Exploring the Space of Colourings with Kempe Changes

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Abstract

Kempe changes were introduced in 1879 in an attempt to prove the 4-colour theorem. They are a convenient if not crucial tool to prove various colouring theorems. Here, we consider how to navigate from a colouring to another through Kempe changes. When is it possible? How fast?

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Kempe changes were introduced in 1879 in an attempt to prove the 4-colour theorem [4]. They are a convenient if not crucial tool to prove various colouring theorems, most notably the 4-colour theorem [1] (every planar graph is 4-colourable) and Vizing's edge colouring theorem [7] (the edges of any graph can be partitioned into at most $\Delta + 1$ matchings, where Δ is the maximum degree of a vertex in the graph). Given a coloured graph, a Kempe change consists in considering two colours a and b and a vertex coloured a, then swapping colours a and b in the maximal (a, b)-coloured component containing the specified vertex. Here, we consider how to navigate from a colouring to another through a series of Kempe changes. When is it possible? How fast? A seminal conjecture of Vizing from 1965 [8] states that in any graph, from any edge-colouring we can reach an optimal one through a well-chosen series of Kempe changes. While this remained a major challenge for decades, being only proved for graphs with maximum degree 3 or 4 [5, 2], then last year for triangle-free graphs [3], Narboni recently provided a full proof of the conjecture [6]. This notably implies that given at least one more colour than the optimal number, one can navigate from any edge-colouring to any other. The extra colour is necessary. How these results extend to the context of multigraphs remains widely open.

We will also discuss the number of steps necessary to navigate from any vertex-colouring of a k-degenerate graph to any other, when the number of colours is sufficiently large compared to the degeneracy, as well as the Kempe equivalent of Hadwiger's conjecture and whether Kempe changes can be useful in the context of graphs with a forbidden minor. If time permits, we will see how this tool can be used for efficient sampling of random colourings of a graph and for counting the number of distinct colourings.

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