On Finding Constrained Independent Sets in Cycles

Ishay Haviv
School of Computer Science, The Academic College of Tel Aviv-Yaffo, Israel

Abstract

A subset of $[n] = \{1, 2, \dots, n\}$ is called stable if it forms an independent set in the cycle on the vertex set $[n]$. In 1978, Schrijver proved via a topological argument that for all integers n and k with $n \geq 2k$, the family of stable k-subsets of $[n]$ cannot be covered by $n - 2k + 1$ intersecting families. We study two total search problems whose totality relies on this result.

In the first problem, denoted by SCHRIJVER(n, k, m), we are given an access to a coloring of the stable k-subsets of $[n]$ with $m = m(n, k)$ colors, where $m \leq n - 2k + 1$, and the goal is to find a pair of disjoint subsets that are assigned the same color. While for $m = n - 2k + 1$ the problem is known to be PPA-complete, we prove that for $m < d \cdot \lfloor \frac{n}{2k+d-2} \rfloor$, with d being any fixed constant, the problem admits an efficient algorithm. For $m = \lfloor n/2 \rfloor - 2k + 1$, we prove that the problem is efficiently reducible to the KNESER problem. Motivated by the relation between the problems, we investigate the family of *unstable* k-subsets of $[n]$, which might be of independent interest.

In the second problem, called Unfair Independent Set in Cycle, we are given ℓ subsets $V_1, \dots, V_\ell$ of $[n]$, where $\ell \leq n - 2k + 1$ and $|V_i| \geq 2$ for all $i \in [\ell]$, and the goal is to find a stable k-subset S of $[n]$ satisfying the constraints $|S \cap V_i| \leq |V_i|/2$ for $i \in [\ell]$. We prove that the problem is PPA-complete and that its restriction to instances with $n = 3k$ is at least as hard as the Cycle plus Triangles problem, for which no efficient algorithm is known. On the contrary, we prove that there exists a constant c for which the restriction of the problem to instances with $n \geq c \cdot k$ can be solved in polynomial time.

2012 ACM Subject Classification Mathematics of computing → Graph theory; Mathematics of computing → Combinatorial algorithms; Theory of computation → Problems, reductions and completeness

Keywords and phrases Schrijver graph, Kneser graph, Stable sets, PPA-completeness

Digital Object Identifier 10.4230/LIPIcs.ICALP.2023.73

Category Track A: Algorithms, Complexity and Games

Funding *Ishay Haviv*: Research partly supported by the Israel Science Foundation (grant No. 1218/20).

1 Introduction

For integers n and k with $n \geq 2k$, the Kneser graph $K(n, k)$ is the graph whose vertices are all the k-subsets of $[n] = \{1, 2 \dots, n\}$, where two such sets are adjacent in the graph if they are disjoint. The graph $K(n, k)$ admits a proper vertex coloring with $n - 2k + 2$ colors. This indeed follows by assigning the color i, for each $i \in [n - 2k + 1]$, to all the vertices whose minimal element is i, and the color $n - 2k + 2$ to the remaining vertices, those contained in $[n] \setminus [n - 2k + 1]$. In 1978, Lovász [22] proved, settling a conjecture of Kneser [20], that fewer colors do not suffice, that is, the chromatic number of the graph satisfies $\chi(K(n, k)) = n - 2k + 2$. Soon later, Schrijver [28] strengthened Lovász's result by proving that the subgraph $S(n, k)$ of $K(n, k)$ induced by the stable k-subsets of $[n]$, i.e., the vertices of $K(n, k)$ that form independent sets in the cycle on the vertex set $[n]$, has the same chromatic number. It was further shown in [28] that the graph $S(n, k)$ is vertex-critical, in the sense that any removal of a vertex from the graph decreases its chromatic number.

50th International Colloquium on Automata, Languages, and Programming (ICALP 2023).
Editors: Kousha Etessami, Uriel Feige, and Gabriele Puppis; Article No. 73; pp. 73:1–73:16
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

It is interesting to mention that despite the combinatorial nature of Kneser's conjecture [20], Lovász's proof [22] relies on the Borsuk–Ulam theorem [6], a fundamental result in the area of algebraic topology. Several alternative proofs and extensions were provided in the literature over the years (see, e.g., [24, 25]). Although they are substantially different from each other, they all essentially rely on topological tools.

The computational search problem associated with Kneser graphs, denoted by KNESER, was proposed by Deng, Feng, and Kulkarni [7] and is defined as follows. Its input consists of integers n and k with $n \geq 2k$ and an access to a coloring of the vertices of $K(n,k)$ with $n-2k+1$ colors. The goal is to find a monochromatic edge in the graph, i.e., two disjoint k-subsets of $[n]$ that are assigned the same color by the given coloring. Since the number of colors used by the input coloring is strictly smaller than the chromatic number of $K(n,k)$ [22], it follows that this search problem is total, in the sense that every input is guaranteed to have a solution. Note that the input coloring may be given as an oracle access that provides the color of any queried vertex, and that an algorithm for the problem is considered efficient if its running time is polynomial in n. In other variants of the problem, the input coloring is given by some succinct representation, e.g., a Boolean circuit or an efficient Turing machine. The computational search problem SCHRIJVER is defined similarly, where the input represents a coloring of the vertices of $S(n,k)$ with $n-2k+1$ colors, and the goal is to find a monochromatic edge, whose existence is guaranteed by the aforementioned result of Schrijver [28].

The computational complexity of the SCHRIJVER problem was determined in [15], where it was shown to be complete in the complexity class PPA. This complexity class, introduced in 1994 by Papadimitriou [26], is known to capture the complexity of several additional total search problems whose totality is based on the Borsuk–Ulam theorem, e.g., Consensus Halving, Bisecting Sandwiches, and Splitting Necklaces [12]. Note that this line of PPA-completeness results is motivated not only from the computational complexity perspective, but also from a mathematical point of view, as one may find those results as an indication for the necessity of topological arguments in the existence proof of the solutions of these problems. As for the KNESER problem, it is an open question whether it is also PPA-hard, as was suggested by Deng et al. [7]. We remark that its complexity is related to that of the Agreeable Set problem from the area of resource allocation (see [23, 16]). The KNESER and SCHRIJVER problems were also investigated in the framework of parameterized algorithms [16, 17], where it was shown that they admit randomized fixed-parameter algorithms with respect to the parameter k, namely, algorithms whose running time is $n^{O(1)} \cdot k^{O(k)}$ on input colorings of $K(n,k)$ and $S(n,k)$.

Before turning to our results, let us mention another computational search problem, referred to as the CYCLE-PLUS-TRIANGLES problem. Its input consists of an integer k and a graph on $3k$ vertices, whose edge set is the disjoint union of a Hamilton cycle and k pairwise vertex-disjoint triangles. The goal is to find an independent set of size k in the graph. The existence of a solution for every input of the problem follows from a result of Fleischner and Stiebitz [13], which settled in the early nineties a conjecture of Du, Hsu, and Hwang [9] as well as its strengthening by Erdös [10]. Their proof in fact shows that every such graph is 3-choosable, and thus 3-colorable, so in particular, it contains an independent set of size k. Here, however, the existence of a solution for every input of the problem is known to follow from several different arguments. While the proof of [13] relies on the polynomial method in combinatorics (see also [3]), an elementary proof was given slightly later by Sachs [27], and another proof, based on the chromatic number of $S(n,k)$, was provided quite recently by Aharoni et al. [1]. Yet, none of these proofs is constructive, in the sense that they do not suggest an efficient algorithm for the Cycle-Plus-Triangles problem. The question of whether the problem admits an efficient algorithm was asked by several authors and is still open (see, e.g., [14, 1, 4]). Interestingly, the approach of [1] implies that the problem is not harder than the restriction of the Schrijver problem to colorings of $S(n,k)$ with $n = 3k$.

1.1 Our Contribution

In this paper, we introduce two total search problems concerned with finding stable sets under certain constraints. The totality of the problems relies on the chromatic number of the graph $S(n,k)$ [28]. We study these problems from algorithmic and computational perspectives. In what follows, we describe the two problems and our results on each of them.

1.1.1 The Generalized Schrijver Problem

We start by considering a generalized version of the Schrijver problem, which allows the number of colors used by the input coloring to be any prescribed number. Let Schrijver(n,k,m) denote the problem which asks to find a monochromatic edge in $S(n,k)$ for an input coloring that uses $m = m(n,k)$ colors. Note that every input of the problem is guaranteed to have a solution whenever $m \leq n-2k+1$, and that for $m = n-2k+1$, the problem coincides with the standard Schrijver problem.

The Schrijver(n,k,m) problem obviously becomes easier as the number of colors m decreases. For example, it is not difficult to see that for $m = \lfloor n/k \rfloor - 1$, the problem can be solved efficiently, in time polynomial in n. Indeed, the clique number of the graph $S(n,k)$ is $\lfloor n/k \rfloor$, which is strictly larger than m, so by querying the input coloring for the colors of the vertices of a clique of maximum size, one can find two adjacent vertices with the same color. Our first result extends this observation and essentially shows that the Schrijver(n,k,m) problem can be solved efficiently for any number of colors m satisfying $m = O(n/k)$.

▶ **Theorem 1.** *For every integer $d \geq 2$, there exists an algorithm for the* Schrijver(n,k,m) *problem with $m < d \cdot \lfloor \frac{n}{2k+d-2} \rfloor$ whose running time is $n^{O(d)}$.*

Our next result relates the generalized Schrijver(n,k,m) problem to the Kneser problem.

▶ **Theorem 2.** Schrijver$(n,k,\lfloor n/2 \rfloor - 2k+1)$ *is polynomial-time reducible to* Kneser*.*

The simple proof of Theorem 2 involves a proper coloring of the subgraph of $K(n,k)$ induced by the *unstable* k-subsets of $[n]$, i.e., the vertices of $K(n,k)$ that do not form vertices of $S(n,k)$. This graph, which we denote by $U(n,k)$, can be properly colored using $\lceil n/2 \rceil$ colors. Indeed, every unstable k-subset of $[n]$ includes an odd element, hence by assigning to each vertex of $U(n,k)$ some odd element that belongs to its set, we obtain a proper coloring of the graph with the desired number of colors. Since $U(n,k)$ is a subgraph of $K(n,k)$, it follows that for all admissible values of n and k, we have $\chi(U(n,k)) \leq \min(n-2k+2, \lceil n/2 \rceil)$.

Motivated by the reduction given by Theorem 2, we further explore the graph $U(n,k)$, whose study may be of independent interest. We prove that the above upper bound on the chromatic number is essentially tight (up to an additive 1 in certain cases; see Corollary 19 and the discussion that follows it). The proof is topological and applies the Borsuk–Ulam theorem. We further determine the independence number of the graph $U(n,k)$ (see Theorem 20), using a structural result of Hilton and Milner [18] on the largest non-trivial intersecting families of k-subsets of $[n]$.

The motivation for Theorem 2 comes from the fact that the Schrijver problem is known to be PPA-hard, whereas no hardness result is known for the Kneser problem. It would be interesting to figure out whether or not the Schrijver(n, k, m) problem with $m = \lfloor n/2 \rfloor - 2k + 1$ admits an efficient algorithm. While this challenge is left open, the following result shows that the problem is not harder than the restriction of the standard Schrijver problem to colorings of $S(n, k)$ with $n = 4k$.

▶ **Theorem 3.** *If there exists a polynomial-time algorithm for the restriction of the Schrijver problem to colorings of $S(n, k)$ with $n = 4k$, then there exists a polynomial-time algorithm for the Schrijver(n, k, m) problem where $m = \lfloor n/2 \rfloor - 2k + 1$.*

We finally observe that the restriction of Schrijver(n, k, m) with $m = \lfloor n/2 \rfloor - 2k + 1$ to instances satisfying $n = \Omega(k^4)$ admits an efficient randomized algorithm. This essentially follows from the fixed-parameter algorithm presented in [17] (see Section 3 for details).

1.1.2 The Unfair Independent Set in Cycle Problem

The second problem studied in this paper is the Unfair Independent Set in Cycle problem, denoted by Unfair-IS-Cycle and defined as follows. Its input consists of two integers n and k with $n \geq 2k$ and ℓ subsets $V_1, \ldots, V_\ell$ of $[n]$, where $\ell \leq n - 2k + 1$ and $|V_i| \geq 2$ for all $i \in [\ell]$. The goal is to find a stable k-subset S of $[n]$ that satisfies the constraints $|S \cap V_i| \leq |V_i|/2$ for $i \in [\ell]$. The name of the problem essentially borrows the terminology of [1], where a set is said to fairly represent a set V_i if it includes at least roughly half of its elements, hence the desired stable set in the Unfair-IS-Cycle problem is required to *unfairly* represent each of the given sets V_i. It is not difficult to show, using the chromatic number of $S(n, k)$, that every input of the Unfair-IS-Cycle problem has a solution (see Lemma 13). Note that the requirement that the input sets satisfy $|V_i| \geq 2$ for all $i \in [\ell]$ is discussed in Section 2.4.

It is natural to compare the definition of the Unfair-IS-Cycle problem to that of the Fair Independent Set in Cycle problem, denoted by Fair-IS-Cycle and studied in [15] (see Definition 11). While the goal in the former is to find a stable subset of $[n]$ with a prescribed size k that includes *no more* than half of the elements of each V_i, the goal in the latter is, roughly speaking, to find a stable subset of $[n]$, of an arbitrary size, that includes *at least* half of the elements of each V_i. The specification of the size k in the inputs of Unfair-IS-Cycle makes the problem non-trivial and allows us to study it for various settings of the quantities n and k.

The following result shows that the complexity of the Unfair-IS-Cycle problem is perfectly captured by the class PPA. This is established using the Schrijver and Fair-IS-Cycle problems which are PPA-complete [15].

▶ **Theorem 4.** *The Unfair-IS-Cycle problem is PPA-complete.*

We next consider some restrictions of the Unfair-IS-Cycle problem to instances in which the integer n is somewhat larger than $2k$. On the one hand, the restriction of the problem to instances with $n = 3k$ is at least as hard as the Cycle-Plus-Triangles problem, for which no efficient algorithm is known (see Proposition 15). On the other hand, we prove that on instances whose ratio between n and k is above some absolute constant, the problem can be solved in polynomial time.

▶ **Theorem 5.** *There exists a constant $c > 0$, such that there exists a polynomial-time algorithm for the restriction of the Unfair-IS-Cycle problem to instances with $n \geq c \cdot k$.*

The proof of Theorem 5 is based on a probabilistic argument with alterations, which is derandomized into a deterministic algorithm using the method of conditional expectations (see, e.g., [5, Chapters 3 and 16.1]). The approach is inspired by a probabilistic argument of Kiselev and Kupavskii [19], who proved that for $n \geq (2+o(1)) \cdot k^2$, every proper coloring of the Kneser graph $K(n,k)$ with $n-2k+2$ colors has a trivial color class (all of whose members share a common element).

1.2 Outline

The rest of the paper is organized as follows. In Section 2, we collect some definitions and results that will be used throughout the paper. In Section 3, we study the generalized SCHRIJVER problem and prove Theorems 1, 2, and 3. In Section 4, we study the UNFAIR-IS-CYCLE problem and prove Theorems 4 and 5. Finally, in Section 5, we consider the family of unstable k-subsets of $[n]$ and study the chromatic and independence numbers of the graph $U(n,k)$. Some proofs are omitted and can be found in the full version of this paper.

2 Preliminaries

2.1 Kneser and Schrijver Graphs

For integers n and k, let $\binom{[n]}{k}$ denote the family of all k-subsets of $[n]$. A subset of $[n]$ is called *stable* if it does not include two consecutive elements nor both 1 and n, equivalently, it forms an independent set in the cycle on the vertex set $[n]$ with the natural order along the cycle. Otherwise, the set is called *unstable*. The family of stable k-subsets of $[n]$ is denoted by $\binom{[n]}{k}_{\text{stab}}$. The Kneser graph and the Schrijver graph are defined as follows.

▶ **Definition 6.** *For integers n and k with $n \geq 2k$, the* Kneser graph $K(n,k)$ *is the graph on the vertex set $\binom{[n]}{k}$, where two sets $A, B \in \binom{[n]}{k}$ are adjacent if they satisfy $A \cap B = \emptyset$. The* Schrijver graph $S(n,k)$ *is the subgraph of $K(n,k)$ induced by the vertices of $\binom{[n]}{k}_{\text{stab}}$.*

Obviously, the number of vertices in $K(n,k)$ is $\binom{n}{k}$. The number of vertices in $S(n,k)$ is given by the following lemma (see, e.g., [16, Fact 4.1]).

▶ **Lemma 7.** *For all integers n and k with $n \geq 2k$, the number of stable k-subsets of $[n]$ is $\frac{n}{k} \cdot \binom{n-k-1}{k-1}$.*

As usual, we denote the independence number of a graph G by $\alpha(G)$, and its chromatic number by $\chi(G)$. The chromatic numbers of $K(n,k)$ and $S(n,k)$ were determined, respectively, by Lovász [22] and by Schrijver [28], as stated below.

▶ **Theorem 8** ([22, 28])**.** *For all integers n and k with $n \geq 2k$,*

$$\chi(K(n,k)) = \chi(S(n,k)) = n - 2k + 2.$$

2.2 Intersecting Families

A family $\mathcal{F}$ of sets is called *intersecting* if for every two sets $A, B \in \mathcal{F}$ it holds that $A \cap B \neq \emptyset$. Note that a family of k-subsets of $[n]$ is intersecting if and only if it forms an independent set in the graph $K(n,k)$. An intersecting family $\mathcal{F}$ is said to be *trivial* if there exists an element that belongs to all members of $\mathcal{F}$. Otherwise, the family $\mathcal{F}$ is *non-trivial*. The famous Erdös-Ko-Rado theorem [11] asserts that the largest size of an intersecting family of k-subsets of $[n]$ is $\binom{n-1}{k-1}$, which is attained by the maximal trivial intersecting families. The following result of Hilton and Milner [18] determines the largest size of a non-trivial intersecting family in this setting and characterizes the extremal families attaining it.

▶ **Theorem 9** (Hilton–Milner Theorem [18]). *For integers $k \geq 3$ and $n \geq 2k$, let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a non-trivial intersecting family. Then,*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Moreover, if $n > 2k$ then equality holds if and only if there exist an element $i \in [n]$ and a k-subset A of $[n]$ with $i \notin A$ such that $\mathcal{F} = \left\{ F \in \binom{[n]}{k} \;\middle|\; i \in F,\ F \cap A \neq \emptyset \right\} \cup \{A\}$, or $k = 3$ and there exists a 3-subset A of $[n]$ such that $\mathcal{F} = \left\{ F \in \binom{[n]}{3} \;\middle|\; |F \cap A| \geq 2 \right\}$.

2.3 Complexity Classes

The complexity class TFNP consists of the total search problems in NP, i.e., the search problems in which every input has a solution, where a solution can be verified in polynomial time. The complexity class PPA (Polynomial Parity Argument [26]) consists of the problems in TFNP that can be reduced in polynomial time to a problem called LEAF. The definition of the LEAF problem is not needed in this paper, but we mention it briefly below for completeness.

The LEAF problem asks, given a graph with maximum degree 2 and a leaf (i.e., a vertex of degree 1), to find another leaf in the graph. The input graph, though, is not given explicitly. Instead, the vertex set of the graph is defined to be $\{0,1\}^n$ for some integer n, and the graph is succinctly represented by a Boolean circuit that for a vertex of the graph computes its (at most two) neighbors. Note that the size of the graph might be exponential in the size of its description.

2.4 Computational Problems

We gather here several computational problems that will be studied and used throughout the paper. We start with a computational search problem associated with Schrijver graphs. This problem is studied in Section 3.

▶ **Definition 10** (Generalized Schrijver Problem). *For $m = m(n,k)$, the SCHRIJVER(n,k,m) problem is defined as follows. The input is a coloring $c : \binom{[n]}{k}_{\text{stab}} \to [m]$ of the vertices of the graph $S(n,k)$ with m colors, and the goal is to find a monochromatic edge, i.e., two vertices $A, B \in \binom{[n]}{k}_{\text{stab}}$ such that $A \cap B = \emptyset$ and $c(A) = c(B)$. In the black-box input model, the coloring c is given as an oracle access that given a vertex A outputs its color $c(A)$. In the white-box input model, the coloring c is given by a Boolean circuit that for a vertex A computes its color $c(A)$. For $m = n - 2k + 1$, the problem SCHRIJVER(n,k,m) is denoted by SCHRIJVER.*

The KNESER problem is defined similarly to the SCHRIJVER problem. Here, the input coloring $c : \binom{[n]}{k} \to [n-2k+1]$ is defined on the entire vertex set of $K(n,k)$. By Theorem 8, every input of the SCHRIJVER and KNESER problems is guaranteed to have a solution. Moreover, whenever $m = m(n,k) \leq n - 2k + 1$, every input of the SCHRIJVER(n,k,m) problem has a solution as well.

We remark that algorithms for the SCHRIJVER(n,k,m) problem are considered in this paper with respect to the black-box input model. The running time of such an algorithm is referred to as polynomial if it is polynomial in n. Observe that a polynomial-time algorithm for the SCHRIJVER(n, k, m) problem in the black-box input model yields an algorithm for the analogue problem in the white-box input model, whose running time is polynomial as well (in the input size). For computational complexity results, like reductions and PPA-completeness, we adopt the more suitable white-box input model. For example, the SCHRIJVER problem in the white-box input model was shown in [15] to be PPA-complete.

Another search problem studied in [15] is the following.

▶ **Definition 11** (Fair Independent Set in Cycle Problem). *In the FAIR-IS-CYCLE problem, the input consists of integers n and m along with a partition $V_1, \ldots, V_m$ of $[n]$ into m sets. The goal is to find a stable subset S of $[n]$ satisfying $|S \cap V_i| \geq \frac{1}{2} \cdot |V_i| - 1$ for all $i \in [m]$.*

The existence of a solution for every input of the FAIR-IS-CYCLE problem was proved in [1]. It was shown in [15] that the FAIR-IS-CYCLE problem is PPA-complete, even restricted to instances in which each part V_i of the given partition has an odd size larger than 2.

We next define the UNFAIR-IS-CYCLE problem, studied in Section 4.

▶ **Definition 12.** *The input of the UNFAIR-IS-CYCLE problem consists of two integers n and k with $n \geq 2k$ and ℓ subsets $V_1, \ldots, V_\ell$ of $[n]$, where $\ell \leq n - 2k + 1$ and $|V_i| \geq 2$ for all $i \in [\ell]$. The goal is to find a stable k-subset S of $[n]$ that satisfies the constraints $|S \cap V_i| \leq |V_i|/2$ for $i \in [\ell]$.*

Note that Definition 12 requires the sets $V_1, \ldots, V_\ell$ of an instance of the UNFAIR-IS-CYCLE problem to satisfy $|V_i| \geq 2$ for all $i \in [\ell]$. This requirement is justified by the observation that if $|V_i| = 1$ for some $i \in [\ell]$, then any solution for the instance does not include the single element of V_i. Hence, by removing this element from the given sets and from the ground set, such an instance can be reduced to an instance with ground set of size smaller by one. By repeatedly applying this reduction, one can get a "core" instance that fits Definition 12.

We observe that the UNFAIR-IS-CYCLE problem is total. The argument relies on the chromatic number of the graph $S(n, k)$.

▶ **Lemma 13.** *Every instance of the UNFAIR-IS-CYCLE problem has a solution.*

Proof. Consider an instance of the UNFAIR-IS-CYCLE problem, i.e., integers n and k with $n \geq 2k$ and ℓ subsets $V_1, \ldots, V_\ell$ of $[n]$, where $\ell \leq n - 2k + 1$ and $|V_i| \geq 2$ for all $i \in [\ell]$. For every $i \in [\ell]$, let

$$\mathcal{F}_i = \left\{ S \in \binom{[n]}{k}_{\text{stab}} \;\middle|\; |S \cap V_i| > |V_i|/2 \right\},$$

and notice that every two sets of $\mathcal{F}_i$ have a common element of V_i, hence $\mathcal{F}_i$ is an intersecting family. However, by Theorem 8, the chromatic number of $S(n, k)$ is $n - 2k + 2$, hence the family of stable k-subsets of $[n]$ cannot be covered by fewer than $n - 2k + 2$ intersecting families. By $\ell \leq n - 2k + 1$, this implies that there exists a set $S \in \binom{[n]}{k}_{\text{stab}}$ that does not belong to any of the families $\mathcal{F}_i$, hence it satisfies $|S \cap V_i| \leq |V_i|/2$ for all $i \in [\ell]$. This implies that S is a valid solution for the given instance, and we are done. ◀

We end this section with the definition of the CYCLE-PLUS-TRIANGLES problem.

▶ **Definition 14** (Cycle plus Triangles Problem). *In the CYCLE-PLUS-TRIANGLES problem, the input consists of an integer k and a graph G on $3k$ vertices, whose edge set is the disjoint union of a Hamilton cycle and k pairwise vertex-disjoint triangles. The goal is to find an independent set in G of size k.*

The existence of a solution for every input of the CYCLE-PLUS-TRIANGLES problem follows from a result of [13] (see also [27, 1]).

3 The Generalized Schrijver Problem

In this section, we prove our results on the SCHRIJVER(n,k,m) problem (see Definition 10). We start with Theorem 1.

Proof of Theorem 1. Fix some integer $d \geq 2$. For integers n and k with $n \geq 2k$, put $t = \lfloor \frac{n}{2k+d-2} \rfloor$ and $m = d \cdot t - 1$, and consider an instance of the SCHRIJVER(n,k,m) problem, i.e., a coloring $c : \binom{n}{k}_{\text{stab}} \to [m]$ of the vertices of $S(n,k)$. The definition of t allows us to consider t pairwise disjoint subsets $J_1, \ldots, J_t$ of $[n]$, where each of the subsets includes $2k+d-2$ consecutive elements. For each $i \in [t]$, let $\mathcal{S}_i$ denote the family of all stable k-subsets of J_i with respect to the natural cyclic order of J_i (where the largest element precedes the smallest one), and notice that $\mathcal{S}_i \subseteq \binom{n}{k}_{\text{stab}}$. Consider the algorithm that given an oracle access to a coloring c as above, queries the oracle for the colors of all the sets of $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_t$, and returns a pair of disjoint sets from this collection that are assigned the same color by c.

For correctness, we show that the collection of sets $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_t$ necessarily includes two vertices that form a monochromatic edge. Indeed, since the number of colors used by the coloring c does not exceed $d \cdot t - 1$, it follows that either there exist distinct $i, j \in [t]$ for which a vertex of $\mathcal{S}_i$ and a vertex of $\mathcal{S}_j$ have the same color, or there exists an $i \in [t]$ for which the vertices of $\mathcal{S}_i$ are colored using fewer than d colors. For the former case, notice that for distinct i and j, every vertex of $\mathcal{S}_i$ is disjoint from every vertex of $\mathcal{S}_j$, hence the collection includes two vertices that form a monochromatic edge. For the latter case, let $i \in [t]$ be an index for which the vertices of $\mathcal{S}_i$ are colored using fewer than d colors. Observe that the subgraph of $S(n,k)$ induced by $\mathcal{S}_i$ is isomorphic to the graph $S(2k+d-2,k)$, hence by Theorem 8, its chromatic number is $(2k+d-2) - 2k + 2 = d$. Since the vertices of $\mathcal{S}_i$ are colored using fewer than d colors, it follows that they include two vertices that form a monochromatic edge, and we are done.

We finally analyze the running time of the algorithm. By Lemma 7, the number of vertices in the graph $S(n,k)$ is

$$\frac{n}{k} \cdot \binom{n-k-1}{k-1} = \frac{n}{k} \cdot \binom{n-k-1}{n-2k} \leq n \cdot (n-k-1)^{n-2k} \leq n^{n-2k+1}.$$

Since the subgraph of $S(n,k)$ induced by each $\mathcal{S}_i$ is isomorphic to $S(2k+d-2,k)$, it follows that the total number of queries that the algorithm makes does not exceed $t \cdot (2k+d-2)^{d-1} \leq n^{O(d)}$. This implies that in running time $n^{O(d)}$, it is possible to enumerate all the sets of $\mathcal{S}_1 \cup \cdots \cup \mathcal{S}_t$, to query the oracle for their colors, and to find the desired monochromatic edge. This completes the proof. ◀

We consider now the SCHRIJVER(n,k,m) problem with $m = \lfloor n/2 \rfloor - 2k + 1$. We first prove Theorem 2 that says that the problem is efficiently reducible to the KNESER problem (whose definition is given in Section 2.4).

Proof of Theorem 2. Put $m = \lfloor n/2 \rfloor - 2k + 1$, and let $c : \binom{[n]}{k}_{\text{stab}} \to [m]$ be an instance of the SCHRIJVER(n,k,m) problem. Consider the reduction that maps such a coloring c to a coloring $c' : \binom{[n]}{k} \to [n-2k+1]$ of the vertices of $K(n,k)$ defined as follows. For every set $A \in \binom{[n]}{k}$, if A is unstable then it includes an odd element, so denote its smallest odd element by $2i-1$, and define $c'(A) = i$. Notice that this i satisfies $1 \leq i \leq \lceil n/2 \rceil$. Otherwise, A is a stable k-subset of $[n]$, and we define $c'(A) = c(A) + \lceil n/2 \rceil$. Notice that $m + \lceil n/2 \rceil = n-2k+1$, hence the colors used by c' are all in $[n-2k+1]$, as needed for an instance of the KNESER problem. Notice further that given a Boolean circuit that computes the coloring c, it is possible to efficiently produce a Boolean circuit that computes the coloring c'.

For correctness, we simply show that any solution for the produced instance of the KNESER problem is also a solution for the given instance of the SCHRIJVER(n, k, m) problem. To see this, consider a solution for the former, i.e., two disjoint k-subsets A and B of $[n]$ with $c'(A) = c'(B)$. By the definition of c', the color assigned by c' to A and B cannot be some $i \le \lceil n/2 \rceil$ because this would imply that the element $2i - 1$ belongs to both A and B, which are disjoint. It thus follows that A and B are stable k-subsets of $[n]$ satisfying $c'(A) = c(A) + \lceil n/2 \rceil$ and $c'(B) = c(B) + \lceil n/2 \rceil$. By $c'(A) = c'(B)$, it follows that $c(A) = c(B)$, hence A and B form a monochromatic edge in $S(n, k)$ and thus a solution for the given instance of the SCHRIJVER(n, k, m) problem. This completes the proof. ◀

The reduction presented in the proof of Theorem 2 extends a given coloring of $S(n, k)$ to a coloring of the entire graph $K(n, k)$. To do so, it uses a proper coloring with $\lceil n/2 \rceil$ colors of the subgraph $U(n, k)$ of $K(n, k)$ induced by the *unstable* k-subsets of $[n]$. However, in order to obtain a coloring of $K(n, k)$ with $n - 2k + 1$ colors, as required for instances of the KNESER problem, one has to reduce from the SCHRIJVER(n, k, m) problem with $m = \lfloor n/2 \rfloor - 2k + 1$. This suggests the question of whether $U(n, k)$ can be properly colored using fewer colors. Motivated by this question, we study some properties of this graph in Section 5, where we essentially answer this question in the negative (see Corollary 19 and the discussion that follows it).

We next show that the SCHRIJVER(n, k, m) problem with $m = \lfloor n/2 \rfloor - 2k + 1$ is not harder than the restriction of the standard SCHRIJVER problem to colorings of $S(n, k)$ with $n = 4k$. This confirms Theorem 3.

Proof of Theorem 3. Suppose that there exists a polynomial-time algorithm, called Algo, for the restriction of the SCHRIJVER problem to colorings of $S(n, k)$ with $n = 4k$. Such an algorithm is able to efficiently find a monochromatic edge in the graph $S(4k, k)$ given an access to a coloring of its vertices with fewer than $\chi(S(4k, k))$ colors. By Theorem 8, it holds that $\chi(S(4k, k)) = 2k + 2$. Suppose without loss of generality that the algorithm Algo queries the oracle for the colors of the two vertices of the monochromatic edge that it returns.

For integers n and k with $n \ge 4k$, put $m = \lfloor n/2 \rfloor - 2k + 1$, and let $c : \binom{n}{k}_{\text{stab}} \to [m]$ be an instance of the SCHRIJVER(n, k, m) problem, i.e., a coloring of the vertices of $S(n, k)$ with m colors. We present an algorithm that finds a monochromatic edge in $S(n, k)$. It may be assumed that $n > 8k$. Indeed, otherwise it holds that $m \le 2k + 1 < \chi(S(4k, k))$, hence a monochromatic edge can be found by running the given algorithm Algo on the restriction of the coloring c to the subgraph of $S(n, k)$ induced by the stable k-subsets of $[4k]$. Since this graph is isomorphic to $S(4k, k)$, Algo is guaranteed to find a monochromatic edge in this subgraph, which also forms a monochromatic edge in the entire graph $S(n, k)$.

Now, put $t = \lfloor \frac{n}{4k} \rfloor$, and let $J_1, \ldots, J_t$ be t pairwise disjoint subsets of $[n]$, where each of the subsets includes $4k$ consecutive elements. For each $i \in [t]$, let $\mathcal{S}_i$ denote the family of all stable k-subsets of J_i with respect to the natural cyclic order of J_i (where the largest element precedes the smallest one). Observe that the subgraph of $S(n, k)$ induced by the vertices of each $\mathcal{S}_i$ is isomorphic to $S(4k, k)$. Observe further that

$$t \cdot (2k + 2) > \left(\frac{n}{4k} - 1 \right) \cdot (2k + 2) = \frac{n}{2} + \frac{n}{2k} - 2k - 2 > \left\lfloor \frac{n}{2} \right\rfloor - 2k + 1 = m, \tag{1}$$

where the last inequality holds because $n > 8k$.

Consider the algorithm that given an oracle access to a coloring c as above, for each $i \in [t]$, simulates the algorithm Algo on the restriction of the coloring c to the subgraph of $S(n, k)$ induced by the vertices of $\mathcal{S}_i$. If all the vertices queried throughout the ith simulation have at most $2k + 1$ distinct colors, then the algorithm returns the monochromatic edge returned by Algo. Otherwise, for each $i \in [t]$, the algorithm uses the queries made in the ith simulation of Algo to produce a set $\mathcal{F}_i \subseteq \mathcal{S}_i$ of $2k+2$ vertices with distinct colors. Then, the algorithm finds a monochromatic edge that involves two vertices of $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_t$ and returns it. This completes the description of the algorithm. Since the running time of Algo is polynomial, the described algorithm can be implemented in polynomial time.

Let us prove the correctness of the algorithm. Suppose first that for some $i \in [t]$, all the vertices queried throughout the ith simulation of Algo have at most $2k+1$ distinct colors (including the two vertices of the returned edge). In this case, the answers of the oracle in the ith simulation is consistent with some coloring with at most $2k+1$ colors of the subgraph of $S(n,k)$ induced by $\mathcal{S}_i$. Since this graph is isomorphic to $S(4k,k)$, whose chromatic number is $2k+2$, Algo is guaranteed to find in the graph a monochromatic edge, which is also a monochromatic edge in $S(n,k)$, and thus a valid output of the algorithm. Otherwise, for each $i \in [t]$, the attempt to simulate Algo on the subgraph of $S(n,k)$ induced by $\mathcal{S}_i$ provides a set $\mathcal{F}_i$ of $2k+2$ vertices of $\mathcal{S}_i$ with distinct colors. By (1), the total number m of colors used by the coloring c is smaller than $t \cdot (2k+2)$. This implies that there exist distinct indices $i, j \in [t]$ for which a vertex of $\mathcal{F}_i$ and a vertex of $\mathcal{F}_j$ have the same color. Since the vertices of $\mathcal{S}_i$ are disjoint from those of $\mathcal{S}_j$, these two vertices form a monochromatic edge in $S(n,k)$ and form a valid output of the algorithm. ◀

We end this section with the observation that there exists an efficient randomized algorithm for the Schrijver(n,k,m) problem with $m = \lfloor n/2 \rfloor - 2k + 1$ on instances with $n = \Omega(k^4)$. This follows from the paper [17], which yields that for such n and k, the Schrijver(n,k,m) problem is essentially reducible to the Schrijver$(n-1,k,m-1)$ problem in randomized polynomial time (with exponentially small failure probability). By applying this reduction $m-1$ times, it follows that the Schrijver(n,k,m) problem with $m = \lfloor n/2 \rfloor - 2k + 1$ where $n = \Omega(k^4)$, is efficiently reducible to the Schrijver$(\lceil n/2 \rceil + 2k, k, 1)$ problem, which can obviously be solved efficiently.

4 The Unfair Independent Set in Cycle Problem

In this section, we study the Unfair-IS-Cycle problem (see Definition 12).

4.1 Hardness

We prove now Theorem 4, which asserts that Unfair-IS-Cycle is PPA-complete.

Proof of Theorem 4. We first show that the Unfair-IS-Cycle problem belongs to PPA. To do so, we show a polynomial-time reduction to the Schrijver problem in the white-box input model, which lies in PPA [15] (see Definition 10).

Consider an instance of the Unfair-IS-Cycle problem, i.e., integers n and k with $n \geq 2k$ and ℓ subsets $V_1, \ldots, V_\ell$ of $[n]$, where $\ell \leq n - 2k + 1$ and $|V_i| \geq 2$ for all $i \in [\ell]$. For such an instance, the reduction produces a Boolean circuit that given a stable k-subset A of $[n]$, outputs the smallest index $i \in [\ell]$ such that $|A \cap V_i| > |V_i|/2$ if such an i exists, and outputs ℓ otherwise. Note that this circuit represents a coloring $c : \binom{[n]}{k}_{\text{stab}} \to [\ell]$ of the vertices of the graph $S(n,k)$ with $\ell \leq n - 2k + 1$ colors, hence it is an appropriate instance of the Schrijver problem. Clearly, the Boolean circuit that computes c can be constructed in polynomial time.

For correctness, we show that a solution for the constructed Schrijver instance can be used to efficiently find a solution for the given Unfair-IS-Cycle instance. Consider a monochromatic edge of $S(n,k)$, i.e., two disjoint sets $A, B \in \binom{[n]}{k}_{\text{stab}}$ with $c(A) = c(B)$.

Since A and B are disjoint, it is impossible that $|A \cap V_i| > |V_i|/2$ and $|B \cap V_i| > |V_i|/2$ for some $i \in [\ell]$. By the definition of the coloring c, it follows that $c(A) = c(B) = \ell$, hence $|A \cap V_i| \leq |V_i|/2$ and $|B \cap V_i| \leq |V_i|/2$ for all $i \in [\ell - 1]$. Moreover, at least one of A and B intersects V_ℓ at no more than $|V_\ell|/2$ elements, and thus forms a valid solution for the given Unfair-IS-Cycle instance. Since it is possible to check in polynomial time which of the sets A and B satisfies this requirement, the proof of the membership of Unfair-IS-Cycle in PPA is completed.

We next prove that the Unfair-IS-Cycle problem is PPA-hard. To do so, we reduce from the Fair-IS-Cycle problem (see Definition 11). We use here the fact, proved in [15], that this problem is PPA-hard even when it is restricted to the instances in which the parts of the given partition have odd sizes larger than 2. Consider such an instance of the Fair-IS-Cycle problem, i.e., integers n and m along with a partition $V_1, \ldots, V_m$ of $[n]$ such that $|V_i|$ is odd and satisfies $|V_i| \geq 3$ for all $i \in [m]$. Notice that n and m have the same parity, and define $k = \frac{n-m}{2}$. Our reduction simply returns the integers n and k, which clearly satisfy $n \geq 2k$, and the sets $V_1, \ldots, V_m$. Note that $|V_i| \geq 2$ for all $i \in [m]$ and that the number m of sets is $n - 2k$. Since the latter does not exceed $n - 2k + 1$, this is a valid instance of the Unfair-IS-Cycle problem.

For correctness, we show that a solution for the constructed Unfair-IS-Cycle instance is also a solution for the given Fair-IS-Cycle instance. Let S be a solution for the Unfair-IS-Cycle instance, i.e., a stable k-subset of $[n]$ such that for all $i \in [m]$ it holds that $|S \cap V_i| \leq |V_i|/2$. Since the sizes of the sets $V_1, \ldots, V_m$ are odd, it follows that $|S \cap V_i| \leq \frac{|V_i|-1}{2}$ for all $i \in [m]$. Since the sets $V_1, \ldots, V_m$ form a partition of $[n]$, it further follows that

$$|S| = \sum_{i \in [m]} |S \cap V_i| \leq \sum_{i \in [m]} \frac{|V_i|-1}{2} = \frac{n-m}{2} = k. \tag{2}$$

However, by $|S| = k$, we derive from (2) that $|S \cap V_i| = \frac{|V_i|-1}{2}$ for all $i \in [m]$. This implies that S is a stable k-subset of $[n]$ satisfying $|S \cap V_i| \geq |V_i|/2 - 1$ for all $i \in [m]$, hence it forms a valid solution for the given Fair-IS-Cycle instance. This completes the proof. ◀

Given the PPA-hardness of the Unfair-IS-Cycle problem, it is interesting to identify the range of the parameters n and k for which the hardness holds. One can verify, using properties of the hard instances constructed in [15], that the hardness given in Theorem 4 holds for instances with $n = (2 + o(1)) \cdot k$, where the $o(1)$ term tends to 0 as n and k tend to infinity. The following simple result shows that for $n = 3k$ the problem is at least as hard as the Cycle-Plus-Triangles problem, whose tractability is an open question (see Definition 14). The proof can be found in the full version of this paper.

▶ **Proposition 15.** *The Cycle-Plus-Triangles problem is polynomial-time reducible to the restriction of the Unfair-IS-Cycle problem to instances that consist of k sets of size 3 that form a partition of $[n]$ where $n = 3k$.*

4.2 Algorithms

We next prove Theorem 5, which states that the Unfair-IS-Cycle problem can be solved efficiently on instances with $n \geq c \cdot k$ for some absolute constant c.

Proof of Theorem 5. We start by presenting a randomized algorithm, based on a probabilistic argument with alterations, and then derandomize it using the method of conditional expectations.

Consider an instance of the UNFAIR-IS-CYCLE problem, i.e., integers n and k with $n \geq 2k$ and ℓ subsets $V_1, \ldots, V_\ell$ of $[n]$, where $\ell \leq n - 2k + 1$ and $|V_i| \geq 2$ for all $i \in [\ell]$. Put $r_i = |V_i| \geq 2$ for each $i \in [\ell]$. Suppose further that $n \geq c \cdot k$ for a sufficiently large constant c to be determined later. Let $p = 2k/n \leq 2/c$, and consider the following randomized algorithm.

1. Pick a random subset A of $[n]$ by including in A every element of $[n]$ independently with probability p.
2. Remove from A every element $j \in [n]$ that satisfies $\{j, j+1\} \subseteq A$ (where for $j = n$, the element $j+1$ is considered as 1). Let A' denote the obtained set.
3. For every $i \in [\ell]$ that satisfies $|A' \cap V_i| > r_i/2$, remove from A' arbitrary $|A' \cap V_i| - \lfloor r_i/2 \rfloor$ elements of V_i. Let A'' denote the obtained set.
4. If $|A''| \geq k$, then return an arbitrary k-subset of A''. Otherwise, return "failure".

We first claim that unless the algorithm returns "failure", it returns a valid output. Indeed, Item 2 of the algorithm guarantees that the set A' is stable. Further, Item 3 guarantees that its subset A'' satisfies $|A'' \cap V_i| \leq \lfloor r_i/2 \rfloor$ for all $i \in [\ell]$. Therefore, in the case where $|A''| \geq k$, any k-subset of A'' returned in Item 4 of the algorithm is a valid solution for the given UNFAIR-IS-CYCLE instance.

We next estimate the expected size of the set A'' produced by the algorithm. The set A chosen in Item 1 of the algorithm includes every element of $[n]$ with probability p. Hence, its expected size satisfies $\mathbf{E}[|A|] = p \cdot n$. In Item 2 of the algorithm, the probability of every element of $[n]$ to be removed from A is equal to the probability that both the element and its successor modulo n belong to A, which is p^2. By linearity of expectation, this implies that the expected size of the set A' satisfies $\mathbf{E}[|A'|] = (p - p^2) \cdot n$. It remains to estimate the expected number of elements removed from A' in Item 3 of the algorithm. Observe that for each $i \in [\ell]$, the algorithm removes from A' the smallest possible number of elements of V_i ensuring that the obtained set A'' includes at most $\lfloor r_i/2 \rfloor$ of them. Therefore, the number of removed elements of V_i does not exceed the number of subsets of V_i of size $\lfloor r_i/2 \rfloor + 1$ that are contained in A (because it suffices to remove one element from each of them). It thus follows that the expected number of elements of V_i that are removed from A' in Item 3 of the algorithm is at most

$$\binom{r_i}{\lfloor r_i/2 \rfloor + 1} \cdot p^{\lfloor r_i/2 \rfloor + 1} \leq 2^{r_i} \cdot p^{\lfloor r_i/2 \rfloor + 1} \leq (4p)^{\lfloor r_i/2 \rfloor + 1} \leq (4p)^2,$$

where in the last inequality we use the assumption $r_i \geq 2$ and the fact that $p \leq 1/4$ (which holds for any sufficiently large choice of the constant c). It therefore follows, using again the linearity of expectation, that the expected size of A'' satisfies

$$\mathbf{E}[|A''|] \geq (p - p^2) \cdot n - \ell \cdot (4p)^2 \geq (p - 17p^2) \cdot n \geq k,$$

where the second inequality holds by $\ell \leq n$, and the last inequality by the definition of $p = 2k/n$, assuming again that $n \geq c \cdot k$ for a sufficiently large constant c (say, $c = 68$). This implies that there exists a random choice for the presented randomized algorithm for which it returns a valid solution.

We next apply the method of conditional expectations to derandomize the above algorithm. Let us start with a few notations. For a set $S \subseteq [n]$, define

$$f(S) = |S| - |\{j \in [n] \mid \{j, j+1\} \subseteq S\}| - \sum_{i \in [\ell]} \left| \left\{ B \subseteq S \cap V_i \;\middle|\; |B| = \lfloor r_i/2 \rfloor + 1 \right\} \right|. \quad (3)$$

In words, $f(S)$ is determined by subtracting from the size of S the number of pairs of consecutive elements in S (modulo n) as well as the number of subsets of $S \cap V_i$ of size $\lfloor r_i/2 \rfloor + 1$ for each $i \in [\ell]$. For a vector $x \in \{0, 1, *\}^n$, let S_x denote a random subset of $[n]$ such that for every $i \in [n]$, if $x_i = 1$ then $i \in S_x$, if $x_i = 0$ then $i \notin S_x$, and if $x_i = *$ then i is chosen to be included in S_x independently with probability $p = 2k/n$. We refer to the vector x as a *partial choice* of a subset of $[n]$. We further define a potential function $\phi : \{0, 1, *\}^n \to \mathbb{R}$ that maps every vector $x \in \{0, 1, *\}^n$ to the expected value of $f(S)$ where S is chosen according to the distribution S_x, that is, $\phi(x) = \mathbf{E}[f(S_x)]$.

We observe that given a partial choice $x \in \{0, 1, *\}^n$, the value of $\phi(x)$ can be calculated efficiently, in time polynomial in n. Indeed, to calculate the expected value of $f(S_x)$, it suffices, by linearity of expectation, to calculate the expected value of each of the three terms in (3) evaluated at the set S_x. It is easy to see that the expected value of the first term is

$$|\{j \in [n] \mid x_j = 1\}| + p \cdot |\{j \in [n] \mid x_j = *\}|,$$

and that the expected value of the second term is

$$\left|\left\{j \in [n] \middle| x_j = x_{j+1} = 1\right\}\right| \quad + \quad p \cdot \left|\left\{j \in [n] \middle| \{x_j, x_{j+1}\} = \{1, *\}\right\}\right| \\ + \quad p^2 \cdot \left|\left\{j \in [n] \middle| x_j = x_{j+1} = *\right\}\right|.$$

As for the third term, by linearity of expectation, it suffices to determine the expected value of

$$\left|\left\{B \subseteq S_x \cap V_i \;\middle|\; |B| = \lfloor r_i/2 \rfloor + 1\right\}\right|$$

for $i \in [\ell]$. Letting $s_i = |\{j \in V_i \mid x_j = *\}|$ and $t_i = |\{j \in V_i \mid x_j = 1\}|$, one can check that the required expectation is precisely

$$\sum_{m=0}^{\lfloor r_i/2 \rfloor + 1} \binom{s_i}{m} \cdot \binom{t_i}{\lfloor r_i/2 \rfloor + 1 - m} \cdot p^m.$$

Since all the terms can be calculated in time polynomial in n, so can $\phi(x)$.

We describe a deterministic algorithm that finds a set $S \subseteq [n]$ satisfying $f(S) \geq k$. Given such a set, the algorithm is completed by applying Items 2, 3, and 4 of the algorithm presented above. Indeed, by applying Items 2 and 3 we obtain a stable set S'' such that $|S'' \cap V_i| \leq r_i/2$ for all $i \in [\ell]$. The fact that $f(S) \geq k$ guarantees that this set S'' satisfies $|S''| \geq k$, hence Item 4 returns a valid solution.

To obtain the desired set $S \subseteq [n]$ with $f(S) \geq k$, our algorithm maintains a partial choice $x \in \{0, 1, *\}^n$ satisfying $\phi(x) \geq k$. We start with $x = (*, \ldots, *)$, for which the analysis of the randomized algorithm guarantees that $\phi(x) \geq k$, provided that $n \geq c \cdot k$ for a sufficiently large constant c. We then choose the entries of x, one by one, to be either 0 or 1. In the ith iteration, in which $x_1, \ldots, x_{i-1} \in \{0, 1\}$, the algorithm evaluates ϕ at the two partial choices $x_{i \leftarrow 0} = (x_1, \ldots, x_{i-1}, 0, *, \ldots, *)$ and $x_{i \leftarrow 1} = (x_1, \ldots, x_{i-1}, 1, *, \ldots, *)$, and continues to the next iteration with one of them which maximizes the value of ϕ. By the law of total expectation, it holds that $\phi(x) = p \cdot \phi(x_{i \leftarrow 1}) + (1 - p) \cdot \phi(x_{i \leftarrow 0})$, implying that the choice of the algorithm preserves the inequality $\phi(x) \geq k$. At the end of the process, we get a vector $x \in \{0, 1\}^n$ with $\phi(x) \geq k$, which fully determines the desired set S with $f(S) \geq k$. Since the evaluations of ϕ can be calculated in time polynomial in n, the algorithm can be implemented in polynomial time. This completes the proof. ◀

Given the above result, it would be interesting to determine the smallest constant c for which the UNFAIR-IS-CYCLE problem can be solved efficiently on instances with $n \geq c \cdot k$. Of particular interest is the restriction of the problem to instances with $n = 3k$ and with pairwise disjoint sets of size 3, because as follows from Proposition 15, an efficient algorithm for this restriction would imply an efficient algorithm for the CYCLE-PLUS-TRIANGLES problem. Interestingly, it turns out that the restriction of the UNFAIR-IS-CYCLE problem to instances with $n = 4k$ and with pairwise disjoint sets of size 4 does admit an efficient algorithm. This is a consequence of the following result derived from an argument of Alon [2] (see also [4]).

▶ **Proposition 16.** *There exists a polynomial-time algorithm that given an integer k and a partition of $[4k]$ into k subsets $V_1, \ldots, V_k$ with $|V_i| = 4$ for all $i \in [k]$, finds a partition of $[4k]$ into four stable k-subsets S_1, S_2, S_3, S_4 of $[4k]$ such that $|S_j \cap V_i| = 1$ for all $j \in [4]$ and $i \in [k]$.*

5 Unstable Sets

In this section, we consider two subgraphs of the Kneser graph $K(n, k)$ induced by families of unstable k-subsets of $[n]$. These subgraphs are defined as follows.

▶ **Definition 17.** *Let n and k be integers with $n \geq 2k$. Let $\widetilde{U}(n, k)$ denote the subgraph of $K(n, k)$ induced by the family of all k-subsets of $[n]$ that include a pair of consecutive elements (where the elements n and 1 are* not *considered as consecutive for $n > 2$). Let $U(n, k)$ denote the subgraph of $K(n, k)$ induced by the family of all k-subsets of $[n]$ that include a pair of consecutive elements modulo n, i.e., the family of unstable k-subsets of $[n]$.*

5.1 Chromatic Number

We study now the chromatic numbers of the graphs $U(n, k)$ and $\widetilde{U}(n, k)$. It is worth mentioning here that a result of Dol'nikov [8] generalizes the lower bound of Lovász [22] on the chromatic number of $K(n, k)$ to general graphs, using a notion called colorability defect (see also [24, Chapter 3.4] and [21]). This generalization implies a tight lower bound of $n - 2k + 2$ on the chromatic number of $K(n, k)$ and a somewhat weaker lower bound of $n - 4k + 4$ on the chromatic number of $S(n, k)$ (see, e.g., [25]). It turns out, though, that this generalized approach of [8] does not yield any meaningful bounds on the chromatic numbers of the graphs from Definition 17.

The following theorem determines the exact chromatic number of the graph $\widetilde{U}(n, k)$.

▶ **Theorem 18.** *For all integers n and k with $n \geq 2k$,*

$$\chi(\widetilde{U}(n, k)) = \min(n - 2k + 2, \lfloor n/2 \rfloor).$$

The proof of Theorem 18 relies on a topological argument and can be found in the full version of the paper. We derive the following result on the chromatic number of $U(n, k)$.

▶ **Corollary 19.** *For all integers n and k with $n \geq 2k$,*

$$\min(n - 2k + 2, \lfloor n/2 \rfloor) \leq \chi(U(n, k)) \leq \min(n - 2k + 2, \lceil n/2 \rceil).$$

Proof. For the upper bound, apply first Theorem 8 to obtain that

$$\chi(U(n, k)) \leq \chi(K(n, k)) = n - 2k + 2.$$

Next, since every vertex of $U(n,k)$ includes two consecutive elements modulo n, it must include an odd element. By assigning to every such vertex its minimal odd element, we obtain a proper coloring of $U(n,k)$ with $\lceil n/2 \rceil$ colors, hence $\chi(U(n,k)) \leq \lceil n/2 \rceil$. This completes the proof of the upper bound. The lower bound follows by combining Theorem 18 with the fact that $\widetilde{U}(n,k)$ is an induced subgraph of $U(n,k)$. ◀

We conclude this section with a discussion on the tightness of Corollary 19. Notice that the upper and lower bounds provided in Corollary 19 coincide whenever the integer n is even or satisfies $n \leq 4k-4$. For other values of n and k the two bounds differ by 1. Yet, it turns out that the proof technique of Theorem 18 can be used to show that the upper bound in Corollary 19 is tight for all integers n that are congruent to 1 modulo 4. This leaves us with a gap of 1 between the upper and lower bounds in Corollary 19 only for those integers n and k, where n is congruent to 3 modulo 4 and satisfies $n \geq 4k-1$.

We further observe that for an odd integer n and for every proper coloring of $U(n,k)$ that includes a trivial color class (all of whose members share a common element), the number of used colors is at least the upper bound in Corollary 19. Indeed, the restriction of such a coloring to the vertices that do not include the common element of the trivial color class is a proper coloring of a graph isomorphic to $\widetilde{U}(n-1,k)$, so by Theorem 18 it uses at least $\min(n-2k+1,(n-1)/2)$ colors. Together with the additional color of the trivial color class, the total number of colors is at least $\min(n-2k+2, \lceil n/2 \rceil)$, as claimed.

5.2 Independence Number

We next determine the largest size of an independent set in the graph $U(n,k)$. The proof uses the Hilton–Milner theorem (Theorem 9) and can be found in the full version of the paper.

▶ **Theorem 20.** *For all integers $k \geq 2$ and $n \geq 2k$, it holds that*

$$\alpha(U(n,k)) = \binom{n-1}{k-1} - \binom{n-k-1}{k-1}.$$

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