 2)$-category presented by $\mathfrak{M}$ as explained in Remark 28. Let $\Uparrow \mathbf{Cat}^{(2)}$ denote the $(\infty, 2)$-category of $(\Uparrow \mathfrak{U})$-small $(\infty, 1)$-categories. Let $\mathbf{Logos}^{(2)}_{\mathrm{LexAcc}} \subset \Uparrow \mathbf{Cat}^{(2)}$ denote the locally full subcategory whose 0-cells are the ∞-logoses and whose 1-cells are the accessible functors preserving finite limits. For an $(\infty, 2)$-category $\mathcal{C}$, let $\mathcal{C}^{\mathrm{op}(1,2)}$ denote the $(\infty, 2)$-category obtained from $\mathcal{C}$ by reversing the directions of 1-cells and 2-cells.

► **Construction 46.** Let $\mathcal{L}$ be an ∞-logos. For a LAM $\mathfrak{m}$ in the internal language of $\mathcal{L}$, the *externalization* of $\mathcal{U}_{\mathfrak{m}}$ is the full subcategory of $\mathcal{L}$ spanned by the $\mathfrak{m}$-modal types. For a function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ from a mode sketch $\mathfrak{M}$ in the internal language of $\mathcal{L}$, the externalizations of $\mathcal{U}_{\mathfrak{m}(i)}$'s and the functions $\bigcirc^{\mathfrak{m}(j)}_{\mathfrak{m}(i)}$ and $\eta^{\mathfrak{m}(i);\mathfrak{m}(k)}_{\mathfrak{m}(j)}$ form a functor $|\mathfrak{M}|^{\mathrm{op}(1,2)} \to \Uparrow \mathbf{Cat}^{(2)}$ which we call the *externalization* of the diagram $\{\mathcal{U}_{\mathfrak{m}(i)}\}_{i:\mathfrak{M}}$. It turns out that this functor factors through $\mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$ by verifying that the LAMs in the internal language of $\mathcal{L}$ correspond to the lex, accessible *localizations* of $\mathcal{L}$ [33, Section 8].

► **Construction 47.** Let I be a small $(\infty, 2)$-category and $\mathcal{C} : I^{\mathrm{op}(1,2)} \to \Uparrow \mathbf{Cat}^{(2)}$ a functor. The *oplax limit of* $\mathcal{C}$ is the $(\infty, 1)$-category $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i$ described as follows. An object x in $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i$ consists of: an object $x_i \in \mathcal{C}_i$ for any object $i \in I$; a morphism $x_\alpha : x_i \to \mathcal{C}_\alpha(x_j)$ for any morphism $\alpha : i \to j$ in I; some coherence data. A morphism $u : x \to y$ in $\mathrm{opLaxLim}_{i \in I} \mathcal{C}_i$ consists of: a morphism $u_i : x_i \to y_i$ for any object $i \in I$; a homotopy u_α filling the square

$$\begin{array}{ccc} x_i & \xrightarrow{\;u_i\;} & y_i \\ {\scriptstyle x_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle y_\alpha} \\ \mathcal{C}_\alpha(x_j) & \xrightarrow[\;\mathcal{C}_\alpha(u_j)\;]{} & \mathcal{C}_\alpha(y_j) \end{array}$$

for any morphism $\alpha : i \to j$ in I; some coherence data. See [33, Section 9] for more explicit construction.

▶ **Theorem 48.** *For any mode sketch $\mathfrak{M}$, we have an equivalence between the following spaces:*

- *the space of models of $\mathfrak{M}$;*
- *the space of functors $|\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$.*

Moreover, when a model $(\mathcal{L}, \mathfrak{m})$ of $\mathfrak{M}$ corresponds to a functor $\mathcal{K} : |\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$, the following hold.

1. $\mathcal{L} \simeq \mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i$
2. *$\mathcal{K}$ is the externalization of the diagram $\{\mathcal{U}_{\mathfrak{m}(i)}\}_{i:\mathfrak{M}}$ in the internal language of $\mathcal{L}$.*

Sketch of proof. Let $\mathcal{K} : |\mathfrak{M}|^{\mathrm{op}(1,2)} \to \mathbf{Logos}^{(2)}_{\mathrm{LexAcc}}$ be a functor. We define $\mathcal{L} = \mathrm{opLaxLim}_{i \in |\mathfrak{M}|} \mathcal{K}_i$. For a cosieve S on $\mathfrak{M}$, we define $\psi_{\mathcal{K}}(S) \in \mathcal{L}$ by

$$\psi_{\mathcal{K}}(S)_i = \begin{cases} \mathbf{1} & \text{if } i \in S \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

The other components are uniquely determined by the universal properties of initial and final objects. This determines a lattice morphism $\psi_{\mathcal{K}} : \mathsf{coSieve}(\mathfrak{M}) \to \mathsf{Prop}$ in the internal language of $\mathcal{L}$. By Theorem 37, this corresponds to a function $\mathfrak{m} : \mathfrak{M} \to \mathsf{LAM}$ satisfying Axioms A and C. One can show that the induced diagram $\{\mathcal{U}_{\mathfrak{m}(i)}\}_{i:\mathfrak{M}}$ is interpreted as the given diagram $\mathcal{K}$, from which it follows that $\mathfrak{m}$ also satisfies Axiom B. Thus, $\mathcal{L}$ is part of a model of $\mathfrak{M}$. This construction is an equivalence by externalizing the argument of exhibiting $\mathcal{U} \simeq \mathcal{U}_{\bigvee_{\mathfrak{M}} \mathfrak{m}}$ as the oplax limit of the diagram $\{\mathcal{U}_{\mathfrak{m}(i)}\}_{i:\mathfrak{M}}$ (Section 3.3). ◀

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