Strongly Finitary Monads for Varieties of Quantitative Algebras

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Abstract

Quantitative algebras are algebras enriched in the category Met of metric spaces or UMet of ultrametric spaces so that all operations are nonexpanding. Mardare, Plotkin and Panangaden introduced varieties (aka 1-basic varieties) as classes of quantitative algebras presented by quantitative equations. We prove that, when restricted to ultrametrics, varieties bijectively correspond to strongly finitary monads T on UMet. This means that T is the left Kan extension of its restriction to finite discrete spaces. An analogous result holds in the category CUMet of complete ultrametric spaces.

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1 Introduction

Quantitative algebraic reasoning was formalized in a series of articles of Bacci, Mardare, Panangaden and Plotkin [5, 15, 16, 6] as a tool for studying computational effects in probabilistic computation. Those papers work with algebras in the category Met of metric spaces or CMet of complete metric spaces. *Quantitative algebras* are algebras acting on a (complete) metric space A so that every n-ary operation is a nonexpanding map from A^n, with the maximum metric, to A. If the underlying metric is an ultrametric, we speak about *ultra-quantitative algebras*. Mardare et al. introduced quantitative equations, which are formal expressions $t =_\varepsilon t'$ where t and t' are terms and $\varepsilon \geq 0$ is a rational number. A quantitative algebra A satisfies this equation iff for every interpretation of the variables the elements of A corresponding to t and t' have distance at most ε. A *variety* (called 1-basic variety in [15]) is a class of quantitative algebras presented by a set of quantitative equations. Classical varieties of algebras are well known to correspond bijectively to *finitary monads* $\mathbf{T}$ on Set (preserving directed colimits): every variety is isomorphic to the category $\mathsf{Set}^{\mathbf{T}}$ of

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algebras for $\mathbf{T}$, and vice versa. The question whether an analogous correspondence holds for quantitative algebras has been posed in [1] and [17]. For ultra-quantitative algebras we answer this by working with enriched (i.e. locally nonexpanding) monads on the category Met of metric spaces, and its full subcategories UMet of ultrametric spaces and CUMet of complete ultrametric spaces. An enriched monad is *strongly finitary* if it is a left Kan extension of its restriction to finite discrete spaces. We characterize these monads as the enriched finitary monads preserving precongruences. Every strongly finitary monad on Met, UMet or CUMet is proved to be the free-algebra monad of a variety of quantitative algebras (Theorem 52 and Theorem 56).

For UMet and CUMet we also prove the converse: for every variety of ultra-quantitative algebras the free-algebra monad is strongly finitary (Theorem 47). We conclude that varieties bijectively correspond to strongly finitary monads on UMet or CUMet. It is an open problem whether this also holds for Met.

Related Work

A closely related result holds for partially ordered algebras (with nonexpanding operations). Here varieties are presented by inequations between terms. Kurz and Velebil [13] proved that they bijectively correspond to strongly finitary monads on the category Pos of posets.

The main tool of Mardare et at. ([15, 16]) are *ω-basic equations*: for a finite set of expressions $x_i =_{\delta_i} y_i$ (where x_i, y_i are variables and $\delta_i \geq 0$) and for terms t and t' one writes $x_i =_{\delta_i} y_i \vdash t =_\varepsilon t'$. An algebra A satisfies this equation if, for every interpretation f of the variables satisfying $d(f(x_i), f(y_i)) \leq \delta_i$ for all i, the elements corresponding to t and t' have distance at most ε. A class of quantitative algebras presented by such equations is called an *ω-basic variety*. Unfortunately, the free-algebra monad of an ω-basic variety need not be finitary ([1], Example 4.1). Monads on UMet corresponding to ω-basic varieties were characterized in [1], Corollary 4.15.

Full proofs of the results presented in this extended abstract can be found in [3].

2 Strongly Finitary Functors

In this section we introduce strongly finitary functors, and present some of their properties. Later we prove a bijective correspondence of varieties and strongly finitary monads for UMet and CUMet.

▶ **Assumption 1.** Throughout our paper we work with categories and functors enriched over a symmetric monoidal closed category $(\mathscr{V}, \otimes, I)$. We recall these concepts shortly. Our leading examples of $\mathscr{V}$ are metric spaces, ultrametric spaces and partially ordered sets.

▶ **Definition 2** ([8], 6.12)**.** *A* symmetric monoidal closed category *is given by a category $\mathscr{V}$, a bifunctor $\otimes : \mathscr{V} \times \mathscr{V} \to \mathscr{V}$ and an object I. Moreover, natural isomorphisms are given expressing that $\otimes$ is commutative and associative, and has the unit I (all up to coherent natural isomorphisms). Finally, for every object Y a right adjoint of the functor $- \otimes Y : \mathscr{V} \to \mathscr{V}$ is given. We denote it by $[Y, -]$ and denote the morphism corresponding to $f : X \otimes Y \to Z$ by $\widehat{f} : Y \to [X, Z]$.*

Often $\otimes$ is the categorical product and I the terminal object; then $\mathscr{V}$ is *cartesian closed.*

▶ **Example 3.**

(1) $\mathscr{V} = \mathsf{Pos}$, the category of posets, is cartesian closed, $[X, Y]$ is the poset of all monotone maps $f : X \to Y$ ordered pointwise. Here $\widehat{f} = \mathsf{curry} f$ is the curried form of f.

(2) $\mathscr{V} = \mathsf{Met}$, the category of *(extended) metric spaces* and nonexpanding maps. Objects are metric spaces defined as usual, except that the distance ∞ is allowed. Nonexpanding maps are those maps $f : X \to Y$ with $d(x, x') \geq d(f(x), f(x'))$ for all $x, x' \in X$.
A product of metric spaces $X \times Y$ is the metric space on the cartesian product with the *maximum metric*

$$d((x, y), (x', y')) = \max\{d(x, x'), d(y, y')\}.$$

This category is not cartesian closed: curryfication is not bijective. However, Met is symmetric closed monoidal w.r.t. the *tensor product* $X \otimes Y$ which is the cartesian product with the *addition metric*

$$d((x, y), (x', y')) = d(x, x') + d(y, y').$$

Here $[X, Y]$ is the metric space $\mathsf{Met}(X, Y)$ of all morphisms $f : X \to Y$ with the *supremum metric*: the distance of $f, g : X \to Y$ is

$$d(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

And I is the singleton space.

(3) The cartesian closed category UMet of (extended) ultrametric spaces is the full subcategory of Met on spaces satisfying the following stricter triangle inequality:

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

Here the curryfication of morphisms $f : X \times Y \to Z$ to $\widehat{f} : Y \to [X, Z]$ is bijective.

(4) The category CMet of complete metric spaces is the full subcategory of Met on spaces with limits of all Cauchy sequences. It has the same symmetric closed monoidal structure as above: if X and Y are complete spaces, then so are $X \otimes Y$ and $[X, Y]$.
Analogously to (3) the category CUMet of complete ultrametric spaces is cartesian closed.

▶ **Convention 4.** By a *category* $\mathscr{C}$ we always mean a category enriched over $\mathscr{V}$. It is given by

(1) a class $\mathrm{ob}\mathscr{C}$ of objects,

(2) an object $\mathscr{C}(X, Y)$ of $\mathscr{V}$ (called the hom-object) for every pair X, Y in $\mathrm{ob}\mathscr{C}$,

(3) a 'unit' morphism $u_X : I \to \mathscr{C}(X, X)$ in $\mathscr{V}$ for every object $X \in \mathrm{ob}\mathscr{C}$, and

(4) 'composition' morphisms

$$c_{X,Y,Z} : \mathscr{C}(X, Y) \otimes \mathscr{C}(Y, Z) \to \mathscr{C}(X, Z)$$

for all $X, Y, Z \in \mathrm{ob}\mathscr{C}$, subject to commutative diagrams expressing the associativity of composition and the fact that u_X are units of composition. For details see [8], 6.2.1.

▶ **Example 5.**

(1) If $\mathscr{V} = \mathsf{Met}$ then $\mathscr{C}$ is an ordinary category in which every hom-set $\mathscr{C}(X, Y)$ carries a metric such that composition is nonexpanding. Analogously for $\mathscr{V} = \mathsf{CMet}$ or UMet.

(2) If $\mathscr{V} = \mathsf{Pos}$ then each hom-set $\mathscr{C}(X, Y)$ carries a partial order such that composition is monotone.

Let us recall the concept of an *enriched functor* $F : \mathscr{C} \to \mathscr{C}'$ for (enriched) categories $\mathscr{C}$ and $\mathscr{C}'$. It assigns

(1) an object $FX \in \mathrm{ob}\mathscr{C}'$ to every object $X \in \mathrm{ob}\mathscr{C}$, and

(2) a morphism $F_{X,Y} : \mathscr{C}(X, Y) \to \mathscr{C}'(FX, FY)$ of $\mathscr{V}$ to every pair $X, Y \in \mathrm{ob}\mathscr{C}$ so that the expected diagrams expressing that F preserves composition and identity morphisms commute.

▶ **Convention 6.** By a *functor* we always mean an enriched functor. We use 'ordinary functor' in the few cases where a non-enriched functor is meant.

▶ **Example 7.**

(1) For categories enriched over Met a functor $F : \mathscr{C} \to \mathscr{C}'$ is an ordinary functor which is *locally nonexpanding*: given $f, g \in \mathscr{C}(X, Y)$ we have $d(f, g) \geq d(Ff, Fg)$. Analogously for CMet or UMet.

(2) For categories enriched over Pos functors F are the *locally monotone* ordinary functors: given $f \leq g$ in $\mathscr{C}(X, Y)$, we get $Ff \leq Fg$ in $\mathscr{C}(FX, FY)$.

▶ Remark 8.

(1) In general one also needs the concept of an enriched natural transformation between parallel (enriched) functors. However, if $\mathscr{V}$ is one of the categories of Example 3, this concept is just that of an ordinary natural transformation between the underlying ordinary functors.

(2) Given two categories $\mathscr{D}, \mathscr{C}$, we denote by $[\mathscr{D}, \mathscr{C}]$ the category of all functors $F : \mathscr{D} \to \mathscr{C}$ enriched by assigning to every pair of functors $F, G : \mathscr{D} \to \mathscr{C}$ an appropriate object $[F, G]$ of $\mathscr{V}$ of all natural transformations.
In case $\mathscr{V}$ = Met, UMet or CMet the distance of $\tau, \tau' : F \to G$ in $[F, G]$ is $\sup_{X \in \mathrm{ob}\mathscr{D}} d(\tau_X, \tau'_X)$.

▶ **Notation 9.**

(1) Every set X is considered as a *discrete poset*: $x \sqsubseteq x'$ iff $x = x'$. This is the coproduct $\coprod_X I$ in Pos. Analogously, X is considered as a *discrete metric space*: all distances of $x \neq x'$ are ∞. This is the coproduct $\coprod_X I$ in Met (and also in UMet and CUMet).

(2) For the category $\mathsf{Set_f}$ of finite sets and mappings we define a functor

$$K : \mathsf{Set_f} \to \mathscr{V}, \qquad X \mapsto \coprod_X I.$$

Thus for $\mathscr{V}$ = Met, CMet, UMet or Pos it assigns to every finite set the corresponding discrete object.

(3) Let us recall the concept of the (enriched) *left Kan extension* of a functor $F : \mathscr{A} \to \mathscr{C}$ along a functor $K : \mathscr{A} \to \mathscr{C}$ [11]: this is an endofunctor $\mathrm{Lan}_K F : \mathscr{C} \to \mathscr{C}$ endowed with a universal natural transformation $\tau : F \to (\mathrm{Lan}_K F) \cdot K$. The universal property states that given a natural transformation $\sigma : F \to G \cdot K$ for any endofunctor $G : \mathscr{C} \to \mathscr{C}$, there exists a unique natural transformation $\overline{\sigma} : \mathrm{Lan}_K F \to G$ with $\sigma = \overline{\sigma} K \cdot \tau$. The functor $\mathrm{Lan}_K F$ is unique up to a natural isomorphism.

▶ **Definition 10** (Kelly and Lack [12]). *An endofunctor F of $\mathscr{V}$ is* strongly finitary *if it is a left Kan extension of its restriction $F \cdot K$ to $\mathsf{Set_f}$. Shortly: $F = \mathrm{Lan}_K(F \cdot K)$.*

▶ **Example 11.**

1. For every natural number n the endofunctor $(-)^n$ of the n-th power is strongly finitary on Met, UMet and CUMet.
2. A coproduct of strongly finitary functors is strongly finitary.

▶ **Theorem 12** ([12]). *If $\mathscr{V}$ is cartesian closed, then strongly finitary endofunctors are closed under composition.*

▶ **Open Problem 13.** Are all strongly finitary endofunctors on Met closed under composition?

In order to characterize strong finitarity for endofunctors on $\mathscr{V} = \mathsf{Met}$, UMet and CMet, we apply Kelly's concept of density presentation that we now recall. For that we first shortly recall weighted colimits.

▶ **Definition 14** ([8, 11]).

(1) *A* weighted diagram *in a category $\mathscr{C}$ is given by a functor $D : \mathscr{D} \to \mathscr{C}$ together with a weight $W : \mathscr{D}^{op} \to \mathscr{V}$. A* weighted colimit *is an object $C = colim_W D$ of $\mathscr{C}$ together with isomorphisms in $\mathscr{V}$:*

$$\psi_X : \mathscr{C}(C, X) \to [\mathscr{D}^{op}, \mathscr{C}](W, \mathscr{C}(D-, X))$$

natural in $X \in \mathrm{ob}\mathscr{C}$.

(2) *The* unit *of this colimit is the natural transformation $\nu = \psi_C(id_C) : W \to \mathscr{C}(D-, C)$.*

(3) *A functor $F : \mathscr{C} \to \mathscr{C}'$* preserves *this colimit if $colim_W(F \cdot D) = FC$ with the unit having components $F\nu_d$ for $d \in \mathscr{D}$.*

In all categories of Example 3 weighted colimits (for all $\mathscr{D}$ small) exist.

▶ **Example 15.** *(Conical) directed colimits* are the special case where $\mathscr{D}$ is a directed poset (every finite subset has an upper bound), and the weight W is trivial: the constant functor with value $\mathbf{1}$ (the terminal object).

(1) In Pos directed colimits are formed on the level of the underlying sets. They commute with finite products.

(2) Directed colimits in Met, UMet and CMet also exist, but they are not formed on the level of the underlying sets. For example, consider the diagram of metric space $A_n = \{0, 1\}$ with $d_n(0, 1) = 2^{-n}$, where the connecting maps are $id : A_n \to A_{n+1}$ $(n < \omega)$. The colimit is a singleton space.

▶ **Lemma 16.** *In* Met*,* UMet *and* CUMet *every space is a directed colimit of all of its finite subspaces.*

▶ **Theorem 17.** *Directed colimits in* Met*,* UMet *or* CMet *commute with finite products.*

Proof sketch.

(1) For a directed diagram $(D_i)_{i \in I}$ in Met, cocones $c_i : D_i \to C$ forming a colimit were characterized in [4], Lemma 2.4, by the following properties: (a) $C = \bigcup_{i \in I} c_i[D_i]$, and (b) for every $i \in I$, given $y, y' \in D_i$ we have $d(c_i(y), c_i(y')) = \inf_{j \geq i} d(f_j(y), f_j(y'))$, where $f_j : D_i \to D_j$ denotes the connecting map.
Given another directed diagram $(D'_i)_{i \in I}$ with a cocone $c'_i : D'_i \to C'$ satisfying (a) and (b), it is our task to prove that the cocone $c_i \times c'_i : D_i \times D'_i \to C \times C'$ satisfies (a), (b), too. Since I is directed, (a) is clear, and (b) needs just a short computation.

(2) The argument for UMet is the same.

(3) For directed colimits in CMet the characterization of colimit cocones is analogous: (b) is unchanged, and in (a) one states that $\bigcup_{i \in I} c_i[D_i]$ is dense in C. The proof is then analogous to (1). ◀

▶ **Definition 18.** *A functor is* finitary *if it preserves directed colimits.*

▶ **Example 19.**

(1) An endofunctor of Set is strongly finitary iff it is finitary.

(2) An endofunctor of Pos is strongly finitary iff it is finitary and preserves reflexive coinserters, see [2].

▶ **Notation 20.** Let $K : \mathscr{A} \to \mathscr{C}$ be a functor. We denote by $\widetilde{K} : \mathscr{C} \to [\mathscr{A}^{op}, \mathscr{V}]$ the functor with $\widetilde{K}C = \mathscr{C}(K-, C)$.

For example, the functor $K : \mathsf{Set_f} \to \mathsf{Met}$ yields $\widetilde{K} : \mathsf{Met} \to [\mathsf{Set_f}^{op}, \mathsf{Met}]$ taking a metric space M to the functor $M^{(-)} : \mathsf{Set_f}^{op} \to \mathsf{Met}$ of finite powers of M.

▶ **Definition 21** ([11])**.** *A* density presentation *of a functor $K : \mathscr{A} \to \mathscr{C}$ is a collection of weighted colimits in $\mathscr{C}$ such that*
(a) *$\widetilde{K}$ preserves those colimits, and*
(b) *$\mathscr{C}$ is the (iterated) closure of the image $K[\mathscr{A}]$ under those colimits.*

▶ **Example 22.** A density presentation of the functor $K : \mathsf{Set_f} \to \mathsf{Met}$ (Notation 9) is given by all directed colimits and all *precongruences* (a name borrowed from [9]) which we now present. They express every metric space as a colimit of discrete spaces. (The weight used for precongruence is, however, not discrete.)

▶ **Notation 23.** For every metric space M let $|M|$ denote its underlying set (a discrete metric space).

▶ **Definition 24.**
(1) *We define the* basic weight $W_0 : \mathscr{D}_0^{op} \to \mathsf{Met}$ *as follows. The category $\mathscr{D}_0$ consists of*
  a. *the linearly ordered set of all rational numbers $\varepsilon \geq 0$,*
  b. *two parallel cocones of it $\lambda_\varepsilon, \rho_\varepsilon : \varepsilon \to a$, and*
  c. *a morphism $\sigma_\varepsilon : a \to \varepsilon$ splitting that pair: $\lambda_\varepsilon \cdot \sigma_\varepsilon = id = \rho_\varepsilon \cdot \sigma_\varepsilon$ (for all ε). The posets $\mathscr{D}_0(\lambda_\varepsilon, \rho_\varepsilon)$ are all discrete.*

  The values of W_0 are $W_0 a = \{0\}$ and $W_0 \varepsilon = \{l, r\}$ with $d(l, r) = \varepsilon$. The morphisms $W_0\lambda_\varepsilon, W_0\rho_\varepsilon : \{0\} \to \{l, r\}$ are given by $0 \mapsto l$, $0 \mapsto r$, respectively, and $W_0\sigma_\varepsilon$ is clear.
(2) *For every metric space M we define its* precongruence *as the weighted diagram $D_M : \mathscr{D}_0 \to \mathsf{Met}$ with the basic weight W_0, where $D_M a = |M|$ and $D_M\varepsilon \subseteq |M| \times |M|$ is the discrete space of all pairs of distance at most ε. Here $D\lambda_\varepsilon, D\rho_\varepsilon : D_M\varepsilon \to |M|$ are the projections π_l and π_r, respectively, and $D_0\sigma_\varepsilon : |M| \to D_M\varepsilon$ is the diagonal. The diagram D_M assigns to the morphism $\varepsilon \leq \varepsilon'$ the inclusion map of the subset $D_M\varepsilon$ of $D_M\varepsilon'$.*

▶ **Proposition 25.** *Every metric space M is the weighted colimit of its precongruence in* Met*.*

Proof. For every space X, to give a natural transformation $\tau : W_0 \to [\mathscr{D}_0^{op}, \mathsf{Met}](D_M-, X)$ means to specify a map $f = \tau_a(0) : |M| \to X$ together with maps $\tau_\varepsilon(l), \tau_\varepsilon(r) : D_M\varepsilon \to X$ such that $\tau_\varepsilon(l) = f \cdot \pi_l$ and $\tau_\varepsilon(r) = f \cdot \pi_r$. Thus τ is determined by f, and the last equations are equivalent to $f : M \to X$ being nonexpanding. The desired isomorphism ψ_X of Definition 14 is given by $\psi_X(\tau) = f$. ◀

▶ Remark 26. To define precongruences in UMet, we just use the codomain restrictions $W_0 : \mathscr{D}_0^{op} \to \mathsf{UMet}$ and $D_M : \mathscr{D}_0 \to \mathsf{UMet}$. Again, every ultrametric space is the weighted colimit of its precongruence in UMet. Analogously for CUMet.

▶ **Example 27.** The categories Met, UMet, CMet and CUMet have a density presentation of K (Notation 9) consisting of all directed diagrams and precongruences of finite spaces. Indeed, in Definition 21 Condition (a) follows from Example 15. For Condition (b) observe that finite metric spaces are obtained from $\mathsf{Set_f}$ as colimits of precongruences by Proposition 25, and every metric space is a directed colimit of all of its finite subspaces in Met. Analogously for the three subcategories of Met.

The importance of the concept of density presentation for our paper stems from the following result of Kelly:

▶ **Theorem 28** ([11], Theorem 5.29)**.** *Given a density presentation of a functor $K : \mathscr{A} \to \mathscr{C}$, an endofunctor T of $\mathscr{C}$ fulfils $T = \mathrm{Lan}_K(T \cdot K)$ iff it preserves the colimits of that presentation.*

▶ **Corollary 29.** *An endofunctor of* Met*,* UMet*,* CMet *or* CUMet *is strongly finitary iff it preserves directed colimits and colimits of precongruences.*

This follows from Theorem 28 and the example above.

3 Varieties of Quantitative Algebras

We now prove that varieties of ultra-quantitative algebras bijectively correspond to *strongly finitary monads* on UMet. These are monads carried by a strongly finitary endofunctor. Throughout this section $\Sigma = (\Sigma_n)_{n\in\mathbb{N}}$ denotes a signature, and V is a specified countable set of variables.

▶ **Notation 30.**

(1) Following Mardare, Panangaden and Plotkin [15], a *quantitative algebra* is a metric space A endowed with a nonexpanding operation $\sigma_A : A^n \to A$ for every $\sigma \in \Sigma_n$ (w.r.t. the maximum metric (Example 3)). We denote by Σ-Met the category of quantitative algebras and nonexpanding homomorphisms. Its forgetful functor is denoted by U_Σ : Σ-Met $\to$ Met.

(2) If A is an ultrametric space we speak about an *ultra-quantitative algebra* and denote the corresponding category by Σ-UMet.

(3) Analogously, a *complete ultra-quantitative algebra* is an ultra-quantitative algebra carried by a complete metric space. The category Σ-CUMet is the corresponding full subcategory of Σ-UMet. We again use U_Σ : Σ-CUMet $\to$ CUMet for the forgetful functor.

▶ **Example 31.**

(1) A free quantitative algebra on a metric space M is the usual algebra $T_\Sigma M$ of *terms* on variables from $|M|$. That is, the smallest set containing $|M|$ and such that for every n-ary symbol σ and every n-tuple of terms t_i $(i < n)$ we obtain a composite term $\sigma(t_i)_{i<n}$. To describe the metric, let us introduce the following equivalence $\sim$ on $T_\Sigma M$ (*similarity* of terms): it is the smallest equivalence turning all variables of $|M|$ into one class, and such that $\sigma(t_i)_{i<n} \sim \sigma'(t'_i)_{i<n'}$ holds iff $\sigma = \sigma'$ and $t_i \sim t'_i$ for all $i < n$. The metric of $T_\Sigma M$ extends that of M as follows: $d(t, t') = \infty$ if t is not similar to t'. For similar terms $t = \sigma(t_i)$ and $t' = \sigma(t'_i)$ we put $d(t, t') = \max_{i<n} d(t_i, t'_i)$.

(2) If M is an ultrametric space, the space $T_\Sigma M$ is clearly ultrametric, too. This is the free quantitative algebra in Σ-UMet.

(3) If M is a complete space, $T_\Sigma M$ is also complete, and this is the free quantitative algebra on M in Σ-CMet.

In particular, if we consider the specified set V of variables as a discrete metric space, then $T_\Sigma V$ is the discrete algebra of usual terms. For every algebra A and every interpretation of variables $f : V \to A$ (in Met, UMet or CUMet) we denote by $f^\sharp : T_\Sigma V \to A$ the corresponding homomorphism: it interprets terms in A.

▶ **Definition 32** ([15]). *By a* quantitative equation *(aka* 1*-basic quantitative equation) is meant a formal expression* $t =_\varepsilon t'$ *where* t, t' *are terms in* $T_\Sigma V$ *and* $\varepsilon \geq 0$ *is a rational number. An algebra* A *in* Σ-Met *(*Σ-UMet *or* Σ-CUMet*)* satisfies *that equation if for every interpretation* $f : V \to A$ *we have* $d(f^\sharp(t), f^\sharp(t')) \leq \varepsilon$*. We write* $t = t'$ *in case* $\varepsilon = 0$*.*

By a variety*, aka* 1*-basic variety, of quantitative (or ultra-quantitative or complete ultra-quantitative) algebras is meant a full subcategory of* Σ-Met *(or* Σ-UMet *or* Σ-CUMet*, resp.) specified by a set of quantitative equations.*

▶ **Example 33.**

(1) *Quantitative monoids* are given by the usual signature: a binary symbol $\cdot$ and a constant e, and by the usual equations: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, $e \cdot x = x$, and $x \cdot e = x$.

(2) *Almost commutative monoids* are quantitative monoids in which the distance of ab and ba is always at most 1. They are presented by the quantitative equation $x \cdot y =_1 y \cdot x$.

(3) *Quantitative semilattices* are commutative, idempotent quantitative monoids, see [15], Section 9.1.

▶ **Proposition 34** (See [15]). *Every variety* $\mathcal{V}$ *of quantitative algebras has free algebras: the forgetful funtor* $U_\mathcal{V} : \mathcal{V} \to$ Met *has a left adjoint* $F_\mathcal{V}$: Met $\to \mathcal{V}$*.*

▶ **Notation 35.** We denote by $\mathbf{T}_\mathcal{V}$ the free-algebra monad of a variety $\mathcal{V}$ on Met. Its underlying functor is $T_\mathcal{V} = U_\mathcal{V} \cdot F_\mathcal{V}$. As usual, $\mathsf{Met}^{\mathbf{T}_\mathcal{V}}$ denotes the Eilenberg-Moore category of algebras for $\mathbf{T}_\mathcal{V}$.

▶ **Example 36.** For $\mathcal{V} = \Sigma$-Met we have seen the monad T_Σ in Example 31: $T_\Sigma M$ is the metric space of all terms over M. Observe that T_Σ is a coproduct of endofunctors $(-)^n$, one summand for each similarity class of terms on n variables over M (which is independent of the choice M). Thus $\mathbf{T}_\Sigma$ is a strongly finitary monad: see Example 11.

▶ Remark 37.

(1) Recall the comparison functor $K_\mathcal{V} : \mathcal{V} \to \mathsf{Met}^{\mathbf{T}_\mathcal{V}}$: it assigns to every algebra A of $\mathcal{V}$ the algebra on $U_\mathcal{V} A$ for $\mathbf{T}_\mathcal{V}$ given by the unique homomorphism $\alpha : F_\mathcal{V} U_\mathcal{V} A \to A$ extending $id_{U_\mathcal{V} A}$. More precisely: $K_\mathcal{V} A = (U_\mathcal{V} A, U_\mathcal{V} \alpha)$.

(2) By a *concrete category* over Met is meant a category $\mathcal{V}$ together with a faithful 'forgetful' functor $U_\mathcal{V} : \mathcal{V} \to$ Met. For example a variety, or $\mathsf{Met}^{\mathbf{T}}$ for every monad $\mathbf{T}$. A *concrete functor* is a functor $F : \mathcal{V} \to \mathcal{W}$ with $U_\mathcal{V} = U_\mathcal{W} F$. For example, the comparison functor $K_\mathcal{V}$.

▶ **Proposition 38.** *Every variety* $\mathcal{V}$ *of quantitative algebras is concretely isomorphic to the category* $\mathsf{Met}^{\mathbf{T}_\mathcal{V}}$*: the comparison functor* $K_\mathcal{V} : \mathcal{V} \to \mathsf{Met}^{\mathbf{T}_\mathcal{V}}$ *is a concrete isomorphism. Analogously for* UMet *and* CUMet*.*

Proof. For classical varieties (over Set) this is proved in [14], Theorem VI.8.1. The proof for Met in place of Set is analogous. ◀

▶ **Example 39** ([15], Theorem 9.3). For the variety $\mathcal{V}$ of quantitative semilattices (Example 3.4 (3)) the monad $\mathbf{T}_\mathcal{V}$ assigns to a metric space M the space of all finite subsets of M with the Hausdorff metric:

$$d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

Here, $d(a, B) = \inf_{b \in B} d(a, b)$. In particular, $d(A, \emptyset) = \infty$ for all $A \neq \emptyset$.

▶ **Notation 40.**

(1) Given a natural number n denote by $[n]$ the signature of one n-ary symbol δ. If a term $t \in T_\Sigma V$ contains at most n variables (say, all variables of t are among $x_0, \dots, x_{n-1}$), we obtain a monad morphism $\bar{t} : \mathbf{T}_{[n]} \to \mathbf{T}_\Sigma$ as follows. For every space M the function $\bar{t}_M$ takes a term s using the single symbol δ and substitutes each occurence of δ by $t(x_0, \dots, x_{n-1})$. More precisely: $\bar{t}_M : T_{[n]}M \to T_\Sigma M$ is defined by $x_i \mapsto x_i$ $(i < n)$ and $\delta(s_0, \dots, s_{n-1}) \mapsto t(\bar{t}_M(s_0), \dots, \bar{t}_M(s_{n-1}))$.

(2) Every metric space A defines the continuation monad $\langle A, A\rangle$ on Met assigning to $X \in \mathsf{Met}$ the space $\langle A, A\rangle X = [[X, A], A]$. More precisely: the functor $[-, A] : \mathsf{Met} \to \mathsf{Met}^{op}$ is self-adjoint, and $\langle A, A\rangle$ is the monad corresponding to that adjunction.

(3) Let $\mathbf{T}$ be a monad on Met and $\alpha : TA \to A$ an algebra for it. We denote by $\widehat{\alpha}_X : TX \to \langle A, A\rangle X$ the morphism which is adjoint to the following composite

$$[X, A] \otimes TX \xrightarrow{T(-)\otimes TX} [TX, TA] \otimes TX \xrightarrow{ev} TA \xrightarrow{\alpha} A.$$

▶ **Theorem 41** ([10]). *Given an algebra $\alpha : TA \to A$ for a monad $\mathbf{T}$ on Met, UMet or CUMet, the morphisms $\widehat{\alpha}_X$ above form a monad morphism $\widehat{\alpha} : \mathbf{T} \to \langle A, A\rangle$. Moreover, every monad morphism from $\mathbf{T}$ to $\langle A, A\rangle$ has that form for a unique algebra (A, α).*

▶ **Lemma 42.** *Let A be a Σ-algebra expressed as a monad algebra $\alpha : T_\Sigma A \to A$. It satisfies a quantitative equation $l =_\varepsilon r$ iff the distance of $\widehat{\alpha} \cdot \bar{l}, \widehat{\alpha} \cdot \bar{r} : \mathbf{T}_{[n]} \to \langle A, A\rangle$ is at most ε.*

▶ **Notation 43.**

1. The category of finitary monads on Met (and monad morphisms) is denoted by $\mathsf{Mnd}_\mathsf{f}(\mathsf{Met})$. It is enriched via the supremum metric: the distance of morphisms $\sigma, \tau : \mathbf{T} \to \mathbf{T}'$ in $\mathsf{Mnd}_\mathsf{f}(\mathsf{Met})$ is $\sup_{X \in \mathsf{Met}} d(\sigma_X, \tau_X)$. We use the same enrichment for its full subcategory of strongly finitary monads, denoted by $\mathsf{Mnd}_\mathsf{sf}(\mathsf{Met})$.
2. Analogously for monads on UMet we use $\mathsf{Mnd}_\mathsf{f}(\mathsf{UMet})$ and $\mathsf{Mnd}_\mathsf{sf}(\mathsf{UMet})$. Again for CUMet we use $\mathsf{Mnd}_\mathsf{f}(\mathsf{CUMet})$ and $\mathsf{Mnd}_\mathsf{sf}(\mathsf{CUMet})$.

▶ **Lemma 44.** *The category $\mathsf{Mnd}_\mathsf{f}(\mathsf{UMet})$ has weighted colimits, and $\mathsf{Mnd}_\mathsf{sf}(\mathsf{UMet})$ is closed under them.*

Proof sketch.

(1) The category $\mathsf{Mnd}_c(\mathsf{UMet})$ of countably accessible monads, i.e., monads preserving countably directed colimits (enriched again by the supremum metric), is locally countably presentable as an enriched category, thus it has weighted colimits.

(2) Both $\mathsf{Mnd}_\mathsf{f}(\mathsf{UMet})$ and $\mathsf{Mnd}_\mathsf{sf}(\mathsf{UMet})$ are coreflective subcategories of $\mathsf{Mnd}_c(\mathsf{UMet})$. The coreflection of a countably accessible monad $\mathbf{T}$ in $\mathsf{Mnd}_\mathsf{sf}(\mathsf{UMet})$ is given by the left Kan extension $\widetilde{T} = \mathrm{Lan}_K(T \cdot K)$. Analogously for $\mathsf{Mnd}_\mathsf{f}(\mathsf{UMet})$: let $\overline{K} : \mathsf{UMet}_\mathsf{f} \to \mathsf{UMet}$ be the full embedding of all finite metric spaces. The coreflection is $\widetilde{T} = \mathrm{Lan}_{\overline{K}}(T \cdot \overline{K})$. ◀

▶ Remark 45.

1. The same result holds for the base category CUMet.
2. Unfortunately, we do not know whether the above result holds for Met. The problem is that for the coreflection of a monad $\mathbf{T}$ in $\mathsf{Mnd}_\mathsf{sf}(\mathsf{Met})$ to be given by $\widetilde{T} = \mathrm{Lan}_K T \cdot K$, we need to know that $\widetilde{T} \cdot \widetilde{T}$ is strongly finitary. Whereas this holds in every cartesian closed category by [12], thus in UMet and CUMet, we do not know whether it also holds for monads on Met.
3. The categories Met and UMet have a factorization system $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ consists of surjective morphisms and $\mathcal{M}$ of isometric embeddings, i.e., morphisms preserving distances.

▶ **Lemma 46.** *Every monad morphism $\alpha : \mathbf{T}_\Sigma \to \mathbf{S}$ in the category* $\mathsf{Mnd_f(UMet)}$ *factorizes as a morphism $\mathbf{T}_\Sigma \to \overline{\mathbf{S}}$ with surjective components followed by a morphism $\overline{\mathbf{S}} \to \mathbf{S}$ whose components are isometric embeddings.*

▶ **Theorem 47.** *For every variety $\mathcal{V}$ of ultra-quantitative algebras the free-algebra monad $\mathbf{T}_\mathcal{V}$ is strongly finitary on* UMet*.*

Proof sketch.

(1) Let $\mathcal{V}$ be given by a signature Σ and quantitative equations $l_i =_{\varepsilon_i} r_i$ $(i \in I)$, each containing n_i variables. For every $i \in I$ we consider the signature $[n(i)]$ of one symbol δ_i of arity $n(i)$. Then the terms l_i, r_i yield the corresponding monad morphisms $\bar{l}_i, \bar{r}_i : \mathbf{T}_{[n(i)]} \to \mathbf{T}_\Sigma$ of Notation 40. An algebra $\alpha : T_\Sigma A \to A$ lies in $\mathcal{V}$ iff the distance of $\widehat{\alpha} \cdot \bar{l}_i, \widehat{\alpha} \cdot \bar{r}_i : \mathbf{T}_{[n(i)]} \to \langle A, A\rangle$ is at most ε_i for each i (Lemma 42).

(2) We verify that $\mathbf{T}_\mathcal{V}$ is a weighted colimit of strongly finitary monads in $\mathsf{Mnd_f(UMet)}$. Then $\mathbf{T}_\mathcal{V}$ is strongly finitary by Lemma 44. The domain $\mathscr{D}$ of the weighted diagram $D : \mathscr{D} \to \mathsf{Mnd_f(UMet)}$ is the discrete category I (indexing the equations) enlarged by a new object a, and by morphisms $\lambda_i, \rho_i : i \to a$ (for every $i \in I$) of distance ε_i. Then put $Di = \mathbf{T}_{[n(i)]}$ and $Da = \mathbf{T}_\Sigma$; further $D\lambda_i = \bar{l}_i$ and $D\rho_i = \bar{r}_i$. The weight $W : \mathscr{D}^{op} \to \mathsf{Met}$ takes i to the space $\{l, r\}$ with $d(l, r) = \varepsilon_i$ and a to $\{0\}$. We define $W\lambda_i(0) = l$ and $W\rho_i(0) = r$. The monads $\mathbf{T}_\Sigma$ and $\mathbf{T}_{[n(i)]}$ are strongly finitary by Example 36. Proving that $\mathbf{T}_\mathcal{V} = \mathit{colim}_W D$ will finish the proof by Lemma 44.
We denote by $\mathbf{T}$ the weighted colimit $\mathbf{T} = \mathit{colim}_W D$ in $\mathsf{Mnd_f(UMet)}$. The proof is concluded by proving that $\mathcal{V}$ is isomorphic, as a concrete category, to the category $\mathsf{UMet}^\mathbf{T}$ of algebras for $\mathbf{T}$. Then $\mathbf{T}$ is the free-algebra monad of $\mathcal{V}$. For $\mathbf{T}$ we have the unit $\nu : W \to [\mathscr{D}^{op}, \mathsf{Mnd_f(UMet)}](D-, \mathbf{T})$ (Definition 14). Its component ν_a assigns to 0 a monad morphism $\gamma = \nu_a(0) : \mathbf{T}_\Sigma \to \mathbf{T}$, whereas for $i \in I$ the component ν_i is given by $l \mapsto \gamma\cdot\bar{l}_i$ and $r \mapsto \gamma\cdot\bar{r}_i$. Since ν_i is nonexpanding, we conclude that $\gamma\cdot\overline{\lambda}_i, \gamma\cdot\overline{\rho}_i : \mathbf{T}_{[n(i)]} \to \mathbf{T}$ have distance at most ε_i. We thus obtain a functor $E : \mathsf{UMet}^\mathbf{T} \to \mathcal{V}$ assigning to every algebra $\alpha : TA \to A$ the Σ-algebra corresponding to $\alpha \cdot \gamma_A : T_\Sigma A \to A$: it satisfies $l_i =_{\varepsilon_i} r_i$ due to Lemma 42. Moreover, γ has surjective components, which can be derived from Lemma 46. Therefore, E is a concrete isomorphism, which concludes the proof. ◀

▶ Remark 48. The same result holds for varieties of quantitative algebras in CUMet.

▶ **Open Problem 49.** Is the free-algebra monad of every variety of quantitative algebras strongly finitary on Met?

▶ **Construction 50.** In the reverse direction we assign to every strongly finitary monad $\mathbf{T} = (T, \mu, \eta)$ on Met, UMet or CUMet a variety $\mathcal{V}_\mathbf{T}$, and prove that $\mathbf{T}$ is its free-algebra monad.

For every morphism $k : X \to A$ in Met together with an algebra $\alpha : TA \to A$, let us denote by

$$k^* = \alpha \cdot Tk : TX \to A$$

the corresponding homomorphism in $\mathsf{Met}^\mathbf{T}$. Recall our fixed set $V = \{x_i \mid i \in \mathbb{N}\}$ of variables, and form, for each $n \in \mathbb{N}$, the finite discrete space $V_n = \{x_i \mid i < n\}$. The signature we use has as n-ary symbols the elements of the space TV_n:

$$\Sigma_n = |TV_n| \text{ for } n \in \mathbb{N}.$$

The variety $\mathcal{V}_\mathbf{T}$ is given by the following quantitative equations, where each symbol $\sigma \in \Sigma_n$ is considered as the term $\sigma(x_0, \dots, x_{n-1})$, and n, m range over $\mathbb{N}$:

(1) $\sigma =_\varepsilon \sigma'$ for all $\sigma, \sigma' \in \Sigma_n$ with $d(\sigma, \sigma') \leq \varepsilon$ in TV_n.
(2) $k^*(\sigma) = \sigma(k(x_i))_{i<n}$ for all $\sigma \in \Sigma_n$ and all maps $k : V_n \to \Sigma_m$ in Set.
(3) $\eta_{V_n}(x_i) = x_i$ for all $i < n$.

▶ **Lemma 51.** *Every algebra $\alpha : TA \to A$ in $\mathsf{Met}^{\mathbf{T}}$ yields an algebra A in $\mathcal{V}_{\mathbf{T}}$ with operations $\sigma_A : A^n \to A$ defined by*

$$\sigma_A(a(x_i)) = a^*(\sigma) \textit{ for all } \sigma \in \Sigma_n \textit{ and } a : V_n \to A.$$

Moreover, every homomorphism in $\mathsf{Met}^{\mathbf{T}}$ is also a Σ-homomorphism between the corresponding algebras in $\mathcal{V}_{\mathbf{T}}$.

Proof sketch.

(a) The mapping σ_A is nonexpanding: given $d((a_i)_{i<n}, (b_i)_{i<n}) = \varepsilon$ in A^n, the corresponding maps $a, b : V_n \to A$ fulfil $d(a, b) = \varepsilon$. Since T is enriched, this yields $d(Ta, Tb) \leq \varepsilon$. Finally α is nonexpanding and $a^* = \alpha \cdot Ta$, $b^* = \alpha \cdot Tb$, thus $d(a^*, b^*) \leq \varepsilon$. In particular $d(a^*(\sigma), b^*(\sigma)) \leq \varepsilon$.

(b) The quantitative equations (1)-(3) hold:

Ad (1) Given $l, r \in TV_n$ with $d(l, r) \leq \varepsilon$, then for every map $a : V_n \to A$ we have $d(a^*(l), a^*(r)) \leq \varepsilon$. Thus $d(l_A(a_i), r_A(a_i)) \leq \varepsilon$ for all $(a_i) \in A^n$.

Ad (2) Given $a : V_n \to A$ we prove $(k^*(\sigma))_A(a_j) = \sigma_A(k(x_i))(a_j)$. The left-hand side is $a^*(k^*(\sigma)) = (a^*k)^*(\sigma)$ since $a^* \cdot k^* = (a^* \cdot k)^*$ holds in general. The right-hand one is $a^*(\sigma_A(k(x_i))) = (a^*k)^*(\sigma)$, too.

Ad (3) Recall that $\alpha \cdot \eta_A = id$ and $Ta \cdot \eta_{V_n} = \eta_A \cdot a$ for every map $a : V_n \to A$. Therefore

$$\begin{aligned}(\eta_{V_n}(x_i))_A(a_j) &= a^*(\eta_{V_n}(x_i))\\ &= \alpha \cdot Ta \cdot \eta_{V_n}(x_i)\\ &= a(x_i) = a_i.\end{aligned}$$

(c) Given a morphism $h : (A, \alpha) \to (B, \beta)$ in $\mathsf{Met}^{\mathbf{T}}$ (i.e., $h \cdot \alpha = \beta \cdot Th$) we are to prove that $h \cdot \sigma_A = \sigma_B \cdot h^n$ for all $\sigma \in TV_n$. This follows easily from $h \cdot a^* = (h \cdot a)^*$ for each $a : V_n \to A$. ◀

▶ **Theorem 52.** *Every strongly finitary monad $\mathbf{T}$ on UMet is the free-algebra monad of the variety $\mathcal{V}_{\mathbf{T}}$.*

Proof. For every ultrametric space M we need to prove that the Σ-algebra associated with (TM, μ_M) in Lemma 51 is free in $\mathcal{V}_{\mathbf{T}}$ w.r.t. the universal map η_M. Then the theorem follows from Proposition 38.

We have two strongly finitary monads, $\mathbf{T}$ and the free-algebra monad of $\mathcal{V}_{\mathbf{T}}$ (Theorem 47). Thus, it is sufficient to prove the above for finite discrete spaces M. Then this extends to all finite spaces because we have $M = colim_{W_0} D_M$ (Lemma 25) and both monads preserve this colimit by Theorem 28. Since they coincide on all finite discrete spaces, they coincide on all finite spaces. Finally, the above extends to all spaces M: by Lemma 16 we have a directed colimit $M = \underset{i\in I}{colim}\, M_i$ of the diagram of all finite subspaces M_i $(i \in I)$ which both monads preserve.

Given a finite discrete space M, we can assume without loss of generality $M = V_n$ for some $n \in \mathbb{N}$. For every algebra A in $\mathcal{V}_{\mathbf{T}}$ and an interpretation $f : V_n \to A$, we prove that there exists a unique Σ-homomorphism $\overline{f} : TV_n \to A$ with $f = \overline{f} \cdot \eta_{V_n}$.

Existence. Define $\overline{f}(\sigma) = \sigma_A(f(x_i))_{i<n}$ for every $\sigma \in TV_n$. The equality $f = \overline{f} \cdot \eta_{V_n}$ follows since A satisfies the equations (3) above: $\eta_{V_n}(x_i) = x_i$, thus the operation of A corresponding to $\eta_{V_n}(x_i)$ is the i-th projection. The map $\overline{f}$ is nonexpanding: given $d(l,r) \leq \varepsilon$ in TV_n, the algebra A satisfies the equation (1) above: $l =_\varepsilon r$. Therefore given an n-tuple $f : V_n \to A$ we have

$$d(l_A(f(x_i)), r_A(f(x_i))) \leq \varepsilon.$$

To prove that $\overline{f}$ is a Σ-homomorphism, take an m-ary operation symbol $\tau \in TV_m$. We prove $\overline{f} \cdot \tau_{V_m} = \tau_A \cdot \overline{f}^m$. This means that every $k : V_m \to TV_n$ fulfils

$$\overline{f} \cdot \tau_{V_m}(k(x_j))_{j<m} = \tau_A \cdot \overline{f}^m(k(x_j))_{j<m}.$$

The definition of $\overline{f}$ yields that the right-hand side is $\tau_A(k(x_j)_A(f(x_i)))$. Due to equation (2) in Construction 50 with τ in place of σ, this is $k^*(\tau)_A(f(x_i))$. The left-hand side yields the same result since

$$\tau_A \cdot \overline{f}^m(k(x_j)) = \tau_A(k(x_j))_A(f(x_i)) = k^*(\tau)_A(f(x_i)).$$

Uniqueness. Let $\overline{f}$ be a nonexpanding Σ-homomorphism with $f = \overline{f} \cdot \eta_{V_n}$. In TV_n the operation σ asigns to $\eta_{V_n}(x_i)$ the value σ. (Indeed, for every $a : n \to |TV_n|$ we have $\sigma_{TV_n}(a_i) = a^*(\sigma) = \mu_{V_n} \cdot Ta(\sigma)$. Thus due to $\mu \cdot T\eta = id$ we get $\sigma_{TV_n}(\eta_{V_n}(x_i)) = \mu_{V_n} \cdot T\eta_{V_n}(\sigma) = \sigma$.) Since $\overline{f}$ is a homomorphism, we conclude

$$f(\sigma) = \sigma_A(\overline{f} \cdot \eta_{V_n}(x_i)) = \sigma_A(f(x_i))$$

which is the above formula. ◀

▶ **Corollary 53.** *Varieties of ultra-quantitative algebras correspond bijectively, up to isomorphism, to strongly finitary monads on* UMet.

Indeed, a stronger result can be deduced from Theorems 47 and 52: let $\mathsf{Var}(\mathsf{UMet})$ denote the category of varieties of quantitative algebras and concrete functors (Remark 37 (2)). Recall that $\mathsf{Mnd_{sf}}(\mathsf{UMet})$ denotes the category of strongly finitary monads.

▶ **Theorem 54.** *The category* $\mathsf{Var}(\mathsf{UMet})$ *of varieties of ultra-quantitative algebras is equivalent to the dual of the category* $\mathsf{Mnd_{sf}}(\mathsf{UMet})$ *of strongly finitary monads on* UMet.

Proof. Morphisms $\varphi : \mathbf{S} \to \mathbf{T}$ between monads in $\mathsf{Mnd_{sf}}(\mathsf{UMet})$ bijectively correspond to concrete functors $\overline{\varphi} : \mathsf{UMet}^{\mathbf{T}} \to \mathsf{UMet}^{\mathbf{S}}$ ([7], Theorem 3.3): $\overline{\varphi}$ assigns to an algebra $\alpha : TA \to A$ of $\mathsf{UMet}^{\mathbf{T}}$ the algebra $\alpha \cdot \varphi_A : SA \to A$ in $\mathsf{UMet}^{\mathbf{S}}$. We know that for every variety $\mathcal{V}$ the comparison functor $K_{\mathcal{V}}$ is invertible (Proposition 38). This yields a functor $\Phi : \mathsf{Var}(\mathsf{UMet})^{op} \to \mathsf{Mnd_{sf}}(\mathsf{UMet})$ assigning to a variety $\mathcal{V}$ the monad $\mathbf{T}_{\mathcal{V}}$ (Theorem 47). Given a concrete functor $F : \mathcal{V} \to \mathcal{W}$ between varieties, there is a unique monad morphism $\varphi : \mathbf{T}_{\mathcal{W}} \to \mathbf{T}_{\mathcal{V}}$ such that $\overline{\varphi} = K_{\mathcal{W}} \cdot F \cdot K_{\mathcal{V}}^{-1} : \mathsf{UMet}^{\mathbf{T}_{\mathcal{V}}} \to \mathsf{UMet}^{\mathbf{T}_{\mathcal{W}}}$. We define $\Phi F = \varphi$ and get a functor which is clearly full and faithful. Thus Theorem 52 implies that Φ is an equivalence of categories. ◀

4 Varieties of Complete Quantitative Algebras

If we take CUMet as our base category, the development of Section 3 works for Σ-CUMet as well. The main difference is in Lemma 46: instead of the factorization system in UMet of Remark 45, we use the factorization system in CUMet where $\mathcal{E}$ = dense morphisms $f : A \to B$

($f[A]$ is a dense subset of B) and $\mathcal{M}$ = isometric embeddings of closed subspaces. Another difference is that for the enrichment of the category $\mathsf{Mnd_f}(\mathsf{CUMet})$ of finitary monads (cf. Notation 43) we must verify that the metric space of monad morphisms (with the supremum metric) is complete; this is easy.

By Example 31 (2) for every complete space M the space $T_\Sigma M$ is complete. The resulting monad $\mathbf{T}_\Sigma$ on the category CUMet is strongly finitary (as in Example 36).

▶ **Example 55.** We describe the monad $\mathbf{T}$ of free complete ultra-quantitative semilattices. It assigns to every complete ultrametric space M the space TM of all compact subsets with the Hausdorff metric (Example 39).

This holds for separable complete spaces: see [15], Theorem 9.6. To extend this result to all complete spaces, first observe that the subset Z of TM of all finite sets is dense. Indeed, every compact set $K \subseteq M$ lies in the closure of Z: given $\varepsilon > 0$, let $K_0 \subseteq K$ be a finite set such that ε-balls with centers in K_0 cover K. Then $K_0 \in Z$ and the Hausdorff distance of K_0 and K is at most ε.

Given a complete ultrametric space M, let X_i $(i \in I)$ be the collection of all countable subsets. Each closure $\overline{X}_i$ is a complete separable space, and $M = \bigcup_{i \in I} \overline{X}_i$ is a directed colimit preserved by T. Since $T\overline{X}_i$ is the space of all compact subsets of $\overline{X}_i$, and since finite subsets of M form a dense set, we conclude that TM is the space of all compact subsets of M.

Every variety $\mathcal{V}$ of complete ultrametric quantitative algebras yields a monad $\mathbf{T}_\mathcal{V}$ on CUMet which is strongly finitary, and $\mathcal{V}$ is isomorphic to $\mathsf{UMet}^{\mathbf{T}_\mathcal{V}}$. The proof is analogous to that of Theorem 47, just at the end we use the above factorization system of CUMet. The proof that every strongly finitary monad on CUMet is the free-algebra monad of a variety is completely analogous to that of Theorem 52. We thus obtain

▶ **Theorem 56.** *The category* $\mathsf{Var}(\mathsf{CUMet})$ *of varieties of complete ultra-quantitative algebras is equivalent to the dual of the category* $\mathsf{Mnd_{sf}}(\mathsf{CUMet})$ *of strongly finitary monads on* CUMet.

5 Conclusions and Open Problems

Varieties (aka 1-basic varieties) of quantitative algebras of Mardare et al. [15, 16], restricted to ultrametrics, correspond bijectively to strongly finitary monads on the category UMet. This is the main result of our paper. It is in surprising contrast to the fact that ω-varieties in op. cit. (where distance restrictions on finitely many variables in equations are imposed) do not even yield finitary monads in general, as demonstrated in [1].

For varieties in Met we do not whether the same is true.

▶ **Open Problem 57.** Is the free-algebra monad of every variety of quantitative algebras strongly finitary?

Our proof would show this is the case provided that strongly finitary endofunctors on Met are closed under composition.

For varieties of complete ultra-quantitative algebras the same result holds: they correspond bijectively to strongly finitary monads on CUMet. This relates the quantitative algebraic reasoning of Mardare et al. closely to the classical equational reasoning of universal algebra where varieties are known to correspond to finitary (= strongly finitary) monads on Set [14].

▶ **Open Problem 58.** Characterize monads on Met or CMet corresponding to ω-varieties of quantitative algebras.

In [1] a partial answer has been given: enriched monads on UMet corresponding to ω-varieties of ultra-quantitative algebras are precisely the enriched monads preserving

(1) directed colimits of split monomorphisms and

(2) surjective morphisms.

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