Monadic NIP in Monotone Classes of Relational Structures

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Abstract

We prove that for any monotone class of finite relational structures, the first-order theory of the class is NIP in the sense of stability theory if, and only if, the collection of Gaifman graphs of structures in this class is nowhere dense. This generalises results previously known for graphs to relational structures and answers an open question posed by Adler and Adler (2014). The result is established by the application of Ramsey-theoretic techniques and shows that the property of being NIP is highly robust for monotone classes. We also show that the model-checking problem for first-order logic is intractable on any monotone class of structures that is not (monadically) NIP. This is a contribution towards the conjecture that the hereditary classes of structures admitting fixed-parameter tractable model-checking are precisely those that are monadically NIP.

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1 Introduction

The development of stability theory in classical model theory, originating with Shelah's classification programme fifty years ago [19, 2], has sought to distinguish *tame* first-order theories from *wild* ones. A key discovery is that combinatorial configurations serve as dividing lines in this classification.

Separately, in the development of finite model theory, there has been in interest in investigating *tame* classes of finite structures. Here tameness can refer to *algorithmic tameness*, meaning that algorithmic problems that are intractable in general may be tractable on a tame class; or it can refer to *model-theoretic tameness*, meaning that the class enjoys some desirable model-theoretic properties that are absent in the class of all finite structures. See [6] for an exposition of these notions of tameness. The tame classes that arise in this context are often based on notions taken from the study of sparse graphs [15] and usually extended to classes of relational structures beyond graphs by applying them to the *Gaifman graphs* of such structures.

In the context of algorithmic tameness of sparse classes, this line of work culminated in the major result of Grohe et al. [10] showing that the problem of model checking first-order sentences is fixed-parameter tractable (FPT) on any class of graphs that is *nowhere dense*. This generalized a sequence of earlier results showing the tractability of the model checking problem on classes of graphs satisfying other notions of sparsity. Moreover, it is also known [13] that this is the limit of tractability for *monotone* classes of graphs. That is to say that (under reasonable assumptions) any monotone class of graphs in which first-order model checking is FPT is necessarily nowhere dense. These results underline the centrality of the notion of nowhere denseness in the study of sparse graph classes.

A significant line of recent research has sought to generalize the methods and results on tame sparse classes of graphs to more general classes that are not necessarily sparse. Interestingly, this has tied together notions of tameness arising in finite model theory and those in classical model theory. Notions arising from stability theory play an increasingly important role in these considerations (see [16, 8], for example). Central to this connection is the realisation that for well-studied notions of sparseness in graphs, the first-order theory of a sparse class $\mathcal{C}$ is stable. Thus, stability-theoretic notions of tameness, applied to the theory of a class of finite structures, generalize the notions of tameness emerging from the theory of sparsity.

A key result connecting the two directions is that a monotone class of finite graphs is stable if, and only if, it is nowhere dense. This connection between stability and combinatorial sparsity was established in the context of infinite graphs by Podewski and Ziegler [17] and extended to classes of finite graphs by Adler and Adler [1]. Indeed, for monotone classes of graphs, stability is a rather robust concept as the theory of such a class is stable if, and only if, it is NIP (that is, it does not have the independence property) and these conditions on monotone classes are in turn equivalent to it being monadically stable and monadically NIP (these notions are formally defined in Section 2 below).

A question posed by Adler and Adler is whether their result can be extended from graphs to structures in any finite relational language. We settle this question in the present paper by establishing Theorem 1 below. In the following $\mathsf{Gaif}(\mathcal{C})$ (respectively $\mathsf{Inc}(\mathcal{C})$) denotes the collection of Gaifman graphs (resp. incidence graphs) of structures in the class $\mathcal{C}$. Note that the extension from graphs to relational structures requires considerable combinatorial machinery in the form of Ramsey-theoretic results, which we detail in later sections. We also relate the characterization to the tractability of the classes. In summary, our key results are stated in the following theorem. See Section 2 for all the relevant definitions.

▶ **Theorem 1.** *Let $\mathcal{C}$ be a monotone class of finite structures in a finite relational language. Then, the following are equivalent:*

1. *$\mathcal{C}$ is NIP;*
2. *$\mathcal{C}$ is monadically NIP;*
3. *$\mathcal{C}$ is stable;*
4. *$\mathcal{C}$ is monadically stable;*
5. $\mathsf{Gaif}(\mathcal{C})$ *is nowhere dense;*
6. $\mathsf{Inc}(\mathcal{C})$ *is nowhere dense; and*
7. *(assuming* $\mathsf{AW}[*] \neq \mathsf{FPT}$*) $\mathcal{C}$ admits fixed-parameter tractable model checking.*

Moreover, the equivalence of the first six notions also holds for classes containing infinite structures.

Thus, for monotone classes of relational structures, the picture is clear. Beyond monotone classes, not every NIP class is stable or monadically NIP. However, it has been conjectured [22, 9] that for any hereditary class $\mathcal{C}$ of structures, the model checking problem on $\mathcal{C}$ is fixed-parameter tractable if, and only if, $\mathcal{C}$ is NIP. This has previously been established for monotone classes of graphs (by the results of Adler and Adler, combined with those of Grohe et al.) and for hereditary classes of ordered graphs by results of Bonnet et al.[3]. Our results also extend the classes for which this conjecture is verified to all monotone classes of relational structures.

We establish some necessary definitions and notation in Sections 2 and 3. The proof of Theorem 1 occupies the next two sections. The equivalence of the first four notions for any monotone class $\mathcal{C}$ is due to Braunfeld and Laskowski [4]. The equivalence of the fifth and sixth notions follows by results in sparsity theory (see [15]) which we recall in Section 6. We, therefore, establish the equivalence of the first with the fifth and the seventh. In Section 4 we show that if $\mathsf{Gaif}(\mathcal{C})$ is not nowhere dense, then $\mathcal{C}$ admits a formula with the independence property. That nowhere density of $\mathsf{Gaif}(\mathcal{C})$ implies tractability is implicit in [10]. We establish the converse of this statement in Section 5. Finally, we give an argument that $\mathsf{Gaif}(\mathcal{C})$ being nowhere dense implies monadic stability in Section 6.

2 Preliminaries

We assume familiarity with first-order logic and the basic concepts of model theory. We have tried to make this paper as self-contained as possible, but refer the reader to [11] for background and undefined notation. Throughout this paper, $\mathcal{L}$ denotes a finite, first-order, relational language. We write $\mathrm{ar}(R)$ for the arity of each relation symbol $R \in \mathcal{L}$. Tuples of elements or variables are denoted by overlined letters and given a tuple $\bar{a}$ and $k \leq |\bar{a}|$, we write $\bar{a}(k)$ to denote the k-th element of $\bar{a}$. Often we abuse notation and treat tuples as unordered sets; whether we refer to the ordered tuple or the unordered set should be clear from the context. For $n \in \mathbb{N}$, we write $[n]$ for the set $\{1, \dots, n\}$.

We adopt the convention of allowing finitely many constant symbols (i.e. parameters) in $\mathcal{L}$-formulas. Syntactically, these are to be understood as additional free variables, while semantically these have a fixed interpretation in every $\mathcal{L}$-structure. This is purely a notational convenience and has no effect on the applicability of our results. By a further abuse of notation, we do not distinguish between a parameter p and its interpretation p^M in an $\mathcal{L}$-structure, M.

2.1 Graphs and relational structures

An $\mathcal{L}$-structure is denoted by $(M, R^M)_{R\in\mathcal{L}}$, where M is its underlying set and $R^M \subseteq M^{\mathrm{ar}(R)}$ is the interpretation of the relation symbol $R \in \mathcal{L}$ in M. We write $\mathfrak{C}(\mathcal{L})$ for the class of all $\mathcal{L}$-structures. By abusing notation, often we do not distinguish between an $\mathcal{L}$-structure and its underlying set. For an $\mathcal{L}$-structure M and a subset $A \subseteq M$ we denote by $M[A]$ the substructure of M induced by A, i.e. the structure on domain A with $R^A = R^M \cap A$ for all $R \in \mathcal{L}$. A pointed $\mathcal{L}$-structure is a pair $(M, \bar{m})$ where $\bar{m}$ is a tuple of $|\bar{m}|$ labelled points of M. By the *equality type* of a tuple $\bar{m}$ from an $\mathcal{L}$-structure M, we mean the set $\Delta_=(\bar{m})$ of atomic formulas $\eta(\bar{x})$ using only the equality symbol such that $M \models \eta(\bar{m})$.

A *homomorphism* from an $\mathcal{L}$-structure M to an $\mathcal{L}$-structure N is a map $f : M \to N$ satisfying such that for all relation symbols $R \in \mathcal{L}$ and tuples $\bar{m} \in M^{\mathrm{ar}(R)}$, if $\bar{m} \in R^M$ then $f(\bar{m}) \in R^N$. A homomorphism of pointed structures $f : (M, \bar{m}) \to (N, \bar{n})$ is understood as a homomorphism $f : M \to N$ of the underlying $\mathcal{L}$-structures such that $f(\bar{m}) = \bar{n}$.

By a graph G we mean an $\{E\}$-structure such that $E^G \subseteq G^2$ is a symmetric, irreflexive binary relation. We write $E(G)$ rather than E^G for the edge set of a graph. Given a graph G and $r \in \mathbb{N}$, we write $G^{(r)}$ for the *r-subdivision of G*, i.e. the graph obtained by replacing every edge of G by a path of length $r+1$. We denote by K_n the complete graph on n vertices and by $K_{t,t}$ the complete bipartite graph with parts of size t. We write $G = (U, V; E)$ for a bipartite graph with parts U and V and edge set $E \subseteq U \times V$, and write $\mathfrak{B}$ for the class of all bipartite graphs.

We recall two ways of constructing a graph from a given relational structure M. First, the *Gaifman graph* of M which is the graph on vertex set M, whose edges are precisely the pairs (u, v) such that u and v appear together in a relation of M. Second, the *Incidence graph* of M which is the the bipartite graph with elements of M in one part, all tuples in all relations in the other part, and edges denoting membership of an element u in a tuple $\bar{v}$. More formally:

▶ **Definition 2** (Gaifman/Incidence graph). *Given an $\mathcal{L}$-structure $(M, R^M)_{M\in\mathcal{L}}$ we define the* Gaifman graph *of M, denoted* $\mathsf{Gaif}(M)$*, to be the graph on vertex set M with edges:*

$$E := \{(x, y) : \exists R \in \mathcal{L} \exists v_1, \dots, v_{\mathrm{ar}(R)-2} \exists \sigma \in \mathcal{S}_{\mathrm{ar}(R)}(\sigma(x, y, v_1, \dots, v_{\mathrm{ar}(i)-2}) \in R^M)\},$$

where $\mathcal{S}_n$ the symmetric group on n elements. Moreover, we define the Incidence graph *of M, denoted* $\mathsf{Inc}(M)$*, to be the bipartite graph $(M, \bigsqcup_{R\in\mathcal{L}} M^R, E')$, where*

$$E' := \{(x, \bar{z}) : x \in \bar{z}\}$$

For a class of relational structures $\mathcal{C}$, all in the same language, we define the Gaifman class *of $\mathcal{C}$ to be* $\mathsf{Gaif}(\mathcal{C}) := \{\mathsf{Gaif}(M) : M \in \mathcal{C}\}$*. Likewise, we define* $\mathsf{Inc}(\mathcal{C}) := \{\mathsf{Inc}(M) : M \in \mathcal{C}\}$*.*

2.2 Sparsity and stability

Throughout this paper, $\mathcal{C}$ refers to a class of $\mathcal{L}$-structures or graphs. We write $\mathrm{Th}(\mathcal{C})$ for the common theory of the class, i.e. the set of all first-order $\mathcal{L}$-sentences that hold in all structures in $\mathcal{C}$. We say that a class $\mathcal{C}$ is:

- *hereditary*, if $\mathcal{C}$ is closed under induced substructures, i.e. if $(M, R^M)_{R\in\mathcal{L}} \in \mathcal{C}$ then $(M', R^M \cap M')_{R\in\mathcal{L}} \in \mathcal{C}$ for any $M' \subseteq M$.
- *monotone*, if $\mathcal{C}$ is closed under weak substructures, i.e. if $(M, R^M)_{R\in\mathcal{L}} \in \mathcal{C}$ then $(M', R^{M'})_{R\in\mathcal{L}} \in \mathcal{C}$ for any $M' \subseteq M$ and $R^{M'} \subseteq R^M$.

▶ **Definition 3.** *Let $\mathcal{C}$ be a class of graphs. We say that $\mathcal{C}$ is* nowhere dense *if for every $r \in \mathbb{N}$ there is some $n \in \mathbb{N}$ such that for all $G \in \mathcal{C}$ we have that $K_n^{(r)}$ is not a subgraph of G.*

Nowhere density was introduced by Nešetřil and Ossona de Mendez [14], as a structural property of classes of finite graphs that generalises numerous well-behaved classes, including graphs of bounded degree, planar graphs, graphs excluding a fixed minor and graphs of bounded expansion. Nowhere dense classes play an important role in algorithmic graph theory, as several computationally hard problems become tractable when restricted to such classes.

Let us now recall some core notions of tameness from classification theory, adapted from the context of infinite structures to that of classes of (not necessarily infinite) structures.

▶ **Definition 4** (Order/Independence Property). *Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures. We say that an $\mathcal{L}$-formula $\phi(\bar{x}, \bar{y})$ has:*

1. The Order Property in $\mathcal{C}$ *if for all $n \in \mathbb{N}$ there is some $M_n \in \mathcal{C}$ and sequences $(\bar{a}_i)_{i\in[n]}$ and $(\bar{b}_j)_{j\in[n]}$ of tuples from M_n such that:*

$$M_n \vDash \phi(\bar{a}_i, \bar{b}_j) \text{ if, and only if, } i < j.$$

2. The Independence Property in $\mathcal{C}$ *if for all bipartite graphs* $G = (U, V; E) \in \mathfrak{B}$ *there is some* $M_G \in \mathcal{C}$ *and sequences of tuples* $(\bar{a}_i)_{i \in U}$ *and* $(\bar{b}_j)_{j \in V}$ *such that:*

$$M_G \vDash \phi(\bar{a}_i, \bar{b}_j) \text{ if, and only if, } (i, j) \in E.$$

We say that $\mathcal{C}$ *is* stable *if no formula has the order property in* $\mathcal{C}$*. We say that* $\mathcal{C}$ *is* NIP *(No Independence Property) if no formula has the independence property in* $\mathcal{C}$*.*

An easy application of compactness reveals that a class $\mathcal{C}$ is stable (resp. NIP) if, and only if, all completions of $\mathrm{Th}(\mathcal{C})$ are stable (resp. NIP) in the standard model-theoretic sense (see for instance [20] for the standard model-theoretic definitions).

Given a class $\mathcal{C}$ of $\mathcal{L}$-structures and an expansion $\mathcal{L}' = \mathcal{L} \cup \{P_i : i \in I\}$ by unary predicates, we say that a class $\mathcal{C}'$ of $\mathcal{L}'$-structures is a *monadic expansion* of $\mathcal{C}$ if $\mathcal{C} = \{M' \restriction_{\mathcal{L}} : M' \in \mathcal{C}'\}$, where for an $\mathcal{L}'$-structure M' we write $M' \restriction_{\mathcal{L}}$ for the $\mathcal{L}$*-reduct* of M', i.e. the $\mathcal{L}$-structure obtained from M' by simply forgetting each relation symbol not in $\mathcal{L}$. In other words, $\mathcal{C}'$ is a monadic expansion of $\mathcal{C}$ if, for each structure $M \in \mathcal{C}$, $\mathcal{C}'$ contains at least one copy of M expanded with unary predicates which are interpreted freely, and no other structures.

▶ **Definition 5** (Monadic Stability/NIP). *Let* $\mathcal{C}$ *be a class of* $\mathcal{L}$*-structures. We say that* $\mathcal{C}$ *is* monadically stable *(resp.* monadically NIP*) if all monadic expansions* $\mathcal{C}'$ *of* $\mathcal{C}$ *are stable (resp. NIP).*

The relationship between sparsity and stability is captured by the following theorem, which was established by Podewski and Ziegler [17], in the context of infinite graphs, and much later translated to the context of graph classes by Adler and Adler [1].

▶ **Theorem 6** (Adler, Adler [1]; Podewski, Ziegler [17]). *Let* $\mathcal{C}$ *be a nowhere dense class of graphs. Then* $\mathcal{C}$ *is monadically stable. Moreover, the following are equivalent when* $\mathcal{C}$ *is monotone:*
1. $\mathcal{C}$ *is NIP;*
2. $\mathcal{C}$ *is monadically NIP;*
3. $\mathcal{C}$ *is stable;*
4. $\mathcal{C}$ *is monadically stable;*
5. $\mathcal{C}$ *is nowhere dense.*

Furthermore, Adler and Adler asked if Theorem 6 can be generalised to arbitrary relational structures with finite signature. Recently, Braunfeld and Laskowski established a collapsing phenomeon akin to Theorem 6 for relational structures.

▶ **Theorem 7** (Braunfeld, Laskowski, [4]). *Let* $\mathcal{C}$ *be a hereditary class of structures. Then* $\mathcal{C}$ *is monadically NIP (resp. monadically stable) if, and only if,* $\mathcal{C}$ *is NIP (resp. stable). Moreover, if* $\mathcal{C}$ *is monotone then* $\mathcal{C}$ *is NIP if, and only if, it is stable.*

In light of the above, Theorem 1 answers the question of Adler and Adler affirmatively by connecting the picture arising in Theorem 7 with the sparsity-theoretic properties of the Gaifman class.

2.3 Model-checking

By model-checking on a class $\mathcal{C}$ we refer to the following parametrised decision problem:

GIVEN: A FO-sentence ϕ and a structure $M \in \mathcal{C}$.
PARAMETER: $|\phi|$.
DECIDE: Whether or not M satisfies ϕ.

▶ **Definition 8.** *We say that $\mathcal{C}$ is* tractable*, or that the model-checking problem on a class $\mathcal{C}$ is fixed-parameter tractable, if there is an algorithm that decides on input (M, ϕ) whether $G \models \phi$, in time $f(|\phi|) \cdot |M|^{\mathcal{O}(1)}$ for some computable function f.*

Model-checking on the class of all graphs is complete with respect to the complexity class $\mathsf{AW}[*]$, which is conjectured to strictly contain the class FPT. We shall assume throughout that $\mathsf{AW}[*] \neq \mathsf{FPT}$.

All hereditary classes of graphs and relational structures that are known to admit tractable model-checking are NIP. Moreover, the robustness of NIP in hereditary classes hints at its potential necessity for tractability. This is the basis of the following conjecture:

▶ **Conjecture 9** ([22, 9, 3]). *Let $\mathcal{C}$ be a hereditary class of relational structures. Then $\mathcal{C}$ is tractable if, and only if, $\mathcal{C}$ is NIP.*

There is good evidence for a positive answer to this conjecture. Indeed, it is known to hold for:

- Monotone classes of graphs, where NIP coincides with nowhere density [10];
- Hereditary classes of ordered graphs, where NIP coincides with bounded twin-width [21, 3].

Although it is not explicitly stated in this form, a careful examination of the argument of [10] reveals that the following holds.

▶ **Theorem 10** (Grohe, Kreutzer, Siebertz, [10]). *Let $\mathcal{C}$ be a class of relational structures such that $\mathsf{Gaif}(\mathcal{C})$ is nowhere dense. Then $\mathcal{C}$ admits fixed-parameter tractable model-checking.*

2.4 Interpretations

Interpretations in classical model theory allow us to find structures in some language in a definable way inside a definable quotient of structures in some other language, mimicking, for instance, the way one can find the rational numbers inside the integers.

In our case, we focus on a restricted version of interpretations, which we call *simple interpretations* (possibly with parameters). Intuitively, a class of $\mathcal{L}'$-structures $\mathcal{D}$ can be interpreted in a class of $\mathcal{L}$-structures $\mathcal{C}$ if there is a uniform way of defining every structure in $\mathcal{D}$, in some (Cartesian power of some) structure from $\mathcal{C}$. More formally:

▶ **Definition 11** (Simple interpretation). *Let $\mathcal{L}, \mathcal{L}'$ be two finite relational languages. A* simple interpretation with parameters *$I : \mathfrak{C}(\mathcal{L}) \to \mathfrak{C}(\mathcal{L}')$ consists of the following data:*

- *A* domain formula *$\delta(\bar{x}, \bar{v}) \in \mathcal{L}$ and, a function d which to each $M \in \mathfrak{C}(\mathcal{L})$ associates a tuple $\bar{d}(M)$ from $M^{|\bar{v}|}$.*
- *For each k-ary relation symbol $R(y_1, \ldots, y_k) \in \mathcal{L}'$ an* interpreting formula *$\phi_R(x_1, \ldots, x_k, \bar{v}_R) \in \mathcal{L}$, where $|\bar{x}_i| = |\bar{x}|$, for each $i \in [k]$, and a function c_R which to each $M \in \mathfrak{C}(\mathcal{L})$ associates a tuple $\bar{c}_R(M)$ from $M^{|\bar{v}_R|}$.*

In order to make our discussion of interpretations easier, we adopt the following notation. Given $M \in \mathfrak{C}(\mathcal{L})$ we write $I(M)$ for the $\mathcal{L}'$ structure on the set $\delta(M) := \{a \in M : M \vDash \delta(a, \bar{d}(M))\}$ with:

$I(M) \vDash R(a_1, \ldots, a_k)$ if, and only if, $M \vDash \phi_R(a_1, \ldots, a_k, \bar{c}_R(M))$,

for each k-ary relation symbol $R \in \mathcal{L}'$ and $a_1, \ldots, a_k \in \delta(M, \bar{d}(M))$. This dually gives a map $\widehat{I} : \mathcal{L}' \to \mathcal{L}$ mapping $\mathcal{L}'$-formulas to $\mathcal{L}$-formulas with parameters, such that for any $\mathcal{L}'$-sentence ϕ we have that:

$M \models \widehat{I}(\phi)$ if, and only if, $I(M) \models \phi$.

In order to be able to reduce the problem of FO model-checking from one class of structures to another, possibly in a different language, we are interested in interpretations that can be computed in polynomial time. More precisely we define the following notion:

▶ **Definition 12** (Polynomial interpretation). *Given classes of structures $\mathcal{C} \subseteq \mathfrak{C}(\mathcal{L})$ and $\mathcal{D} \subseteq \mathfrak{C}(\mathcal{L}')$ we say that $\mathcal{D}$ is* polynomially interpreted in $\mathcal{C}$, with parameters, *if there are:*

1. *A simple interpretation with parameters, $I : \mathfrak{C}(\mathcal{L}) \to \mathfrak{C}(\mathcal{L}')$, as in Definition 11, such that the functions d and $(c_R)_{R\in\mathcal{L}'}$ are computable in polynomial time; and*
2. *a polynomial-time computable map $f : \mathcal{D} \to \mathcal{C}$ such that for all $D \in \mathcal{D}$ we have that $D = I(f(D))$.*

In this case, we write $\mathcal{D} \leq_{\mathsf{P}} \mathcal{C}$.

The next lemma justifies why polynomial interpretations are particularly useful.

▶ **Lemma 13.** *The relations $\leq_{\mathsf{P}}$ is a quasi-order on the collection of classes of structures in finite relational languages. Moreover $\leq_{\mathsf{P}}$ preserves tractability, i.e. if $\mathcal{C}$ is tractable and $\mathcal{D} \preceq_{\mathsf{P}} \mathcal{C}$, then $\mathcal{D}$ is tractable.*

Proof. The first part of the lemma is immediate, so let us only discuss the second part. We reduce the problem of model checking in $\mathcal{D}$ to model checking in $\mathcal{C}$. Given an $\mathcal{L}'$-sentence ϕ and an $\mathcal{L}'$-structure $M \in \mathfrak{C}(\mathcal{L}')$, we can compute, by assumption, in polynomial time an $\mathcal{L}$-structure $f(D) \in \mathcal{C}$ such that $M = I(f(D))$. By assumption, we can also compute $I(f(D))$ in polynomial time, since the parameters in the domain and interpreting formulas are computable from M in polynomial time. Then, we have that:

$$f(D) \models \widehat{I}(\phi) \text{ if, and only if, } I(f(D)) = M \models \phi,$$

where $\widehat{I}(\phi)$ is obtained, essentially, as in the discussion after Definition 11, which can clearly be done in polynomial time, from ϕ. Since $\mathcal{C}$ is tractable, it follows that $\mathcal{D}$ is tractable. ◀

2.5 Ramsey Theory

A core technique that is used repeatedly in our arguments is that if a finite structure is large enough, then patterns in it are inevitable. This is the main idea of Ramsey theory, the relevant tools from which we recall here. The notation we use is standard, given a set S and $k \in \mathbb{N}$ we write $[S]^{(k)}$ for the collection of all k-element subsets of S.

▶ **Theorem 14** (Ramsey's Theorem, [18]). *There is a computable function $\mathcal{R} : \mathbb{N}^3 \to \mathbb{N}$ such that for all $m, k, r \in \mathbb{N}$ and for every colouring $\chi : [\mathcal{R}(m, k, r)]^{(k)} \to [r]$ there exists some $S \subseteq [\mathcal{R}(m, k, r)]$ of size m which is monochromatic.*

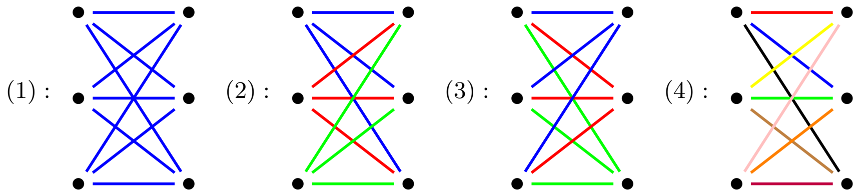
Another standard theorem from Ramsey theory that we make use of is the following well-known variant of Theorem 14:

▶ **Theorem 15** (Bipartite Ramsey Theorem). *There is a computable function $\mathcal{P} : \mathbb{N}^2 \to \mathbb{N}$ such that for all $m, r \in \mathbb{N}$ and all edge colourings of the complete bipartite graph $K_{\mathcal{P}(m,r),\mathcal{P}(m,r)}$ with r colours, there are subsets A¸B of the two parts, both of size m, which induce a monochromatic copy of $K_{m,m}$.*

We also need to make use of the following Ramsey-theoretic result, where the number of colours is allowed to be possibly infinite. Of course, in this case, we cannot expect to find monochromatic subsets. Nonetheless, we can ensure that the behaviour of the colouring falls into one of few "canonical" cases on a large enough set. The original canonical Ramsey theorem is due to Erdős and Rado [7], but for the purposes of this paper, we are only interested in the bipartite version in its effective form.

▶ **Theorem 16** (Bipartite Canonical Ramsey Theorem, [12]). *There is a computable function $\mathcal{K} : \mathbb{N} \to \mathbb{N}$ such that for every $n \in \mathbb{N}$ and every edge-colouring of the complete bipartite graph $K_{\mathcal{K}(n),\mathcal{K}(n)}$ there exist subsets X,Y of the two parts, both of size n, such that one of the following occurs for all $x, x' \in X$ and $y, y' \in Y$:*

1. $\chi(x, y) = \chi(x', y')$;
2. $\chi(x, y) = \chi(x', y')$ *if, and only if,* $x = x'$;
3. $\chi(x, y) = \chi(x', y')$ *if, and only if,* $y = y'$;
4. $\chi(x, y) = \chi(x', y')$ *if, and only if,* $x = x'$ *and* $y = y'$.



Henceforth, we shall say that an edge colouring of a complete bipartite graph is *canonical of type* 1 (resp. 2, 3, 4) if it satisfies condition 1 (resp. 2, 3, 4) from Theorem 16 for all edges. More generally, we say that such a colouring is *canonical* whenever it is canonical of any type.

3 Path formulas

Recall that a formula $\phi(\bar{x})$ is called *primitive positive* if it has the form $\exists \bar{y} \psi(\bar{x}, \bar{y})$, where ψ is a conjunction of atomic formulas. Primitive positive formulas are also known as *conjunctive queries* in the database theory literature. The following association of a canonical structure with a primitive positive formula and conversely a canonical such formula with a finite structure goes back to Chandra and Merlin [5].

▶ **Definition 17** (Canonical structures). *Given a primitive positive formula $\phi(\bar{x}) = \exists \bar{y} \psi(\bar{x}, \bar{y})$ we define a pointed $\mathcal{L}$-structure $(\mathcal{M}_\phi, \bar{x})$ whose domain is the set $\{v_1, \dots, v_r\}$ of variables of ϕ, and where each $R \in \mathcal{L}$ is interpreted as follows:*

$$\mathcal{M}_\phi \vDash R(v_1, \dots, v_n) \text{ if, and only if, } R(v_1, \dots, v_n) \text{ appears as a conjunct in } \psi(\bar{x}, \bar{y}).$$

The pointed elements $\bar{x}$ precisely correspond to the free variables of ϕ. This structure is unique, up to isomorphism, and we call it the canonical structure of ϕ.

Similarly, for every pointed $\mathcal{L}$-structure $(A, \bar{x})$ we may associate a primitive positive formula $\phi_A(\bar{x})$ so that $(\mathcal{M}_{\phi_A}, \bar{x}) = (A, \bar{x})$. We call this formula the *canonical formula* of $(A, \bar{x})$. Let $\phi(\bar{x})$ be a primitive positive formula and $(\mathcal{M}_\phi, \bar{x})$ its canonical structure. It is easy to see that for any $\mathcal{L}$-structure A and $\bar{a} \in A$ we have that $A \models \phi(\bar{a})$ if, and only if, there exists a homomorphism (of pointed structures) $h : (\mathcal{M}_\phi, \bar{x}) \to (A, \bar{a})$.

In our analysis, we argue that whenever a monotone class of relational structures has the independence property then this is witnessed by a certain kind of primitive positive formula. In the case of graphs, it is implicit in the work of Adler and Adler that the canonical structure of this primitive positive formula is a path in the standard graph-theoretic sense, i.e. a tuple $(x_1, \dots, x_n)$ of pairwise distinct elements such that $E(x_i, x_{i+1})$ for all $i \in [n-1]$.

In this section, we introduce the analogue of (graph) paths that witnesses the independence property in general relational structures. We start with the following rather technical definition.

▶ **Definition 18** (Path). *By a* path of length n*, we mean an* $\mathcal{L}$*-structure* $\mathbf{P}$ *consisting of a sequence of pairwise disjoint tuples each consisting of pairwise different elements* $\bar{e}_1, \dots, \bar{e}_n$ *such that:*

- $\mathbf{P} = \bigcup_{i\in[n]} \bar{e}_i$*;*
- $|\bar{e}_i \cap \bar{e}_{i+1}| = 1$*, for all* $i < n$*;*
- $\bar{e}_i \not\subseteq \bar{e}_{i+1}$ *and* $\bar{e}_{i+1} \not\subseteq \bar{e}_i$*, for all* $i < n$*;*
- $\bar{e}_i \cap \bar{e}_j = \emptyset$*, for all* $j \in [n] \setminus \{i-1, i, i+1\}$*;*
- $R_i(\bar{e}_i)$*, for exactly one relation symbol* $R_i \in \mathcal{L}$*;*
- $R(\bar{a}) \implies \bar{a} = \bar{e}_i$ *for some* $i \in [n]$*, for all relation symbols* $R \in \mathcal{L}$ *and all tuples* $\bar{a} \in \mathbf{P}$*.*

We write $S(\boldsymbol{P}) = \bar{e}_1 \backslash \bar{e}_2$ *and call these the starting vertices, while we write* $F(\mathbf{P}) = \bar{e}_n \backslash \bar{e}_{n-1}$ *and call these the finishing vertices. We refer to the tuples* $\bar{e}_i$ *as the steps of the path, and to the singletons in* $\bar{e}_i \cap \bar{e}_{i+1}$ *as the joints of the path.*

Given a primitive positive formula $\phi(\bar{x}, \bar{y}, \bar{z})$ (where $\bar{z}$ is possibly empty), we say that ϕ is a *path formula* if there are $x_0 \in \bar{x}$ and $y_0 \in \bar{y}$ such that $\mathcal{M}_\phi$ is a path with $x_0 \in S(\mathcal{M}_\phi)$ and $y_0 \in F(\mathcal{M}_\phi)$. Similarly, we call ϕ a *simple path formula* if $\bar{x} \subseteq S(\mathcal{M}_\phi)$ and $\bar{y} \subseteq F(\mathcal{M}_\phi)$.

Note that technically, no graph G can be a path under the above definition. Indeed, the last condition ensures that $E(G)$ cannot be symmetric as no permutation of a tuple appearing in a relation R can appear in any other relation from $\mathcal{L}$. To avoid confusion, we always refer to paths in the standard graph-theoretic sense as *graph paths*.

Intuitively, a path formula $\phi(\bar{x}, \bar{y})$ plays the role of a higher arity graph path from $\bar{x}$ to $\bar{y}$. However, under enough symmetry, it is possible that we cannot definably tell the direction of ϕ, i.e. $\bar{x}$ and $\bar{y}$ look the same within ϕ. This is formalised in the following definition, and is important in the proof of Theorem 27.

▶ **Definition 19** (Symmetric path). *A* symmetric path *is a path* $\mathbf{P}$ *of length* n*, such that* $R_i = R_{n+1-i}$ *for all* $i \in [n]$*. A* symmetric path formula $\phi(\bar{x}, \bar{y}, \bar{z})$ *is a simple path formula with* $|\bar{x}| = |\bar{y}| = m$ *such that* $\mathcal{M}_\phi$ *is a symmetric path and there is an automorphism* f *of* $\mathcal{M}_\phi$ *which maps* $\bar{x} = (x_1, \dots, x_m) \mapsto (y_{\sigma(1)}, \dots, y_{\sigma(m)})$ *and* $\bar{y} = (y_1, \dots, y_m) \mapsto (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(m)})$*, for some* $\sigma \in \mathcal{S}_m$ *which is not the identity permutation. Moreover, if* ϕ *contains parameters then these must be fixed by* f*.*

Given an $\mathcal{L}$-structure and a graph path in $\mathsf{Gaif}(M)$, we may produce a path formula that describes a "type" for this path. This idea is captured by the following definition which is relevant for the proof of Lemma 21.

▶ **Definition 20** (Path type). *Let* M *be an* $\mathcal{L}$*-structure, and* $S = (u_1, \dots, u_n)$ *a graph path in* $\mathsf{Gaif}(M)$*. For every* $i \in [n-1]$ *we may associate a relation symbol* $R_i \in \mathcal{L}$*, elements* $v_{i,1}, \dots, v_{i,\mathrm{ar}(R_i)}$*, and a permutation* $\sigma_i \in \mathcal{S}_{\mathrm{ar}(R_i)}$ *such that* $\sigma_i(u_i, u_{i+1}, \bar{v}_i) \in R_i^M$*. Then we call the formula*

$$\begin{aligned}&\phi(x,y,z_2,\dots,z_{n-1}) = \\ &\qquad \exists \bar{v}_i \dots \bar{v}_{n-1}(R_1(\sigma_1(x,z_2,\bar{v}_1)) \wedge R_2(\sigma_2(z_2,z_3,\bar{v}_2)) \wedge \dots \wedge R_{n-1}(\sigma_{n-1}(z_{n-1},y,\bar{v}_{n-1})))\end{aligned}$$

a path type *for the graph path* $u_1, \dots, u_n$*.*

It is easy to see that whenever $S = (u_1, \dots, u_n)$ is a graph path in $\mathsf{Gaif}(M)$, then there is a path type ϕ for S such that $M \models \phi(u_1, u_n, u_2, \dots, u_{n_1})$. Clearly, this is not uniquely determined by S, as for the same graph path $u_1, \dots, u_n$ in $\mathsf{Gaif}(M)$ we can possibly obtain different sequences of relations $R_i, \dots R_{i-1}$ and permutations $\sigma_1, \dots, \sigma_{i-1}$ as in Definition 20.

4 From somewhere density to IP

The main result in this section is Theorem 23, where we prove that for any monotone class $\mathcal{C}$ of relational structures whose Gaifman class is somewhere dense, there is a path formula (in the sense of Definition 18) which codes the edge relation of all bipartite graphs uniformly over $\mathcal{C}$.

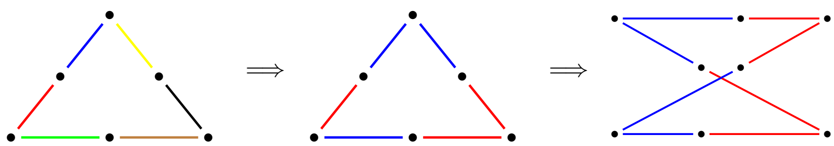
We work towards this theorem via two preparatory lemmas, which have the benefit of applying to classes that are not necessarily monotone. Intuitively, Lemma 21 tells us that if $\mathcal{C}$ is a monotone class of relational structure whose Gaifman class is somewhere dense, then we can find a path formula that codes the edge relation of all finite complete bipartite graphs in $\mathcal{C}$.

▶ **Lemma 21.** *Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures such that $\mathsf{Gaif}(\mathcal{C})$ is somewhere dense. Then there is a path formula $\phi(x, y, \bar{z}) = \exists \bar{w} \psi(x, y, \bar{z}, \bar{w})$ of length ≥ 2 whose joints are precisely the variables in $\bar{z}$, and for each $n \in \mathbb{N}$ there is some $M_n \in \mathcal{C}$ and pairwise distinct elements $(a_i)_{i\in[n]}, (b_j)_{j\in[n]}, (\bar{c}_{i,j})_{(i,j)\in[n]^2}$ from M_n such that*

$$M_n \models \phi(a_i, b_j, \bar{c}_{i,j}), \text{ for all } i, j \in [n].$$

Proof. If $\mathsf{Gaif}(\mathcal{C})$ is somewhere dense, then there exists $r \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ there is some $M_n \in \mathsf{Gaif}(\mathcal{C})$ with $K_n^{(r)} \leq \mathsf{Gaif}(M_n)$. Without loss of generality, we may assume that $r \geq 1$. Indeed, if $r = 0$ then $K_n^1 \leq K_{n^2} \leq \mathsf{Gaif}(M_{n^2})$ so we may pass to a subsequence of $(M_n)_{n\in\mathbb{N}}$ and relabel the indices appropriately.

For every $i < j$ from $[n]$ let $S_{i,j}^n$ be the graph path in $\mathsf{Gaif}(M_n)$ corresponding to the r-subdivision of the edge (i, j) from K_n, directed from i to j. Let $q \in \mathbb{N}$ be the maximum arity of a relation symbol $R \in \mathcal{L}$. Observe that there are at most $p = (|\mathcal{L}| \times q!)^{r+1}$ path types for each graph path $S_{i,j}^n$. By Ramsey's theorem we may find for each n some $\Sigma_n \subseteq [\mathcal{R}(n, 2, q)]$ of size n such that $S_{i,j}^{\mathcal{R}(n,2,q)}$ have the same path type for all $i < j$ from Σ_n. By passing to a subsequence of $(M_n)_{n\in\mathbb{N}}$ and relabelling indices, we may therefore assume that all the $S_{i,j}^n$ have the same path type. Let this be ϕ_n. Since there are only finitely many possible path types for every n, we may prune the sequence $(M_n)_{n\in\mathbb{N}}$ once again to ensure that the same path type $\phi(x, y, \bar{z})$ is obtained for all $n \in \mathbb{N}$. By definition, the joints of $\mathcal{M}_\phi$ are precisely the variables in $\bar{z}$, while $\mathcal{M}_\phi$ has length ≥ 2 since $r \geq 1$.



Work in M_{2n} and let $(a_i)_{i\in[n]}$ be the elements corresponding to $1, \dots, n$ from $K_{2n}^{(r)}$, and $(b_j)_{j\in[n]}$ be those corresponding to $n+1, \dots, 2n$. Moreover, let $\bar{c}_{i,j}$ be the tuples obtained by removing a_i and b_j from the beginning and end respectively of the graph path $S_{i,n+j}^{2n}$. It is clear that the elements $(a_i)_{i\in[n]}, (b_j)_{j\in[n]}, (\bar{c}_{i,j})_{i,j\in[n]}$ are pairwise distinct. Since the path type of $S_{i,n+j}^{2n}$ is equal to ϕ for all $i, j \in [n]$, it follows that

$$M_{2n} \models \phi(a_i, b_j, \bar{c}_{i,j}), \text{ for all } i, j \in [n].$$

We finally pass to the subsequence $(M_{2n})_{n\in\mathbb{N}}$ and relabel. ◀

Having established that we may encode the edge relation of any complete bipartite graph, we want to use monotonicity in order to encode the edge relation of arbitrary bipartite graphs, and consequently, to witness the independence property. To achieve this, we must ensure that the tuples used in the encoding are "sufficiently disjoint" so that the removal of the desired relations does in fact translate to the removal of an encoded edge. The following lemma is a step toward this.

▶ **Lemma 22.** *Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures such that $\mathsf{Gaif}(\mathcal{C})$ is somewhere dense. Then there is a path formula $\phi(\bar{x}, \bar{y}, \bar{z}) = \exists \bar{w} \psi(\bar{x}, \bar{y}, \bar{z}, \bar{w})$ of length ≥ 2 with parameters $\bar{p}$ whose joints are precisely the elements of $\bar{z}$, and for every $n \in \mathbb{N}$ there is some $M_n \in \mathcal{C}$ and tuples $(\bar{a}_i)_{i\in[n]}$, $(\bar{b}_j)_{j\in[n]}$, $(\bar{c}_{i,j})_{(i,j)\in[n]^2}$, $(\bar{d}_{i,j})_{i,j\in[n]^2}$ from M_n such that the following hold for all $i, i', j, j' \in [n]$:*

1. *$M_n \models \psi(\bar{a}_i, \bar{b}_j, \bar{c}_{i,j}, \bar{d}_{i,j})$;*
2. *$\bar{a}_i(k) \neq \bar{a}_{i'}(k)$, for $i \neq i'$ and all $k \in [|\bar{x}|]$;*
3. *$\bar{b}_j(k) \neq \bar{b}_{j'}(k)$, for $j \neq j'$ and all $k \in [|\bar{y}|]$;*
4. *$\bar{c}_{i,j}(k) \neq \bar{c}_{i',j'}(k)$ and $\bar{c}_{i,j}(k) \neq \bar{c}_{i,j}(l)$, for $(i, j) \neq (i', j')$ and all $k \neq l$ from $[|\bar{z}|]$;*
5. *$\bar{d}_{i,j}(k) \neq \bar{d}_{i',j'}(k)$, for $(i, j) \neq (i', j')$ and all $k \in [|\bar{w}|]$.*

Proof. Let $\phi(x, y, \bar{z}) = \exists \bar{w} \psi(x, y, \bar{z}, \bar{w})$ and $(M_n)_{n\in\mathbb{N}}$ be as in Lemma 21. For clarity, we write $(a^n_i)_{i\in[n]}, (b^n_j)_{j\in[n]}, (\bar{c}^n_{i,j})_{(i,j)\in[n]^2}$ to denote the elements of M_n from the same lemma. For each $n \in \mathbb{N}$, and for each pair $(i, j) \in [n]^2$, pick a tuple $\bar{d}^n_{i,j}$ of elements from M_n consisting of some arbitrarily fixed existential witnesses to $M_n \models \phi(a^n_i, b^n_j, \bar{c}^n_{i,j})$, i.e. $M_n \models \psi(a^n_i, b^n_j, \bar{c}^n_{i,j}, \bar{d}^n_{i,j})$ for all $i, j \in [n]$.

Let $m = |\bar{d}_{i,j}|$. By m applications of Theorem 16, we may assume that whether $\bar{d}^n_{i,j}(k) = \bar{d}^n_{i',j'}(k)$ depends on one of the four canonical cases from that theorem, and not on n. Indeed, for every $n \in \mathbb{N}$ and each $k \in [m]$, define colourings $\chi_{n,k}(i, j) = \bar{d}^n_{i,j}(k)$ of the edges of $K_{n,n}$. Let $\mathcal{K} : \mathbb{N} \to \mathbb{N}$ be the computable function guaranteed by Theorem 16 and write $\mathcal{K}^m$ for the composition of $\mathcal{K}$ with itself m times. It follows that the complete bipartite graph with parts of size $\mathcal{K}^m(n)$ contains subsets A_n, B_n of the two parts of size n, which induce a copy of $K_{n,n}$ on which $\chi_{\mathcal{K}^m(n),k}$ is canonical for all $k \in [m]$. We may thus restrict the argument on the subsequence $(M_{\mathcal{K}^m(n)})_{n\in\mathbb{N}}$ and the elements $a^{\mathcal{K}^m(n)}_i, b^{\mathcal{K}^m(n)}_j, \bar{c}^{\mathcal{K}^m(n)}_{i,j}, \bar{d}^{\mathcal{K}^m(n)}_{i,j}$ for $i \in A_n$ and $j \in B_n$ and relabel appropriately. For every $n \in \mathbb{N}$, after the relabelling, we have thus obtained a tuple $\bar{t}_n \in [4]^m$ such that $\chi_{n,k}$ is canonical of type $\bar{t}_n(k)$. Since there are only finitely many such $\bar{t}_n$, by the pigeonhole principle we may consider a subsequence of $(M_n)_{n\in\mathbb{N}}$ for which $\bar{t}_n$ is constant and equal to some $\bar{t} \in [4]^m$, and relabel once more.

We now proceed to sequentially remove elements from the tuples $\bar{d}^n_{i,j}$, and to either name them by a parameter, or to append them to one of a^n_i or b^n_j. Since $\bar{t}$ is constant for all n, exactly the same process is carried out to all tuples $\bar{d}_{i,j}$, and so we may concurrently move the corresponding variables from ϕ. So, if we fall into Case 1 for some k, i.e. if $\bar{t}(k) = 1$, then $\bar{d}_{i,j}(k)$ is the same for all i, j, and so we may name it by a parameter and remove it from every $\bar{h}_{i,j}$. If we fall into Case 2, then $\bar{d}_{i,j}(k) = \bar{d}_{i',j'}(k)$ if, and only if, $i = i'$. Then, for every $i \in [n]$ we may remove the common element $\bar{d}_{i,j}(k)$ from each $\bar{d}_{i,j}$ and append it to a_i, turning it into a tuple $\bar{a}_i$. We then adjust ϕ accordingly by shifting the corresponding variable v_k from $\bar{v}$ to x, which also becomes a tuple $\bar{x}$. Case 3 is symmetric to Case 2, only now we append $\bar{d}_{i,j}(k)$ to $\bar{b}_j$ and shift the variable v_k to $\bar{y}$. We may therefore assume that we fall into Case 4 for all the remaining $k \in [m]$.

We argue that the resulting formula and tuples satisfy the requirements of the lemma. Clearly, $M_n \models \phi(\bar{a}_i, \bar{b}_j, \bar{c}_{i,j}, \bar{d}_{i,j})$ for all $n \in \mathbb{N}$ and $i, j \in [n]$. Condition 2 is also satisfied, since the original singletons $(a_i)_{i\in[n]}$ were pairwise disjoint, while for every $i \neq i'$ and $k \in [m]$ the elements $\bar{d}_{i,j}(k)$ and $\bar{d}_{i',j}(k)$, appended to a_i and $a_{i'}$ respectively, come from an instance of Case 2, and are therefore pairwise distinct. Likewise, condition 3 is satisfied. Since we have not interfered with the tuples $\bar{c}_{i,j}$ in the above process and these contain pairwise distinct elements by Lemma 21, Condition 4 is also satisfied. Finally, Condition 5 is trivially satisfied since the elements remaining in $\bar{d}_{i,j}$ fall into Case 4. ◀

▶ **Theorem 23.** *Let $\mathcal{C}$ be a monotone class of $\mathcal{L}$-structures such that* $\mathsf{Gaif}(\mathcal{C})$ *is somewhere dense. Then there is a path formula $\phi(\bar{x}, \bar{y}) = \exists \bar{w} \psi(\bar{x}, \bar{y}, \bar{w})$ with parameters $\bar{p}$ and for each bipartite graph $G = (U, V; E) \in \mathfrak{B}$ there is some $M_G \in \mathcal{C}$ and sequences of tuples $(\bar{a}_u)_{u \in U}$ $(\bar{b}_v)_{v \in V}$, $(\bar{h}_{u,v})_{(u,v) \in E}$ from M_G such that:*

1. *$M_G \models \phi(\bar{a}_u, \bar{b}_v)$ if, and only if, $(u, v) \in E$ (so, in particular $\mathcal{C}$ is not NIP);*
2. *If $(u, v) \in E$ then $M_G \models \psi(\bar{a}_u, \bar{b}_v, \bar{h}_{u,v})$;*
3. *The equality type of $\bar{p}_{u,v} = \bar{a}_u {}^\frown \bar{b}_v {}^\frown \bar{h}_{u,v}$ is constant for all $(u, v) \in E(G)$;*
4. *Any two tuples in $\{\bar{a}_u, \bar{b}_v, \bar{h}_{u,v} : u \in U, v \in V\}$ are disjoint and do not intersect the parameters $\bar{p}$.*

Proof. Let $\phi(x, y, \bar{z}) = \exists \bar{w} \psi(x, y, \bar{z}, \bar{w})$, with parameters $\bar{p}$, and $(M_n)_{n \in \mathbb{N}}$ be as in Lemma 22. For clarity, we again write $(a_i^n)_{i \in [n]}, (b_j^n)_{j \in [n]}, (\bar{c}_{i,j}^n)_{(i,j) \in [n]^2}$ to denote the elements from that lemma coming from M_n. Consider the tuples $\bar{p}_{i,j}^n = \bar{a}_i^n {}^\frown \bar{b}_j^n {}^\frown \bar{c}_{i,j}^n {}^\frown \bar{d}_{i,j}^n$, and let $q = |\bar{p}_{i,j}^n|$. Observe that for every $n \in \mathbb{N}$, at most $q \cdot |\bar{p}|$ many tuples $\bar{p}_{i,j}^n$ intersect the parameters $\bar{p}$ because of the conditions in Lemma 22. By working with suitably large n and avoiding these tuples, we may relabel so that no $\bar{p}_{i,j}^n$ intersects $\bar{p}$.

For $i, j, k, l \in [n]$, we say that the tuples $\bar{p}_{i,j}^n$ and $\bar{p}_{k,l}^n$ *intersect trivially* whenever

$$\bar{p}_{i,j} \cap \bar{p}_{k,l} = \begin{cases} \bar{p}_{i,j}, & \text{if } i = k \wedge j = l \\ \bar{a}_i, & \text{if } i = k \wedge j \neq l \\ \bar{b}_j, & \text{if } i \neq k \wedge j = l \\ \emptyset, & \text{otherwise.} \end{cases}$$

Letting $f(n) = q \cdot (n-1)^2 + n$, we claim that for all $n \in \mathbb{N}$ and all $m \geq f(n)$ we may find a set $A_n \subseteq [f(n)]$ of size n so that $\bar{p}_{i,j}^m$ and $\bar{p}_{k,l}^m$ intersect trivially for all $i, j, k, l \in A_n$.

We show this by induction. Indeed, for $n = 1$ this is trivially true as $A_1 = [1]$ works for all $m \geq 1$. Suppose that the claim holds for $n - 1$ and fix $m \geq f(n)$. Since $f(n) \geq f(n-1)$, by the induction hypothesis there is some $A_{n-1} \subseteq [f(n-1)] \subseteq [f(n)]$ of size $n - 1$ so that $\bar{p}_{i,j}^m$ and $\bar{p}_{k,l}^m$ intersect trivially for all $i, j, k, l \in A_{n-1}$. Notice, that because of Lemma 22, for every fixed $\bar{p}_{i,j}$, there are at most q tuples $\bar{p}_{k,l}$ that do not intersect trivially with it. Hence, there are at most $q \cdot (n-1)^2$ elements $l \in [f(n)]$ such that $\bar{p}_{i,j}^m$ and $\bar{p}_{k,l}^m$ do not intersect trivially for all $i, j, k \in A_{n-1}$. Since $[f(n)]$ contains an additional n elements, we are guaranteed to find some $l \in [f(n)]$, which is not one of the $n - 1$ elements of A_{n-1}, such that $\bar{p}_{i,j}^m$ and $\bar{p}_{k,l}^m$ intersect trivially for all $i, j, k \in A_{n-1}$. We may therefore let $A_n = A_{n-1} \cup \{l\}$.

Hence, we may consider the subsequence $(M_{f(n)})_{n \in \mathbb{N}}$ and relabel the tuples appropriately, so that all tuples $\bar{p}_{i,j}^n, \bar{p}_{k,l}^n$ intersect trivially for all $n \in \mathbb{N}$ and $i, j, k, l \in [n]$. Furthermore, by an application of Theorem 15, we may assume that the tuples $\bar{p}_{i,j}^n$ have the same equality type for all $i, j \in [n]$ and all $n \in \mathbb{N}$. More precisely, for every pair $(i, j) \in [n]^2$ let $\Delta_n(i, j) := \Delta_=(\bar{p}_{i,j}^n)$. Letting $q = |\bar{p}_{i,j}|$, it is easy to see that there are at most $p = 2^{q^2}$ sets $\Delta_n(i, j)$. It follows by Theorem 15, that there are subsets A, B of $[\mathcal{P}(n, 2, p)]$ of size n such that $\Delta_{\mathcal{P}(n,2,p)}(i, j)$ is constant for all $i \in A$, $j \in B$. Hence, we may relabel appropriately so that $\Delta_n(i, j)$ is constant for all $i, j \in [n]$. Since there are only finitely many such sets, the pigeonhole principle implies that we may prune the sequence $(M_n)_{n \in \mathbb{N}}$ so that $\Delta_n(i, j)$ is uniformly constant for all $n \in \mathbb{N}$.

It follows that no tuple $\bar{a}_i^n$ can intersect a tuple $\bar{b}_j^n$. Indeed, since the equality types are constant, and in particular $\Delta_n(i, j) = \Delta_n(i, j')$, if $\bar{a}_i^n$ and $\bar{b}_j^n$ had an element in common then $\bar{b}_j^n(k) = \bar{b}_{j'}^n(k)$ for some k and all $j' \neq j$, contradicting the assumptions of Lemma 22. Likewise, no tuple $\bar{h}_{i,j}^n = \bar{c}_{i,j}^n {}^\frown \bar{d}_{i,j}^n$ can intersect the tuples $\bar{a}_i$ or $\bar{b}_j$. Since the tuples $\bar{p}_{i,j}^n$ intersect trivially, this implies that any two tuples $\{\bar{a}_i^n, \bar{b}_j^n, \bar{h}_{i,j}^n : i, j \in A_n\}$ are pairwise disjoint, and furthermore do not intersect the parameters $\bar{p}$.

For every $n \in \mathbb{N}$, consider the weak substructure $M'_n \leq M_n$ consisting of the elements in $\bar{p}^n_{i,j}$ and the parameters $\bar{p}$, and containing solely the relations necessary to witness $M_n \models \psi(\bar{a}^n_i, \bar{b}^n_j, \bar{h}^n_{i,j})$. By monotonicity, $M'_n \in \mathcal{C}$. Notice that every tuple appearing in a relation of M'_n contains at least one element of $\bar{c}^n_{i,j}$ for some $i, j \in [n]$. Indeed, the elements of $\bar{c}_{i,j}$ correspond precisely to the joints of the paths $\phi(\bar{a}_i, \bar{b}_j)$, and since $\mathcal{M}_\phi$ has length ≥ 2 every path has at least one joint.

Finally, given $G = (U, V; E)$ with $U = V = [n]$, let $M_G \in \mathcal{C}$ be the induced substructure of M'_n obtained by removing $\bar{h}^n_{i,j}$ for all $(i,j) \notin E$. Since the tuples in $\{\bar{a}^n_i, \bar{b}^n_j, \bar{h}^n_{i,j} : i, j \in [n]\}$ are pairwise disjoint, it follows that $\bar{h}^n_{i,j} \in M_G$ for $(i,j) \in E(G)$. Hence, letting $\phi'(\bar{x}, \bar{y}) = \exists \bar{z} \phi(\bar{x}, \bar{y}, \bar{z})$, we see that $M_G \models \phi'(\bar{a}^n_i, \bar{b}^n_j)$ for all $(i,j) \in E(G)$. Moreover, $M_G \models \neg\phi(\bar{a}_i, \bar{b}_j)$ for $(i,j) \notin E(G)$. Indeed, since the elements of $\bar{c}^n_{i,j}$ are not in M_G for $(i,j) \notin E(G)$, the above observation implies that $M_G \models \neg\phi(\bar{a}_i, \bar{b}_j)$. ◀

Note that all of the above can be proved by working with an appropriate infinite model of $\mathrm{Th}(\mathcal{C})$ obtained by compactness, and applying the infinite versions of the different Ramsey theorems. We have chosen to give a finitistic proof, which is admittedly more involved, so that everything is carried out effectively. Therefore, if we assume that the *VC-dimension* of formulas in the class is computable, we may compute given r the maximum size of an r-subdivided clique occurring in the Gaifman graph of a structure in $\mathcal{C}$.

▶ **Definition 24.** *We say that a class $\mathcal{C}$ of structures is effectively NIP if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all formulas $\phi(\bar{x}, \bar{y})$ and all structures $M \in \mathcal{C}$ there is no $n > f(|\phi|)$ and $(\bar{a}_i)_{i \in [n]}, (\bar{b}_J)_{J \subseteq [n]}$ with*

$$M \models \phi(\bar{a}_i, \bar{b}_J) \iff i \in J.$$

Recall that a class of graphs $\mathcal{C}$ is called *effectively nowhere dense* whenever there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$ and for all $G \in \mathcal{C}$ we have that $K^{(r)}_{f(r)}$ is not a subgraph of G.

▶ **Corollary 25.** *Let $\mathcal{C}$ be a monotone and (monadically) NIP class of $\mathcal{L}$-structures in a finite relational language. Then $\mathsf{Gaif}(\mathcal{C})$ is nowhere dense. Moreover, if $\mathcal{C}$ is effectively NIP then $\mathsf{Gaif}(\mathcal{C})$ is effectively nowhere dense.*

5 Intractability

In this section, we prove that any monotone class of relational structures whose Gaifman class is somewhere dense polynomially interprets the class of all bipartite graphs, and is therefore intractable. Towards this, we first strengthen Theorem 23 to obtain a simple path formula ϕ as well as a computable function $\Phi : \mathfrak{B} \to \mathcal{C}$ such that ϕ codes the edge relation of G in $\Phi(G)$.

▶ **Lemma 26.** *Let $\mathcal{C}$ be a monotone class of $\mathcal{L}$-structures such that $\mathsf{Gaif}(\mathcal{C})$ is somewhere dense. Then there is a simple path formula $\phi(\bar{x}, \bar{y})$ with parameters $\bar{p}$ and a polynomial time computable function $\Phi : \mathfrak{B} \to \mathcal{C}$, such that for each bipartite graph $G = (U, V; E) \in \mathfrak{B}$ there are tuples $(\bar{a}_u)_{u \in U}$ $(\bar{b}_v)_{v \in V}$, $(\bar{h}_{u,v})_{(u,v) \in E}$ from $\Phi(G)$ satisfying:*

$$\Phi(G) \models \phi(\bar{a}_u, \bar{b}_v) \text{ if, and only if, } (u, v) \in E.$$

Given ϕ, the interpretation of the parameters $\bar{p}$ in $\Phi(G)$ can be computed in constant time from $G \in \mathfrak{B}$.

Proof. Let ϕ and $(M_G)_{G\in\mathfrak{B}}$ be as in Theorem 23. Consider the path $\mathcal{M}_\phi$. Observe that either there is a step $\bar{e}_i$ such that both $\bar{e}_i \cap \bar{x} = \bar{x}' \neq \emptyset$ and $\bar{e}_i \cap \bar{y} = \bar{y}' \neq \emptyset$, or there are $i < j$ and steps $\bar{e}_i$, $\bar{e}_j$ such that $\bar{e}_i \cap \bar{y} = \emptyset, \bar{e}_j \cap \bar{x} = \emptyset$ and $\bar{e}_i \cap \bar{x} = \bar{x}' \neq \emptyset, \bar{e}_j \cap \bar{y} = \bar{y}' \neq \emptyset$ and for all $k \in \{i+1, \ldots, j-1\}$ we have that $\bar{e}_k \cap \bar{x} = \bar{e}_k \cap \bar{y} = \emptyset$. Consider the induced substructure $\mathcal{M}'$ of $\mathcal{M}_\phi$ consisting solely of the step $\bar{e}_i$ in the first case or the steps $\bar{e}_i, \ldots, \bar{e}_j$ in the second, and let $\phi'(\bar{x}', \bar{y}') = \exists \bar{w}' \psi'(\bar{x}', \bar{y}', \bar{w}')$ be the canonical formula of $(\mathcal{M}', \bar{x}', \bar{y}')$. Clearly, ϕ' is a simple path formula, and it follows by construction that for each $G \in \mathfrak{B}$ we may pick minimal subtuples $\bar{a}'_u \subseteq \bar{a}_u, \bar{b}'_v \subseteq \bar{b}_v, \bar{c}'_{u,v} \subseteq \bar{c}_{u,v}\bar{h}'_{u,v} \subseteq \bar{h}_{u,v} \in M_G$ for all $u \in U, v \in V$ such that :

- $M_G \models \phi'(\bar{a}'_u, \bar{b}'_v)$ if, and only if $(u,v) \in E$, and
- $(u,v) \in E$ implies $M_G \models \phi'(\bar{a}'_u, \bar{b}'_v, \bar{c}'_{u,v}, \bar{h}'_{u,v})$.

Clearly, these new subtuples are mutually disjoint and do not intersect any of the parameters $\bar{p}' \subseteq \bar{p}$ that appear in ϕ'. We finally let M'_G be the induced substructure of M_G consisting solely of these subtuples. Since the equality type of all tuples $\bar{p}'_{u,v} = \bar{a}_u^\frown \bar{b}_v^\frown \bar{h}_{u,v}$ is uniformly constant by Theorem 23, it follows that M_G may be computed from $G = (U, V; E)$ by adding disjoint tuples $(\bar{a}^G_u)_{u\in U}, (\bar{b}^G_v)_{v\in V}, (\bar{h}^G_{u,v})_{(u,v)\in E(G)}, \bar{p}^G$ of appropriate equality types to represent vertices and existential witnesses, and the relations specified by ϕ' to represent the edges. Clearly, the tuple $\bar{p}^G$ which interprets the parameters of ϕ' is obtained in constant time from G. ◀

With this, we proceed to show intractability for monotone classes with somewhere Gaifman class. Our proof is essentially based on the proof of [13, Theorem 6.1], which covers the case of graphs. There, monotonicity and somewhere density imply that for some $r \in \mathbb{N}$ we may find an r-subdivided copy of any finite graph G in our class. The aim is then to definably distinguish the native points of G from the subdivision points. Assuming this, G can be simply interpreted, defining an edge between two native points if there is a path of length r between them. The idea is to distinguish points by their degrees; however, while all subdivision points have degree two, other points in G may as well have degree two. To address this, we first pre-process G to obtain a graph G' by adding two pendant vertices to each non-isolated vertex. Then, G may be definably recovered from G', and moreover, given an r-subdivision of G', we can definably distinguish the subdivision points and the remaining points by their degrees. Our construction is essentially the same, although the degree of a subdivision point is bounded by the length of paths in the subdivision, rather than by two. Moreover, we ought to ensure that the formula coding paths is not symmetric, so as to avoid accidentally creating two disjoint copies of the graph we wish to interpret.

▶ **Theorem 27.** *Let $\mathcal{C}$ be a monotone class of $\mathcal{L}$-structures such that* $\mathsf{Gaif}(\mathcal{C})$ *is somewhere dense, and assume that* $\mathsf{AW}[*] \neq \mathsf{FPT}$*. Then* FO *model-checking on $\mathcal{C}$ is not fixed-parameter tractable.*

Proof. Let $\mathcal{C}$ satisfy the above, and assume that $\mathsf{AW}[*] \neq \mathsf{FPT}$. We argue that we may polynomially interpret the class of all bipartite graphs in $\mathcal{C}$.

Let $\phi(\bar{x}, \bar{y})$ be the simple path formula from Lemma 26. Without loss of generality, we may assume that ϕ is not symmetric (in the sense of Definition 19). Indeed, if ϕ is symmetric let $\sigma \in \mathcal{S}_n$ be the non-identity permutation from Definition 19, and consider the formula $\phi'(\bar{x}, \bar{y}) = \phi(\bar{x}, \sigma^{-1}(\bar{y}))$, where σ^{-1} is applied to the indices of $\bar{y}$. Clearly, ϕ' is no longer symmetric, while the tuples $(\bar{a}_u)_{u\in U}$ $(\sigma(\bar{b}_v))_{v\in V}$, $(\bar{h}_{u,v})_{(u,v)\in E}$ still satisfy the conditions in Lemma 26.

Now, let k the length of the path $\mathcal{M}_\phi$ and define the auxiliary map:

$$f : \mathfrak{B} \to \mathfrak{B}$$
$$G = (U, V; E) \mapsto (U', V'; E'),$$

where $U' := U \sqcup \{\dot{u}_{v,1}, \ldots, \dot{u}_{v,k+1} : v \in V\}$, $V' := V \sqcup \{\dot{v}_{u,1}, \ldots, \dot{v}_{u,k+1} : u \in U\}$, and $E' := E \sqcup \{(u, \dot{v}_{u,i}) : u \in U, i \in [k+1]\} \sqcup \{(v, \dot{u}_{v,i}) : v \in V, i \in [k+1]\}$.

This is clearly computable in polynomial time. Given $G = (U, V; E) \in \mathfrak{B}$, consider $\Phi \circ f(G) \in \mathcal{C}$ given from Theorem 23, and let:

$$\theta_U(\bar{x}) := \exists^{>k}\bar{y}\phi(\bar{x}, \bar{y}) \wedge \bar{x} \neq \bar{p} \text{ and } \theta_V(\bar{y}) := \exists^{>k}\bar{x}\phi(\bar{x}, \bar{y}) \wedge \bar{y} \neq \bar{p},$$

where $\bar{p}$ are the parameters of ϕ. Without loss of generality, we may assume that $|\bar{x}| = |\bar{y}|$, for if $m = |\bar{y}| < |\bar{x}| = n$, then we may take $\theta_V(\bar{y}, y_{m+1}, \ldots, y_n)$ to be $\theta_V(\bar{y}) \wedge \bigwedge_{i=m}^{n-1}(y_i = y_{i+1})$, and similarly if $|\bar{x}| < |\bar{y}|$. So, let $\theta(\bar{x}) = \theta_V(\bar{x}) \vee \theta_U(\bar{x})$.

Observe that G is an induced subgraph of $f(G)$, so we may view $\Phi(G)$ as an induced substructure of $\Phi \circ f(G)$. Letting $\bar{p}_{u,v} = \bar{a}_u\frown\bar{b}_v\frown\bar{h}_{u,v}$, it holds that $\bar{p}_{u,v} \cap \bar{p} = \emptyset$ and

$$\bar{p}_{u,v} \cap \bar{p}_{u',v'} = \begin{cases} \bar{a}_u & \text{if } u = u'; \\ \bar{b}_v & \text{if } v = v'; \\ \emptyset & \text{otherwise.} \end{cases}$$

whenever $(u, v) \neq (u', v')$. Hence, the only non-parameter elements that appear more than k times within a path are those in the tuples $\bar{a}_u$ and $\bar{b}_v$ for $u \in U$ and $v \in V$, i.e. those tuples corresponding to the elements of G. Since ϕ is not symmetric, it follows that $\theta(\Phi \circ f(G)) = \{\bar{a}_u, \bar{b}_v : u \in U, v \in V\}$, and so the pair $I = (\theta(\bar{x}), \phi(\bar{x}, \bar{y}))$ is an interpretation with computable parameters such that $I(\Phi \circ f(G)) = G$ for all $G \in \mathfrak{B}$. It follows that $\mathfrak{B} \leq_P \mathcal{C}$, and therefore $\mathcal{C}$ is not tractable. ◀

6 From nowhere density to monadic stability

Here we establish that a class of structures with nowhere dense Gaifman graphs is monadically stable. This argument relies on the extension of Theorem 6 to coloured digraphs, and the equivalence of nowhere density for the classes of Gaifman graphs and incidence graphs.

▶ **Lemma 28** ([15, Proposition 5.7]). *Let $\mathcal{C}$ be a class of structures in a finite relational language. Then* $\mathsf{Gaif}(\mathcal{C})$ *is nowhere dense if, and only if,* $\mathsf{Inc}(\mathcal{C})$ *is nowhere dense.*

We enrich the definition of incidence graphs by colouring the edges to distinguish between the various relations in the original language, and to indicate that a point in the domain corresponds to the i^{th} point of an incident tuple. We also direct the edges from points in the original domain to incident tuples. This will allow us to easily recover the original structure via a simple interpretation.

▶ **Definition 29** (Coloured incidence graphs). *Let M be an $\mathcal{L}$-structure in a finite relational language. Write n for the maximum arity of a relation symbol in $\mathcal{L}$, and let $E_\mathcal{L}$ be the language containing binary relation symbols $\{R_i : i \in [n], R \in \mathcal{L}\}$. We define the* coloured incidence graph *of M to be the $E_\mathcal{L}$-structure $\mathsf{Inc}^\mathsf{c}(M)$ on domain $M \sqcup \bigsqcup_{R \in \mathcal{L}} R^M$ such that for all $R \in \mathcal{L}$ and $i \in [n]$*

$$(u, \bar{v}) \in R_i^{\mathsf{Inc}^\mathsf{c}(M)} \text{ if, and only if, } \bar{v} \in R^M \text{ and } \bar{v}(i) = u.$$

For a class $\mathcal{C}$ of $\mathcal{L}$-structures we write $\mathsf{Inc}(\mathcal{C})$ *for the class* $\{\mathsf{Inc}(M) : M \in \mathcal{C}\}$.

▶ **Theorem 30.** *Let $\mathcal{C}$ be a class of structures in a finite relational language. If $\mathsf{Gaif}(\mathcal{C})$ is nowhere dense, then $\mathcal{C}$ is monadically stable.*

Proof. By Lemma 28, $\mathsf{Gaif}(\mathcal{C})$ being nowhere dense implies that $\mathsf{Inc}(\mathcal{C})$ is nowhere dense. In turn, this implies that $\mathsf{Inc}^{\mathsf{c}}(\mathcal{C})$ is monadically stable by the generalisation of Theorem 6 to coloured directed graphs, mentioned in both [1] and [17]. It is easily observed that $\mathcal{C}$ is simply interpreted in $\mathsf{Inc}^{\mathsf{c}}(\mathcal{C})$ by the formulas $\delta(x) = \neg\exists y \bigvee_{R\in\mathcal{L}} \bigvee_{i\in[n]} R_i(y,x)$ and $\phi_R(x_1,\dots,x_{\mathrm{ar}(R)}) = \exists z \bigwedge_{i\in[\mathrm{ar}(R)]} R_i(x_i,z)$ for $R \in \mathcal{L}$. Since interpretations preserve monadic stability, $\mathcal{C}$ is monadically stable as well. ◀

An alternative proof of the theorem above is indicated in [1], which does not pass through incidence graphs but instead explicitly codes the relations into a graph via gadgets.

The hypothesis in the following corollary is weaker than demanding that $\mathsf{Gaif}(\mathcal{C})$ be nowhere dense, as witnessed by the class of finite cliques with countably many edge colours and no two edges receiving the same colour.

▶ **Corollary 31.** *Let $\mathcal{C}$ be a class of structures in an infinite relational language. If for every reduct to a finite language, $\mathsf{Gaif}(\mathcal{C}^-)$ is nowhere dense, then $\mathcal{C}$ is monadically stable.*

Proof. The failure of monadic stability is witnessed by a single formula in some unary expansion, which only uses finitely many relations. ◀

▶ **Corollary 32.** *Let M be a relational structure such that for every reduct M^- to a finite language, for every $r \in \mathbb{N}$ there is some $n \in \mathbb{N}$ with $K_n^{(r)} \not\leq \mathsf{Gaif}(M^-)$. Then M is monadically stable.*

Proof. Let $\mathcal{C}$ be the class of finite substructures of M. Given the assumption, the previous lemma implies $\mathcal{C}$ is monadically stable. By [4], this implies M is monadically stable. ◀

7 Conclusion

Our paper settles the question of Adler and Adler, showing that tameness for a monotone class of relational structures can be completely recovered from the structural sparsity of its Gaifman class. We believe that many results from the theory of sparse graphs will generalise to relational structures by working with the Gaifman class, and we plan to exhibit such generalisations in future work.

Although this has not been addressed thus far, monotonicity as defined for classes of relational structures does not fully correspond to monotonicity in the standard graph-theoretic sense. Indeed, in the graph-theoretic sense, a monotone class of graphs is one closed under removal of *undirected* edges, that is, simultaneous removal of pairs of relations $E(u,v), E(v,u)$. However, a monotone class of $\{E\}$-structures is one where we can remove any E relation (so possibly we can turn an undirected edge into a directed one). In future work, we aim to address this subtle difference by introducing *symmetrically monotone* classes, so that our results can extend to broader classes of relational structures, such as classes of undirected hypergraphs closed under removal of hyperedges.

Finally, our paper makes a significant contribution towards Conjecture 9, settling it for the case of monotone classes of structures. While the machinery used in this paper will certainly assist in tackling the full conjecture, we believe that new techniques are required for this task. Here, it is important to understand the role of linear orders in the collapse of monadic NIP and bounded twin-width for hereditary classes of ordered graphs, and to identify which model-theoretic conditions generalise this phenomenon to arbitrary hereditary graph classes.

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