Forward and Backward Steps in a Fibration

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Abstract

Distributive laws of various kinds occur widely in the theory of coalgebra, for instance to model automata constructions and trace semantics, and to interpret coalgebraic modal logic. We study steps, which are a general type of distributive law, that allow one to map coalgebras along an adjunction. In this paper, we address the question of what such mappings do to well known notions of equivalence, e.g., bisimilarity, behavioural equivalence, and logical equivalence.

We do this using the characterisation of such notions of equivalence as (co)inductive predicates in a fibration. Our main contribution is the identification of conditions on the interaction between the steps and liftings, which guarantees preservation of fixed points by the mapping of coalgebras along the adjunction. We apply these conditions in the context of lax liftings proposed by Bonchi, Silva, Sokolova (2021), and generalise their result on preservation of bisimilarity in the construction of a belief state transformer. Further, we relate our results to properties of coalgebraic modal logics including expressivity and completeness.

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1 Introduction

The theory of coalgebras provides a general perspective on state-based systems, parametric in an endofunctor which models the type of system [19]. Accordingly, many interesting constructions on state-based systems arise as functors between categories of coalgebras.

These functors between categories of coalgebras often arise as liftings of left or right adjoints between the underlying base categories. Such liftings are central to, for instance, coalgebraic approaches to trace semantics and determinisation [13, 20, 33, 6, 22, 32] as well as testing semantics and algebraic semantics of modal logics [27, 31, 5, 24, 8, 23].

A central question is how these constructions on coalgebras affect behavioural equivalence. For instance, determinisation of automata turns a coalgebra on, e.g., the category Set of sets and functions, into a coalgebra on the category of Eilenberg-Moore algebras for a monad, so that the canonical notion of behavioural equivalence changes from bisimilarity to language semantics. Subsequently, the algebraic structure may be forgotten again, turning the determinised coalgebra back into a Set coalgebra, and this simple operation does not

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affect behavioural equivalence. But when determinisation is not given by a distributive law, such as in the construction of belief-state transformers in [4], proving that this “forgetting” preserves and reflects behavioural equivalence can be non-trivial (see *op. cit.*, [35]).

A different type of example of a construction given by a lifting of an adjoint is the algebraic semantics of modal logic, where the semantics yields a transformation that takes a coalgebra (e.g., a Kripke model) and turns it into an algebra (here viewed as a coalgebra on an opposite category, for uniformity of the presentation). As we show later, one form of preservation of behavioural equivalence amounts to adequacy and expressivity of the logic.

We propose an abstract framework to analyse whether a coalgebra lifting of an adjoint preserves behavioural equivalence. The basic infrastructure is as follows.

- We use functor liftings in fibrations, which is a standard approach to define coalgebraic bisimilarity [17] and other (co)inductive predicates [14]. This approach to define coinductive predicates beyond bisimilarity has recently been used, for instance, in general expressivity proofs of modal logics [29, 26], closely connected to the current paper.
- We use the notion of a *step* to lift left and right adjoints to categories of coalgebras. Steps are a variant of distributive laws (also known as morphisms of endofunctors) over a left or right adjoint, named as such in [32] but widely used before. They are relevant in all of the above-mentioned examples on language semantics, determinisation and modal logic.

This paper connects steps and fibrations, to speak generally about preservation of coinductive (and inductive) predicates by coalgebra constructions. The key technical idea is to use a variant of fibred adjunctions [21]. We start with an adjunction and a step, and assume a fibration and functor lifting on both sides of the adjunction to formulate the coinductive predicates that we wish to relate. We then lift this adjunction to the total categories of the fibrations involved [16]. This setting allows us to formulate sufficient conditions for preservation of coinductive predicates by coalgebra constructions induced by steps.

There are two main variants of this abstract story: one that starts from a step that lifts the *left* adjoint to coalgebras, and one for lifting the *right* adjoint. The first allows us, for instance, to speak about adequacy and expressivity of modal logics, without referring to initial algebras. This connects to recent work that uses Galois connections [1], and in fact we recover those Galois connections from our adjunctions between fibrations. We also study an example that has not occurred in previous abstract frameworks for expressivity: proving expressivity of a logic by relating it to *apartness* instead of bisimilarity [12, 11].

The second variant – constructions arising as liftings of right adjoints – includes preservation of bisimilarity on belief-state transformers [4]. More generally, it follows from our results that any right adjoint in a *lax lifting* situation preserves and reflects bisimilarity (assuming split monos instead of injectivity), generalising the result for belief-state transformers. By using opposite categories we also get a very different example in this context, which connects preservation of coinductive predicates to completeness of coalgebraic modal logic.

2 Preservation of coinductive predicates in lattices

Before moving to the general theory of fibrations and steps, we start with introductory examples on preservation of (co)inductive predicates in the context of lattices, forming a special (and well-known) case of steps on Galois connections. In subsequent sections, we will use the structure of steps, being certain natural transformations allowing us to transform coalgebras along an adjunction. A similar structure is already known in order theory, where we may consider inequalities between compositions of monotone maps, as in:

$$b \circlearrowright \Delta \underset{g}{\overset{f}{\underset{\bot}{\rightleftarrows}}} \Gamma \circlearrowleft l \qquad (1)$$

with Δ and Γ lattices, and f, g, b, l monotone maps. We call the inequality $bg \leq gl$ a *forward* step, and $gl \leq bg$ a *backward* step. Central to our study of steps is the following standard result found in, e.g., [9], relating them to preservation of greatest fixed points (which we denote here using the operator ν).

▶ **Lemma 1.** *Given the setting of* (1)*:*
1. *If $gl \leq bg$, then $g(\nu l) \leq \nu b$;*
2. *If $bg \leq gl$, then $g(\nu l) \geq \nu b$.*

Now, if $bg = gl$ in the setting of (1), the inequalities combine to give $g(\nu l) = \nu b$. This has been shown more generally in the context of coalgebras in [17], where the equality $bg = gl$ is instead a natural isomorphism $BG \xrightarrow{\sim} GL$, with F, G, B, L functors. It is shown in *op. cit.* that this allows the lifting of the adjunction $F \dashv G$ to coalgebras (generalising post-fixed points) so that the right adjoint preserves the final coalgebra (generalising the greatest fixed point) as right adjoints preserve limits.

2.1 Example: Closed and Convex Relations

We will first consider two instances where the lattices consist of relations on sets on one side, and relations with either topological or convex structure on the other, i.e.:

$$b \circlearrowright \mathsf{Rel}_X \underset{u}{\overset{c}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{CRel}_{\mathbb{X}} \circlearrowleft v \qquad b \circlearrowright \mathsf{Rel}_A \underset{u}{\overset{h}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{ConRel}_{\mathbb{A}} \circlearrowleft d \tag{2}$$

where $\mathsf{CRel}_{\mathbb{X}}$ consists of closed relations on a compact Hausdorff space $\mathbb{X}$ and $\mathsf{ConRel}_{\mathbb{A}}$ consists of convex relations on a convex algebra $\mathbb{A}$. The monotone maps v, d will be such that the post-fixed points are bisimulations and the greatest fixed points are bisimilarity on systems with topological ($\mathbb{X}$) or convex ($\mathbb{A}$) structure, and the maps b characterise bisimulations/bisimilarity on systems where this structure has been forgotten (X and A respectively). Lemma 1, thus, tells us how bisimilarity on each side can be related via the right adjoint.

These settings arise in examples of ultrafilter extensions for coalgebras and the transformation of probabilistic automata into belief-state transformers. In the first instance, the closed relations can in fact be restricted to Stone topological spaces (those compact Hausdorff spaces which are zero-dimensional), where we consider coalgebras for the Vietoris functor in Stone. It is shown in [2] that these coalgebras correspond to descriptive frames, which arise in the first stage of the construction of the ultrafilter extension of a Kripke frame given in [28]. The second stage given there is to transport these back to a coalgebra in Set. The construction of a belief-state transformer from a probabilistic automaton (PA) has a similar structure, where the second stage is to transport a system with extra algebraic structure back to a Set coalgebra. In each case, it is important that behavioural equivalence is preserved and reflected in the second stage, shown in [2, 4] respectively for the above examples. These results are recovered already in [35]. However, the approach taken there does not immediately apply to the examples of adequacy and expressivity of modal logics, so we prefer the conditions given in the current paper for their generality.

2.2 Example: Expressivity

Another example relates to work on expressivity of coalgebraic logics [29, 1, 24, 23], where we wish to relate bisimilarity and logical equivalence (or indistinguishability). The lattices involved are equivalence relations on the carrier X of a coalgebra and predicates on 2^X.

$$b \circlearrowright \mathsf{ERel}_X \underset{g}{\overset{f}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Pred}_{2^X}^{\mathrm{op}} \circlearrowleft l \tag{3}$$

This gives us the setting shown in (3), where we define $g(\Gamma \subseteq 2^X) = \{(x, y) \mid \forall S \in \Gamma.\, x \in S \iff y \in S\}$. In words, we relate those elements which are in exactly the same sets of Γ. Next, the action of f on an equivalence relation R is to give those subsets which are closed under R, i.e., they are a union of equivalence classes of R. More formally, we have $f(R) = \{\Gamma \subseteq X \mid \forall (x, y) \in R.\, x \in \Gamma \iff y \in \Gamma\}$.

The monotone map b is taken to be such that the greatest fixed point is bisimilarity. The map l is dual to the map whose *least* fixed point we can think of as those predicates obtainable as the interpretation of a formula of a modal logic. In essence, these *are* the formulas which we generate from some propositional constants and applications of the operators of our logic.

Applying g to these "reachable" predicates gives an equivalence relating states which satisfy exactly the same formulas. This is exactly logical equivalence, and the above picture then allows us to relate this to bisimilarity. Namely, if $g(\nu l) \geq \nu b$, then bisimilarity implies logical equivalence, which is precisely adequacy of the logic. If, conversely, $g(\nu l) \leq \nu b$, then logical equivalence implies bisimilarity, called expressivity of the logic.

Now, Lemma 1 gives us a way to obtain these inclusions via inequalities capturing the interaction between the logic and behaviour in a rather general way. Later, we will show in more detail how these conditions relate to existing approaches to the semantics of coalgebraic modal logic and the properties of adequacy and expressivity.

3 Fibrations and Bisimulations

We give the basic definitions related to fibrations (for details see [21]).

Given a functor $p\colon \mathcal{E} \to \mathcal{C}$, a morphism $b\colon R \to S$ in $\mathcal{E}$ is *(p-)Cartesian* over $f\colon X \to Y$ in $\mathcal{C}$, if $pb = f$ and for every $c\colon T \to S$ s.t. $pc = f \circ g$ for some $g\colon pT \to X$, there is a unique $d\colon T \to R$ with $c = b \circ d$. A functor $p\colon \mathcal{E} \to \mathcal{C}$ is now a (Grothendieck) fibration if for all objects $S \in \mathcal{E}$ and arrows $f\colon X \to pS$, there is a Cartesian arrow $b\colon R \to S$ in $\mathcal{E}$ with $pb = f$ (and thus also $pR = X$). We say that R is *above* pR and $b\colon R \to S$ is *above* $pb\colon pR \to pS$. We will also call $\mathcal{C}$ the base category and $\mathcal{E}$ the total category of the fibration.

For an object $X \in \mathcal{C}$, the *fibre above* X is the category $\mathcal{E}_X$ whose objects are those objects in $\mathcal{E}$ above X, and arrows are above the identity on X. A choice of Cartesian lifting for every $f\colon X \to Y$ in $\mathcal{C}$ is called a *cleavage*, and any cleavage defines, for each such f, a *reindexing* functor $f^*\colon \mathcal{E}_Y \to \mathcal{E}_X$ defined on objects exactly by the choice of Cartesian arrow $\overline{f}(S)\colon f^*(S) \to S$. We assume below that reindexing functors have left adjoints $\coprod_f \dashv f^*$ (called *direct-image*). This is equivalent to the condition that both $p\colon \mathcal{E} \to \mathcal{C}$ and $p^{\mathrm{op}}\colon \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ are fibrations, in which case, p is also called a *bifibration*.

Given fibrations $p\colon \mathcal{E} \to \mathcal{C}$ and $q\colon \mathcal{F} \to \mathcal{D}$, a *morphism of fibrations* from p to q is a pair of functors $(\overline{F}\colon \mathcal{E} \to \mathcal{F}, F\colon \mathcal{C} \to \mathcal{D})$ such that $q \circ \overline{F} = F \circ p$. In that case, for every object we have a restriction $\overline{F}_X\colon \mathcal{E}_X \to \mathcal{E}_{FX}$, denoted by $\overline{F}$ if the type is clear from the context. We will also call $\overline{F}$ a lifting of F. If $\overline{F}$ preserves Cartesian morphisms, it is called *fibred*. This is equivalent to having the equation $\overline{F}_X \circ f^* = (Ff)^* \circ \overline{F}_Y$ for all morphisms $f\colon X \to Y$.

We will work with fibrations with the additional assumption that the fibres form complete lattices and reindexing preserves meets, i.e., the fibrations have *fibred* meets:

▶ **Assumption 2.** *We assume that for any fibration $p\colon \mathcal{E} \to \mathcal{C}$, the fibres $\mathcal{E}_X$ form complete lattices and reindexing preserves meets. We will also call such a fibration a* $\mathsf{CLat}_\wedge$*-fibration.*

This ensures that the fibrations have many desirable properties, while being general enough for our purposes of defining coinductive predicates. In particular, such fibrations are always bifibrations. For a more detailed treatment of $\mathsf{CLat}_\wedge$-fibrations see, e.g., [25, 34].

3.1 Subobject and Relation fibrations

Take a category $\mathcal{C}$ which is complete and well-powered (subobjects of a given object form a set). Then, the category $\mathsf{Pred}(\mathcal{C})$ is defined as follows: objects are subobjects $f\colon S \rightarrowtail X$, i.e., equivalence classes of monos; and morphisms are maps $u\colon X \to Y$ in $\mathcal{C}$ such that there is a (necessarily unique) arrow making the diagram on the left in (4) commute. Then the functor $p\colon \mathsf{Pred}(\mathcal{C}) \to \mathcal{C}$ sending a subobject $f\colon S \rightarrowtail X$ to X, is a fibration, with reindexing given by pullbacks, referred to as the *predicate* fibration. Since the base category is complete and well-powered, the fibres $\mathsf{Pred}(\mathcal{C})_X$ are complete lattices [7, Cor. 4.2.5]. Since reindexing is defined by pullback, it preserves meets, so that p is a $\mathsf{CLat}_\wedge$-fibration.

$$
\begin{array}{ccc}
S & \dashrightarrow & T \\
\downarrow & & \downarrow \\
X & \xrightarrow{u} & Y
\end{array}
\qquad
\begin{array}{ccccc}
\mathsf{Rel}(\mathcal{C}) & \longrightarrow & \mathsf{Rel}'(\mathcal{C}) & \longrightarrow & \mathsf{Pred}(\mathcal{C}) \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\mathcal{C} & \xrightarrow{\Delta} & \mathcal{C}\times\mathcal{C} & \xrightarrow{\times} & \mathcal{C}
\end{array}
\tag{4}
$$

As $\mathcal{C}$ is complete, it furthermore has products, so we can form the relation fibration $\mathsf{Rel}(\mathcal{C})$ via the pullbacks as on the right in (4). The fibration $\mathsf{Rel}'(\mathcal{C})$ consists of relations on *all* pairs of objects $(X, Y) \in \mathcal{C} \times \mathcal{C}$, whereas we will use the fibration $\mathsf{Rel}(\mathcal{C})$ consisting of relations for which $X = Y$. By this definition, we obtain that an object of $\mathsf{Rel}(\mathcal{C})$ is a subobject $R \rightarrowtail X \times X$ of the product of X with itself. The functor part of the fibration sends a relation $R \rightarrowtail X \times X$ to X and a morphism to the underlying arrow $u\colon X \to Y$, analogously to (4). By the same argument as for $\mathsf{Pred}(\mathcal{C})$, $\mathsf{Rel}(\mathcal{C})$ is a $\mathsf{CLat}_\wedge$-fibration.

▶ **Example 3.** In Set, subobjects are just subsets $U \subseteq X$, and reindexing is inverse image, i.e., $f^*(V \subseteq Y) = \{x \in X \mid f(x) \in V\}$. Similarly, relations are subsets $R \subseteq X \times X$, with $f^*(S \subseteq Y \times Y) = \{(x_1, x_2) \in X \times X \mid (f(x_1), f(x_2)) \in S\}$. Notice Set is complete, and the collection of subsets of a set is its powerset which is again a set. Each powerset is thus a complete lattice with join and meet given by union and intersection, respectively.

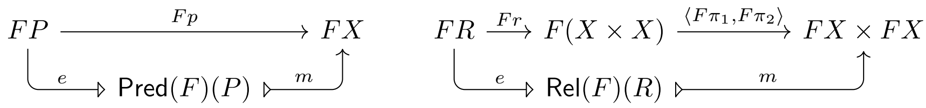
For a monad $T\colon \mathsf{Set} \to \mathsf{Set}$, let $\mathsf{EM}(T)$ be the category of Eilenberg-Moore algebras. Then $\mathsf{Pred}(\mathsf{EM}(T))$ consists of subalgebras, and $\mathsf{Rel}(\mathsf{EM}(T))$ consists of congruences, i.e., relations that are closed under the algebra structure (not necessarily equivalence relations).

We can restrict $\mathsf{Rel}(\mathcal{C})$ to the category $\mathsf{ERel}(\mathcal{C})$ of equivalence relations, defined internally (e.g., [21]), and define reindexing and meets for equivalence relations in the same way as for relations since these are defined as pullbacks. This turns $\mathsf{ERel}(\mathcal{C}) \to \mathcal{C}$ into a $\mathsf{CLat}_\wedge$-fibration.

3.2 Predicate and Relation liftings

Here we recall a method for lifting functors to predicates and relations based on factorisation systems. For a factorisation system $(\mathcal{E}, \mathcal{M})$, we refer to elements of $\mathcal{E}$ as *abstract epis* and write them as $\cdot \twoheadrightarrow \cdot$, and maps in $\mathcal{M}$ *abstract monos* written as $\cdot \rightarrowtail \cdot$. As an example, for Set we can take $\mathcal{E}$ to be the class of all surjective functions, and $\mathcal{M}$ to be the class of all injective functions. The factorisation of a function $f\colon X \to Y$ is the image factorisation, where $e\colon X \twoheadrightarrow \mathrm{Im}(f)$ acts as f and $m\colon \mathrm{Im}(f) \rightarrowtail Y$ embeds the image of f into the original codomain Y. Another important example is that of *regular categories*, where maps factorise as a *regular* epi (i.e., an epi which is the coequaliser of some parallel pair of morphisms) followed by a mono. In fact, the existence of such factorisations is part of the defining property of a regular category.

Assuming a category $\mathcal{D}$ with a factorisation system $(\mathcal{E}, \mathcal{M})$ such that all maps in $\mathcal{M}$ are monos, we can define the (canonical) predicate and relation liftings $\mathsf{Pred}(F)$ and $\mathsf{Rel}(F)$ of a functor $F\colon \mathcal{C} \to \mathcal{D}$ via the following factorisations, where $p\colon P \rightarrowtail X$ is a predicate and $r\colon R \rightarrowtail X \times X$ a relation:



(5)

By the assumption that all maps in $\mathcal{M}$ are monos, the above constructions define functors $\mathsf{Pred}(F)\colon \mathsf{Pred}(\mathcal{C}) \to \mathsf{Pred}(\mathcal{D})$ and $\mathsf{Rel}(F)\colon \mathsf{Rel}(\mathcal{C}) \to \mathsf{Rel}(\mathcal{D})$ respectively, with the actions on arrows defined by orthogonality.

3.3 Invariants and Coinductive Predicates

We will now recall the notion of coinductive invariants and predicates, defined as post and greatest fixed points of certain monotone maps respectively (see also [19, 14]). Assumption 2 ensures that the monotone maps always have such fixpoints.

Assuming a fibration $p\colon \mathcal{E} \to \mathcal{C}$, a coalgebra $f\colon X \to BX$ with X in $\mathcal{C}$, and a lifting $\overline{B}\colon \mathcal{E} \to \mathcal{E}$ of $B\colon \mathcal{C} \to \mathcal{C}$, we can define a monotone map $f^* \circ \overline{B}_X\colon \mathcal{E}_X \to \mathcal{E}_X$ using reindexing. Instantiating this to the category Set, and the fibration with $\mathsf{Rel}(\mathsf{Set})$ as total category, we can consider the canonical relation lifting (5) of B, given explicitly by $\mathsf{Rel}(B)(R) = \{(y_1, y_2) \in BX \times BX \mid \exists z \in BR. B\pi_1(z) = y_1 \wedge B\pi_2(z) = y_2\}$. As mentioned earlier, reindexing for the relation fibration is given by pullbacks, which amounts to taking the inverse image, so that:

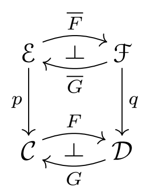
$$f^* \circ \mathsf{Rel}(B)(R) = \{(x_1, x_2) \in X \times X \mid \exists z \in BR. B\pi_1(z) = f(x_1) \wedge B\pi_2(z) = f(x_2)\} \quad (6)$$

Taking a post-fixed point $R \leq f^* \circ \mathsf{Rel}(B)(R)$ of such a monotone map (also called an *invariant*), we recover the usual notion of *coalgebraic bisimulation*. The greatest fixed point $\nu(f^* \circ \mathsf{Rel}(B)(-))$ is then bisimilarity.

More generally, for a lifting $\overline{B}$ of B, we call such a greatest fixed point the *coinductive predicate* defined by $\overline{B}$ on f. This covers many more examples than bisimilarity. For a simple instance, take B to be the powerset functor $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$, and $\overline{\mathcal{P}}\colon \mathsf{Pred} \to \mathsf{Pred}$ with $\overline{B}(P \subseteq X) = \{S \subseteq X \mid P \cap S \neq \emptyset\}$. A $\mathcal{P}$-coalgebra is a transition system, and the coinductive predicate on it defined by $\overline{\mathcal{P}}$ is the set of all states which have an infinite path. For other examples of coinductive predicates defined in this way see, e.g., [14, 3, 34, 18, 25].

4 Lifting adjunctions in a fibration

Let $F \dashv G\colon \mathcal{D} \to \mathcal{C}$ be an adjunction, and assume fibrations $p\colon \mathcal{E} \to \mathcal{C}$ and $q\colon \mathcal{F} \to \mathcal{D}$. Further, assume an adjunction $\overline{F} \dashv \overline{G}\colon \mathcal{F} \to \mathcal{E}$, as in (7). If we have $q \circ \overline{F} = F \circ p$, $p \circ \overline{G} = G \circ q$, and the unit and counit of the adjunction $\overline{F} \dashv \overline{G}$ are above the unit and counit of the adjunction $F \dashv G$ respectively, then we say that $\overline{F} \dashv \overline{G}$ is a *lifting of the adjunction* $F \dashv G$ (alternatively, this is an adjunction in $\mathsf{Cat}^{\to}$, the 2-category of functors and commuting squares [16]).

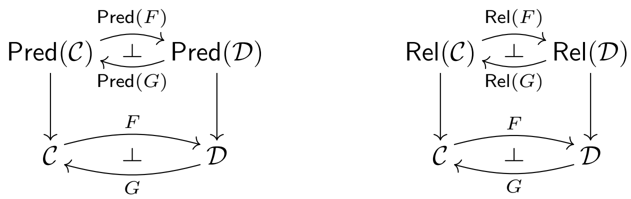


(7)

This definition differs from the usual notion of a *fibred adjunction*, as we do not assume fibredness of either adjoint. However, it has been shown in [36, Lemma 4.5] for fibrations over a single base category, and later generalised in [16, Lemma 3.3.3] to fibrations over arbitrary base categories, that the right adjoint in a lifting of an adjunction is in fact always fibred. Dually, we have that the left adjoint is co-fibred (i.e., it preserves op-Cartesian maps).

A family of instances is given by predicate and relation liftings.

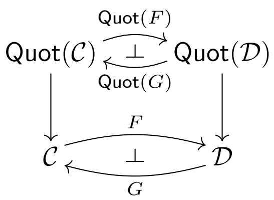
▶ **Lemma 4.** *Let $\mathcal{C}, \mathcal{D}$ be complete and well-powered categories with factorisation systems $(\mathcal{E}_1, \mathcal{M}_1), (\mathcal{E}_2, \mathcal{M}_2)$ with $\mathcal{M}_1$ and $\mathcal{M}_2$ consisting of monos. Given an adjunction $F \dashv G \colon \mathcal{D} \to \mathcal{C}$ such that $F(\mathcal{E}_1) \subseteq \mathcal{E}_2$, the predicate and relation liftings form liftings of the adjunction:*



(8)

From these liftings, especially the case for predicates, we can lift also to quotients and equivalence relations given some extra conditions. For a category $\mathcal{C}$, we denote by $\mathsf{Quot}(\mathcal{C})$ the category of co-subobjects of objects of $\mathcal{C}$, that is, equivalence classes of epimorphisms. This is exactly the category $\mathsf{Pred}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$, so that also $\mathsf{Pred}(\mathcal{C}^{\mathrm{op}}) \simeq \mathsf{Quot}(\mathcal{C})^{\mathrm{op}}$ and $\mathsf{Quot}(\mathcal{C}^{\mathrm{op}}) \simeq \mathsf{Pred}(\mathcal{C})^{\mathrm{op}}$. Further, we can define quotient lifting dually to predicate lifting. The following is then the dual of the above result.

▶ **Corollary 5.** *Let $\mathcal{C}, \mathcal{D}$ be co-complete and co-well-powered categories with factorisation systems $(\mathcal{E}_1, \mathcal{M}_1), (\mathcal{E}_2, \mathcal{M}_2)$ with $\mathcal{E}_1, \mathcal{E}_2$ consisting of epis. Then, given an adjunction $F \dashv G \colon \mathcal{D} \to \mathcal{C}$, s.t. $G(\mathcal{M}_1) \subseteq \mathcal{M}_2$, the quotient liftings form a lifting of the adjunction:*



(9)

As discussed above, predicates and quotients in the opposite category are (as objects) exactly quotients and predicates in the original category respectively. In the following result, we take a dual adjunction, so that the lifting gives an adjunction between predicates and quotients. We further give conditions under which the quotients correspond to equivalence relations (ERel). We then have adjunctions between predicates and equivalence relations, which we require for our applications to modal logic in Section 6.

▶ **Corollary 6.** *Let $\mathcal{C}$ and $\mathcal{D}$ be complete, well-powered and co-complete, co-well-powered categories respectively, with factorisation systems $(\mathcal{E}_1, \mathcal{M}_1), (\mathcal{E}_2, \mathcal{M}_2)$ with $\mathcal{M}_1$ consisting of monos, and $\mathcal{E}_2$ consisting of epis. Suppose also an adjunction $F \dashv G \colon \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$, such that $F(\mathcal{E}_1) \subseteq \mathcal{M}_2$. If $\mathcal{D}$ is an (Barr) exact category in which all epis are regular, we obtain a lifting of the adjunction as on the left below. If instead $\mathcal{C}$ is exact and all epis are regular we obtain the lifting of the adjunction as on the right:*

$$
\begin{array}{ccc}
\mathsf{Pred}(\mathcal{C}) & \underset{\mathsf{Pred}(G)}{\overset{\mathsf{Pred}(F)}{\rightleftarrows}}\;\bot & \mathsf{ERel}(\mathcal{D})^{\mathrm{op}} \\
\downarrow & & \downarrow \\
\mathcal{C} & \underset{G}{\overset{F}{\rightleftarrows}}\;\bot & \mathcal{D}^{\mathrm{op}}
\end{array}
\qquad
\begin{array}{ccc}
\mathsf{ERel}(\mathcal{C}) & \underset{\mathsf{Quot}(G)}{\overset{\mathsf{Quot}(F)}{\rightleftarrows}}\;\bot & \mathsf{Pred}(\mathcal{D})^{\mathrm{op}} \\
\downarrow & & \downarrow \\
\mathcal{C} & \underset{G}{\overset{F}{\rightleftarrows}}\;\bot & \mathcal{D}^{\mathrm{op}}
\end{array}
\tag{10}
$$

Our goal is now to relate liftings of adjunctions to adjunctions defined between fibres. In [21] it is shown how this can be done for fibrations over a single base category. Also studied in [15, 16] is how adjunctions between fibrations with distinct base categories arise from adjunctions between fibrations with a common base category.

▶ **Lemma 7.** *Suppose we have a lifted adjunction as in* (7)*. Then we also have the following adjunctions between fibres, for all objects X of $\mathcal{C}$ and Y of $\mathcal{D}$, where η and ε are the unit and counit of the adjunction $F \dashv G$ respectively.*

$$
\mathcal{E}_X \underset{\eta_X^* \circ \overline{G}}{\overset{\overline{F}}{\rightleftarrows}}\;\bot\; \mathcal{F}_{FX}
\qquad\qquad
\mathcal{E}_{GY} \underset{\overline{G}}{\overset{\coprod_{\varepsilon_Y} \circ \overline{F}}{\rightleftarrows}}\;\bot\; \mathcal{F}_Y
\tag{11}
$$

Returning to the example of adjunctions for predicate and relation liftings (Lemma 4), Lemma 7 allows us to obtain adjunctions between fibres, which are of interest when we study invariants and coinductive predicates in the coming section. In particular, we recover the adjunctions from Section 2.

▶ **Example 8** (Eilenberg-Moore)**.** For the case of an adjunction monadic over Set, each category (Set and $\mathsf{EM}(T)$ for a monad T) has a ($\mathsf{RegEpi}, \mathsf{Mono}$)-factorisation system as they are regular. Also, the abstract epis are preserved by left adjoints and these categories are complete and well-powered. We thus obtain a lifting of any monadic adjunction to predicates, relations and quotients by Lemma 4 and Corollary 5. Furthermore, the adjunction between fibres as on the right in Equation (11) then exactly instantiates to the adjunctions discussed in Section 2.1 for the cases of compact Hausdorff spaces and convex algebras. The left adjoint in each case takes the closure or convex hull of a relation on a set.

In fact, each of these "local" adjunctions implies the existence of the other. Note that we do not assume a (global) lifted adjunction, so we must assume (co-)fibredness explicitly.

▶ **Lemma 9.** *Suppose we have an adjunction $F \dashv G\colon \mathcal{D} \to \mathcal{C}$ and bifibrations $p\colon \mathcal{E} \to \mathcal{C}$, $q\colon \mathcal{F} \to \mathcal{D}$. Also, suppose $\overline{G}$ is a fibred lifting of G and $\overline{F}$ is a co-fibred lifting of F. Then we have a adjunctions $\overline{F}_X \vdash \eta_X^* \circ \overline{G}_{FX}$ for all X iff we have adjunctions $\coprod_{\varepsilon_Y} \circ \overline{F}_{GY} \vdash \overline{G}_Y$ for all Y, that is, the adjunctions in* (11) *can be derived from each other.*

Due to results of [16, 15] on factorisation of fibred adjunctions, we can also go from the existence of adjunctions between fibres (above all objects of our base category) to an adjunction between the total categories. As mentioned before, we drop the requirement of fibredness as much as possible.

▶ **Lemma 10.** *Suppose we have an adjunction $F \dashv G\colon \mathcal{D} \to \mathcal{C}$ and fibrations $p\colon \mathcal{E} \to \mathcal{C}$ and $q\colon \mathcal{F} \to \mathcal{D}$. Then the following are equivalent*

1. *A lifting of the adjunction to $\overline{F} \dashv \overline{G}\colon \mathcal{F} \to \mathcal{E}$*
2. *A fibred lifting $\overline{G}$ of G and for each object Y of $\mathcal{D}$, a left adjoint to $\overline{G}\colon \mathcal{F}_Y \to \mathcal{E}_{GY}$*
3. *A fibred lifting $\overline{G}$ of G and for each object Y of $\mathcal{D}$, $\overline{G}\colon \mathcal{F}_Y \to \mathcal{E}_{GY}$ preserves meets.*

This allows us, under certain conditions, to lift an adjunction to *equivalence* relations.

▶ **Lemma 11.** *In the context of Lemma 4 and assuming that we use a factorisation system on $\mathcal{C}$ with all abstract epis being split epi,* $\mathsf{Rel}(G)$ *maps equivalence relations to equivalence relations. Further, its restriction to equivalence relations has a left adjoint* $\overline{F}$*, forming a lifting of the adjunction between base categories.*

▶ Remark 12. The condition on abstract epis of Lemma 11 is stronger than our earlier assumption that the left adjoint preserves abstract epis, as having a factorisation system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{E} \subseteq \mathsf{SplitEpi}$ and $\mathcal{M} \subseteq \mathsf{Mono}$ implies that in fact $\mathcal{E} = \mathsf{SplitEpi}$ (cf. [19, Exercise 4.4.2]) and split epis are absolute in the sense that *all* functors preserve them.

5 Comparing coinductive predicates along steps

In this section, we consider endofunctors in the setting of an adjunction, and will study coalgebras for these endofunctors – and sometimes algebras, viewed as coalgebras in an opposite category. These endofunctors are connected via the notion of a *step* [32], which is a natural transformation that allows one to transport coalgebras along the adjunction. More formally, steps give rise to liftings of the right and left adjoint (depending on which kind of step) to categories of coalgebras. The key question that we address in this section is whether these liftings to categories of coalgebras preserve a coinductive predicate of interest.

▶ **Definition 13.** *Consider an adjunction with endofunctors as follows:*

$$B \circlearrowright \mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D} \circlearrowleft L \quad (F \dashv G) \tag{12}$$

Then a forward step *is a natural transformation* $\delta \colon BG \to GL$ *and a* backward step *is a natural transformation* $\iota \colon GL \to BG$*.*

Due to the adjunction $F \dashv G$, a natural transformation $\delta \colon BG \to GL$ has a *mate* $\hat{\delta} \colon FB \to LF$ given by $\hat{\delta} = \varepsilon_{LF} \circ F\delta F \circ \eta_{FB}$. This then gives rise to liftings of F and G to coalgebras, called *step-induced coalgebra liftings* and denoted $\hat{F} \colon \mathsf{CoAlg}(B) \to \mathsf{CoAlg}(L)$ and $\hat{G} \colon \mathsf{CoAlg}(L) \to \mathsf{CoAlg}(B)$ respectively. These are defined on objects by

$$f \colon X \to BX \mapsto \hat{\delta}_X \circ Ff \colon FX \to LFX \tag{13}$$

$$g \colon Y \to LY \mapsto \iota_Y \circ Gg \colon GY \to BGY \tag{14}$$

On arrows, they act simply as F and G. This is well defined due to functoriality of F and G and naturality of the involved steps.

▶ Remark 14. The names "forward" and "backward" steps are from [35], where they are assumed to be one-sided inverses. In the current paper, we make no such assumption and study forward and backward steps independently from each other. In [32] only what we refer to as a forward step appears. There is a clear asymmetry between the two; forward steps have a mate correspondence, and at least two other equivalent presentations via transposing along the adjunction. For backward steps there seem to be no such equivalent characterisations, as the left adjoint is on the "wrong" side.

▶ **Example 15.** An example of such transformations occurs in a determinisation procedure for probabilistic automata given in [4]. There, the functors B and L are taken to be $B = \mathcal{P}^A$ and $L = \mathcal{P}_c^A$ with A a set of labels and $\mathcal{P}_c$ the convex powerset on $\mathsf{EM}(\mathcal{D})$, equivalent to the category of convex algebras. Note that we allow the empty set in the definition of $\mathcal{P}_c(X) = \{S \subseteq X \mid S \text{ convex}\}$. It is shown in *op. cit.* that there is an injective natural transformation $\iota\colon \mathcal{U} \circ \mathcal{P}_c^A \to \mathcal{P}^A \circ \mathcal{U}$, induced by an analogous transformation for $B = \mathcal{P}$ and $L = \mathcal{P}_c$. Such a transformation without labels simply includes convex subsets into the powerset. This has a componentwise inverse which forms, for each subset, its convex hull.

Aside from this example, which we will revisit later, steps occur, e.g., as the one-step semantics of coalgebraic modal logics (more usually, the mate of a forward step) [31, 24, 32], and have been used to construct ultrafilter extensions of coalgebras [28]. In the case of ultrafilter extensions for powerset coalgebras, the steps are those defining a weak lifting in the sense of Garner [10]; the forward step forms the topological closure of all subsets, and the backward step includes closed subsets into the powerset.

5.1 Comparing coinductive predicates

We have now seen how steps defined for an adjunction with endofunctors on each of the categories allow us to map coalgebras along this adjunction. Further, when we have fibrations on each of the categories of the adjunction, and liftings of the involved functors, we can define predicates on these coalgebras. Next, we will combine these transformations and give conditions on the steps and liftings, which allow us to link predicates on a coalgebra with predicates on the coalgebra obtained by applying a step-induced lifting.

▶ **Assumption 16.** *Throughout this section, we assume a lifting $\overline{F} \dashv \overline{G}\colon \mathcal{F} \to \mathcal{E}$ of an adjunction $F \dashv G\colon \mathcal{D} \to \mathcal{C}$, together with endofunctors B, L on $\mathcal{C}$ and $\mathcal{D}$ and liftings $\overline{B}$ and $\overline{L}$ to $\mathcal{E}$ and $\mathcal{F}$ respectively.*

These assumptions give us coinductive predicates on B-coalgebras, using $\overline{B}$, and on L-coalgebras, using $\overline{L}$. This setting allows us to put conditions on forward and backward steps. These conditions, in turn, allow us to obtain steps at the level of the induced adjunctions between fibres, which puts us back into the setting of Section 2. In particular, it allows us to apply Lemma 1 to preserve the relevant coinductive predicates. We now explain this in more detail for backward and forward steps separately.

5.1.1 Preservation via backward steps

Consider a backward step $\iota\colon GL \to BG$. Given an L-coalgebra (Y, g) together with this backward step and Assumption 16, we have the following setting.

$$(\iota_Y \circ Gg)^* \circ \overline{B} \circlearrowright \mathcal{E}_{GY} \underset{\overline{G}}{\overset{\coprod_{\varepsilon_Y} \circ \overline{F}}{\rightleftarrows}} \mathcal{F}_Y \circlearrowleft g^* \circ \overline{L} \tag{15}$$

The greatest fixed point $\nu(g^* \circ \overline{L})$ is a coinductive predicate on the L-coalgebra (Y, g). The greatest fixed point $\nu((\iota_Y \circ Gg)^* \circ \overline{B})$ is instead a coinductive predicate on the B-coalgebra obtained by applying the lifting $\hat{G}\colon \mathsf{CoAlg}(L) \to \mathsf{CoAlg}(B)$ induced by the step ι to (Y, g). Like in Section 2, we ask whether the right adjoint $\overline{G}_Y$ preserves the greatest fixed point, that is, maps $\nu(g^* \circ \overline{L})$ to $\nu((\iota_Y \circ Gg)^* \circ \overline{B})$.

The following result gives a sufficient condition for constructing a step in the above adjunction between fibres; this condition is in terms of the backward step ι and the liftings.

▶ **Lemma 17.** *For a (backward) step* $\iota\colon GL \to BG$ *and an* L*-coalgebra* $g\colon Y \to LY$*:*
1. *If* $\overline{G} \circ \overline{L} \leq \iota^* \circ \overline{B} \circ \overline{G}$*, then* $\overline{G}_Y \circ g^* \circ \overline{L}_Y \leq (\iota_Y \circ Gg)^* \circ \overline{B}_{GY} \circ \overline{G}_Y$*;*
2. *If* $\iota^* \circ \overline{B} \circ \overline{G} \leq \overline{G} \circ \overline{L}$*, then* $\overline{G}_Y \circ g^* \circ \overline{L}_Y \geq (\iota_Y \circ Gg)^* \circ \overline{B}_{GY} \circ \overline{G}_Y$*.*

We note that the condition of Item 1 is equivalent to having a lifting $\overline{\iota}\colon \overline{G}\,\overline{L} \to \overline{B}\,\overline{G}$ of ι, using the existence of a Cartesian lifting of ι. The inequality of Item 2 often requires further assumptions and more work. We will give some instances where it is satisfied in Section 6. Together, the assumptions are equivalent to $\overline{\iota}$ itself being a Cartesian map. Using Lemmas 1, 7, and 17 we obtain the following preservation result for coinductive predicates.

▶ **Corollary 18.** *Suppose we have a (backward) step* $\iota\colon GL \to BG$*. Then for any* $g\colon Y \to LY$*:*
1. *If* $\overline{G} \circ \overline{L} \leq \iota^* \circ \overline{B} \circ \overline{G}$*, then* $\overline{G}_Y(\nu(g^* \circ \overline{L}_Y)) \leq \nu((\iota_Y \circ Gg)^* \circ \overline{B}_{GY})$*;*
2. *If* $\iota^* \circ \overline{B} \circ \overline{G} \leq \overline{G} \circ \overline{L}$*, then* $\overline{G}_Y(\nu(g^* \circ \overline{L}_Y)) \geq \nu((\iota_Y \circ Gg)^* \circ \overline{B}_{GY})$*.*

Corollary 18 thus gives sufficient conditions for $\overline{G}_Y$ to map the greatest fixed point of the coinductive predicate on an L-coalgebra (Y, g) to the greatest fixed point of the coinductive predicate on the B-coalgebra $\hat{G}(Y, g)$ constructed via ι, in the setting of (15).

▶ Remark 19. It is in fact not necessary that ι is natural; that is, Lemma 17 and Corollary 18 go through even if ι is just a collection of arrows.

5.1.2 Preservation via forward steps

We proceed to focus on forward steps. Recall from Lemma 7 that the lifted adjunction between fibrations induces two types of adjunctions between fibres; for backward steps we used one of them, for forward steps we use the other. We thus work in the following setting:

$$f^*\circ\overline{B} \circlearrowright \mathcal{E}_X \underset{\eta_X^*\circ\overline{G}}{\overset{\overline{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{F}_{FX} \circlearrowleft (\hat{\delta}_X\circ Ff)^*\circ\overline{L} \tag{16}$$

where (X, f) is a B-coalgebra. We have the following result on constructing steps in this adjunction between fibres.

▶ **Lemma 20.** *Suppose we have a (forward) step* $\delta\colon BG \to GL$*, then for* $f\colon X \to BX$*:*
1. *If* $\delta^* \circ \overline{G} \circ \overline{L} \leq \overline{B} \circ \overline{G}$ *and* $\overline{B}$ *is fibred, then* $\eta_X^* \circ \overline{G}_{FX} \circ (\hat{\delta}_X \circ Ff)^* \circ \overline{L}_{FX} \leq f^* \circ \overline{B}_X \circ \eta_X^* \circ \overline{G}_{FX}$*;*
2. *If* $\overline{B} \circ \overline{G} \leq \delta^* \circ \overline{G} \circ \overline{L}$*, then* $\eta_X^* \circ \overline{G}_{FX} \circ (\hat{\delta}_X \circ Ff)^* \circ \overline{L}_{FX} \geq f^* \circ \overline{B}_X \circ \eta_X^* \circ \overline{G}_{FX}$

Similarly to backward steps, we get the following result from Lemmas 1, 7, and 20, giving a sufficient condition for preservation of the coinductive predicate by the right adjoint in (16).

▶ **Corollary 21.** *Suppose we have a (forward) step* $\delta\colon BG \to GL$ *and a lifting of the adjunction as in Equation* (7)*. Then for* $f\colon X \to BX$*:*
1. *If* $\delta^* \circ \overline{G} \circ \overline{L} \leq \overline{B} \circ \overline{G}$ *and* $\overline{B}$ *is fibred, then* $\eta_X^* \circ \overline{G}_{FX}(\nu((\hat{\delta}_X \circ Ff)^* \circ \overline{L}_{FX})) \leq \nu(f^* \circ \overline{B}_X)$*;*
2. *If* $\overline{B} \circ \overline{G} \leq \delta^* \circ \overline{G} \circ \overline{L}$*, then* $\eta_X^* \circ \overline{G}_{FX}(\nu((\hat{\delta}_X \circ Ff)^* \circ \overline{L}_{FX})) \geq \nu(f^* \circ \overline{B}_X)$

▶ Remark 22. Contrary to the case of backward steps (see Remark 19), for forward steps we use naturality, in the proof of Lemma 20. That proof is more involved than that of Lemma 17, emphasising again the asymmetry between forward and backward steps.

▶ Remark 23. We have assumed that the adjunction between base categories lifts to the total categories of the fibrations, even though the results in Corollary 21 and Corollary 18 are about the adjunctions between fibres. Therefore, one might be tempted to only assume these adjunctions between fibres instead of an adjunction between total categories. However, in Lemma 17 and Lemma 20 (on which the aforementioned results rely) we also use that the right adjoint $\overline{G}$ is fibred, and if we additionally assume this then it is equivalent to having an adjunction between the total categories (Lemma 10).

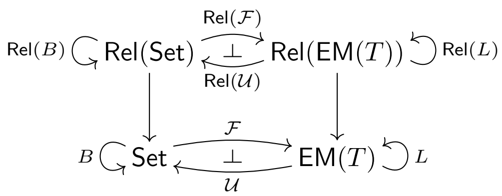
▶ Remark 24. Our focus is on the separate analysis of ι and δ. If we assume instead that: ι and δ both exist; their liftings form an isomorphism $\bar{\iota}\colon \overline{G}\,\overline{L} \simeq \overline{B}\,\overline{G}$; and we have a fibred lifting $\overline{G}$ of G such that its restriction to fibres preserves meets, then we have $\overline{G}_Y(\nu(g^* \circ \overline{L}_Y)) = \nu((\iota_Y \circ Gg)^* \circ \overline{B}_{GY})$ where $\iota = p\bar{\iota}$, for any L-coalgebra (Y, g). Restricting ourselves to fibrations over a single base category, $B = L$, and a lifting of the identity between the total categories, we recover [34, Prop. 6.2].

6 Examples

6.1 Lax liftings

Our first application of the results of the previous section continues on from Example 15, where we are now able to apply Corollary 18 to show the preservation and reflection of bisimilarity by the second stage of the construction given in the example.

That construction goes from probabilistic automata, which combine probabilistic transitions with non-deterministic choice, to belief-state transformers, where the probabilities occur in the state space rather than on the transitions. It has its roots in the generalised determinisation procedure of [33], but requires an alternative approach due to the non-existence of a lifting of the powerset monad to convex algebras. The determinisation starts from a monadic adjunction over Set, and then proceeds in two steps: first a lifting of the left adjoint gives a "determinised" system with algebraic structure, then a lifting of the right adjoint forgets this structure to give a system in Set. Here, we consider the second stage and take a lifting of the adjunction and endofunctors B and L to Rel fibrations as in (17), so that we may apply our earlier results to show preservation and reflection of bisimilarity.



(17)

The lifting of the right adjoint to coalgebras uses a ι which comes from a so-called *lax lifting* [4]. Given a functor $B\colon \mathsf{Set} \to \mathsf{Set}$, a lax lifting of B is a functor $L\colon \mathsf{EM}(T) \to \mathsf{EM}(T)$ such that there is an injective natural transformation $\iota\colon \mathcal{U} \circ L \to B \circ \mathcal{U}$. We show the following result for transformations that are componentwise split mono, and then show how this applies to the example of probabilistic automata.

▶ **Lemma 25.** *The lifting* $\hat{\mathcal{U}}\colon \mathsf{CoAlg}(L) \to \mathsf{CoAlg}(B)$ *induced by a componentwise split mono transformation* $\iota\colon \mathcal{U} \circ L \to B \circ \mathcal{U}$ *preserves and reflects bisimilarity.*

Taking $B = \mathcal{P}^A$ and $L = \mathcal{P}_c^A$ in the setting of Equation (17) (recall also Example 15), we have a componentwise split mono ι because we have an injective transformation, and $\mathcal{U} \circ \mathcal{P}_c^A(X)$ is only empty when A is empty, in which case also $\mathcal{P}^A \circ \mathcal{U}(X)$ is empty. In [35], a similar result is shown for behavioural equivalence instead of bisimilarity in case the functor B preserves weak pullbacks.

6.2 Expressivity

In this subsection, we will establish the well-known expressivity of modal logic with respect to $\mathcal{P}$-bisimilarity using our abstract framework. Note that for simplicity we consider (unlabelled) transition systems modelled as $\mathcal{P}$-coalgebra with $\mathcal{P}\colon \mathsf{Set} \to \mathsf{Set}$ the (full) powerset functor.

We relate bisimilarity to the logic defined by the following grammar:

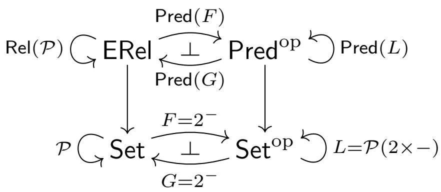
$$\phi \quad ::= \quad \Diamond\Big(\bigwedge_{i\in I} \phi_i \wedge \bigwedge_{i\in J} \neg\phi_i\Big)$$

There are no size restrictions on I and J, so that the collection of formulas forms a proper class. As a consequence, the usual syntax based on initial algebras (living in Set) is not well founded. While this expressivity result is not new – [30] shows expressivity of a similar infinitary modal logic for $\mathcal{P}$-bisimilarity – we include this example as it demonstrates that this fundamental expressivity result of modal logic fits into our general framework. So, these considerations leads to the contravariant adjoint situation between Set and $\mathsf{Set}^{\mathrm{op}}$ as depicted in (18).

Now taking inspiration from [24] where this is done for the finitary case, we consider the coalgebraic modal logic (L, δ), where the syntax is given by the endofunctor $L = \mathcal{P}(2 \times -)$ on Set and the "one-step" semantics $\delta\colon \mathcal{P}2^- \to 2^L$ is defined as follows:

$$\delta_X(S)(U) = \bigvee_{\varphi\in S} \Big(\bigwedge_{(1,x)\in U} \varphi(x) \wedge \bigwedge_{(0,x)\in U} \neg\varphi(x)\Big).$$

Note that the step-induced coalgebra lifting of δ turns a transition system with set X of states into an L-algebra on 2^X (cf. (13)). This gives an abstract notion of definability: precisely those sets $\varphi \in 2^X$ which are "reachable", that is, contained in the least subalgebra of all predicates on LX. So we consider the fibrations Pred and ERel of predicates and equivalence relations (see (18)) on Set (note ERel is chosen since $\mathcal{P}$-bisimilarity is an equivalence).



(18)

Next we define the corresponding liftings of the functor in order to invoke Corollary 21 in proving expressivity of our logic. For a predicate $P \rightarrowtail Y$ we fix $\mathsf{Pred}(2^-)(P) = \{(\varphi, \psi) \mid \forall p \in P.\, \varphi(p) \leftrightarrow \psi(p)\}$ and $\mathsf{Pred}(L)(P) = (\mathcal{P}(2 \times P) \rightarrowtail \mathcal{P}(2 \times Y))$.

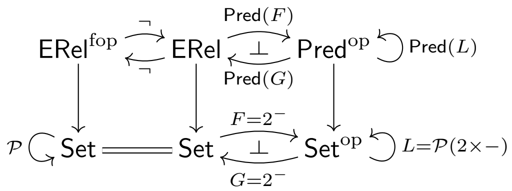
▶ Remark 26. It should be noted that the Galois connection (cf. Section 2) between the lattices $\mathsf{ERel}_X, \mathsf{Pred}^{\mathrm{op}}_{2^X}$ can be reconstructed from the adjunction between the total categories ERel and $\mathsf{Pred}^{\mathrm{op}}$. In particular, Lemma 7 gives: $\overline{F} \dashv \eta^*_X \circ \overline{G}\colon \mathsf{Pred}^{\mathrm{op}}_{FX} \to \mathsf{ERel}_X$. Moreover, $\eta^*_X \circ \overline{G} = g$ (recall g from Section 2).

Now adequacy and expressivity of our logic L follows by proving their corresponding sufficient condition (cf. Corollary 21) as in the following proposition.

▶ **Proposition 27.** *For any $P \rightarrowtail X$, we have $\delta^*(\mathsf{Pred}(G)\,\mathsf{Pred}(L)(P)) = \mathsf{Rel}(\mathcal{P})\,\mathsf{Pred}(G)(P)$.*

6.3 Apartness

In this subsection, we again consider (unlabelled) transition systems and rather show how our framework allows us to prove the dual of the Hennessy-Milner theorem: *two states are $\mathcal{P}$-apart [12, 11] (i.e., not bisimilar) iff there is a distinguishing formula between them.*



(19)

Recall that an apartness relation R on a set X is an irreflexive, symmetric, and co-transitive relation (i.e., $\forall x, y \in X.\, x\ R\ y \to \forall z \in Z.\,(x\ R\ z \vee y\ R\ z)$). Following [12], the fibration of apartness relations on Set can be seen as the fibred opposite of ERel. In particular, the functor $\neg$ maps a tuple (X, R) (when R is an equivalence/apartness on X) to the tuple $(X, \neg R)$. Note that, alternatively, one can also directly recover the above adjoint situation from (10). Moreover, on fibres, the functor $\neg \circ \mathsf{Pred}(G)$ takes a predicate $P \rightarrowtail X$ and produces an apartness relation $P_{\neg G}$ on 2^X given as follows:

$$\varphi P_{\neg G} \varphi' \iff \exists x \in P.\,(\varphi \#_x \varphi' \vee \varphi' \#_x \phi), \qquad \text{where} \quad \varphi \#_x \varphi' \iff \varphi(x) \wedge \neg\varphi'(x).$$

For the lifting $\mathsf{ERel}^{\mathrm{fop}}(\mathcal{P})$, we consider the following definition[1]:

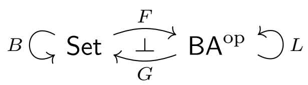
$$U\ \mathsf{ERel}^{\mathrm{fop}}(\mathcal{P})(R)\ V \iff \exists x \in U.\,\forall y \in V.\, x\ R\ y \vee \exists y \in V.\,\forall x \in U.\, x\ R\ y$$

Now we are in the position to use Corollary 21 and establish the dual of Hennessy-Milner theorem, which was also recently shown in [11] though for image-finite transition systems.

▶ **Proposition 28.** *For any* $P \rightarrowtail X$*,* $\mathsf{ERel}^{\mathrm{fop}}(\mathcal{P})(P_{\neg G}) = \delta^*(\neg\,\mathsf{Pred}(G)\,\mathsf{Pred}(L)(P))$.

6.4 Completeness

We now turn to the example of completeness of (finitary) basic modal logic by using a backward step ι. Consider the functor $B = \mathcal{P}$ with the *dual* adjunction of Equation (20) for $F = \mathrm{hom}_{\mathsf{Set}}(-, 2)$ and $G = \mathrm{hom}_{\mathsf{BA}}(-, 2)$.



(20)

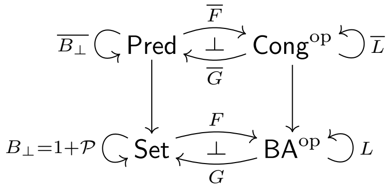
We obtain basic modal logic as coalgebraic modal logic for B using the predicate lifting $\blacksquare : F \to F \circ B$ where for $X \in \mathsf{Set}$ and $U \in FX$ we put $\blacksquare_X(U) = \{V \in BX \mid V \subseteq U\}$. Consider a sound and complete deduction system D for propositional logic. We define modal derivability $\vdash_{ML}$ by extending D with the derivation rules

$$\frac{a \leftrightarrow b \wedge c}{\Box a \leftrightarrow \Box b \wedge \Box c} \qquad\qquad \frac{a \leftrightarrow \top}{\Box a \leftrightarrow \top}$$

We call a set of formulas Φ inconsistent if there are formulas $\varphi_1, \dots, \varphi_n \in \Phi$ such that $\vdash_{ML} \varphi_1 \wedge \cdots \wedge \varphi_n \to \bot$, otherwise Φ is consistent. Our goal is to prove completeness of the logic, i.e., we would like to show that any consistent set of formulas Φ is satisfied in some B-coalgebra. The proof usually proceeds via a canonical model construction, that equips the set of maximally consistent sets of formulas ("theories") with a B-coalgebra structure.

[1] Note that our definition differs from the lifting of an apartness relation given in [12], where the two logical formulae ($\exists x \in U.\,\forall y \in V.\, x\ R\ y$ and $\exists y \in V.\,\forall x \in U.\, x\ R\ y$) are composed incorrectly by conjunction.

We will adjust this by viewing canonical models as fixpoints of a construction that defines models on possibly inconsistent theories and iteratively removes inconsistent theories until only consistent ones are left. An issue is that inconsistent theories will not have a model and thus we cannot define a meaningful B-coalgebra structure on them. Instead, we leave the coalgebra structure "undefined". To model such a partial B-coalgebra structure we switch to $B_\perp$-coalgebras with $B_\perp = 1 + B$. The intuition behind our construction is that the coalgebra structure maps a theory to $\mathrm{inl}(*)$ iff it is inconsistent. Ultimately we are left with a B-coalgebra based on the set of all consistent theories. The full setup is as in Equation (21).



(21)

Here Cong denotes the category of congruences over Boolean algebras, i.e., objects are pairs $(A, \equiv)$ with A being a Boolean algebra and $\equiv\, \subseteq A \times A$ being a congruence on A. It is well known that Cong is isomorphic to the category Quot of quotients of Boolean algebras. Therefore the above situation can be seen to meet the requirements of Cor. 6 and we obtain suitable liftings $\overline{F}$ and $\overline{G}$ of F and G, respectively.

Given a congruence $\equiv\, \subseteq A \times A$, the predicate $\overline{G}(\equiv)$ on GA can be given by $u \in \overline{G}(\equiv)$ iff $\forall a \in u. a \not\equiv \perp$ (equivalently to the "canonical" lifting of Corollary 6, so that we have a left adjoint). Intuitively, $\overline{G}(\equiv)$ contains all ultrafilters that are consistent with respect to $\equiv$. The lifting $\overline{B_\perp}$ of $B_\perp = 1 + B$ is defined using the canonical predicate lifting for B, i.e., for all $t \in B_\perp X$ and a predicate $U \subseteq X$ we have $t \in \overline{B_\perp}(U)$ iff $t = \mathrm{inr}(V)$ for some $V \subseteq U$. Finally, the lifting $\overline{L}$ of L is defined by letting $(LA, \equiv_{LA})$ be the smallest Boolean congruence containing $\Box a \wedge \Box b \equiv_{LA} \Box c$ for $a \wedge b \equiv c$ and $\Box a \equiv_{LA} \top$ for $a \equiv \top$. We turn now to the definition of a suitable backward step $\iota : GL \to B_\perp G$ that will allow us to prove completeness. To this aim we let $u \in GLA$ and consider the following intersection:

$$\mathrm{sem}(u) = \bigcap_{\Box a \in u} \blacksquare\hat{a} \cap \bigcap_{\Box a \notin u} BGA \setminus \blacksquare\widehat{a}$$

where $\hat{a} = \{v \in GA \mid a \in v\}$. We define a ι by selecting for each $u \in GLA$ an element $t \in \mathrm{sem}(u)$ if such an element exists and by putting $\iota_A(u) := \mathrm{inr}(t)$. Otherwise we put $\iota_A(u) = \mathrm{inl}(*)$. Note that with this definition ι will not necessarily be natural, but this is not required in our setting. In addition, using topological machinery, we could ensure naturality of ι by requiring $\iota(u)$ to be closed in the Vietoris topology (cf. e.g. [28]).

We now show that $\iota_A(u) = \mathrm{inl}(*)$ iff $u \notin \overline{G}\,\overline{L}(\equiv) = \overline{G}(\equiv_{LA})$, by case distinction:

Case $u \notin \overline{G}(\equiv_{LA})$ because there exists $a, b, c \in A$ with $a \wedge b \equiv c$ but $\Box a, \Box b \in u$ and $\Box c \notin u$. Then $\mathrm{sem}(u) \subseteq \blacksquare\hat{a} \cap \blacksquare\hat{b} \cap BGA \setminus \blacksquare\hat{c}$ and the latter is empty because any element would need to contain an ultrafilter $v \in GA$ with $a \in v, b \in v$ and $c \notin v$ which contradicts the assumption that $a \wedge b \equiv c$ and $v \in \overline{G}(\equiv_A)$.

Case All other cases with $u \notin \overline{G}(\equiv_{LA})$ can be proven in the same way as the first case.

Case $u \in \overline{G}(\equiv_{LA})$. In this case one can use compactness to argue that $\mathrm{sem}(u)$ is non-empty: by compactness and the definition of $\blacksquare$, if $\mathrm{sem}(u) = \emptyset$, there would need to be some $\Box a \in u$ and $\{\Box a_1, \ldots, \Box a_k\} \subseteq LA \setminus u$ such that $\blacksquare\hat{a} \cap \bigcap_{j=1}^{k} BGA \setminus \blacksquare\widehat{a}_j = \emptyset$ which can be seen to entail that $a \le a_j$ for some $j \in \{1, \ldots, k\}$. By monotonicity of $\Box$ (a well-known consequence of the axiomatisation above) we obtain $\Box a \le_{LA} \Box a_j$. Therefore, as $\Box a \in u$

and $u \in \overline{G}(\equiv_{LA})$ by assumption, we get $\Box a_j \in u$ which is a contradiction. This shows that $\mathrm{sem}(u) \neq \emptyset$ as required. Note that in the standard completeness proof of coalgebraic modal logic this case is the key step, proving so-called one-step completeness of the logic.

The claim above can be used to show that $\overline{G} \circ \overline{L} = \iota^* \circ \overline{B_\bot} \circ \overline{G}$. Furthermore, for a given $g : LA \to A$ we observe that $\nu((\iota_A \circ Gg)^* \circ (\overline{B_\bot})_A)$ consists precisely of those ultrafilters in GA on which $(\iota_A \circ Gg)$ restricts to a B-coalgebra structure, i.e., that cannot reach a state $u \in GA$ for which $(\iota_A \circ Gg)(u) = \mathrm{inl}(*)$. On the other hand, spelling out the definitions one can show that for an L-algebra $g : LA \to A$, $\mu(\coprod_g \circ \overline{L}_A)$ yields the least congruence $\equiv$ over A that satisfies the modal axioms. Corollary 18 then implies that any ultrafilter $u \in GA$ satisfying $\nu((\iota_A \circ Gg)^* \circ (\overline{B_\bot})_A)$ is $\equiv$-consistent ("soundness") and that any $\equiv$-consistent ultrafilter of A is satisfiable ("completeness"). Here satisfiable simply means that there is B-coalgebra structure defined on u. To establish a precise connection with with the standard notions of soundness and completeness, one would need to define the usual semantics of $\Box$ via a forward step δ. Standard completeness then follows when starting from the Boolean algebra consisting of all modal formulas quotiented by equivalence in propositional logic.

7 Related and future work

In [35] we studied a preservation result assuming both a forward step δ and a backward step ι, which form one-sided inverses, that is, $\delta \circ \iota = \mathsf{id}$. In the current paper, we treat preservation of coinductive predicates by forward and backward steps as separate cases, which we realise by formulating the conditions in a purely fibrational way instead of assuming inverses. This allows us, for instance, to provide a general preservation result for lax liftings (Section 6.1), which can not be obtained from the results in *op. cit.*: the latter requires a *natural* inverse δ, which is not part of the assumptions of a lax lifting (only componentwise inverses are assumed), and can in fact be non-trivial to provide; for instance, in [35] the argument for existence went via weak distributive laws. Moreover, in the current paper we are more general by moving from the relation fibration to general $\mathsf{CLat}_\wedge$-fibrations; this allows us, for instance, to use fibrations of predicate and equivalence relation fibrations, as we do in the analysis of expressivity and completeness.

In [29] a general approach to expressivity of logics with respect to coinductive predicates is proposed. In that paper, there is a fibration only on one of the two categories, and the coinductive predicate of interest is related to logical equivalence. Logical equivalence is defined there via the semantics of the logic, which is in turn obtained via the universal property of an initial algebra. In contrast, in the current paper, we do not use initial algebras and instead obtain logical equivalence by characterising "modally definable" on the coalgebra of interest, which yields a notion of logical equivalence by applying the right adjoint in a Galois connection between equivalence relations and predicates. This Galois connection was also used in [23], and in the recent [1] as the starting point for proving expressivity. Here, instead, we obtain this Galois connection from an adjunction between fibrations.

Future Work. In [1] it was shown how the functional characterising bisimilarity can be synthesised from a "logic" function. Using the notations of this paper, this meant defining $\overline{B}$ in terms of $\overline{L}$. This question and its symmetric one (constructing $\overline{L}$ from $\overline{B}$) are of interest at the global level of contravariant adjunctions. An answer to these questions would pave the way not only for sufficient conditions for expressivity, but also provide the means to establish them in a more structured manner. Last, it would be interesting to try and apply our results on comparing coinductive predicates and lifting adjunctions in the quantitative setting of (pseudo-)metrics.

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