Low Sample Complexity Participatory Budgeting

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Abstract

We study low sample complexity mechanisms in participatory budgeting (PB), where each voter votes for a preferred allocation of funds to various projects, subject to project costs and total spending constraints. We analyse the distortion that PB mechanisms introduce relative to the minimum-social-cost outcome in expectation. The Random Dictator mechanism for this problem obtains a distortion of 2. In a special case where every voter votes for exactly one project, [11] obtain a distortion of 4/3. We show that when PB outcomes are determined as any convex combination of the votes of two voters, the distortion is 2. When three uniformly randomly sampled votes are used, we give a PB mechanism that obtains a distortion of at most 1.66, thus breaking the barrier of 2 with the smallest possible sample complexity.

We give a randomized Nash bargaining scheme where two uniformly randomly chosen voters bargain with the disagreement point as the vote of a voter chosen uniformly at random. This mechanism has a distortion of at most 1.66. We provide a lower bound of 1.38 for the distortion of this scheme. Further, we show that PB mechanisms that output a median of the votes of three voters chosen uniformly at random, have a distortion of at most 1.80.

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1 Introduction

More than 1500 cities around the globe have begun adopting *Participatory Budgeting* (PB) [22, 13], a process through which residents can vote directly on a city government's use of public funds. Residents might, for example, vote directly on how to allocate a budget of reserved funds between projects like street repairs or library renovations. PB has been shown to promote government transparency, resident engagement, and good governance [23].

[1] In alphabetical order.

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We study a PB setup similar to [12] where each vote is an allocation of funds to projects (we call it a "preferred budget") subject to the constraint that the sum of allocations to all projects is equal to one. Projects have a fixed cost, and allocations to any project cannot exceed its cost. However, allocations less than the project's cost are allowed. ([12] consider all project costs equal to one). In this model, therefore, every vote and the outcome of the PB election can be represented as a point on the unit simplex.

We study the *distortion* (Definition 5) that PB mechanisms introduce in expectation relative to the social cost minimizing allocation in the worst case of PB instances, following the lines of [1]. We adopt the ℓ_1 distance as the cost function where a voter with preferred budget a experiences a cost of $d(a, b) = \|a - b\|_1$ from an outcome budget b (Definition 2).

Several preference elicitation methods have been studied for PB [7, 2, 15, 5]. Policymakers must then transform a list of votes into a real-world allocation of funds. Furthermore, even though there may be an "optimal" allocation (under natural notions of social welfare), this allocation may be intractable to compute [19, 21] or difficult to reliably estimate if turnout is low [8]. In some situations, policymakers need to obtain a quick estimate of the budgetary region in which preferences may lie. In these cases, and when running a fully-fledged PB election is costly or difficult, low-sample complexity PB mechanisms are an attractive choice.

Low-sample complexity preference elicitation mechanisms have also been of interest recently in computational social choice [11, 10, 1, 9] – in this work, we give low-sample complexity mechanisms (using the preferred budgets of a small number of sampled voters) for PB, which achieve a distortion of less than 2. Note that 2 is a natural barrier for the distortion in this problem since the *Random Dictator* mechanism achieves a distortion of 2 in our model of PB. The Random Dictator mechanism chooses the outcome as the preferred budget of a uniformly randomly chosen voter. From Theorem 5 of [1], its distortion is at most 2, and from our Lemma 7, it is 2. We further prove that a mechanism that chooses any linear combination of two randomly sampled votes (*Random Diarchy*) also attains a distortion of 2 (Lemma 8). Another low sample-complexity mechanism, *Random Referee* [10], asks a randomly chosen voter ("the referee") to choose one out of two possible outcomes, which are random samples from the preferred budgets of the voters. This mechanism also attains a distortion of at least 2 in our setup (Lemma 9). We give a PB mechanism which samples three voters uniformly at random and attains a distortion of at most 1.66.

1.1 Our Contributions

When the PB mechanism samples three voters uniformly at random, we show that aggregation schemes that choose a median of their preferred budgets achieve a distortion of at most 1.80. We refer to such schemes as the *median schemes* and denote this class of schemes by $\mathcal{M}$.

We then turn to the case where two uniformly randomly chosen voters can come together and "bargain" with a third voter's preferred budget (again chosen uniformly at random) as the "disagreement point." We formulate the bargaining rules for the voters via the well-studied Nash bargaining framework [6]. When these bargaining rules can be further specified by a randomized rule (§4.2), we show that the distortion of the resulting mechanism is at most 1.66 (Theorem 35). We call this mechanism the *randomized Nash bargaining scheme* $\mathfrak{n}_{rand}$.

A key technical tool we use is the analysis of *pessimistic distortion* (PD) (Definition 26) first proposed by [10]. PD is a form of distortion where the comparison is made with a counterfactual which chooses a separate outcome for every small subset of voters (of a fixed size κ), thereby attaining a lower social cost than the true "optimal". In this work, we use $\kappa = 6$. This choice is due to computational constraints. We show that the PD with $\kappa = 6$ is an upper bound on the distortion of our proposed mechanisms with any number of voters n.

We then reduce the problem of computing the PD into a set of linear programs for the median schemes $\mathcal{M}$ and bilinear programs of constant size for the randomized Nash bargaining scheme $\mathfrak{n}_{rand}$. For this, we use a projection of the preferred budgets of voters into a space (we call it the *incremental allocation space* (§3)) that captures the common preferences of a subset of voters relative to other voters. In the median schemes, funds are allocated to projects ensuring that the final outcome is the median of the preferred budgets of three randomly sampled voters. In the randomized Nash bargaining scheme $\mathfrak{n}_{rand}$, the expected funds allocated to a project satisfy additional proportionality constraints (§4.2), resulting in bilinear programs. Since the proportionality constant is not fixed, this results in another variable in the optimization formulation. The problem has a complex combinatorial structure due to the nuances of Nash bargaining. However, we are able to exploit symmetries of the problem, enabling us to solve it efficiently. Since the bilinear programs are of a constant size (depends on κ, which we set to 6), we can solve these in fixed time.

Same as *Random Dictator* and *Random Referee*, our PB mechanism $\mathfrak{n}_{rand}$ also naturally respects project interactions such as complementarity and substitution as long as the voters are aware of these interactions. This is because the bargaining outcome between two voters is guaranteed to be Pareto optimal for them. We describe this point in detail in §8.

1.2 Related Work

The *sequential deliberation* (SD) mechanism for social choice was proposed in [11] where the two uniformly randomly chosen voters deliberate in each round under the rules of Nash bargaining, and the outcome for every round is the disagreement point for the next round. The SD for one round corresponds to the randomized Nash bargaining scheme $\mathfrak{n}_{rand}$. They analyzed the mechanism in median spaces, which include median graphs and trees, and found an upper bound of the distortion of the mechanism to be 1.208. They also analyze the distortion in the *budget space* (or unit simplex) in a special setting where each voter only approved funds for a single project. In this case, they show that the distortion in the equilibrium of SD is 4/3. This paper extends their work in the case of the unit simplex, such that voters in our model do not have to restrict their vote to one project.

The authors of [16] study a model where voters' opinions evolve via deliberations in small groups over multiple rounds. Opinions in their model correspond to preferred budgets in our model; however, unlike preferred budgets, opinions change as a result of deliberations. They study the distortion in single-winner elections setting and show that it is bounded by $O\left(1+\sqrt{\frac{\log n}{n}}\right)$ when voters deliberate in groups of 3 (n is the number of voters).

The work most closely related to ours is [10]; they study the *random referee* mechanism. We use their technique of analyzing the PD of 6 voters. However, they apply this technique where the underlying decision space is the Euclidean plane and use the underlying geometric structure to perform a grid search. In contrast, we study the PD with Nash bargaining, which leads to a complex structure of outcomes that we capture in linear or bilinear programs.

The authors of [9] analyze low sample-complexity randomized mechanisms for PB. They obtain constant factor guarantees for higher moments of distortion, and the distortion bound they provide is much larger than 2. Several additional results and research directions in PB are described in the survey [4].

1.3 Future Directions

A natural direction for future work is to analyze the distortion for multiple rounds of deliberation in our model, with every round's outcome serving as the next round's disagreement point. Another interesting modelling question is to study the deliberation or bargaining process with more than two agents participating together. Closing the gap of the distortion of $\mathfrak{n}_{rand}$ also remains an interesting open problem.

1.4 Roadmap

We describe the model and preliminaries in §2, introduce a projection operation and give some technical results in §3, characterize the outcome of different schemes in §4. We derive the distortion of the class of median schemes in §5. We derive the distortion under $\mathfrak{n}_{rand}$ in §6. We give empirical results on real-world Participatory Budget (PB) data in §7, and discuss project interactions in §8.

2 Model and Preliminaries

Suppose we have m projects and are required to design a budget. A *budget* denotes the fraction of the total funds that are to be spent on each project. Projects have a maximum possible allocation or "project costs." All votes respect these *project costs*, and consequently, the outcomes of all our mechanisms also respect the project costs.[2] For notational simplicity, we drop the project costs from the model henceforth and operate under the assumption that project costs are 1. All our results trivially follow for general project costs.

▶ **Definition 1.** *Let b_j denote the funds allotted to project j in budget b. We define the* budget simplex *as the set of valid budgets i.e.,* $\mathbb{B} = \{b \in \mathbb{R}^m | \sum_{j=1}^m b_j = 1$ *and* $b_j \geq 0, \forall j \in [m]\}$.

There are n voters, each with a *preferred budget* $v_i \in \mathbb{B}$. A *vote profile* P denotes the list of preferred budgets of all voters, i.e, $P = (v_1, v_2, \ldots, v_n)$. The funds allotted to project j by voter i is $v_{i,j}$. A vote profile defines an *instance of PB.* The outcome of an instance of PB is a budget in $\mathbb{B}$. Voters adopt the ℓ_1 distance as the cost function. (Not to be confused with the project costs, which is a different concept here.)

▶ **Definition 2.** *For $a, b \in \mathbb{B}$, the cost of an outcome b for a voter with preferred budget a is* $d(a, b) = \sum_{j=1}^m |a_j - b_j|$. *The sum of cost over all budgets,* $\sum_{i\in[n]} d(v_i, b)$, *is the social-cost of budget b.*

We define the *overlap utility* which is closely related to the *cost.* Note that this notion of overlap utility has been studied in knapsack voting [15, 12].

▶ **Definition 3** (Overlap Utility)**.** $u(a, b) = \sum_{j=1}^m \min(a_j, b_j)$.

▶ **Lemma 4.** *For budgets $a, b \in \mathbb{B}$, $d(a, b) = 2 - 2u(a, b)$.*

A proof is in Appendix A.13 of the extended version [17]. Lemma 4 implies that for a voter, maximizing overlap utility is the same as minimizing the cost. Note that overlap utility is *symmetric*, i.e, $u(a, b) = u(b, a)$.

2.1 Distortion

Here we define *distortion,* which we use as a metric to quantify how good a outcome is in comparison to the optimal solution for minimizing social-cost. We define distortion through the *cost* $d(\cdot, \cdot)$.

▶ **Definition 5.** *The **distortion** of budget b for vote profile P is*

$$\text{Distortion}_P(b) = \frac{\sum_{v\in P} d(v, b)}{\min_{b^*\in\mathbb{B}} \sum_{v\in P} d(v, b^*)}.$$

[2] There is no constraint on the minimum allocation to a project other than that it must be non-negative.

Let $h(P)$ be the output of mechanism h for vote profile P.

▶ **Definition 6.** *The distortion of a class of voting mechanisms* $\mathcal{H}$ *is:* $\text{Distortion}(\mathcal{H}) = \sup_{n\in\mathbb{Z}^+,\ P\in\mathbb{B}^n,\ h\in\mathcal{H}} \mathbb{E}[\text{Distortion}_P(h(P))]$.

Note that distortion is defined as a supremum over all instances of PB and all mechanisms in class $\mathcal{H}$.[3] The expectation is over the randomness of the mechanism, which also includes the randomness in the selection of voters.

The distortion of a voting mechanism is widely used to evaluate its performance regarding how close its output is to the social cost-minimizing outcome in expectation [1, 20, 3, 14, 10]. The *Random Dictator* [1] voting mechanism has a distortion of 2, as shown in Lemma 7. A proof is given in Appendix A.1 in the extended version [17].

▶ **Lemma 7.** *Any aggregation method constrained to choose its outcome as the preferred budget of a uniformly randomly chosen voter has distortion* 2*.*

Now, consider a mechanism that chooses the outcome via the deliberation between two voters chosen uniformly at random with preferred budgets a and b. Within this class, we consider mechanisms constrained to choose the outcome as a convex combination of budgets a and b.

Now, consider a mechanism constrained to choose the outcome as a linear combination of budgets a and b where a and b denote the preferred budgets of randomly sampled voters. That is, $\alpha(P)a + (1-\alpha(P))b$ for $\alpha(P) \in [0, 1]$. Note that $\alpha(P)$ may be optimized over the entire vote profile.[4] We refer to this class of mechanisms as *Random Diarchy* and denote it by $\mathcal{Q}$. Interestingly, the distortion of $\mathcal{Q}$ is 2, the same as that of *Random Dictator*.

▶ **Lemma 8.** *For Random Diarchy* $\inf_{\mathfrak{q}\in\mathcal{Q}} \text{Distortion}(\mathfrak{q}) = 2$.

A proof is given in Appendix A.2 in the extended version [17]. We further show that *Random Referee* scheme described in [11] where one of the two preferred budgets of the bargaining voters is chosen based on the preferred budget of third sampled voter also has a distortion ratio of at least 2 in Lemma 9, proven in Appendix A.3 in the extended version [17]. We denote the class of such mechanisms by $\mathcal{R}$.

▶ **Lemma 9.** *For Random Referee* $\inf_{\mathfrak{q}\in\mathcal{R}} \text{Distortion}(\mathfrak{q}) \geq 2$.

2.2 Model of preference aggregation

Let us define the mechanism formally in steps and we call it **Triadic scheme**.

1. Pick a voter i uniformly at random and set the disagreement point c as the preferred budget of voter i.
2. Now choose two voters a and b uniformly at random with replacement and they bargain with c as the disagreement point.

All our theoretical results in this paper are for the outcome of the triadic scheme. However, as discussed in [11], we can extend this bargaining scheme to multiple rounds by setting the outcome of the previous round as the disagreement point for the next round and sampling the two bargaining voters uniformly at random without replacement. We provide empirical results for this setup for multiple rounds (upto 10 rounds) in § 7.

[3] We will often study the distortion of a single mechanism, i.e., not a class. In that case, Distortion(h) simply denotes the distortion of the mechanism h.
[4] All such outcomes maximize the sum of the overlap utilities of the deliberating agents.

Our bound on pessimistic distortion assumes that the voters are chosen with replacement as done in [10]. This directly gives us a bound on the distortion when voters are sampled without replacement. It is easy to see that the difference in these bounds is of $O(\frac{1}{n})$. The case where two or more identical voters are sampled out of three defaults to the *Random Dictator* mechanism, which has constant distortion – the probability of this event is of $O(\frac{1}{n})$.

We consider bargaining schemes satisfying one or more of the following constraints, namely a) Pareto efficiency, b) Invariance to Affine Transformation, c) Symmetry, and d) Independence of Irrelevant Alternatives. Bargaining schemes that satisfy all of these constraints are the class of Nash bargaining schemes denoted by $\mathcal{N}$ [6].

▶ **Definition 10.** *An outcome of $\mathcal{N}(a, b, c)$, the* ***Nash bargaining*** *between two voters with preferred budgets a and b and the disagreement point c, is a budget z which maximizes the Nash product $(u(a, z) - u(a, c)) \times (u(b, z) - u(b, c))$, subject to individual rationality $u(a, z) \geq u(a, c)$ and $u(b, z) \geq u(b, c)$, and in case of a tie between possible outcomes, maximizes $u(c, z)$.*

The fact that $\mathcal{N}$ breaks ties in favor of the disagreement point is crucial for the distortion of triadic scheme with bargaining schemes in $\mathcal{N}$ to be smaller than 2. It is also crucial for the membership of $\mathcal{N}$ in a class of bargaining schemes that maximize the sum of overlap utilities of the bargaining agents and the disagreement point. We now define this class of bargaining schemes.

▶ **Definition 11.** *$\mathcal{M}$ is the class of* ***median schemes*** *if any outcome $z \in \mathcal{M}(a, b, c)$ maximises the sum of utilities with budgets a, b and c i.e. $u(z, a) + u(z, b) + u(z, c)$.*

The following important result is proved in Appendix A.11 in the extended version [17].

▶ **Theorem 12.** *Every scheme in $\mathcal{N}$ is also a median scheme i.e. $\mathcal{N} \subseteq \mathcal{M}$*

3 Incremental allocation space

We now give a function that captures the marginal preferences of a subset of voters S regarding the allocation to project j, relative to the preference of the other voters (i.e., $P \setminus S$). This function will be useful as an analytical tool in the paper. Specifically,

▶ **Definition 13.** *Given a vote profile $P = (v_1, v_2, \ldots, v_n)$, and project j, the* ***incremental project allocation*** *$X_{j,P} : 2^{[n]} \to [0, 1]$ maps a subset of budgets S to*

$$X_{j,P}(S) = \max\left(\left(\min_{i \in S} v_{i,j}\right) - \left(\max_{i \in P \setminus S} v_{i,j}\right), 0\right).$$

Here max and min over $\emptyset$ are defined as 0 and 1, respectively. $X_{j,P}(S)$ denotes the amount by which the budgets in S *all* agree on *increasing* the allocation to project j above the *maximum* allocation to j by any budget in $P \setminus S$. Summing this quantity over all projects $j \in [m]$ gives us $X_P(S)$, which is defined in the following.

▶ **Definition 14.** *For a vote profile P, the* ***incremental allocation*** *$X_P : 2^{[n]} \to \mathbb{R}$ is $X_P(S) = \sum_{j=1}^{m} X_{j,P}(S)$ for all $S \subseteq P$.*

We use $X_P(.)$ in §5 since its complexity is dependent only on the number of voters n and not on the number of projects m. This helps us give results valid for arbitrarily large values of m. We illustrate the functions $X_{j,P}(\cdot)$ and $X_P(\cdot)$ in the following example.

▶ **Example 15.** Consider an instance of PB with three projects and a vote profile P with three budgets $a = \langle 1, 0, 0\rangle$, $b = \langle 0, 1, 0\rangle$, and $c = \langle 0.25, 0.25, 0.5\rangle$. [5] Then, $X_{1,P}(a) = 0.75$. This is because the budget a has allocation 1 to project 1, out of which only 0.75 is *incremental* on top of $\max(b_1, c_1)$. Also, $X_{2,P}(a) = X_{3,P}(a) = 0$. As a result, $X_P(a) = 0.75$. Similarly $X_{2,P}(b) = X_P(b) = 0.75$. Also, $X_{3,P}(c) = X_P(c) = 0.5$. Further, $X_P(ac) = 0.25$. This is because the subset $\{a, c\}$ has a minimum allocation of 0.25 to project 1 among themselves. It is also incremental since $b_1 = 0$. Further, we have $X_{1,P}(abc) = X_{2,P}(abc) = X_{3,P}(abc) = X_P(abc) = 0$. This is because the group of all three budgets has no allocation that is common to all. Finally, $X_P(\emptyset) = X_{3,P}(\emptyset) = 0.5$ because no budget allocated funds more than 0.5 to project 3.

We use $\mathcal{P}(P)$ to denote the power set of P. We now give an important corollary regarding the function $X_{j,P}(\cdot)$.

▶ **Corollary 16.** $\sum_{S\in\mathcal{P}(P)} X_{j,P}(S) = 1,\ \forall j \in [m]$.

A proof is given in Appendix A.8 in the extended version [17]. Corollary 16 says that every incremental allocation to project j by budgets in S adds up to 1 when summed over all subsets S (this includes the empty set; $X_{j,P}(\emptyset) > 0$ implies that no voter allocated the full 1 unit budget to project j).

3.1 Projection On Incremental Allocations

We now give a *projection* of $X_{j,P}$ from P to $Q \subseteq P$ to get $X_{j,Q}$. This operation has two applications in this paper. First, it enables us to study the allocations of an outcome z relative to the vote profile P by making projections from $P \cup \{z\}$ to P. Second, it is used to study the outcomes of bargaining with a subset $Q \subseteq P$ of voters with respect to the entire vote profile P via projections from P to Q.

▶ **Lemma 17.** *For any vote profile P and $Q \subseteq P$, the* ***projection from P to Q*** *is $X_{j,Q}(S) = \sum_{\hat{S}\in\mathcal{P}(P\setminus Q)} X_{j,P}(\hat{S} \cup S)$ for all $S \in \mathcal{P}(Q)$, and all $j \in [m]$. Summing over $j \in [m]$, $X_Q(S) = \sum_{\hat{S}\in\mathcal{P}(P\setminus Q)} X_P(\hat{S} \cup S)$.*

A proof is given in Appendix A.9 in the extended version [17]. Lemma 17 captures an important technical fact. To calculate the incremental project allocation function on S over a vote profile $Q \subseteq P$, i.e., $X_Q(S)$, we may sum $X_P(\cdot)$ over all subsets of budgets in P which contain all elements of S but no element of $Q \setminus S$. Note that here $S \subseteq Q \subseteq P$.

We now consider the problem of analyzing an outcome z with the help of the incremental allocation function. Towards this, we define the function $Z_{j,P}(S)$ with respect to an outcome z with the help of the projection operation described in Lemma 17.

▶ **Definition 18.** *For vote profile P and budget z, define $Z_{j,P} : 2^{[n]} \to [0, 1]$ as $Z_{j,P}(S) = X_{j,P\cup\{z\}}(S \cup \{z\})\ \forall S \subseteq P$.*

Recall from Definition 13 that $X_{j,P}(S)$ denotes the amount by which *all* budgets in S want to increase the allocation to project j over the maximum allocation to j by any budget in $P \setminus S$. The quantity $Z_{j,P}(S)$ denotes the amount by which the outcome budget z "accepts" this preference of S. Naturally, $Z_{j,P}(S) \le X_{j,P}(S)$.

Analogous to summing $X_{j,P}(S)$ over all $j \in [m]$ to get $X_P(S)$, we can sum $Z_{j,P}(S)$ over all $j \in [m]$ to get $Z_P(S)$.

[5] For brevity, we omit braces and commas in the argument of X.

▶ **Definition 19.** *For vote profile P and budget z, define $Z_P(S) : 2^{[n]} \to [0,1]$ as $Z_P(S) = \sum_{j=1}^{m} Z_{j,P}(S)\ \forall S \subseteq P$.*

$Z_P(S)$ informally denotes the amount by which outcome budget z "accepts" the preference of S for increasing allocations above the allocations of $P \setminus S$ across all projects. See that $Z_P(S) \leq X_P(S)$.

▶ **Corollary 20.** *For any vote profile P and budget z, $Z_{j,P}(S) \leq X_{j,P}(S)$ for all $j \in [m]$. Summing over $j \in [m]$, $Z_P(S) \leq X_P(S)$.*

Proof. Follows directly from Lemma 17 since $X_{j,P}(S) = Z_{j,P}(S) + X_{j,P\cup\{z\}}(S)$ (we are projecting from $P \cup \{z\}$ to P). ◀

▶ **Corollary 21.** *$\sum_{S\in\mathcal{P}(P)} Z_{j,P}(S) = z_j$ for all vote profiles P and $z \in \mathbb{B}$. Summing over all projects $j \in [m]$, we get $\sum_{S\in\mathcal{P}(P)} Z_P(S) = 1$.*

Proof. We have $z_j = X_{j,\{z\}}(\{z\})$ [Definition 13]. Apply Lemma 17 by doing a projection from $P \cup \{z\}$ to $\{z\}$. ◀

This result captures, in the incremental common budget space, the fact that the total funds allocated by a budget z to projects $j \in [m]$ is 1. The following example illustrates $Z_{j,P}(S)$.

▶ **Example 22.** Consider vote profile $P = \{a, b, c\}$ with two projects. Let the budgets a, b, and c be $\langle 0.2, 0.8\rangle$, $\langle 0.5, 0.5\rangle$, and $\langle 0.8, 0.2\rangle$ respectively. Let the outcome budget z be $\langle 0.4, 0.6\rangle$. In this case, $X_{2,P}(ab) = 0.3$ and $Z_{2,P}(ab) = 0.3$ since the excess allocation by outcome z to project 2 over the allocation by budget c (i.e., 0.4) is larger than the least excess allocation to project 2 by budgets a and b over allocation in budget c (i.e., $X_{2,P}(ab)$ which is 0.3). In other words, the entire incremental allocation to project 2 by budgets a and b is accepted by outcome z. However, $X_{2,P}(a) = 0.3$ but $Z_{2,P}(a) = 0.1$ since the incremental allocation to project 2 by budget z over budgets b and c is 0.1. Thus only a partial incremental allocation to project 2 by budget a is "accepted" by budget z.

4 Overview of Median and Nash bargaining schemes

Recall the triadic mechanism from § 2.2 and we characterize its outcome. Let the disagreement point be c and the preferred budgets of the agents chosen randomly for the mechanism be a and b. For simplicity of notation, we denote $X_{\{a,b,c\}}(S)$ by $\mathfrak{X}(S)$ for S being any subset of $\{a, b, c\}$[6]. We also denote the outcome budget of the bargaining by z and $Z_{(a,b,c)}(S) = X_{\{a,b,c,z\}}(S\cup\{z\})$ by $\mathcal{Z}(S)$ for $S \subseteq \{a, b, c\}$.

4.1 Overview of class of schemes $\mathcal{M}$ and $\mathcal{N}$

In Figure 1, we illustrate the incremental allocations $\{\mathfrak{X}(S)\}_{S\subseteq\{a,b,c\}}$ with budgets a, b, and c on a Venn diagram. Recall from Definition 19 that $\mathcal{Z}(S)$ denotes what incremental allocation from $\mathfrak{X}(S)$ is "accepted" by outcome z. For the construction of $\mathcal{Z}(\cdot)$, the bargaining agents first select all the allocations "agreed" to by at least two of the three budgets. In Figure 1, this corresponds to the area of the overlaps. Now, we have two cases, i.e. the total allocation to $\mathcal{Z}$ is less than 1 or exceeds 1. We denote the difference between 1 and the total allocation to $\mathcal{Z}$ by Excess.

[6] Note that we do not consider $X_P(\cdot)$ in this section where P is the set of the preferred budgets of all the voters, even those not involved in the bargaining.

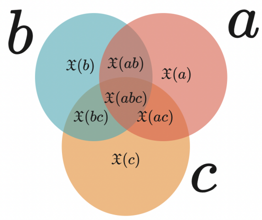

Figure 1 Incremental allocations with two preferred budgets of bargaining agents a and b, and disagreement point c.

Consider the case when the total allocation to $\mathcal{Z}(S)$ is less than 1. Here, the agents need to make further allocations worth EXCESS. Under the class of median schemes $\mathcal{M}$ [described in §4.3], they may select project allocations from $\mathfrak{X}(a), \mathfrak{X}(b)$, and $\mathfrak{X}(c)$ arbitrarily into the outcome z and thus into $\mathcal{Z}(a), \mathcal{Z}(b)$ and $\mathcal{Z}(c)$. In Figure 1, this corresponds to the area covered by exactly one of the budgets. However, under Nash bargaining schemes $\mathcal{N}$, they select allocations worth $\frac{\text{EXCESS}}{2}$ from each of $\mathfrak{X}(a)$ and $\mathfrak{X}(b)$.

Now, consider the case when the total allocation to $\mathcal{Z}(S)$ is more than 1. In this case, under median schemes, $\mathcal{M}$, the participating agents select total project allocations worth EXCESS arbitrarily from $\mathfrak{X}(ab), \mathfrak{X}(bc)$ and $\mathfrak{X}(ca)$ and remove allocations to these projects. In Figure 1, this corresponds to the area of the overlap of exactly two budgets. However, under Nash bargaining schemes $\mathcal{N}$, they select allocations worth $\frac{|\text{EXCESS}|}{2}$ from each of $\mathfrak{X}(ac)$ and $\mathfrak{X}(bc)$ and remove allocations to these projects from the outcome z.

The following lemma characterizes the overlap of the outcome $z \in \mathcal{N}(a, b, c)$ with the budgets a, b, and c, in terms of the incremental allocation functions $\mathfrak{X}(\cdot)$ and $\mathcal{Z}(\cdot)$.

▶ **Lemma 23.** *For any preferred budgets of bargaining agents a and b, disagreement point c, and outcome z of $\mathcal{N}(a, b, c)$,*

$$\begin{aligned}
\mathcal{Z}(abc) &= \mathfrak{X}(abc), \quad \mathcal{Z}(ab) = \mathfrak{X}(ab),\\
\mathcal{Z}(ac) &= \mathfrak{X}(ac) + \min(\textit{Excess}/2, 0),\\
\mathcal{Z}(bc) &= \mathfrak{X}(bc) + \min(\textit{Excess}/2, 0),\\
\mathcal{Z}(a) &= \mathcal{Z}(b) = \max(0, \textit{Excess}/2),\\
\mathcal{Z}(c) &= \mathcal{Z}(\emptyset) = 0.
\end{aligned}$$

Where, $\textit{Excess} = (1 - \mathfrak{X}(abc) - \mathfrak{X}(ab) - \mathfrak{X}(ac) - \mathfrak{X}(bc))$.

Proof Sketch. In Nash bargaining, no part of z is such that it is not preferred by both a and b. That is, $\mathcal{Z}(c) = \mathcal{Z}(\emptyset) = 0$. Otherwise, we could construct a new outcome z' that reallocated the funds from $\mathcal{Z}(c)$ or $\mathcal{Z}(\emptyset)$ to $\mathcal{Z}(a)$ and $\mathcal{Z}(b)$. This would increase $u(a, z)$ and $u(b, z)$ and thus z would not be Pareto optimal. The parts of z that benefit both a and b must be maximized. That is, $\mathcal{Z}(abc) = \mathfrak{X}(abc)$ and $\mathcal{Z}(ab) = \mathfrak{X}(ab)$. Otherwise, we could construct a new outcome z' that reallocates funds from any other project to the project that benefits both a and b, thus showing that z is not Pareto optimal. The remaining part of the proof is technical and is in Appendix A.12 in the extended version [17]. ◀

We give an explanation of the construction of the Nash bargaining solution z (and correspondingly $\mathcal{Z}$) in three steps.[7]

[7] The steps are only for illustration purposes. There is no chronology or structure required in bargaining processes. We can only characterize the outcome.

Step 1: The voters with preferred budgets a and b mutually decide to allocate funds to projects that benefit both of them. This means, for all projects $j \in [m]$, $z_j = \min(a_j, b_j)$. In terms of $\mathfrak{X}(\cdot)$ and $\mathcal{Z}(\cdot)$, this corresponds to $\mathcal{Z}(abc) = \mathfrak{X}(abc)$ and $\mathcal{Z}(ab) = \mathfrak{X}(ab)$. At this point, $\mathcal{Z}(\cdot)$ is zero for all other subsets of $\{a, b, c\}$.

Step 2: At this point, the total allocation to projects in the bargaining outcome z may be less than 1. The bargaining agents now allocate more funds to the projects $j \in [m]$ for which $z_j < \max(a_j, b_j)$ *and* $z_j < c_j$. Now z_j is set to the "median" of (a_j, b_j, c_j) for *all* projects $j \in [m]$. In terms of $\mathfrak{X}(\cdot)$ and $\mathcal{Z}(\cdot)$, this corresponds to setting $\mathcal{Z}(ac) = \mathfrak{X}(ac)$ and $\mathcal{Z}(bc) = \mathfrak{X}(bc)$.

Step 3: Now, two possibilities arise for the total amount of funds allocated in z so far, i.e., the bargaining agents have either over-spent or under-spent the total funds. These cases are central to the analysis in the paper and will be revisited several times.

Case 1: The total funds currently allocated in z is at most 1, i.e., $\mathcal{Z}(ab) + \mathcal{Z}(bc) + \mathcal{Z}(ac) + \mathcal{Z}(abc) \leq 1$. This is same as:

$$\mathfrak{X}(ab) + \mathfrak{X}(bc) + \mathfrak{X}(ac) + \mathfrak{X}(abc) \leq 1. \tag{1}$$

Recall the definition of EXCESS in Lemma 23. In this case, since there is a positive EXCESS, the bargaining agents now allocate more funds to projects with $z_j < \max(a_j, b_j)$. Since in Nash bargaining we assume equal importance of the overlap utilities of both the bargaining agents, they divide the EXCESS equally. They incrementally fund projects with $z_j < a_j$ and the projects with $z_j < b_j$ with EXCESS/2 amount each. They ensure that $z_j \leq \max(a_j, b_j)$. The precise manner of doing so is not important to satisfy the axioms of Nash bargaining. In terms of $\mathcal{Z}(\cdot)$, this corresponds to setting $\mathcal{Z}(a) = \mathcal{Z}(b) = \text{EXCESS}/2$.

Case 2: The total funds currently allocated in z exceeds 1, i.e., $\mathcal{Z}(ab) + \mathcal{Z}(bc) + \mathcal{Z}(ac) + \mathcal{Z}(abc) \geq 1$. This is same as:

$$\mathfrak{X}(ab) + \mathfrak{X}(bc) + \mathfrak{X}(ac) + \mathfrak{X}(abc) \geq 1. \tag{2}$$

If we are in this case, then the bargaining agents have overspent the funds and EXCESS is negative. They need to remove $-$EXCESS amount of allocations from z. Recall that at this point, z_j is set to the median of (a_j, b_j, c_j) for *all* projects $j \in [m]$. They remove funds from projects with $(z_j > a_j)$ and the projects with $(z_j > b_j)$ with EXCESS/2 amount each. They ensure that $z_j \geq \min(a_j, b_j)$. The precise manner of doing so is not important to satisfy the axioms of Nash bargaining. In terms of $\mathcal{Z}(\cdot)$, this corresponds to setting $\mathcal{Z}(ac) = \mathfrak{X}(ac) + \text{EXCESS}/2$, and $\mathcal{Z}(bc) = \mathfrak{X}(bc) + \text{EXCESS}/2$.

We now give a randomized way of allocating the EXCESS funds in **Step 3** while satisfying the axioms of Nash bargaining.

4.2 Randomised Nash bargaining solution $\mathfrak{n}_{\text{rand}}$

Case 1: Denote $s_j^a = \max\{a_j - z_j, 0\}$ for all projects j.[8] To projects with $s_j^a > 0$, allocate incremental funds r_j^a at random such that $\mathbb{E}[r_j^a]$ is proportional to s_j^a. The sum of r_j^a over all $j \in [m]$ is EXCESS/2 and no incremental allocation r_j^a is more than s_j^a.[9] A similar process is followed for projects j with $z_j < b_j$ by defining $s_j^b = \max\{b_j - z_j, 0\}$ and making incremental allocations r_j^b summing to EXCESS/2, $\mathbb{E}[r_j^b]$ proportional to s_j^b, and with $r_j^b \leq s_j^b$.

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[8] This precisely corresponds to $X_{j,Q}(a)$ in the incremental allocation space.

[9] The randomness of this process is the same as the hypergeometric distribution with (discretized) s_j^a balls corresponding to each project $j \in [m]$ in an urn, and we pick (discretized) EXCESS/2 balls without replacement to provide incremental allocations.

Case 2: Denote $t_j^a = \max\{z_j - a_j, 0\}$ for all projects $j \in [m]$.[10] From projects with $t_j^a > 0$, remove r_j^a amount of previously allocated funds at random such that $\mathbb{E}[r_j^a]$ is proportional to t_j^a. The sum of r_j^a over all $j \in [m]$ is $-\text{EXCESS}/2$ and with $r_j^a \le t_j^a$. A similar process is followed for projects with $z_j > b_j$ by defining $t_j^b = \max\{z_j - b_j, 0\}$ and removing allocations r_j^b from project j summing to $\text{EXCESS}/2$, $\mathbb{E}[r_j^b]$ proportional to t_j^b, and with $r_j^b \le t_j^b$.

We now give a characterization of median schemes $\mathcal{M}$ in terms of $\mathcal{Z}$ [recall that $\mathcal{N} \subseteq \mathcal{M}$ from Theorem 12 in §2.2].

4.3 Median schemes $\mathcal{M}$

▶ **Theorem 24.** *For any budgets $a, b, c \in \mathbb{B}$, a budget $z \in \mathbb{B}$ is in $\mathcal{M}(a, b, c)$ if and only if it satisfies the following conditions.*

1. $\mathcal{Z}(abc) = \mathfrak{X}(abc)$ *and* $\mathcal{Z}(\emptyset) = 0$.
2. *In **Case 1**:* $\mathcal{Z}(ab) = \mathfrak{X}(ab)$, $\mathcal{Z}(bc) = \mathfrak{X}(bc)$, $\mathcal{Z}(ca) = \mathfrak{X}(ca)$.
3. *In **Case 2**:* $\mathcal{Z}(a) = \mathcal{Z}(b) = \mathcal{Z}(c) = 0$.

The proof of this theorem is technical and is given in Appendix A.10 in the extended version [17].

Note that all the conditions on the outcomes of the bargaining schemes in $\mathcal{M}$ are *symmetric* in all three of $\{a, b, c\}$. However, outcomes in $\mathcal{N}$ also satisfy some additional conditions which may not be symmetric in all three of $\{a, b, c\}$.

We now give a lower bound on Distortion($\mathcal{N}$). Since $\mathcal{M}$ contains $\mathcal{N}$, this bound also applies to Distortion($\mathcal{M}$). Moreover, the same bound also holds for the distortion of $\mathfrak{n}_{\text{rand}}$.

▶ **Theorem 25.** $\text{Distortion}(\mathcal{M}) \ge \text{Distortion}(\mathcal{N}) > 1.38$.
Also, $\text{Distortion}(\mathfrak{n}_{rand}) > 1.38$.

Proof. The proof is by the following example of a PB instance. Suppose there are $n_A + n_B$ voters and $n_A + 1$ projects for some $n_A, n_B \ge 1$. Let o_i denote the budget where the i-th project receives allocation 1 and all the other projects get allocation 0. Each voter i in group A ($i \in [n_A]$) prefers budget o_i. Each voter i in group B ($i \in [n_A + n_B] \setminus [n_A]$) prefers budget o_{n_A+1}. The analysis of this example is in Appendix A.4 in the extended version [17] where we set $n_A = 2200$; $n_B = 3000$. ◀

We now give upper bounds of the distortion of $\mathcal{M}$.

5 Distortion Of Schemes in $\mathcal{M}$

To find an upper bound of the distortion of triadic scheme with any bargaining scheme, we use a technique introduced in [10], called *pessimistic distortion* (PD). In this technique, we first analyze the distortion for a small group of voters, call it PD, and then show that the distortion over all voters cannot be more than the PD. Specifically, in this paper, we analyze the PD for a group of 6 voters. The idea is that we allow the counterfactual solution to choose a separate "optimal" budget for every 6-tuple of voters, thereby attaining a smaller social cost than a common outcome for all voters. On the other hand, for our mechanism, we consider the expected social cost under one outcome. This is why the distortion calculated is *pessimistic*. Formally:

[10] This precisely corresponds to $X_{j,Q}(bc)$ in the incremental allocation space.

▶ **Definition 26.** *The **pessimistic distortion (PD)** of the class of mechanisms $\mathcal{M}$ with triadic scheme with 6 voters is:*

$$PD(\mathcal{M}) = \sup_{P\in\mathbb{B}^6;\ h\in\mathcal{M}} \frac{\frac{1}{20}\sum_{Q\in\mathcal{C}([6],3)} \frac{1}{3}\sum_{i\in[6]\setminus Q} d(h(Q), P_i)}{\min_{p\in\mathbb{B}} \frac{1}{6}\sum_{i\in[6]} d(p, P_i)}.$$

Here $\mathcal{C}(S,k)$ denotes the set of all k-combinations of set S.[11]

Notice that in the definition of PD, we only consider the cost for the non-bargaining agents (same as in [10]). We illustrate the PD in Figure 2, where the bargaining is over budgets $\{a, b\}$, the disagreement point is c, and the cost is computed only for $\{d, e, f\}$, the budgets not involved in the bargaining. This definition is more pessimistic than considering all agents' costs. Further, since the outcome of $\mathcal{M}$ is *symmetric* in $\{a, b, c\}$, we can use any *combination* Q of three voters to compute the outcome of bargaining without designating one of the budgets as the disagreement point. The next result, proved in Appendix A.14 in

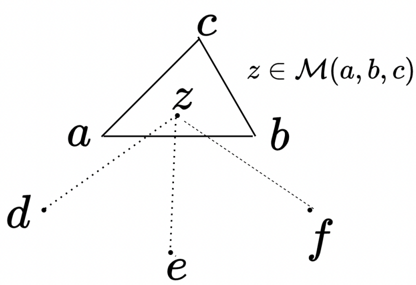


Figure 2 Illustration of PD where a, b, c are sampled for the mechanism $\mathcal{M}$, and $\{d, e, f\}$ are the other budgets for which we measure the cost of outcome z.

the extended version [17], is that the distortion of any bargaining scheme in $\mathcal{M}$ with triadic scheme cannot be more than its PD with triadic scheme with only 6 voters.

▶ **Lemma 27.** Distortion$(\mathcal{M}) \leq PD(\mathcal{M})$.

We now give a representation of the overlap utilities $u(\cdot,\cdot)$ (equivalently the cost $d(\cdot,\cdot)$), in terms of the incremental allocations $X_P(S)$. This representation is of technical importance for proofs.

▶ **Lemma 28.** *For budgets $\{a, b\}$, and a vote profile P that includes $\{a, b\}$, we have $u(a, b) = X_{(ab)}(ab) \overset{(1)}{=} \sum_{\hat{S}\in\mathcal{P}(P\setminus\{a,b\})} X_P(\hat{S}\cup\{a, b\})$.*

Proof. From Definition 3, we have $u(a, b) = \sum_{j=1}^{m} \min(a_j, b_j)$. From Definition 13 we have $\sum_{j=1}^{m} \min(a_j, b_j) = \sum_{j=1}^{m} X_{j,(a,b)}(ab) = X_{(ab)}(ab)$. Now apply Lemma 17 with $Q = S = \{a, b\}$, to obtain equality (1). ◀

[11] For simplicity of notation, we use $\mathfrak{n}(Q)$ in place of $\mathfrak{n}(P_{Q_1}, P_{Q_2}, P_{Q_3})$ in PD.

Lemma 28 shows that the overlap utility between two budgets a, b is the same as the sum of what a, b, and all subsets of the other budgets in P have in common via the incremental allocation function $X_P(S)$. For example, if $P = (a, b, c, d)$, then $u(a, b) = X_P(ab) + X_P(abc) + X_P(abd) + X_P(abcd)$.

Lemma 28 is useful for the proof of the following important result, which is an upper bound for $PD(\mathcal{M})$.

▶ **Lemma 29.** *$PD(\mathcal{M}) \leq 1.80$.*

We give a sketch of the proof here. The detailed proof is in Appendix A.15 in the extended version [17].

Proof Sketch. Let p^Q denote a budget obtained on bargaining with budgets in set Q using a bargaining scheme in $\mathcal{M}$. Note that mechanisms in $\mathcal{M}$ are *symmetric* in Q therefore, we do not need to designate a disagreement point in Q for analysis.

$$PD(\mathcal{M}) = \sup_{P\in\mathbb{B}^6;\ h\in\mathcal{M}} \frac{\frac{1}{60}\sum_{Q\in\mathcal{C}([6],3)}\sum_{i\in[6]\setminus Q} d(h(Q), P_i)}{\frac{1}{6}\min_{v\in\mathbb{B}}\sum_{i\in[6]} d(v, P_i)},$$
$$\leq \sup_{P\in\mathbb{B}^6} \frac{\frac{1}{60}\sum_{Q\in\mathcal{C}([6],3)}\sup_{p^Q\in\mathcal{M}(Q)}\sum_{i\in[6]\setminus Q} d(p^Q, P_i)}{\frac{1}{6}\min_{v\in\mathbb{B}}\sum_{i\in[6]} d(v, P_i)}.$$

Suppose that $PD(\mathcal{M}) > 1.80$. Then the following optimization problem has an optimal objective value strictly greater than 0.

$$\begin{aligned}
\text{maximize} \quad & \frac{1}{60}\sum_{Q\in\mathcal{C}([6],3)}\sum_{i\in[6]\setminus Q} d(p^Q, P_i) - 1.80\cdot\frac{1}{6}\sum_{i\in[6]} d(v, P_i), \\
\text{subject to} \quad & P \in \mathbb{B}^6, \\
& p^Q \in \mathcal{M}(Q) \qquad \forall Q \in \mathcal{C}([6],3), \\
& v \in \mathbb{B}.
\end{aligned} \tag{3}$$

To convert this problem into a linear program, we map it to the incremental allocation space of the set of 6 budgets $P = \{P_1, P_2, \ldots, P_6\}$. Denote $X_P(\cdot)$ by $X(\cdot)$ for simplicity of notation in the optimization programs. Similar to Definition 19, we define $V(S) = X_{(P\cup\{v\})}(S\cup\{v\})$ via the "optimal" budget v and $Z^Q(S) = X_{(P\,\cup\,\{p^Q\})}(S\,\cup\,\{p^Q\})$ using the outcome of our mechanism p^Q, for each $Q \in \mathcal{C}([6], 3)$.

By Lemma 4, we write the cost in terms of the overlap utility $d(p^Q, P_i) = 2 - 2u(p^Q, P_i)$, which, by Lemma 28 and the definition of $Z^Q(S)$, equals $2 - 2\sum_{S\in\mathcal{P}(P\setminus P_i)} Z^Q(S\cup P_i)$. Similarly, we have $d(v, P_i) = 2-2\sum_{S\in\mathcal{P}(P\setminus P_i)} V(S\cup P_i)$. To make the $p^Q \in \mathcal{M}(Q)$ constraints linear, we use case analysis.

Consider a given $Q = \{q_1, q_2, q_3\} \in C([6], 3)$ and a budget $p^Q \in \mathbb{B}$. Let $\mathfrak{X}(S) = X_Q(S)$ and $\mathcal{Z}(S) = X_{(Q\cup\{p^Q\})}(S\cup\{p^Q\})$. Theorem 24 implies that $p^Q \in \mathcal{M}(Q)$ if and only if the following holds:

- **Case 1**: If $\mathfrak{X}(q_1q_2q_3) + \mathfrak{X}(q_1q_2) + \mathfrak{X}(q_1q_3) + \mathfrak{X}(q_2q_3) \geq 1$, $\mathcal{Z}(q_1) = \mathcal{Z}(q_2) = \mathcal{Z}(q_3) = 0$.
- **Case 2**: If $\mathfrak{X}(q_1q_2q_3) + \mathfrak{X}(q_1q_2) + \mathfrak{X}(q_1q_3) + \mathfrak{X}(q_2q_3) \leq 1$, $\mathcal{Z}(q_1q_2) = \mathfrak{X}(q_1q_2), \mathcal{Z}(q_1q_3) = \mathfrak{X}(q_1q_3), \mathcal{Z}(q_2q_3) = \mathfrak{X}(q_2q_3)$.

We break each $p^Q \in \mathcal{M}(Q)$ constraint into two cases. Since there are $\binom{6}{3}$ such constraints in the optimization problem, there are $2^{\binom{6}{3}}$ cases overall. We represent each case by a binary string of length 20 where a 0 or 1 at each position denotes whether the triplet Q corresponding to that position is in **Case 1** or **Case 2**.

However, most of these $2^{\binom{6}{3}}$ cases are not unique up to the permutation of preferred budgets, i.e. when the preferred budgets of different voters are permuted, we may move from one case to another. Since these cases have the same objective value, we do not need to solve all the cases. Exploiting further symmetries, we have 2136 unique cases, each of which is formulated as a linear program with precise details in Appendix A.15 in the extended version [17]. We obtain the optimal value for each case to be 0 hence, a contradiction. ◀

Using Lemmas 27 and 29, we get the following key result.

▶ **Theorem 30.** Distortion($\mathcal{M}$) ≤ 1.80.

6 Distortion of $\mathfrak{n}_{\text{rand}}$

Recall the randomized Nash bargaining scheme $\mathfrak{n}_{\text{rand}}$ explained in § 4.2. In this section, we derive an upper bound for it. Towards this, we first define a *hypothetical* bargaining scheme $\tilde{\mathfrak{n}}_{\text{rand}}$. This scheme is hypothetical because it assumes that the bargaining agents use some knowledge about the preferred budgets of the non-bargaining agents to break ties among potential outcomes. We then show in Lemma 32 that the Distortion of $\mathfrak{n}_{\text{rand}}$ is at most as much as that of $\tilde{\mathfrak{n}}_{\text{rand}}$. We then bound the Distortion of $\tilde{\mathfrak{n}}_{\text{rand}}$ by its *expected pessimistic distortion* (EPD), a quantity similar in essence to the PD. We define the EPD in Definition 33. Our main technical contribution in this section is the analysis of the EPD of $\tilde{\mathfrak{n}}_{\text{rand}}$, which we do by expressing it as the solution of a bilinear optimization problem.

6.1 Construction of bargaining solution in $\tilde{\mathfrak{n}}_{\text{rand}}$

Recall Definition 18 of $Z_{j,P}(\cdot)$ for an outcome budget z. Also recall that $Z_{j,P}(\cdot)$ satisfies Corollaries 20 and 21. For $\tilde{\mathfrak{n}}_{\text{rand}}$, we characterize the outcome in the incremental allocation space; denoted by $\tilde{Z}_{j,P}(\cdot)$. Same as $Z_{j,P}(\cdot)$, $\tilde{Z}_{j,P}(\cdot)$ also satisfies Corollaries 20 and 21, i.e.,

$$0 \leq \tilde{Z}_{j,P}(S) \leq X_{j,P}(S) \forall S \in \mathcal{P}(P) \text{ and all } j \in [m]. \tag{4}$$

$$\sum_{j=1}^{m} \tilde{Z}_{j,P}(S) = \tilde{Z}_P(S) \quad \text{and,} \quad \sum_{S \in \mathcal{P}(P)} \tilde{Z}_P(S) = 1. \tag{5}$$

Before describing the construction of $\tilde{\mathfrak{n}}_{\text{rand}}$, we now give the following result on the overlap utility $u(a, z)$ of outcome budget z and any budget $a \in P$ in terms of $Z_P(S)$.

▶ **Lemma 31.** *For a vote profile P, a budget $a \in P$, and any budget z, the overlap utility is $u(a, z) = \sum_{S \in \mathcal{P}(P) | S \ni a} Z_P(S)$.*

Proof. In Lemma 28, use z for b, a for a, and $P \cup \{z\}$ for P. ◀

By Lemma 31, $u(v, \tilde{Z}_P) = \sum_{S \in \mathcal{P}(P) | S \ni a} \tilde{Z}_P(S)$.[12] Similarly, the cost can be given by $d(v, \tilde{Z}_P) = 2 - 2u(v, \tilde{Z}_P)$.

Let c be the disagreement point, and $\{a, b\}$ be the preferred budgets of the agents chosen to bargain. Denote $Q = \{a, b, c\}$. For the construction of $\tilde{Z}_P(\cdot)$, we first do **Step 1** and **Step 2** from § 4. We then have for all $j \in [m]$, $\tilde{Z}_{j,P}(S) = X_{j,P}(S)$ for all $S \in \mathcal{P}(P)$ such that S contains at least 2 elements of Q and $\tilde{Z}_{j,P}(S) = 0$ for all other $S \in \mathcal{P}(P)$. We then encounter either **Case 1** or **Case 2**, as in § 4.

[12] Note the overload in the notation of the overlap utility; it was initially defined for a pair of budgets v and z, here we define it for v and $\tilde{Z}$ where $\tilde{Z}$ captures z.

Case 1: Here we need to allocate more funds to projects. Recall the construction of z for $\mathfrak{n}_{\text{rand}}$ in § 4.2. Recall the random incremental allocations r_j^a and r_j^b used in $\mathfrak{n}_{\text{rand}}$. For the incremental allocations in $\tilde{\mathfrak{n}}_{\text{rand}}$ we construct $\alpha_{j,P}(S) = r_j^a \cdot (X_{j,P}(S)/X_{j,Q}(a))$[13] for all $\{S \mid a \in S; b,c \notin S\}$ for all projects $j \in [m]$. Intuitively, this may be thought of as a proportional selection of projects from every subset of budgets S. Similarly we construct $\beta_{j,P}(S) = r_j^b \cdot (X_{j,P}(S)/X_{j,Q}(b))$ for all $\{S \mid b \in S; a,c \notin S\}$ and all projects $j \in [m]$.

Now, set $\tilde{Z}_{j,P}(S) = \tilde{Z}_{j,P}(S) + \alpha_{j,P}(S) \ \ \forall \{S \mid a \in S; b,c \notin S\}$ and $\tilde{Z}_{j,P}(S) = \tilde{Z}_{j,P}(S) + \beta_{j,P}(S) \ \ \forall \{S \mid b \in S; a,c \notin S\}$ and $\forall\, j \in [m]$.

Case 2: In this case we need to remove allocations from projects. Recall the construction of z for $\mathfrak{n}_{\text{rand}}$ in § 4.2. Recall the removals of allocations r_j^a and r_j^b used in $\mathfrak{n}_{\text{rand}}$. For the removals of allocations in $\tilde{\mathfrak{n}}_{\text{rand}}$, we construct $\alpha_{j,P}(S) = r_j^a \cdot (X_{j,P}(S)/X_{j,Q}(bc))$ for all $\{S \mid b,c \in S, a \notin S\}$ for all $j \in [m]$. Similarly we construct $\beta_{j,P}(S) = r_j^b \cdot (X_{j,P}(S)/X_{j,Q}(ac))$ for all $\{S \mid a,c \in S; b \notin S\}$.

Now, set $\tilde{Z}_{j,P}(S) = \tilde{Z}_{j,P}(S) - \alpha_{j,P}(S) \ \forall \{S \mid b,c \in S; a \notin S\}$, and $\tilde{Z}_{j,P}(S) = \tilde{Z}_{j,P}(S) - \beta_{j,P}(S) \ \forall \{S \mid a,c \in S; b \notin S\} \ \forall\, j \in [m]$.

We can now construct $\tilde{Z}_P(S)$ via $\tilde{Z}_P(S) = \sum_{j=1}^{m} \tilde{Z}_{j,P}(S)$. With this, we now construct $\tilde{Z}_Q$ as the outcome of the hypothetical bargaining process, via the projection from P to Q That is, $\tilde{Z}_Q(S) = \sum_{\hat{S} \in \mathcal{P}(P \setminus Q)} \tilde{Z}_P(S \cup \hat{S})$ [recall projection in Lemma 17].

See that $\{\tilde{Z}_{j,P}(.)\}_{j \in [m]}$ satisfies Corollaries 20 and 21. Further, $\tilde{Z}_Q(.)$ satisfies all equations of Lemma 23 [proof in Appendix A.16 in the extended version [17]].

6.2 Distortion under $\mathfrak{n}_{\text{rand}}$

We now bound the distortion of the triadic scheme with bargaining scheme $\mathfrak{n}_{\text{rand}}$ by that of the hypothetical scheme $\tilde{\mathfrak{n}}_{\text{rand}}$. A proof is in the Appendix A.18 in the extended version [17].

▶ **Lemma 32.** Distortion$(\mathfrak{n}_{rand}) \leq$ Distortion$(\tilde{\mathfrak{n}}_{rand})$.

We now follow a similar approach as in §5 and define expected pessimistic distortion under bargaining scheme $\tilde{\mathfrak{n}}_{\text{rand}}$ as follows.

▶ **Definition 33.** *The* ***expected pessimistic distortion*** *of $\tilde{\mathfrak{n}}_{rand}$ with triadic scheme with 6 voters, $EPD(\tilde{\mathfrak{n}}_{rand})$ is*

$$\sup_{P \in \mathbb{B}^6} \frac{\frac{1}{60} \sum_{c \in [6]} \sum_{\substack{\{a,b\} \in \\ \mathcal{C}([6] \setminus \{c\}, 2)}} \frac{1}{3} \sum_{i \in [6] \setminus \{a,b,c\}} \mathbb{E}[d(\tilde{\mathfrak{n}}_{rand}(a,b,c), P_i)]}{\min_{p \in \mathbb{B}} \frac{1}{6} \sum_{i \in [6]} d(p, P_i)}.$$

▶ **Lemma 34.** Distortion$(\tilde{\mathfrak{n}}_{rand}) \leq EPD(\tilde{\mathfrak{n}}_{rand})$.

The proof is similar to Lemma 27 and is in Appendix A.19 in the extended version [17].

▶ **Lemma 35.** $EPD(\tilde{\mathfrak{n}}_{rand}) \leq 1.66$.

[13] Note that $\alpha_{j,P}(S) \leq r_j^a$ since $X_{j,P}(S) \leq X_{j,Q}(a)$ [follows from Lemma 17] and $\sum_{j=1}^{m} \sum_{\substack{S \in \mathcal{P}(P) \\ S \ni a, S \not\ni b,c}} \alpha_{j,P}(S) = \frac{\text{EXCESS}}{2}$ since $\sum_{\substack{S \in \mathcal{P}(P) \\ S \ni a, S \not\ni b,c}} X_{j,P}(S) = X_{j,Q}(S)$ [follows from Lemma 17] and the fact that $\sum_{j=1}^{m} r_j^a = \text{EXCESS}/2$ [as defined in **Case 1** in §4.2].

The proof is similar to that of Lemma 29 and is presented in Appendix A.20 in the extended version [17]. We present the key ideas of the proof here.

Proof Sketch. Recall the construction of $\tilde{Z}_{j,P}(.) \sim \tilde{\mathfrak{n}}_{\text{rand}}(a,b,c)$ and consider **Case 1**. A similar analysis holds for **Case 2** as well.

We show in Appendix A.20 in the extended version [17] that $\mathbb{E}[\tilde{Z}_P(S)] = \gamma_a^1 X_P(S)$ for all $\{S : S \ni a; S \not\ni b, c\}$ and $\mathbb{E}[\tilde{Z}_P(S)] = \gamma_b^1 X_P(S)$ for all $\{S : S \ni b; S \not\ni a, c\}$ for some variables $0 \leq \gamma_a^1, \gamma_b^1 \leq 1$. Here, γ_a^1 and γ_b^1 denote what fraction of allocation from the incremental allocation $X_P(S)$ is "accepted" into $\tilde{Z}_P(S)$. In our optimization problem formulation equation (29) in Appendix A.20 in the extended version [17], we use γ_b^1, γ_a^1 as variables of our optimization formulation, together with $X_P(S)$ and therefore we obtain a bilinear program. We solve it with the Gurobi solver [18]. Similar to the proof of Lemma 27, we remove the cases that are not unique to permutations of voters and use further symmetries of the problem to reduce number of bilinear programs from $2^{\binom{6}{3}}$ to 1244. ◀

Using Lemmas 32, 34, and 35, we get the following result.

▶ **Theorem 36.** Distortion$(\mathfrak{n}_{rand}) \leq 1.66$.

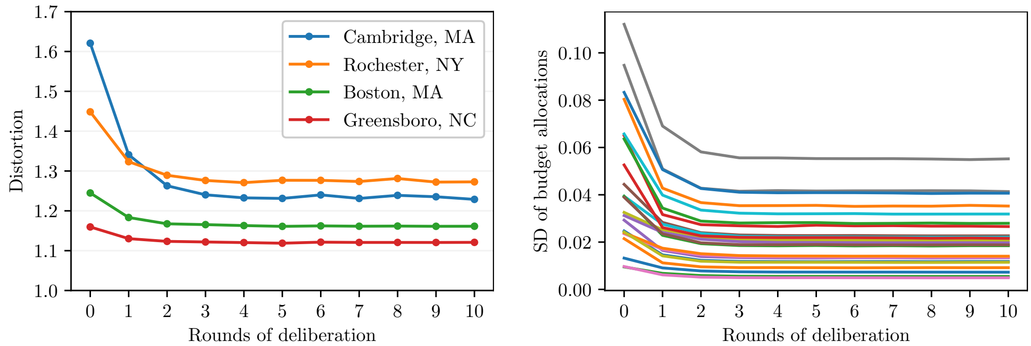
7 Empirical Results

Recall triadic scheme as described in §2.2. We now define a sequential deliberation mechanism that could run bargaining over multiple rounds by setting the disagreement point for each round as the outcome of the previous round as proposed in [11].

1. Pick a voter i uniformly at random. Set the disagreement point for the deliberation c to their preferred budget v_i.
2. Repeat the following process T times,
   a. Pick two voters independently and uniformly at random with replacement. They bargain with c as the disagreement point.
   b. Set the disagreement point c to the outcome of the bargaining.
3. The outcome of the process is c.

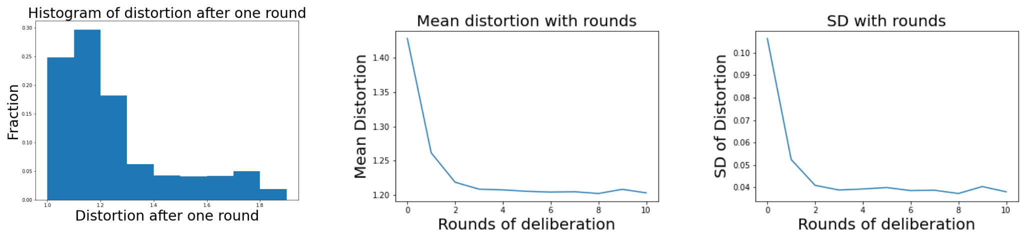
Observe that on setting $T = 1$, we exactly get triadic scheme as §2.2. To evaluate the distortion of sequential deliberation in PB empirically, we ran a simulation from the online participatory budgeting elections in Boston in 2016 ($n = 4,482$), Cambridge in 2015 ($n = 3,273$), Greensboro in 2019 ($n = 512$), and Rochester in 2019 ($n = 1,563$) where the data were obtained from `https://budget.pbstanford.org/`. In these elections, projects had a fixed cost, and voters participated in knapsack voting [15], in which they could choose any number of projects as long as they fitted within the fund limits. Note that in this simulation setup partial project funding is not allowed, unlike the setup in the theoretical model. We further present simulation results in Figures 4a, 4b, 4c on real dataset from a PB (participatory budgeting) process run by a non-profit organisation in Boston in 2016 where they used a fractional allocation setting, more aligned with our theoretical work.

To simulate sequential deliberation, we picked a voter uniformly at random to set their preferred budget as the disagreement point. We then picked another two voters independently and uniformly at random and calculated a Nash bargaining solution between them. We assumed that everyone voted truthfully. We then made the bargaining outcome the new disagreement point and repeated the deliberation process for $T = 10$ rounds. We repeated this entire simulation 10,000 times for each PB election. The average distortion after each round of deliberation is shown in Figure 3a. The point corresponding to 0 rounds of deliberation is



(a) Average distortion. **(b)** SD of distortion.

Figure 3 (a) The average distortion after each round of sequential deliberation in a simulation using the data from PB elections in four cities. The simulation was run 10,000 times for each city. (b) The standard deviation (SD) of the fund allocation to each project in the simulation of sequential deliberation in the PB election in Cambridge. Each line represents a project.



(a) Histogram of triadic scheme. **(b)** Average distortion. **(c)** SD of distortion.

Figure 4 Distortion results on PB platform in Boston under the fractional allocation setup

the first disagreement point and is selected uniformly at random. Since voters did not have to use all the budget available, we added an "unspent" project and allocated the unspent budget of each voter to this project. We normalized the budget to sum to 1 in each election.

The mean and standard deviation of the distortion after each round of sequential deliberation for the fractional allocation setting as in the PB process in Boston is shown in Figures 4b and 4c, respectively. A histogram plot of the distortion after one round of deliberation is in Figure 4a. As before, we observe a quick convergence within three rounds of sequential deliberation with the point corresponding to zero rounds of deliberation being the first disagreement point.

The results from all the PB elections show that the average distortion is quite low, even after only two rounds of deliberation. It also shows that the distortion converges quickly within three rounds. Further, we measured the stability of the fund allocation to the projects after each round of deliberation. We simulated sequential deliberation on the data from the PB in Cambridge $1,000,000$ times, each time with 10 rounds of deliberation. The fund allocation to each project after each round was recorded. The fund allocation's standard deviation (SD) is shown in Figure 3b. We can see that the SD stabilizes after only three rounds of deliberation.

8 Triadic Scheme With Project Interactions

Mathematically, we model project interactions as follows: if projects in group q are **perfect complements** of each other, then the overlap utility that voters can derive from *each* project in q is the minimum funding of any project in q. For example, consider a proposal of buying some computers for the community. Within this, one project is for buying hardware and another one is for buying software. If the software and hardware projects are funded 0.2 and 0.5, then the community members can only use 0.2 each, and the extra funding of 0.3 for the hardware project is wasted [14].

On the other hand, if the projects in group r are **perfect substitutes**, then the utility that voters can derive from group r is the maximum funding of a project in r. Thus, if two companies are paid 0.2 and 0.5 to do the same work, only 0.5 will be used, and 0.2 is wasted.

We now give a formal model of the set of projects. Let m_c denote the number of groups of *perfect complementary* projects, m_s denote the number of groups of *perfect substitute* projects, and m_r denote the number of *regular* projects. Let $s(q)$ denote the number of projects in group q. For groups of perfect complementary and perfect substitute projects, $s(q) \geq 2$ and for regular projects $s(q) = 1$. The total number of projects is $m = \left(\sum_{q=1}^{m_c+m_s} s(q)\right) + m_r$. For simplicity, project groups are arranged such that groups $1, \ldots, m_c$ are perfect complementary, groups $m_c + 1, \ldots, m_c + m_s$ are perfect substitutes, and $m_c + m_s + 1, \ldots, m_c + m_s + m_r$ are regular projects.

Let $f(b)$ be the *efficiency function* which quantifies how much budget b respects the project interactions. Specifically, $f(b)$ takes a budget $b \in \mathbb{R}^m$ and outputs a vector in $\mathbb{R}^{m_c+m_s+m_r}$, where

$$f(b)_q = \begin{cases} s(q) \cdot \min(\{b_j \mid j \in \text{group } q\}) & \text{if } q \in [1, m_c] \quad \text{(perfect complementary groups),} \\ \max(\{b_j \mid j \in \text{group } q\}) & \text{if } q \in [m_c + 1, m_c + m_s], \quad \text{(perfect substitute groups)} \\ \{b_j \mid j \in \text{group } q\} & \text{otherwise.} \quad \text{(regular projects).} \end{cases}$$

For a group of perfect complementary projects, the corresponding output element is the bottle-neck allocation in the group, multiplied by the number of projects in the group. For a group of perfect substitute projects, the corresponding output element is the largest allocation in that group. For regular projects, the corresponding output elements are the same as the allocation to the project. We now give a modified definition of the overlap utility, accounting for project interactions.

▶ **Definition 37.** *The **overlap utility** of budgets a and b, accounting for project interactions is* $u(a, b) = \sum_{q=1}^{m_c+m_s+m_r} \min(f(a)_q, f(b)_q)$.

In the following definition we formally state the requirements for a budget to be consistent with the project interactions.

▶ **Definition 38.** *A budget b **respects the project interactions** if and only if projects in each perfect complementary group are all funded equally, and at most one project in each perfect substitute group is funded at all.*

The following lemma states that the efficiency function $f(b)$ sums to 1 if and only if the budget b *respects the project interactions.*

[14] This is a stylized model and in general, the scale of the funds required for each project can be very different.

▶ **Lemma 39.** *Budget b respects the project interactions iff the efficiency function $f(b)$ satisfies $\sum_q f(b)_q = 1$. Otherwise, $\sum_q f(b)_q < 1$.*

We now give a result that a Pareto improvement exists over a budget that does not respect the project interactions.

▶ **Lemma 40.** *If $\sum_q f(b)_q < 1$, then for some $k \in [m_c + m_s + m_r]$, there exists a budget b' for which $f(b')_k > f(b)_k$ and $f(b')_q \geq f(b)_q$ for all project groups q.*

The proofs of lemmas 39 and 40 are presented in Appendix A.5 and A.6 in [17].

We now give the main result of this section. We show that if either of the budgets of the bargaining agents respect the project interactions (which will be true for rational agents), then the outcome of any median scheme respects project interactions. Since the class of median schemes contains the class of Nash bargaining schemes (Theorem 12), this result also applied to $\mathcal{N}$ and therefore also to our randomized bargaining scheme $\mathfrak{n}_{\text{rand}}$.

▶ **Theorem 41.** *If budget a or b respects the project interactions, then for any budget $c \in \mathbb{B}$, $\mathcal{M}(a, b, c)$ respects the project interactions.*

Proof. Let z be an outcome from $\mathcal{M}(a, b, c)$. Assume without loss of generality that budget a respects the project interactions, and suppose that outcome z does not. By Lemma 39, $\sum_q f(a)_q = 1$ and $\sum_q f(z)_q < 1$. Thus, there exists some k where $f(a)_k > f(z)_k$. By Lemma 40, there exists a budget z' which respects the project interactions and $f(z')_q \geq f(z)_q$ for all project groups q and $f(z')_k > f(z)_k$. The overlap utility functions satisfy:

$$u(a, z') = \sum_q \min(f(a)_q, f(z')_q) > \sum_q \min(f(a)_q, f(z)_q) = u(a, z),$$
$$u(b, z') = \sum_q \min(f(b)_q, f(z')_q) \geq \sum_q \min(f(b)_q, f(z)_q) = u(b, z).$$
$$u(c, z') = \sum_q \min(f(c)_q, f(z')_q) \geq \sum_q \min(f(c)_q, f(z)_q) = u(c, z).$$

This implies that the sum of overlap utilities of a, b, and C with z' is higher than that with z, a contradiction for an outcome of $\mathcal{M}$. ◀

Theorem 41 implies that if every voter has a preferred budget that respects the project interactions, then the outcome of the sequential deliberation mechanism will also respect the project interactions, no matter how many rounds it runs.

9 Conclusion

We study low sample-complexity mechanisms for PB, which are particularly attractive when the policymakers are interested in obtaining a quick estimate of the voter's preferences or when a full-fledged PB election is difficult or costly to conduct. In our PB setup, the distortion of mechanisms that obtain and use the votes of only one uniformly randomly sampled voter is 2. Extending this result, we show that when two voters are sampled, and a convex combination of their votes is used by the mechanism, the distortion cannot be made smaller than 2. We then show that with 3 samples, there is a significant improvement in the distortion – we give a PB mechanism that obtains a distortion of 1.66. Our mechanism builds on the existing works on Nash bargaining between two voters with a third voter's preferred outcome as the disagreement point. We also give a lower bound of 1.38 for our mechanism.

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