Completeness for Categories of Generalized Automata

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Abstract

We present a slick proof of completeness and cocompleteness for categories of F-*automata*, where the span of maps $E \xleftarrow{d} E \otimes I \xrightarrow{s} O$ that usually defines a deterministic automaton of input I and output O in a monoidal category $(\mathcal{K}, \otimes)$ is replaced by a span $E \leftarrow FE \rightarrow O$ for a generic endofunctor $F : \mathcal{K} \rightarrow \mathcal{K}$ of a generic category $\mathcal{K}$: these automata exist in their "Mealy" and "Moore" version and form categories F-Mly and F-Mre; such categories can be presented as strict 2-pullbacks in Cat and whenever F is a left adjoint, both F-Mly and F-Mre admit all limits and colimits that $\mathcal{K}$ admits. We mechanize our main results using the proof assistant Agda and the library `agda-categories`.

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Category (Co)algebraic pearls

Supplementary Material *Software (Source Code)*: `https://github.com/iwilare/categorical-automata`
archived at `swh:1:dir:5e12eb2d97e15118b9018d5b5f8254fd6f6e24f3`

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1 Introduction

One of the most direct representations of *deterministic automata* in the categorical settings consists (cf. [1, 4, 5]) of a span of morphisms $E \xleftarrow{d} E \times I \xrightarrow{s} O$, where the left leg provides a notion of ***d**ynamics* or *next state* function, given a current state E and an input I, and the right leg provides an *final **s**tate* or output O.

According to whether the output morphism depends on both the current state and an input or just on the state, one can then talk about classes of *Mealy* and *Moore automata*, respectively. This perspective of "automata in a category" naturally captures the idea that morphisms of a category can be interpreted as a general abstraction of processes/sequential operations.

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The above notion of deterministic automaton carries over to any monoidal category, on which the various classical notions of automata, e.g., minimization, bisimulation, powerset construction, can be equivalently reconstructed; this is studied, to a large extent, in the monograph [5].

In [1, 7], automata are generalized to the case in which, instead of taking spans from the monoidal product of states and inputs $E \otimes I$, one considers spans $E \leftarrow FE \to O$ for a generic endofunctor $F : \mathcal{K} \to \mathcal{K}$, providing an abstraction for the ambient structure that allows the automata to advance to the "next" state and give an output.

A general theorem asserting that the category of Mealy and Moore automata $\mathsf{Mly}_{\mathcal{K}}(I, O)$, $\mathsf{Mre}_{\mathcal{K}}(I, O)$ in a monoidal category $(\mathcal{K}, \otimes)$ are complete and cocomplete whenever $\mathcal{K}$ is itself complete and cocomplete can be obtained with little conceptual effort, cf. [5, Ch. 11], but the proof given therein is a bit ad-hoc, and provides no intuition for why finite products and terminal objects tend to be so complicated.

With just a little bit more category-theoretic technology, some general considerations can be made about the shape of limits in such settings: colimits and connected limits can be computed as they are computed in $\mathcal{K}$ (as a consequence of the fact that the forgetful functor from the category of machines *creates* them, cf. [16]), whereas products (and in particular the empty product, the terminal object) have dramatically different shapes than those provided in $\mathcal{K}$. The profound reason why this happens is the fact that such a terminal object (which we refer to O_∞) coincides with the terminal coalgebra of a specific endofunctor, which, for Moore or Mealy automata, is respectively given by $A \mapsto O \times RA$ and $A \mapsto RO \times RA$. The complicated shape of the terminal object O_∞ in $\mathsf{Mly}_{\mathcal{K}}(I, O)$ is then explained by Adámek's theorem, which presents the terminal object O_∞ as an (usually intricate) inverse limit in $\mathcal{K}$.

In this paper, we show that under the same assumption of completeness of the underlying category $\mathcal{K}$, the completeness of F-automata can be obtained by requiring that the endofunctor F admits a right adjoint R. The proof we provide follows a slick argument proving the existence of (co)limits by fitting each $\mathsf{Mly}_{\mathcal{K}}(I, O)$ and $\mathsf{Mre}_{\mathcal{K}}(I, O)$ into a strict 2-pullback in Cat, and deriving the result from stability properties of limit-creating functors.

1.1 Outline of the paper

The present short note develops as follows:

- First (Section 2) we introduce the language we will employ and the structures we will study:[1] categories of automata valued in a monoidal category $(\mathcal{K}, \otimes)$ (in two flavours: "Mealy" machines, where one considers spans $E \leftarrow E \otimes I \to O$, and "Moore", where instead one consider pairs $E \leftarrow E \otimes I, E \to O$) and of F-automata, where F is an endofunctor of $\mathcal{K}$ (possibly with no monoidal structure). "Mealy" automata are known as "deterministic automata" in today's parlance, but since we need to distinguish between the two kinds of diagram from time to time, we stick to an older terminology.
- Then (Theorem 3.6), to establish the presence of co/limits of shape $\mathcal{J}$ in categories of F-automata, under the two assumptions that $F : \mathcal{K} \to \mathcal{K}$ is a left adjoint in an adjunction $F \underset{\eta}{\overset{\epsilon}{\dashv}} R$, and that co/limits of shape $\mathcal{J}$ exist in the base category $\mathcal{K}$.
- Last (Subsection 3.1), to address the generalisation to F-machines of the "behaviour as an adjunction" perspective expounded in [18, 19].

[1] An almost identical introductory short section appears in [2], of which the present note is a parallel submission –although related, the two manuscripts are essentially independent, and the purpose of this repetition is the desire for self-containment.

Similarly to the situation for Mealy/Moore machines, where $F = \_ \otimes I$, *discrete* limits in F-Mly and F-Mre exist but tend to have a shape that is dramatically different than the one in $\mathcal{K}$.

A number of examples of endofunctors F that satisfy the previous assumption come from considering F as the (underlying endofunctor of the) comonad LG of an adjunction $L \dashv G \dashv U$, since in that case $LG \dashv UG$: the shape-flat and flat-sharp adjunctions of a cohesive topos [13, 14], or the base-change adjunction $\mathrm{Lan}_f \dashv f^* \dashv \mathrm{Ran}_f$ for a morphism of rings, or more generally, G-modules in representation theory, any essential geometric morphism, or any topological functor $V : \mathcal{E} \to \mathcal{B}$ [3, Prop. 7.3.7] with its fully faithful left and right adjoints $L \dashv V \dashv R$ gives rise to a comodality LV, left adjoint to a modality RV.

The results we get are not particularly surprising; we have not, however, been able to trace a reference addressing the co/completeness properties of F-Mly, F-Mre nor an analogue for the "behaviour as an adjunction" theorems expounded in [18, 19]; in the case $F = \_ \otimes I$ co/completeness results follows from unwieldy ad-hoc arguments (cf. [5, Ch. 11]), whereas in Theorem 3.6 we provide a clean, synthetic way to derive both results from general principles, starting by describing F-Mly and F-Mre as suitable pullbacks in Cat, in Proposition 3.5.

We provide a mechanisation of our main results using the proof assistant Agda and the library `agda-categories`: we will add a small Agda logo (🐓) next to the beginning of a definition or statement whenever it is accompanied by Agda code: this is a hyperlink pointing directly to the formalisation files. The full development is freely available for consultation and is available at `https://github.com/iwilare/categorical-automata`.

2 Automata and F-automata

The only purpose of this short section is to fix notation; classical comprehensive references for this material are [1, 5]; in particular, [1, Ch. III] is entirely devoted to the study of what here are called F-Moore automata, possibly equipped with an "initialization" morphism.

2.1 Mealy and Moore automata

For the entire subsection, we fix a monoidal category $(\mathcal{K}, \otimes, 1)$.

▶ **Definition 2.1** (Mealy machine)**.** (🐓) *A* Mealy machine *in $\mathcal{K}$ of input object I and output object O consists of a triple (E, d, s) where E is an object of $\mathcal{K}$ and d, s are morphisms in a span*

$$\mathfrak{e} := \left(E \xleftarrow{\;d\;} E \otimes I \xrightarrow{\;s\;} O \right) \tag{2.1}$$

▶ Remark 2.2 (The category of Mealy machines). Mealy machines of fixed input and output I, O form a category, if we define a *morphism of Mealy machines* $f : (E, d, s) \to (T, d', s')$ as a morphism $f : E \to T$ in $\mathcal{K}$ such that

$$\begin{array}{ccccc} E & \xleftarrow{\;d\;} & E \otimes I & \xrightarrow{\;s\;} & O \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\otimes I} & & \| \\ T & \xleftarrow[\;d'\;]{} & T \otimes I & \xrightarrow[\;s'\;]{} & O \end{array} \tag{2.2}$$

Clearly, composition and identities are performed in $\mathcal{K}$.

The category of Mealy machines of input and output I, O is denoted as $\mathsf{Mly}_{\mathcal{K}}(I, O)$.

▶ **Definition 2.3** (Moore machine). ( ) *A* Moore machine *in $\mathcal{K}$ of input object I and output object O is a diagram*

$$\mathfrak{m} := \Big(E \xleftarrow{d} E \otimes I \;\; ; \;\; E \xrightarrow{s} O \Big) \tag{2.3}$$

▶ Remark 2.4 (The category of Moore machines). Moore machines of fixed input and output I, O form a category, if we define a *morphism of Moore machines* $f : (E, d, s) \to (T, d', s')$ as a morphism $f : E \to T$ in $\mathcal{K}$ such that

$$\begin{array}{ccc} E & \xleftarrow{d} & E \otimes I \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f \otimes I} \\ T & \xleftarrow{d'} & T \otimes I \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{s} & O \\ {\scriptstyle f}\downarrow & & \| \\ T & \xrightarrow[s']{} & O \end{array} \tag{2.4}$$

2.2 F-Mealy and F-Moore automata

The notion of F*-machine* arises by replacing the tensor $E \otimes I$ in (2.1) with the action FE of a generic endofunctor $F : \mathcal{K} \to \mathcal{K}$ on an object $E \in \mathcal{K}$, in such a way that a Mealy/Moore machine is just a $(\_ \otimes I)$-Mealy/Moore machine; cf. [7, ff. 2.1.3°], or Chapter III of the monograph [1]. This natural idea acts as an abstraction for the structure that allows the machine to advance to the "next" state and give an output, and it leads to the following two definitions (where we do *not* require $\mathcal{K}$ to be monoidal).

▶ **Definition 2.5** (**F**-Mealy machine). ( ) *Let $O \in \mathcal{K}$ be a fixed object. The objects of the category $F\text{-}\mathsf{Mly}_{/O}$ (or simply $F\text{-}\mathsf{Mly}$ when the object O is implicitly clear) of* F-Mealy machines of output O *are the triples (E, d, s) where $E \in \mathcal{K}$ is an object and s, d are morphisms in $\mathcal{K}$ that fit in the span*

$$E \xleftarrow{d} FE \xrightarrow{s} O \tag{2.5}$$

A morphism *of F-Mealy machines $f : (E, d, s) \to (T, d', s')$ consists of a morphism $f : E \to T$ in $\mathcal{K}$ such that*

$$\begin{array}{ccccc} E & \xleftarrow{d} & FE & \xrightarrow{s} & O \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} & & \| \\ T & \xleftarrow{d'} & FT & \xrightarrow[s']{} & O \end{array} \tag{2.6}$$

Unsurprisingly, we can generalise in the same fashion Definition 2.3 to the case of a generic endofunctor $F : \mathcal{K} \to \mathcal{K}$.

▶ **Definition 2.6** (**F**-Moore machine). ( ) *Let $O \in \mathcal{K}$ be a fixed object. The objects of the category $F\text{-}\mathsf{Mre}_{/O}$ (or simply $F\text{-}\mathsf{Mre}$ when the object O is implicitly clear) of* F-Moore machines of output O *are the triples (E, d, s) where $E \in \mathcal{K}$ is an object and s, d are a pair of morphisms in $\mathcal{K}$*

$$E \xleftarrow{d} FE \;\; ; \;\; E \xrightarrow{s} O \tag{2.7}$$

A morphism *of F-Moore machines $f : (E, d, s) \to (T, d', s')$ consists of a morphism $f : E \to T$ in $\mathcal{K}$ such that*

$$\begin{array}{ccc} E & \xleftarrow{d} & FE \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Ff} \\ T & \xleftarrow{d'} & FT \end{array} \qquad \begin{array}{ccc} E & \xrightarrow{s} & O \\ {\scriptstyle f}\downarrow & & \| \\ T & \xrightarrow[s']{} & O \end{array} \tag{2.8}$$

▶ Remark 2.7 (Interdefinability of notions of machine). All the concepts of machine introduced so far are interdefinable, provided we allow the monoidal base $\mathcal{K}$ to change (cf. [7, ff. Proposition 30]): a Mealy machine is, obviously, an F-machine where $F : \mathcal{K} \to \mathcal{K}$ is the functor $\_ \otimes I : E \mapsto E \otimes I$; an F-machine consists of a Mealy machine in a category of endofunctors: in fact, F-machines are precisely the Mealy machines of the form $E \leftarrow F \circ E \to O$, where E, O are constant endofunctors on objects of $\mathcal{K}$ and F is the input object: more precisely, the category of F-machines is contained in the category $\mathsf{Mly}_{([\mathcal{K},\mathcal{K}],\circ)}(F, c_O)$, where c_O is the constant functor on $O \in \mathcal{K}$, as the subcategory of those triples (E, d, s) where E is a constant endofunctor.

3 Completeness and behaviour in F-Mly and F-Mre

The first result that we want to generalise to F-machines is the well-known fact that, considering for example Mealy machines, if $(\mathcal{K}, \otimes)$ has countable coproducts preserved by each $I \otimes \_$, then the span (2.1) can be "extended" to a span

$$E \xleftarrow{d^+} E \otimes I^+ \xrightarrow{s^+} O \tag{3.1}$$

where d^+, s^+ can be defined inductively from components $d_n, s_n : E \otimes I^{\otimes n} \to E, O$.

Under the same assumptions, each Moore machine (2.3) can be "extended" to a span

$$E \xleftarrow{d^*} E \otimes I^* \xrightarrow{s^*} O \tag{3.2}$$

where d^*, s^* can be defined inductively from components $d_n, s_n : E \otimes I^{\otimes n} \to E, O$.[2]

▶ Remark 3.1. In the case of Mealy machines, the object I^+ corresponds to the *free semigroup* on the input object I, whereas for Moore machines one needs to consider the *free monoid* I^*: this mirrors the intuition that in the latter case an output can be provided without any previous input. Note that the extension of a Moore machine gives rise to a span of morphisms from the same object $E \otimes I^*$, i.e., a Mealy machine that accepts the empty string as input.

A similar construction can be carried over in the category of F-Mealy machines, using the F-algebra map $d : FE \to E$ to generate iterates $E \xleftarrow{d_n} F^n E \xrightarrow{s_n} O$, for $n \geq 1$.

From now on, let F be an endofunctor of a category $\mathcal{K}$ that has a right adjoint R. Examples of such arise naturally from the situation where a triple of adjoints $L \dashv G \dashv R$ is given, since we obtain adjunctions $LG \dashv RG$ and $GL \dashv GR$:

- every homomorphism of rings $f : A \to B$ induces a triple of adjoint functor between the categories of A and B-modules (cf. [3, 4.7.4]);
- similarly, every homomorphism of monoids $f : M \to N$ induces a "base change" functor $f^* : N\text{-}\mathsf{Set} \to M\text{-}\mathsf{Set}$ (this is usuall treated as a fact all category theorists know; however, an elementary exposition of this fact can be found in [21, Prop. 4.1.4.11]);
- every essential geometric morphism between topoi $\mathcal{E} \leftrightarrows \mathcal{F}$, i.e. every triple of adjoints $f_! \dashv f^* \dashv f_*$ (cf. [10, 1.16]);
- every topological functor $V : \mathcal{E} \to \mathcal{B}$ [3, Prop. 7.3.7] with its fully faithful left and right adjoints $L \dashv V \dashv R$ (this gives rise to a comodality LV, left adjoint to a modality RV).

[2] Assuming countable coproducts in $\mathcal{K}$, the free *monoid* I^* on I is the object $\sum_{n\geq 0} I^n$; the free *semigroup* I^+ on I is the object $\sum_{\geq 1} I^n$; clearly, if 1 is the monoidal unit of $\otimes$, $I^* \cong 1 + I^+$, and the two objects satisfy "recurrence equations" $I^+ \cong I \otimes I^+$ and $I^* \cong 1 + I \otimes I^*$.

▶ **Construction 3.2** (Dynamics of an $\boldsymbol{F}$-machine)**.** For any given F-Mealy machine

$$E \xleftarrow{d} FE \xrightarrow{s} O \tag{3.3}$$

we define the family of morphisms $s_n : F^n E \to O$ (for $n \geq 1$) inductively, as the composites

$$\begin{cases} s_1 &= FE \xrightarrow{s} O \\ s_2 &= FFE \xrightarrow{Fd} FE \xrightarrow{s} O \\ s_n &= F^n E \xrightarrow{F^{n-1}d} F^{n-1}E \to \cdots \xrightarrow{FFd} FFE \xrightarrow{Fd} FE \xrightarrow{s} O \end{cases} \tag{3.4}$$

Under our assumption that F has a right adjoint R, this is equivalent to the datum of their mates $\bar{s}_n : E \to R^n O$ for $n \geq 1$ under the adjunction $F^n \dashv_{\eta_n} R^n$ obtained by composition, iterating the structure in $F \overset{\epsilon}{\underset{\eta}{\dashv}} R$.

Such a s_n is called the nth *skip map*. Observe that in case $\mathcal{K}$ has countable products, the family of all nth skip maps $(s_n \mid n \in \mathbb{N}_{\geq 1})$ is obviously equivalent to a single map of type $\bar{s}_\infty : E \to \prod_{n\geq 1} R^n O$.

▶ Remark 3.3. Reasoning in a similar fashion, one can define extensions $s : E \to O$, $s \circ d : FE \to E \to O$, $s \circ d \circ Fd : FFE \to O$, etc. for an F-Moore machine.
This is the first step towards the following statement, which will be substantiated and expanded in Theorem 3.6 below:

▷ Claim 3.4. The category F-Mre of Definition 2.6 has a terminal object $\mathfrak{o} = (O_\infty, d_\infty, s_\infty)$ with carrier $O_\infty = \prod_{n\geq 0} R^n O$; similarly, the category F-Mly has a terminal object with carrier $O_\infty = \prod_{n\geq 1} R^n O$. (Note the shift in the index of the product, motivated by the fact that the skip maps for a Moore machine are indexed on $\mathbb{N}_{\geq 0}$, and on $\mathbb{N}_{\geq 1}$ for Mealy.)

The "modern" way to determine the presence of a terminal object in categories of automata relies on the elegant coalgebraic methods in [9]; the interest in such completeness theorems can be motivated essentially in two ways:

- the terminal object O_∞ in a category of machines tends to be "big and complex", as a consequence of the fact that it is often a terminal coalgebra for a suitably defined endofunctor of $\mathcal{K}$, so Adámek's theorem presents it as inverse limit of an op-chain.
- Coalgebra theory allows us to define a *bisimulation* relation between states of different F-algebras (or, what is equivalent in our blanket assumptions, R-coalgebras), which in the case of standard Mealy/Moore machines (i.e., when $F = \_ \otimes I$) recovers the notion of bisimulation expounded in [9, Ch. 3].

The following universal characterisation of both categories as pullbacks in Cat allows us to reduce the whole problem of completeness to the computation of a terminal object, and thus prove Theorem 3.6.

▶ **Proposition 3.5.**

CX1) *the category F-Mly of F-Mealy machines given in Definition 2.5 can be characterised as the top left corner in the pullback square*

$$\begin{array}{ccc} F\text{-Mly} & \xrightarrow{U'} & (F_{/O}) \\ {\scriptstyle V'}\downarrow & \lrcorner & \downarrow{\scriptstyle V} \\ \mathsf{Alg}(F) & \xrightarrow[U]{} & \mathcal{K} \end{array} \tag{3.5}$$

where $F_{/O}$ is the comma category defined by F and the constant functor on O, V is the forgetful functor defined by the universal property of comma categories and U is the canonical forgetful functor of F-algebras.

CX2) *the category F-*Mre *of F-Moore machines given in Definition 2.6 can be characterised as the top left corner in the pullback square*

$$\begin{array}{ccc} F\text{-}\mathsf{Mre} & \xrightarrow{U'} & (\mathcal{K}_{/O}) \\ {\scriptstyle V'}\downarrow & \lrcorner & \downarrow{\scriptstyle V} \\ \mathsf{Alg}(F) & \xrightarrow[U]{} & \mathcal{K} \end{array} \tag{3.6}$$

where V is the forgetful functor from the slice category $\mathcal{K}_{/O}$ to $\mathcal{K}$, sending an arrow to its domain and U is the canonical forgetful functor of F-algebras.

Proof. Straightforward inspection of the definition of both pullbacks. ◀

As a consequence of this characterization, by applying [16, V.6, Ex. 3] we can easily show the following completeness result, provided we recall that in both (3.5) and (3.6) U is monadic, and since F is a left adjoint, V preserves connected limits.

▶ **Theorem 3.6** (Limits and colimits of F-machines). [3]

- *Let $\mathcal{K}$ be a category admitting colimits of shape $\mathcal{J}$; then, F-*Mre *and F-*Mly *have colimits of shape $\mathcal{J}$, and they are computed as in $\mathcal{K}$;*
- *Equalizers (and more generally, all connected limits) are computed in F-*Mre *and F-*Mly *as they are computed in $\mathcal{K}$; if $\mathcal{K}$ has countable products and pullbacks, F-*Mre *and F-*Mly *also have products of any finite cardinality (in particular, a terminal object).*

Proof of Theorem 3.6. It is worth unraveling the content of [16, V.6, Ex. 3], from which the claim gets enormously simplified: the theorem asserts that in any strict pullback square of categories

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{U'} & \mathcal{B} \\ {\scriptstyle V'}\downarrow & \lrcorner & \downarrow{\scriptstyle V} \\ \mathcal{C} & \xrightarrow[U]{} & \mathcal{K} \end{array} \tag{3.7}$$

if U creates, and V preserves, limits of a given shape $\mathcal{J}$, then U' creates limits of shape $\mathcal{J}$. Thus, thanks to Proposition 3.5, all connected limits (in particular, equalizers) are created in the categories of F-Mealy and F-Moore machines by the functors $U' : F\text{-}\mathsf{Mly} \to (F_{/O})$ and are thus computed as in $(F_{/O})$, i.e. as in $\mathcal{K}$; this result is discussed at length in [5, Ch. 10] in the case of $(\_ \otimes I)$-machines, i.e. classical Mealy machines, to prove the following:

- assuming $\mathcal{K}$ is cocomplete, all colimits are computed in F-Mly as they are computed in the base $\mathcal{K}$;
- assuming $\mathcal{K}$ has connected limits, they are computed in F-Mly as they are computed in the base $\mathcal{K}$.

Discrete limits have to be treated with additional care: for classical Moore machines (cf. Definition 2.3) the terminal object is the terminal coalgebra of the functor $A \mapsto A^I \times O$ (cf. [9, 2.3.5]): a swift application of (the analogue of) Adámek's theorem (for a Cartesian category other than Set) yields the object $[I^*, O]$; for classical Mealy machines (cf. Definition 2.1) the terminal object is the terminal coalgebra for $A \mapsto [I, O] \times [I, A]$; similarly, Adámek's theorem yields $[I^+, O]$.

[3] We only provide a mechanization of the proof of existence of finite products: binary products, and a terminal object.

Adámek's theorem then yields the terminal object of F-Mre as the terminal coalgebra for the functor $A \mapsto O \times RA$, which is the $O_{\infty,0}$ of Claim 3.4, and the terminal object of F-Mly as $O_{\infty,1}$ and for $A \mapsto RO \times RA$ (in F-Mly). All discrete limits can be computed when pullbacks and a terminal object have been found, but we prefer to offer a more direct argument to build binary products.

Recall from Construction 3.2 the definition of dynamics map associated to an F-machine $\mathfrak{e} = (E, d, s)$.

Now, our claim is two-fold:

TO1) the object $O_\infty := \prod_{n \geq 1} R^n O$ in $\mathcal{K}$ carries a canonical structure of an F-machine $\mathfrak{o} = (O_\infty, d_\infty, s_\infty)$ such that $\mathfrak{o}$ is terminal in F-Mly;

TO2) given objects $(E, d_E, s_E), (T, d_T, s_T)$ of F-Mly, the pullback

$$\begin{array}{ccc} P_\infty & \longrightarrow & T \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \bar{s}_{T,\infty}} \\ E & \xrightarrow[\bar{s}_{E,\infty}]{} & O_\infty \end{array} \tag{3.8}$$

is the carrier of an F-machine structure that exhibits $\mathfrak{p} = (P_\infty, d_P, s_P)$ as the product of $\mathfrak{e} = (E, d_E, s_E), \mathfrak{f} = (T, d_T, s_T)$ in F-Mly.

In this way, the category F-Mly comes equipped with all finite products; it is easy to prove a similar statement when an infinite number of objects $(\mathfrak{e}_i \mid i \in I)$ is given by using wide pullbacks whenever they exist in the base category.

Observe that the object P_∞ can be equivalently characterized as the single wide pullback obtained from the pullback P_n of $\bar{s}_{E,n}$ and $\bar{s}_{T,n}$ (or rather, an intersection, since each $P_n \to E \times T$ obtained from the same pullback is a monomorphism):

$$\begin{array}{ccc} P_\infty & \longrightarrow & P_n \\ \downarrow & \cdot^{\cdot^{\cdot}} & \downarrow \\ P_m & \longrightarrow & E \times T \end{array} \tag{3.9}$$

Showing the universal property of P_∞ will be more convenient at different times in one or the other definition.

In order to show our first claim in TO1, we have to provide the F-machine structure on O_∞, exhibiting a span

$$O_\infty \xleftarrow{d_\infty} FO_\infty \xrightarrow{s_\infty} O \tag{3.10}$$

On one side, s_∞ is the adjoint map of the projection $\pi_1 : O_\infty \to RO$ on the first factor; the other leg d_∞ is the adjoint map of the projection deleting the first factor, thanks to the identification $RO_\infty \cong \prod_{n \geq 2} R^n O$; explicitly then, we are considering the following diagram:

$$O_\infty \xleftarrow{\epsilon_{O_\infty}} FRO_\infty \xleftarrow{F\pi_{\geq 2}} FO_\infty \xrightarrow{F\pi_1} FRO \xrightarrow{\epsilon_O} O \tag{3.11}$$

To prove the first claim, let's consider a generic object (E, d, s) of F-Mly, i.e. a span

$$E \xleftarrow{d} FE \xrightarrow{s} O \tag{3.12}$$

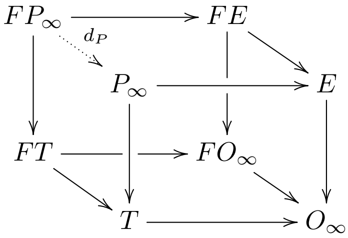
and let's build a commutative diagram

$$\begin{array}{ccccccccc} E & & \xleftarrow{\quad\quad d \quad\quad} & & FE & & \xrightarrow{\quad\quad s \quad\quad} & & O \\ {\scriptstyle u}\downarrow & & & & \downarrow{\scriptstyle Fu} & & & & \| \\ O_\infty & \xleftarrow{\epsilon_{O_\infty}} & FRO_\infty & \xleftarrow{F\pi_{\geq 2}} & FO_\infty & \xrightarrow{F\pi_1} & FRO & \xrightarrow{\epsilon_O} & O \end{array} \tag{3.13}$$

for a unique morphism $u : E \to O_\infty = \prod_{n\geq 1} R^n O$ that we take exactly equal to $\bar{s}_\infty$. The argument that u makes diagram (3.13) commutative, and that it is unique with this property, is now a completely straightforward diagram chasing.

Now let's turn to the proof that the tip of the pullback in (3.8) exhibits the product of $(E, d_E, s_E), (T, d_T, s_T)$ in F-Mly; first, we build the structure morphisms s_P, d_P as follows:

- d_P is the dotted map obtained thanks to the universal property of P_∞ from the commutative diagram



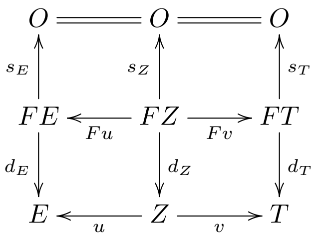
(3.14)

- $s_P : FP_\infty \to O$ is obtained as the adjoint map of the diagonal map $P_\infty \to O_\infty$ in (3.8) composed with the projection $\pi_1 : O_\infty \to RO$.

Let's now assess the universal property of the object

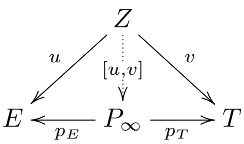
$$P_\infty \xleftarrow{d_P} FP_\infty \xrightarrow{s_P} O \tag{3.15}$$

We are given an object $\mathfrak{z} = (Z, d_Z, s_Z)$ of F-Mly and a diagram



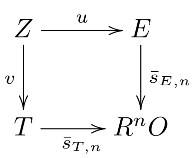
(3.16)

commutative in all its parts. To show that there exists a unique arrow $[u, v] : Z \to P_\infty$



(3.17)

we can argue as follows, using the joint injectivity of the projection maps $\pi_n : O_\infty \to R^n O$: first, we show that each square



(3.18)

is commutative, and in particular that its diagonal is equal to the nth skip map of Z; this can be done by induction, showing that the composition of both edges of the square with the canonical projection $O_\infty \to R^n O$ equals $\bar{s}_{n,Z}$ for all $n \geq 1$. From this, we deduce that there exist maps

$$Z \xrightarrow{z_n} P_n \longrightarrow E \times T \tag{3.19}$$

(cf. (3.9) for the definition of P_n) for every $n \geq 1$, But now, the very way in which the z_ns are defined yields that each such map coincides with $\langle u, v\rangle : Z \to E \times T$, thus Z must factor through P_∞. Now we have to exhibit the commutativity of diagrams

$$\begin{array}{ccccc} Z & \xleftarrow{d_Z} & FZ & \xrightarrow{s_Z} & O \\ \downarrow{\scriptstyle [u,v]} & & \downarrow{\scriptstyle F[u,v]} & & \| \\ P_\infty & \xleftarrow[d_P]{} & FP_\infty & \xrightarrow[s_P]{} & O \end{array} \tag{3.20}$$

and this follows from a straightforward diagram chasing.

This concludes the proof. ◀

▶ Remark 3.7. Spelled out explicitly, the statement that $\mathfrak{o} = (O_\infty, d_\infty, s_\infty)$ is a terminal object amounts to the fact that given any other F-Mealy machine $\mathfrak{e} = (E, d, s)$, there is a unique $u_E : E \to O_\infty$ with the property that

$$\begin{array}{ccccc} E & \xleftarrow{d} & FE & \xrightarrow{s} & O \\ \downarrow{\scriptstyle u_E} & & \downarrow{\scriptstyle Fu_E} & & \| \\ O_\infty & \xleftarrow[d_\infty]{} & FO_\infty & \xrightarrow[s_\infty]{} & O \end{array} \tag{3.21}$$

are both commutative diagrams; a similar statement holds for F-Moore automata.

3.1 Adjoints to behaviour functors

In [18, 19] the author concentrates on building an adjunction between a category of machines and a category collecting the *behaviours* of said machines.

Call an endofunctor $F : \mathcal{K} \to \mathcal{K}$ an *input process* if the forgetful functor $U : \mathsf{Alg}(F) \to \mathcal{K}$ has a left adjoint G; in simple terms, an input process allows to define free F-algebras.[4]

In [18, 19] the author concentrates on proving the existence of an adjunction

$$L : \mathsf{Beh}(F) \underset{\perp}{\rightleftarrows} \mathsf{Mach}(F) : E \tag{3.22}$$

where $\mathsf{Mach}(F)$ is the category obtained from the pullback

$$\begin{array}{ccccc} \mathsf{Mach}(F) & \longrightarrow & & & \mathcal{K}^{\to} \times \mathcal{K}^{\to} \\ \downarrow & \lrcorner & & & \downarrow{\scriptstyle d_1 \times d_0} \\ \mathsf{Alg}(F) & \xrightarrow[U]{} & \mathcal{K} & \xrightarrow[\Delta]{} & \mathcal{K} \times \mathcal{K} \end{array} \tag{3.23}$$

Δ is the diagonal functor, $\mathsf{Beh}(F)$ is a certain comma category on the free F-algebra functor G and d_0, d_1 are the domain and codomain functors from the arrow category.

Phrased in this way, the statement is conceptual enough to carry over to F-Mealy and F-Moore machines (and by extension, to all settings where a category of automata can be presented through a strict 2-pullback in Cat of well-behaved functors –a situation that given (3.5), (3.6), (3.23) arises quite frequently).

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[4] Obviously, this is in stark difference with the requirement that F has an adjoint, and the two requests are independent: if F is a monad, it is always an input process, regardless of F admitting an adjoint on either side.

▶ **Theorem 3.8.** (🗘) *There exists a functor $B : F\text{-}\mathsf{Mre} \to \mathsf{Alg}(F)_{/(O_\infty, d_\infty)}$, where the codomain is the slice category of F-algebras and the F-algebra (O_∞, d_∞) is determined in Claim 3.4. The functor B has a left adjoint L.*

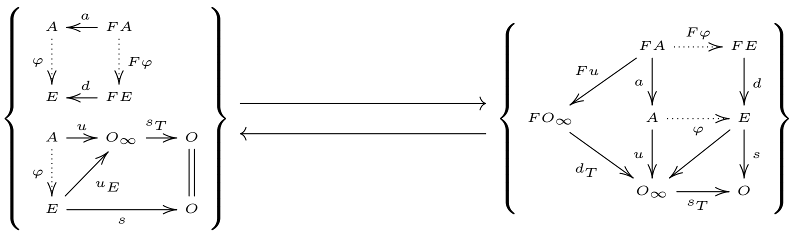
Proof. An object of $\mathsf{Alg}(F)_{/O_\infty}$ is a tuple $((A, a), u)$ where $a : FA \to A$ is an F-algebra with its structure map, and $u : A \to O_\infty$ is an F-algebra homomorphism, i.e. a morphism u such that $d_\infty \circ Fu = u \circ a$.
The functor B is defined as follows:

- on objects $\mathfrak{e} = (E, d, s)$ in F-Mre, as the correspondence sending $\mathfrak{e}$ to the unique map $u_E : E \to O_\infty$, which is an F-algebra homomorphism by the construction in (3.13);
- on morphisms, $f : (E, d, s) \to (F, d', s')$ between F-Moore machines, B acts as the identity, ultimately as a consequence of the fact that the terminality of O_∞ yields at once that $u_F \circ f = u_E$.

A putative left adjoint for B realises a natural bijection

$$F\text{-}\mathsf{Mre}_{/O}\big(L((A, a), u), (E, d, s)\big) \cong \mathsf{Alg}(F)_{/O_\infty}\big(((A, a), u), B(E, d, s)\big) \tag{3.24}$$

between the following two kinds of commutative diagrams:



(3.25)

There is a clear way to establish this correspondence. ◀

▶ Remark 3.9. (🗘) The adjunction in Theorem 3.8 is actually part of a longer chain of adjoints obtained as follows: recall that every adjunction $G : \mathcal{K} \leftrightarrows \mathcal{H} : U$ induces a "local" adjunction $\tilde{G} : \mathcal{K}_{/UA} \leftrightarrows \mathcal{H}_{/A} : \tilde{U}$ where $\tilde{U}(FA, f : FA \to A) = Uf$. Then, if F is an input process, we get adjunctions

$$\mathcal{K}_{/O_\infty} \underset{\tilde{U}}{\overset{\tilde{G}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Alg}(F)_{/(O_\infty, d_\infty)} \underset{B}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} F\text{-}\mathsf{Mre}. \tag{3.26}$$

4 Conclusion and Future Works

Our research is part of a bigger ongoing project [2] aimed to understand automata theory from the point of view of formal category theory [6, 24, 25]. The endeavour has a long history (the work of Naudé that we generalize a bit serves as a remarkable example in this direction), and the technology of category-theoretic approaches is rapidly shifting towards 2-dimensional categories as foundations for complex systems [15, 17, 20]. By leveraging simple universal properties of pullbacks and comma objects in Cat, we have established a way for generating "categories of automata and their behaviour".

In fact, our findings hint at the existence of exciting possibilities for understanding behavior coalgebraically within categories of automata. This approach, well-known and fruitful in the literature, has been extensively studied by Jacobs [9, 8]. We are confident that we can extend this line of research to derive insightful statements in the "internal language" of the category of automata under consideration. For instance, we can examine bisimilarity as an internal equivalence relation in our categories of generalised automata, utilizing the calculus of relations available in every regular category, and categorical algebra, broadly intended. In our opinion, this exploration holds great potential for deepening our understanding of automata theory and its applications.

In future works, we would like to further explore the properties of the adjunctions sketched in this paper, emphasizing on applications. We also plan to delve deeper into the "coalgebraic behavior" perspective, with particular care for its implications in different aspects of automata theory. In [2] we exploit the fact that Mealy automata form a bicategory, building on prior work [11]: it is a banality that the composition of 1-cells in such a bicategory amounts to the so-called *cascade product* between a Mealy machine and a semiautomaton. Among many different direction for future research, an exciting prospect is to prove the Krohn-Rhodes theorem [12, 22, 23] by resorting to bicategorical properties.

Besides providing a guarantee of correctness, formalizing our results in a proof assistant might also pave the way for "concrete" implementations of our theoretical results, where, for instance, the proofs also act as concrete programs that allow the user to convert between different automata in a provably correct way.

Overall, we believe our research started a foundational look to automata theory by offering a novel perspective on known results. As category theorists, we are confident that approaching familiar concepts from a higher vantage point yields invaluable insights, fostering the advancement of the field and unlocking practical applications.

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A Agda formalization

Here we briefly comment here on our use of the Agda proof assistant to formalize some of the main results of this paper.

In most cases we used it to formalize the most tricky aspects of the proofs, without focusing on providing a complete formalization of all results shown in this work, for which the pen-and-paper approach still has a considerable edge in terms of speed and effort. For example, the proof mechanized for Theorem 3.6 concentrates only on explicitly defining terminal and binary products, thus providing only a general insight on how non-connected limits are computed. Our development consists of around 2000 LoC and, thanks to its reusability, has been employed to formalize results in subsequent papers such as [2].

We use the library `agda-categories` as a starting point from which to build and prove further theorems, without having to formalize basic notions of category theory from scratch. Most of the proofs mechanized for this paper are straightforward and follow directly from the universal properties of the objects under consideration; the most difficult part of our development has been to identify the necessary properties to prove facts about inductively defined objects (e.g., the interdependencies between the different lemma needed in Theorem 3.6) and the lack of automation mechanisms to close the proofs, which can end up in particularly long sequences of hom-reasoning steps.

Other minor issues arise from some architectural choices made in the `agda-categories` library, which, following a well-established practice in formalizations of category theory, defines categories as setoid-enriched, i.e., every category incorporates an internal notion of equality between morphisms. This often results in better-behaved but weaker notions of equalities between morphisms that more closely follow the principle of equivalence; for example, in the large category `Cat`, equality of functors is defined as natural isomorphism between functors, rather than strict equality on objects and arrows. This becomes problematic when defining universal objects in `Cat`, such as the (strict) 2-pullbacks used in Proposition 3.5

to characterize the categories F-Mly and F-Mre, since in this picture limits are actually defined up to equivalence of categories –from the theoretical point of view, they are *bilimits*; from the implementation point of view, the weak universal property is due to the lack of uniqueness of identity proofs for arbitrary hom-equalities.

In practice this has been dealt with by working in the (large) category StrictCat where equality of functors is defined strictly, which allows us to recover pullbacks between categories and the characterizations shown in this paper.