Ordinal Measures of the Set of Finite Multisets

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Abstract

Well-partial orders, and the ordinal invariants used to measure them, are relevant in set theory, program verification, proof theory and many other areas of computer science and mathematics. In this article we focus on a common data structure in programming, finite multisets of some well partial order. There are two natural orders one can define on the set of finite multisets of a partial order: the multiset embedding and the multiset ordering. Though the maximal order type of these orders is already known, other ordinal invariants remain mostly unknown. Our main contributions are expressions to compute compositionally the width of the multiset embedding and the height of the multiset ordering. Furthermore, we provide a new ordinal invariant useful for characterizing the width of the multiset ordering.

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Introduction

Measuring partial orders is useful in many domains, from set theory to proof theory, including infinitary combinatorics, program verification, rewriting theory, proof automation and many more.

There are intuitive notions of measure for a partial order when it is finite: its cardinal obviously, but also its height (the length of a maximal chain) or its width (the length of a maximal antichain). Similar notions exist for infinite partial orders, as long as they are *well partial orders* (wpo), i.e., well-founded partial orders with no infinite antichains [10, 12]. Two such notions are the *ordinal height*, which is the order type of a maximal chain, and the *maximal order type* (mot), which is the order type of a maximal linearisation, a notion introduced by De Jongh and Parikh in order to measure hierarchies of functions [6]. These are transfinite measures, hence we call them *ordinal invariants*. Kříž and Thomas introduced alternative characterizations for mot and ordinal height, which naturally led to the definition of a third ordinal invariant, *ordinal width* [11]. Less studied than its counterparts, the width of a wpo relates to its antichains, even though it cannot be defined as the order type of a maximal antichain. While exploring techniques for program termination, Blass and Gurevich rediscovered these characterizations with a game-theoretical point of view [4].

Ordinal invariants of wpos have also been used to prove complexity bounds. In the last decade there has been a flurry of complexity results for the verification of well-structured transition systems (wsts), i.e., transition systems whose set of configurations is a wpo and whose transitions respect this ordering [5]. When a wsts is based on a wpo X of maximal order type ω^α, one can expect the complexity of coverability to be in H_{ω^α} in the Hardy hierarchy, or in F_α in the fast-growing hierarchy [9]. This bound can be refined by looking at controlled antichains instead of controlled bad sequences [14], thus bounding complexity with width instead of maximal order type.

Computing ordinal invariants compositionally. Many wpos underlying wsts are built from classical operations on simpler wpos whose invariants are known. This has spurred new interest in measuring the ordinal invariants of various well-ordered data structures: De Jongh and Parikh computed the mot of the disjoint sum and the Cartesian product of wpos [6]. Schmidt then computed the mot of word embedding and homeomorphic tree embedding on a wpo [13]. Abraham and Bonnet pursued this line of study by computing the height of Cartesian product, but also the width of disjoint sum and lexicographic product [1]. For a complete survey of these results see [8], where Džamonja et al. computed the ordinal invariants of the lexicographic product, but also the height of the multiset word and tree embeddings.

Finite multisets. In this article, we study the ordinal invariants of the set of finite multisets. Multisets, also called "bags", a common data structure in computer science. Informally, a finite multiset over a set X is a finite subset of X where an element can appear finitely many times. For instance, $\langle a, a, b\rangle$ denote the multiset where a appears twice and b once. One can see the set of finite multisets on a wpo as the set of finite words quotiented by the equivalence relation "equality up to some permutation". It comes down to describing a multiset as a word where the order of terms is irrelevant. A finite multiset can be represented by a function from X to $\mathbb{N}$ with finite support, which associates its multiplicity with each element.

Two orderings are classically defined on the finite multisets of any ordered set. The first one is the *multiset ordering*, which often appears in rewriting theory and automation of termination proofs [7]. The other, less-known, ordering is the *multiset embedding*, or term ordering as it is called in [15]. It was presented by Aschenbrenner and Pong as a natural extension of the embedding order over finite words [3].

Some invariants of these two orderings have already been measured: Van der Meeren, Rathjen, and Weiermann [15] built on [17] to compute the mot of the set of finite multisets on a wpo X ordered with the multiset ordering, and provided a new proof for the expression of the mot of the multiset embedding computed in [18]. Džamonja et al. [8] proved that the height of the multiset embedding is equal to the height of the set of finite words ordered with word embedding. It is noteworthy that these three results give expressions that are functional in (i.e., can be expressed as a function of) the mot and height of X. However, the height of the multiset ordering still needs to be determined, and the width remains unstudied for both orderings.

Our contributions. In this article, we provide functional expressions for the width of the multiset embedding (Theorem 2.1) and the height of the multiset ordering (Theorem 3.1).

We further show that the width of the multiset ordering cannot be expressed as a function of the three ordinal invariants (Example 3.2). Nonetheless, we get around this issue by introducing a fourth ordinal invariant, the *friendly order type* (Definition 3.3), in which the width of the multiset ordering is functional (Theorem 3.4). We then proceed to investigate and compute this new ordinal invariant.

1 Definitions and state of the art

1.1 Width, height and maximal order type

A sequence $x_1, \ldots, x_n, \ldots$ on a partial order $(X, \leq_X)$ is *good* when there exist $i < j$ such that $x_i \leq_X x_j$, otherwise it is a *bad* sequence. An *antichain* is a sequence whose elements are pairwise incomparable.

A *well partial order* (wpo) is a partial order that has no infinite bad sequences. Equivalently, a wpo is a partial order that is both well-founded (i.e. no infinite strictly decreasing sequences) and has no infinite antichains.

Let $(X, \leq_X)$ be a wpo. We often write just X when $\leq_X$ is understood. The trees $\mathit{Bad}(X)$, $\mathit{Dec}(X)$ and $\mathit{Ant}(X)$ are defined as the sets of bad sequences, strictly decreasing sequences, and antichains of X, respectively, ordered by inverse prefix order (a sequence is smaller than its prefixes) ([11, 8]). The finiteness of bad sequences, strictly decreasing sequences and antichains in a wpo implies that these trees are well-founded. Therefore, one can define a notion of rank on these trees: a sequence has rank 0 when it cannot be extended; otherwise its rank is the smallest ordinal strictly larger than the ranks of its extensions. The rank of a tree is the rank of the empty sequence (which is the root of the tree).

The *maximal order type* (or mot) of X, denoted by $\boldsymbol{o}(X)$, is defined as the rank of $\mathit{Bad}(X)$. Similarly, the *height* $\boldsymbol{h}(X)$ and the *width* $\boldsymbol{w}(X)$ of X are defined as the ranks of $\mathit{Dec}(X)$ and $\mathit{Ant}(X)$, respectively. Together, $\boldsymbol{o}(X)$, $\boldsymbol{h}(X)$ and $\boldsymbol{w}(X)$ are called the *ordinal invariants* of X.

For any wpo X, $\mathit{Dec}(X)$ and $\mathit{Ant}(X)$ are subtrees of $\mathit{Bad}(X)$. Thus $\boldsymbol{h}(X) \leq \boldsymbol{o}(X)$ and $\boldsymbol{w}(X) \leq \boldsymbol{o}(X)$.

Let $x \perp y$ denote that x and y are incomparable. For a relation $*$ among $\{ \not\geq, <, \perp \}$, we define the residual X_{*x} as $\{ y \in X \ : \ y * x \}$. This definition can be extended to subsets $S \subseteq X$: $X_{*S} \overset{\text{def}}{=} \{ y \in X \ : \ \forall x \in S, \ y * x \}$.

▶ **Example 1.1.** In Figure 1, you can see the residuals at $x = (4, 6)$ of $\mathbb{N} \times \mathbb{N}$ ordered component-wise.

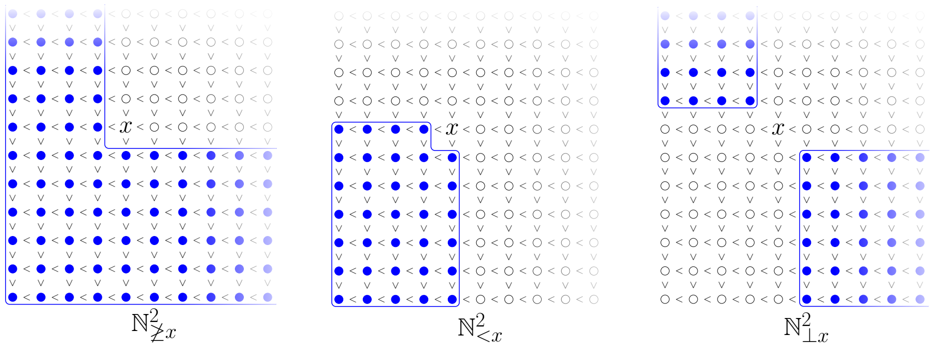

Figure 1 Residuals of $\mathbb{N}^2$ at $(4, 6)$.

Since the rank of the empty sequence is the smallest ordinal strictly larger than the ranks of the sequences of length 1, the definitions of mot, height and width can be reformulated inductively through the following *residual equations*:

$$\boldsymbol{o}(X) = \sup_{x \in X} (\boldsymbol{o}(X_{\not\geq x}) + 1) \tag{Res-o}$$

$$\boldsymbol{h}(X) = \sup_{x \in X} (\boldsymbol{h}(X_{<x}) + 1) \tag{Res-h}$$

$$\boldsymbol{w}(X) = \sup_{x \in X} (\boldsymbol{w}(X_{\perp x}) + 1) \tag{Res-w}$$

With these equations we can compute easily the ordinal invariants of $\mathbb{N}^2$. For instance, observe that $(\mathbb{N}^2_{<x})$ is finite for any $x \in \mathbb{N}^2$, so its height is finite but can be arbitrarily big. Hence $\boldsymbol{h}(\mathbb{N}^2) = \omega$.

1.2 Ordinal arithmetic

We suppose well-known the notions of sum, product, subtraction, natural sum, natural product on ordinals, denoted with $+$, $\cdot$, $-$, $\oplus$, $\otimes$ [2]. However, let us recall some definitions and notations that might be less familiar to the reader.

An ordinal α is *indecomposable* iff for any $\delta, \gamma < \alpha$, we have $\delta \oplus \gamma < \alpha$. Equivalently, α is indecomposable when there is an ordinal β such that $\alpha = \omega^\beta$. α is an *ϵ-number* when $\alpha = \omega^\alpha$.

The *Hessenberg-based product* $\alpha \odot \beta$ is defined inductively as follows [1]:

$$\alpha \odot 0 = 0\,, \quad \alpha \odot (\beta + 1) = (\alpha \odot \beta) \oplus \alpha\,, \quad \alpha \odot \beta = \sup\{\, \alpha \odot \gamma \;:\; \gamma < \beta \,\} \text{ for limit } \beta.$$

This definition ensures that $\alpha \cdot \beta \leq \alpha \odot \beta \leq \alpha \otimes \beta$.

For any ordinal $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, let $\widehat{\alpha} \stackrel{\text{def}}{=} \omega^{\alpha'_1} + \cdots + \omega^{\alpha'_n}$, where α'_i is $\alpha_i + 1$ when α_i is the sum of an ϵ-number and a finite ordinal, otherwise $\alpha'_i = \alpha_i$.

For any ordinals α, β, let $\alpha \mathbin{\hat{\oplus}} \beta \stackrel{\text{def}}{=} \sup\{\, \alpha' \oplus \beta' \;:\; \alpha' < \alpha, \beta' < \beta \,\}$.

1.3 Ordinal invariants of basic data structures

For any wpos P, Q, the *disjoint sum* $P \sqcup Q$ is the disjoint union of P and Q ordered such that elements of P and Q cannot be compared together, whereas the *direct sum* $P + Q$ is the disjoint union of P and Q ordered such that for all $p \in P, q \in Q$, $p \leq q$. For a family of wpos $(A_i)_{i<\alpha}$, let $\Sigma_{i<\alpha} A_i$ denote the direct sum of the A_is along the ordinal α.

The *Cartesian product* $P \times Q$ is the set of pairs $(p, q) \in P \times Q$ where elements are compared component-wise. The *lexicographic product* of P along Q, written $P \cdot Q$, has the same support as $P \times Q$, with a different ordering: $(p, q) \leq_{P \cdot Q} (p', q')$ iff $q <_Q q'$, or $q = q'$ and $p \leq_P p'$.

Sums and products are the most basic operations on wpos one can find. Their ordinal invariants are easy to compute compositionally (see Table 1), with the notable exception of the width of the Cartesian product which cannot be expressed as a function of the ordinal invariants its factors [16].

Table 1 How to compute ordinal invariants compositionally, [8]. See Section 1.2 for definitions of $\hat{\oplus}$ and $\odot$.

Space X	M.O.T. $\boldsymbol{o}(X)$	Height $\boldsymbol{h}(X)$	Width $\boldsymbol{w}(X)$
$A \sqcup B$	$\boldsymbol{o}(A) \oplus \boldsymbol{o}(B)$	$\max(\boldsymbol{h}(A), \boldsymbol{h}(B))$	$\boldsymbol{w}(A) \oplus \boldsymbol{w}(B)$
$A + B$	$\boldsymbol{o}(A) + \boldsymbol{o}(B)$	$\boldsymbol{h}(A) + \boldsymbol{h}(B)$	$\max(\boldsymbol{w}(A), \boldsymbol{w}(B))$
$A \times B$	$\boldsymbol{o}(A) \otimes \boldsymbol{o}(B)$	$\boldsymbol{h}(A) \mathbin{\hat{\oplus}} \boldsymbol{h}(B)$	*(Not functional)*
$A \cdot B$	$\boldsymbol{o}(A) \cdot \boldsymbol{o}(B)$	$\boldsymbol{h}(A) \cdot \boldsymbol{h}(B)$	$\boldsymbol{w}(A) \odot \boldsymbol{w}(B)$

1.4 Comparing wpos

A widely-used and intuitive relation between wpos is the *reflection* relation. A mapping between wpos $f\colon (A, \leq_A) \to (B, \leq_B)$ is a *reflection* if $f(x) \leq_B f(y)$ implies $x \leq_A y$, i.e. it is a morphism from $(A, \not\leq_A)$ to $(B, \not\leq_B)$ Let $A \hookrightarrow B$ denote that there is a reflection from A to B.

However, in this article, we prefer to use the stronger notions of augmentations and substructures.

▶ **Definition 1.2** (Substructure, augmentation). *A* wpo $(A, \leq_A)$ *is a* substructure *of a* wpo $(B, \leq_B)$ *whenever* $A \subseteq B$ *and* $\leq_A$ *is the restriction of* $\leq_B$ *to* A*. This relation is written* $A \leq_{st} B$*. Similarly* $(A, \leq_A)$ *is an* augmentation *of* $(B, \leq_B)$ *whenever* $A = B$ *and* $\leq_B \subseteq \leq_A$*. We write this relation* $A \geq_{aug} B$*.*

Obviously, $A \leq_{\mathrm{st}} B$ or $A \geq_{\mathrm{aug}} B$ imply $A \hookrightarrow B$.

We often abuse these notations and write $A \leq_{\mathrm{st}} B$ (resp. $B \leq_{\mathrm{aug}} A$) to mean that A is isomorphic to a substructure (resp. an augmentation) of B.

We denote by $A \equiv B$ that $(A, \leq_A)$ is isomorphic to $(B, \leq_B)$.

In this article, when we consider a subset Y of a wpo X, it is understood that $Y \leq_{\mathrm{st}} X$, i.e. Y is ordered with $\leq_X$ restricted to the subset.

These notions of augmentations and substructures allow us to compare the ordinal invariants of wpos.

▶ **Lemma 1.3.** *Let* A *and* B *be* wpo*s.*

If $A \leq_{st} B$ *then* $\boldsymbol{i}(A) \leq \boldsymbol{i}(B)$ *for* $\boldsymbol{i} \in \{ \boldsymbol{o}, \boldsymbol{h}, \boldsymbol{w} \}$*.*

If $A \geq_{aug} B$ *then* $\boldsymbol{o}(A) \leq \boldsymbol{o}(B)$ *and* $\boldsymbol{w}(A) \leq \boldsymbol{w}(B)$*. However* $\boldsymbol{h}(A) \geq \boldsymbol{h}(B)$*.*

The substructure and augmentation relations are monotonous through most operations on wpos. For instance, if $A \leq_{\mathrm{st}} A'$, then $A \times B \leq_{\mathrm{st}} A' \times B$.

An ordinal, as defined by Von Neumann, is the linear wpo that contains all smaller ordinals. Thus augmentations and substructures relations can also be used to compare directly ordinals to wpos. The following result is well-known:

▶ **Proposition 1.4.** *For any wpo* X*,* $\boldsymbol{h}(X)$ *and* $\boldsymbol{o}(X)$ *are the largest ordinals such that* $\boldsymbol{h}(X) \leq_{st} X$ *and* $\boldsymbol{o}(X) \geq_{aug} X$*.*

1.5 Orderings on the set of finite multisets

We assume familiarity with finite multisets and the associated operations as used in [17]: union, intersection and subtraction, denoted by $\cup, \cap$ and $\setminus$, respectively. Let $\langle x_1, \ldots, x_n \rangle$ denote the finite multiset that contains the elements $x_1, \ldots, x_n$ (they do not have to be distinct). For any $k \in \mathbb{N}$, $m \times k$ means the union of k copies of m. Let $|m|$ denote the number of elements of a multiset m.

There are two main orderings classically defined on the set of finite multisets $M(X)$ of a partial order X:

▶ **Definition 1.5** (Multiset embedding [18]). *The* multiset embedding *on* $M(X)$*, also known as the term ordering, is defined as:*

$m \leq_\diamond m'$ *iff there exists* $f : m \to m'$ *injective such that for any* $x \in m$*,* $x \leq f(x)$*.*

▶ **Definition 1.6** (Multiset ordering [17]). *The* multiset ordering *on* $M(X)$ *is defined as:*

$$m \leq_r m' \iff m = m' \text{ or } \forall x \in m \setminus (m \cap m'), \exists y \in m' \setminus (m \cap m'), x < y \ .$$

We write $M^\diamond(X)$ for $(M(X), \leq_\diamond)$ and $M^r(X)$ for $(M(X), \leq_r)$.

The multiset ordering and the multiset embedding are both augmentations of the word embedding on X^* the set of finite words on X. Therefore, according to Higman's lemma [10], $M^\diamond(X)$ and $M^r(X)$ are wpos when X is. Moreover $M^\diamond(X) \leq_{\mathrm{aug}} M^r(X)$, as was observed by Aschenbrenner and Pong [3].

Observe that if X is a linear ordering, then $M^r(X)$ is linear, while $M^\diamond(X)$ is not as long as X has more than two elements.

▶ **Proposition 1.7** (Transformation equations)**.** *For any* wpo*s A and B,*

$$M^*(A \sqcup B) \equiv M^*(A) \times M^*(B) \text{ for } * \in \{\diamond, r\} , \tag{Trans-1}$$
$$M^r(A + B) \equiv M^r(A) \cdot M^r(B) , \tag{Trans-2}$$
$$M^\diamond(A + B) \leq_{aug} M^\diamond(A) \cdot M^\diamond(B) . \tag{Trans-3}$$

▶ **Lemma 1.8** (Width of $M(X)$ on Γ_k)**.** *For any $k < \omega$, we denote by Γ_k the* wpo *that contains k incomparable elements. Then $\boldsymbol{w}(M^\diamond(\Gamma_k)) = \boldsymbol{w}(M^r(\Gamma_k)) = \omega^{k-1}$.*

Proof. Since $M^\diamond(\Gamma_1) \equiv M^r(\Gamma_1) \equiv \omega$, Equation (Trans-1) tells us that $M^\diamond(\Gamma_k)$ and $M^r(\Gamma_k)$ are both isomorphic to the k-fold Cartesian product $\omega \times \cdots \times \omega$. This special case of the width of a Cartesian product is known [16]: $\boldsymbol{w}(\omega \times \cdots \times \omega) = \omega^{k-1}$. ◀

The augmentation and substructure relations are monotone with respect to the multiset ordering and multiset embedding:

▶ **Proposition 1.9.** *Let A, B be two* wpo*s. Then $A \leq_{st} B$ implies $M^\diamond(A) \leq_{st} M^\diamond(B)$ and $M^r(A) \leq_{st} M^r(B)$. Moreover, $A \geq_{aug} B$ implies that $M^\diamond(A) \geq_{aug} M^\diamond(B)$ and $M^r(A) \geq_{aug} M^r(B)$.*

Ordinal invariants of the set of finite multisets

Van der Meeren, Rathjen and Weiermann computed the mot of $M^\diamond(X)$ and $M^r(X)$.

▶ **Theorem 1.10** (Mot of multiset embedding [15, 18])**.** *For any* wpo *X, $\boldsymbol{o}(M^\diamond(X)) = \omega^{\widehat{\boldsymbol{o}(X)}}$.*

▶ **Theorem 1.11** (Mot of multiset ordering [15, 17])**.** *For any* wpo *X, $\boldsymbol{o}(M^r(X)) = \omega^{\boldsymbol{o}(X)}$.*

Observe that $\omega^{\boldsymbol{o}(X)} \leq \omega^{\widehat{\boldsymbol{o}(X)}}$, as one would expect since $M^r(X) \geq_{\text{aug}} M^\diamond(X)$. Furthermore, we expect that $\boldsymbol{w}(M^r(X)) \leq \boldsymbol{w}(M^\diamond(X))$, while $\boldsymbol{h}(M^r(X)) \geq \boldsymbol{h}(M^\diamond(X))$.

▶ **Theorem 1.12** (Height of the multiset embedding [8])**.** *Let X be a* wpo*.*
Then $\boldsymbol{h}(M^\diamond(X)) = \boldsymbol{h}^*(X)$, where

$$\boldsymbol{h}^*(X) \stackrel{def}{=} \begin{cases} \boldsymbol{h}(X) & \text{if } \boldsymbol{h}(X) \text{ is infinite and indecomposable,} \\ \boldsymbol{h}(X) \cdot \omega & \text{otherwise.} \end{cases}$$

1.6 A tool to compute the width: Quasi-incomparable subsets

Of all three ordinal invariants, the width is the less studied, since it has been introduced more recently, and also the hardest invariant to study for lack of tools.

A powerful tool to analyse the width of a wpo is the notion of *quasi-incomparable* subsets of a wpo, which was first introduced in [16] for the Cartesian product of several ordinals.

For any subsets Y, Z of X, let $Y \perp Z$ denote that for every $y \in Y, z \in Z$, $y \perp z$.

▶ **Definition 1.13.** *Let A be a* wpo*, and $A_1, \ldots, A_n$ be n subsets of A. Then $(A_i)_{1 \leq i \leq n}$ is a* quasi-incomparable *family of subsets of A iff for any $i < n$, for any finite $Y \subseteq A_1 \cup \cdots \cup A_i$, there exists $A'_{i+1} \subseteq A_{i+1}$ such that $A'_{i+1} \perp Y$ and $A'_{i+1} \equiv A_{i+1}$.*

This definition is slightly more restrictive than the one in [16], which only required that $\boldsymbol{w}(A'_{i+1}) = \boldsymbol{w}(A_{i+1})$.

The idea behind these quasi-incomparable subsets is that sometimes one can slice a wpo A into simpler subsets $A_1, \dots, A_n$ whose width is known, such that $Ant(A_n) + \dots + Ant(A_1)$ is embedded in $Ant(A)$. Intuitively, it means that one can combine antichains of $A_1, \dots, A_n$ into one antichain of A.

This entails a practical relation between the widths of A and its subsets:

▶ **Lemma 1.14** ([16]). *Let $(A_i)_{i\le n}$ be a quasi-incomparable family of subsets of A. Then $\boldsymbol{w}(A) \ge \boldsymbol{w}(A_n) + \dots + \boldsymbol{w}(A_1)$.*

2 Ordinal width of the multiset embedding

In this section we compute the width of $M^\diamond(X)$ for any wpo X, which happens to be functional in the width of X:

▶ **Theorem 2.1** (Width of the multiset embedding). *For any* wpo *X, $\boldsymbol{w}(M^\diamond(X)) = \omega^{\widehat{\boldsymbol{o}(X)}-1}$. (See Section 1.2 for the definition of $\widehat{\alpha}$.)*

It is already known that, in some cases, the width of the multiset embedding reaches its mot.

▶ **Lemma 2.2** ([8]). *If $\boldsymbol{o}(X)$ is infinite and indecomposable, $\boldsymbol{w}(M^\diamond(X)) = \boldsymbol{o}(M^\diamond(X))$.*

We focus for now on the set of finite multisets on a linear wpo, i.e., an ordinal. Let us treat first the case of successor ordinals.

▶ **Lemma 2.3.** *For any successor ordinal $\alpha = \beta + 1$, $\boldsymbol{w}(M^\diamond(\alpha)) \ge \boldsymbol{w}(M^\diamond(\beta)) \cdot \omega$.*

Proof. We denote with $M^\diamond_{>k}(X)$ the subset $\{\, m \in M^\diamond(X) \;:\; |m| > k \,\}$ for any $k \in \mathbb{N}$ of $M^\diamond(X)$ for any wpo X, for any $k < \omega$.

Let $m_n \stackrel{\text{def}}{=} \langle\beta\rangle \times n$ for any $n \in \mathbb{N}$. According to Equation (Res-w),

$$\begin{aligned}\boldsymbol{w}(M^\diamond(\alpha)) &= \sup\{\, \boldsymbol{w}(M^\diamond(\alpha)_{\perp m}) + 1 \;:\; m \in M^\diamond(\alpha) \,\}\\ &\ge \sup\{\, \boldsymbol{w}(M^\diamond(\alpha)_{\perp m_n}) + 1 \;:\; n \in \mathbb{N} \,\}\,.\end{aligned}$$

Let $M_k \stackrel{\text{def}}{=} \{\, \langle\beta\rangle \times (n-k) \cup m \;:\; m \in M^\diamond_{>k}(\beta) \,\}$ for $k \in [1, n]$. These subsets of $M^\diamond(\alpha)$ are actually subsets of $M^\diamond(\alpha)_{\perp m_n}$: for all $m \in \mathsf{M}_k$, $m \perp m_n$ since $|m| > |m_n|$. Observe also that for any $k \in [1, n]$, $M_k \equiv M^\diamond(\beta)$.

Moreover, $(M_k)_{k\in[1,n]}$ is a quasi-incomparable family of subsets of $M^\diamond(\alpha)_{\perp m_n}$: for any $i < n$, for any finite $Y \subset M_1 \cup \dots \cup M_i$, let $s(Y) = max\{|m|, m \in Y\}$. Observe that M_{i+1} contains $M_{i+1} \cap M^\diamond_{>s(Y)}(\beta)$ which is incomparable to Y, and isomorphic to M_{i+1}.

Therefore, $\boldsymbol{w}(M^\diamond(\alpha)_{\perp m_n}) \ge \boldsymbol{w}(M_n) + \dots + \boldsymbol{w}(M_1) = \boldsymbol{w}(M^\diamond(\beta)) \cdot n$ according to Lemma 1.14. Thus $\boldsymbol{w}(M^\diamond(\alpha)) \ge \sup\{\, \boldsymbol{w}(M^\diamond(\beta) \cdot n + 1 \;:\; n \in \mathbb{N} \,\} = \boldsymbol{w}(M^\diamond(\beta)) \cdot \omega$. ◀

▶ **Lemma 2.4.** *For any infinite ordinal α, $\boldsymbol{w}(M^\diamond(\alpha)) = \boldsymbol{o}(M^\diamond(\alpha))$.*

Proof. We already know that $\boldsymbol{w}(M^\diamond(\alpha)) \le \boldsymbol{o}(M^\diamond(\alpha))$. We prove the lower bound by induction on α:

- If α is indecomposable, see Lemma 2.2.

- If $\alpha = \beta + 1$, then according to Lemma 2.3,

$$\begin{aligned} \boldsymbol{w}(M^\diamond(\alpha)) &\geq \boldsymbol{w}(M^\diamond(\beta)) \cdot \omega \\ &= \boldsymbol{o}(M^\diamond(\beta)) \cdot \omega && \text{by induction hypothesis,} \\ &= \omega^{\widehat{\beta}+1} = \omega^{\widehat{\beta+1}} = \boldsymbol{o}(M^\diamond(\alpha)) && \text{according to Theorem 1.10.} \end{aligned}$$

- If $\alpha = \beta + \omega^\rho$ with $\beta, \omega^\rho < \alpha$ and $\rho > 0$, then according to the transformation equation Trans-3, $M^\diamond(\alpha) \leq_{\text{aug}} M^\diamond(\beta) \cdot M^\diamond(\omega^\rho)$. Hence according to Lemma 1.3 and Table 1,

$$\begin{aligned} \boldsymbol{w}(M^\diamond(\alpha)) &\geq \boldsymbol{w}(M^\diamond(\beta)) \odot \boldsymbol{w}(M^\diamond(\omega^\rho)) \\ &= \boldsymbol{o}(M^\diamond(\beta)) \odot \boldsymbol{o}(M^\diamond(\omega^\rho)) && \text{by induction hypothesis,} \\ &= \omega^{\widehat{\beta}} \odot \omega^{\widehat{\omega^\rho}} = \omega^{\widehat{\alpha}} \\ &= \boldsymbol{o}(M^\diamond(\alpha)) && \text{according to Theorem 1.10.} \end{aligned}$$ ◀

We can now prove that Lemma 2.4 generalizes to non-linear wpos.

▶ **Lemma 2.5.** *If $\boldsymbol{o}(X)$ is infinite then $\boldsymbol{w}(M^\diamond(X)) = \boldsymbol{o}(M^\diamond(X))$.*

Proof. Let $\alpha = \boldsymbol{o}(X)$. Then $X \leq_{\text{aug}} \alpha$ from Proposition 1.4, hence $M^\diamond(X) \leq_{\text{aug}} M^\diamond(\alpha)$ according to Lemma 1.3 and Proposition 1.9. Thus

$$\boldsymbol{w}(M^\diamond(\alpha)) \leq \boldsymbol{w}(M^\diamond(X)) \leq \boldsymbol{o}(M^\diamond(X)) \ .$$

Now $\boldsymbol{o}(M^\diamond(X)) = \omega^{\widehat{\alpha}} = \boldsymbol{o}(M^\diamond(\alpha))$ according to Theorem 1.10. Now with Lemma 2.4 $\boldsymbol{w}(M^\diamond(\alpha)) = \boldsymbol{o}(M^\diamond(\alpha))$, hence $\boldsymbol{w}(M^\diamond(X)) = \boldsymbol{o}(M^\diamond(X))$. ◀

We can also compute the width of $M^\diamond(X)$ when X is a finite wpo:

▶ **Lemma 2.6.** *If $\boldsymbol{o}(X)$ is finite, then $\boldsymbol{w}(M^\diamond(X)) = \omega^{\boldsymbol{o}(X)-1}$.*

Proof. Let $k = \boldsymbol{o}(X)$. Then $\Gamma_k \leq_{\text{aug}} X \leq_{\text{aug}} k$, hence $\boldsymbol{w}(M^\diamond(\Gamma_k)) \geq \boldsymbol{w}(M^\diamond(X)) \geq \boldsymbol{w}(M^\diamond(k))$ thanks to Lemma 1.3. According to Lemma 1.8, $\boldsymbol{w}(M^\diamond(\Gamma_k)) = \omega^{k-1}$, and according to Lemma 2.3 applied $(k-1)$ times, $\boldsymbol{w}(M^\diamond(k)) \geq \boldsymbol{w}(M^\diamond(1)) \cdot \omega^{k-1} = \omega^{k-1}$. Therefore $\boldsymbol{w}(M^\diamond(X)) = \omega^{k-1} = \omega^{\boldsymbol{o}(X)-1}$. ◀

This section's main result follows directly from Lemmas 2.5 and 2.6.

Proof of Theorem 2.1. If $\boldsymbol{o}(X)$ is finite, then $\widehat{\boldsymbol{o}(X)} - 1 = \boldsymbol{o}(X) - 1$. On the other hand, if $\boldsymbol{o}(X)$ is infinite, then $\widehat{\boldsymbol{o}(X)} - 1 = \widehat{\boldsymbol{o}(X)}$. ◀

3 Ordinal height and width of the multiset ordering

For the height of $M^r(X)$, we obtain a result similar to Theorem 1.11.

▶ **Theorem 3.1** (Height of the multiset ordering). *Let X be a* wpo. *Then $\boldsymbol{h}(M^r(X)) = \omega^{\boldsymbol{h}(X)}$.*

Proof. Observe that the multiset ordering of any linear ordering is also linear. Thus, for any ordinal α, $M^r(\alpha)$ is isomorphic to ω^α (the function $\langle x_1, \ldots, x_n \rangle \mapsto \omega^{x_1} \oplus \cdots \oplus \omega^{x_n}$ is an isomorphism).

According to Proposition 1.4, $X \geq_{\text{st}} \boldsymbol{h}(X)$, and thus $M^r(X) \geq_{\text{st}} M^r(\boldsymbol{h}(X)) \equiv \omega^{\boldsymbol{h}(X)}$ (Proposition 1.9). Therefore $\boldsymbol{h}(M^r(X)) \geq \omega^{\boldsymbol{h}(X)}$ according to Lemma 1.3. See the proof of the upper bound in Appendix A. ◀

The width of the multiset ordering is harder to compute, as $\boldsymbol{w}(M^r(X))$ is not functional in the ordinal invariants of X. The following example exhibits two wpos X_1 and X_2, with identical ordinal invariants, such that $\boldsymbol{w}(M^r(X_1)) \neq \boldsymbol{w}(M^r(X_2))$.

▶ **Example 3.2.** Let $H \stackrel{\text{def}}{=} \Sigma_{n<\omega}\Gamma_n$. An interesting property of H is that $\boldsymbol{w}(H) = \boldsymbol{h}(H) = \boldsymbol{o}(H) = \omega$. Since $M^r(H) \geq_{\text{st}} M^r(\Gamma_n)$, then $\omega^{n-1} \leq \boldsymbol{w}(M^r(H)) \leq \boldsymbol{o}(M^r(H)) = \omega^\omega$ for all $n < \omega$ according to Lemma 1.8 and Theorem 1.11. Hence $\boldsymbol{w}(M^r(H)) = \omega^\omega$.

Consider $X_1 = H + H$ and $X_2 = H + \omega$, two wpos with the same ordinal invariants: $\boldsymbol{o}(X_i) = \boldsymbol{h}(X_i) = \omega \cdot 2$ and $\boldsymbol{w}(X_i) = \omega$ for $i \in \{1, 2\}$. According to Equation (Trans-2) and Table 1, $\boldsymbol{w}(M^r(X_1)) = \boldsymbol{w}(M^r(H)) \odot \boldsymbol{w}(M^r(H)) = \omega^\omega \odot \omega^\omega = \omega^{\omega\cdot 2}$ and $\boldsymbol{w}(M^r(X_2)) = \boldsymbol{w}(M^r(H)) \odot \boldsymbol{w}(M^r(\omega)) = \omega^\omega \odot 1 = \omega^\omega$.

Fortunately, we uncovered a new ordinal invariant, defined similarly to the usual invariants, in which the width of the multiset ordering is functional.

▶ **Definition 3.3** (Friendly order type). *A bad sequence is* open-ended *if it is empty or of the form sx where s is an open-ended sequence and x has a "friend"* [1] *in the residual $X_{\not\geq s}$, i.e., an element incomparable to x. For any* wpo *X, let $Bad_\perp(X)$ be the subtree of $Bad(X)$ which contains all open-ended bad sequences. As $Bad_\perp(X)$ is a substructure of $Bad(X)$, it has a rank that we denote by $\boldsymbol{o}_\perp(X)$ the* friendly order type *of X (or* fot*).*

This definition can be expressed as the following residual equation:

$$\boldsymbol{o}_\perp(X) = \sup_{x\in X, X_{\perp x}\neq\emptyset} (\boldsymbol{o}_\perp(X_{\not\geq x}) + 1) \tag{Res-f}$$

▶ **Theorem 3.4.** *For any* wpo *X, $\boldsymbol{w}(M^r(X)) = \omega^{\boldsymbol{o}_\perp(X)}$*

Proof. See Appendix B. The proof of Theorem 3.4 is quite technical, and relies on the notion of quasi-incomparable subsets. ◀

4 Computing the friendly order type

Like the usual ordinal invariants, the fot can be computed compositionally for some basic operations on wpos:

▶ **Proposition 4.1.** *For any non empty* wpo *A, B,*
1. $\boldsymbol{o}_\perp(A + B) = \boldsymbol{o}_\perp(A) + \boldsymbol{o}_\perp(B)$,
2. $\boldsymbol{o}_\perp(A \sqcup B) = 1 + (\boldsymbol{o}(A) - 1) \oplus (\boldsymbol{o}(B) - 1)$,

Proof.
1. For any sequences s_A, s_B in $Bad_\perp(A), Bad_\perp(B)$, the concatenation $s_B s_A$ is a sequence of $Bad_\perp(A + B)$. Furthermore, any sequence of $Bad_\perp(A + B)$ is of this form.
2. For any two sequences s_1, s_2, let $s_1 \sqcup\!\sqcup s_2$ denote the set of sequences obtained through shuffling s_1, s_2 together (e.g. $abcad \in aba \sqcup\!\sqcup cd$). Let x_A, x_B be two minimal elements of A and B. For any sequences s_A, s_B in $Bad(A\backslash\{x_A\}), Bad(B\backslash\{x_B\})$, for any $s \in s_A \sqcup\!\sqcup s_B$, we know that s and sx_A and sx_B are in $Bad_\perp(A \sqcup B)$. Reciprocally, from any $s \in Bad_\perp(A \sqcup B)$, there is a partition $s_A \in Bad(A), s_B \in Bad(B)$ such that $s \in s_A \sqcup\!\sqcup s_B$. Furthermore, the natural sum of the ranks of s_A in $Bad(A)$ and s_B in $Bad(B)$ is strictly positive.

[1] Can one be friend with one's superior or inferior? No. Your true friends are those you cannot (and do not have to) compare yourselves with.

Suppose for contradiction sake that s_A and s_B have rank 0 in $Bad(A)$ and $Bad(B)$. Let $s = s'x$. Then $(A \sqcup B)_{\not\geq s} = \emptyset$ and in particular x has no friend in $(A \sqcup B)_{\not\geq s'}$. Thus $s \notin Bad_\perp(A \sqcup B)$, contradiction. ◀

Observe how friendly order type behaves similarly to mot. It is not unusual to have fot coincides with mot, for instance $\boldsymbol{o}_\perp(\omega \sqcup \omega) = \boldsymbol{o}(\omega \sqcup \omega)$ (Proposition 4.1).

To bring this new ordinal invariant closer to familiar grounds, we bound the fot of a wpo X with the mot of a special subset of X, the *stripped* subset.

▶ **Definition 4.2** (Stripped subset). *The* stripped *subset of a* wpo *X, denoted by $str(X)$, is X without its friendless elements:*

$$str(X) \stackrel{def}{=} \{\, x \in X \ : \ X_{\perp x} \neq \emptyset \,\} .$$

Since $Bad_\perp(X)$ is a subtree of $Bad(str(X))$, we know that $\boldsymbol{o}_\perp(X) \leq \boldsymbol{o}(str(X))$. Here is an example where this inequality is strict:

▶ **Example 4.3.** Let $X = \omega \sqcup \{\clubsuit\}$. Here $str(X) = X$, so $\boldsymbol{o}(str(X)) = \omega + 1$. However, in $Bad_\perp(X)$, the singleton $\clubsuit$ has rank 0, and the singleton n for any $n \in \omega$ has rank n. Therefore $\boldsymbol{o}_\perp(X) = \omega < \boldsymbol{o}(str(X))$.

Let us show that $\boldsymbol{o}(str(X))$ also appears in a lower bound on $\boldsymbol{o}_\perp(X)$, by introducing an alternative characterisation of fot as the mot of a specific subset.

A *maximal linearisation* is a monotonic function from a wpo X onto $\boldsymbol{o}(X)$.

▶ **Definition 4.4** (Friendly subset). *A subset X' of X is friendly if there exist a maximal linearisation $\ell : X' \to \boldsymbol{o}(X')$ such that for any bad sequence $s = x_1, \dots x_n$ in X' verifying $\ell(x_1) > \cdots > \ell(x_n)$, s is open-ended. We say that ℓ witnesses the* friendly condition*.*

Observe that every friendly subset of X is a substructure of $str(X)$.

For any ordinal α, let

$$\delta(\alpha) \stackrel{\text{def}}{=} \begin{cases} \alpha \text{ if } \alpha \text{ is limit,} \\ \gamma + \lfloor n/2 \rfloor \text{ if } \alpha = \gamma + n \text{ with } \gamma \text{ limit and } n < \omega. \end{cases}$$

▶ **Theorem 4.5** (Alternative characterisation of $\boldsymbol{o}_\perp(X)$). *Let X be a* wpo*. There exists a friendly subset X' of X which maximizes $\boldsymbol{o}(X')$, and $\boldsymbol{o}_\perp(X) = \boldsymbol{o}(X')$. Furthermore, $\delta(\boldsymbol{o}(str(X))) \leq \boldsymbol{o}_\perp(X) \leq \boldsymbol{o}(str(X))$.*

Proof. See proof in Appendix C. ◀

▶ **Example 4.6** (Following on Example 3.2). Remember that $H \stackrel{\text{def}}{=} \Sigma_{n<\omega}\Gamma_n$. Thus $str(H) = \Sigma_{2\leq n<\omega}\Gamma_n$, and $\boldsymbol{o}(str(H)) = \boldsymbol{o}(H) = \omega$. Consider $X_1 = H + H$ and $X_2 = H + \omega$. Observe that $str(X_1) = str(H) + str(H)$ whereas $str(X_2) = str(H)$. Therefore, according to Theorem 4.5, $\boldsymbol{o}_\perp(X_1) = \omega \cdot 2$ and $\boldsymbol{o}_\perp(X_2) = \omega$.

▶ **Corollary 4.7.** *For any* wpo *X, if $\boldsymbol{o}(X)$ is limit and $\boldsymbol{o}(str(X)) = \boldsymbol{o}(X)$, then $\boldsymbol{o}_\perp(X) = \boldsymbol{o}(X)$.*

The conditions in Corollary 4.7 are often satisfied:

▶ **Proposition 4.8.** *For any* wpo *non-empty X, $\boldsymbol{o}_\perp(M^\diamond(X)) = \boldsymbol{o}(M^\diamond(X))$.*

Proof. Observe that $M^\diamond(X) = M^\diamond(X) \setminus \{\emptyset\}$. Thus $\boldsymbol{o}(str(M^\diamond(X))) = \boldsymbol{o}(M^\diamond(X)) - 1 = \boldsymbol{o}(M^\diamond(X))$ (Theorem 1.10). We conclude with Corollary 4.7. ◀

Conclusion

Table 2 sums up this article's contributions (in the gray cases) amidst the former state of the art.

Table 2 Ordinal invariants of the set of finite multisets.

Invariants	Multiset embedding of X	Multiset ordering of X
Mot $\boldsymbol{o}$	$\omega^{\widehat{\boldsymbol{o}(X)}}$	$\omega^{\boldsymbol{o}(X)}$
Height $\boldsymbol{h}$	$\boldsymbol{h}^*(X)$	$\omega^{\boldsymbol{h}(X)}$
Width $\boldsymbol{w}$	$\omega^{\widehat{\boldsymbol{o}(X)}-1}$	$\omega^{\boldsymbol{o}_\perp(X)}$

These results are part of a more general research program (see [8, 16]) aimed at measuring more precisely and more effectively the complexity of wpos used in well-structured systems, termination proofs, and other algorithmic applications.

Investigating the friendly order type is a subject for further research: How does it relate to other concepts? Can it be computed compositionally for more operations? Can we define a class of wpos where friendly order type always coincides with mot?

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A Proof of Theorem 3.1

We write $m \overset{\not\emptyset}{\leq_r} m'$ (resp. $m \overset{\not\emptyset}{<} m'$, $m \overset{\not\emptyset}{\perp} m'$) when $m \cap m' \neq \emptyset$ and $m \leq_r m'$ (resp $m < m'$, $m \perp m'$). With these new notations, the multiset ordering can be reformulated as follows

▶ **Definition A.1** (Multiset ordering (reformulated)). *$M^r(X) = (M(X), \leq_r)$ is ordered with the* multiset ordering*: $m \leq_r m'$ iff there exists m_1, m'_1, m_2 such that $m = m_1 \cup m_2$, $m' = m'_1 \cup m_2$, and $m_1 \overset{\not\emptyset}{<} m'_1$.*

▶ **Lemma A.2.** *Let $A = \cup_{i \leq n} A_i$ a set partitioned in n subsets, for some $n \in \mathbb{N}$. Let $\leq_A$ a well-partial ordering on A, and $\leq_{A_i}$ the same ordering restricted to the subset A_i for $i \leq n$. Then*

$$\boldsymbol{h}(A, \leq_A) \leq \bigoplus_{i \leq n} \boldsymbol{h}(A_i, \leq_{A_i}) \ .$$

Proof. From any decreasing sequence s on A, one can extract a decreasing sequence s_i by restricting s to A_i for any $i \leq n$. By induction on the rank of s in $Dec(A)$, one shows that $rk(s) \leq \bigoplus_{i \leq n} rk(s_i)$. ◀

Proof of Theorem 3.1. We prove the upper bound by induction on $\boldsymbol{h}(X)$.

If $\boldsymbol{h}(X) = 0$ then $X = \emptyset$ and $\boldsymbol{h}(M^r(\emptyset)) = 1 = \omega^0$.

Suppose that X is not empty. For any non-empty multiset $m \in M^r(X)$, the residual $M^r(X)_{<m}$ can be partitioned as follows:

$$M^r(X)_{<m} = \bigcup_{m_1 + m_2 = m, m_1 \neq \emptyset} \{ \, m' + m_2 \ : \ m' \overset{\not\emptyset}{<} m_1 \, \} \ .$$

Note that this union is a partition of the support of $M^r(X)_{<m}$, it does not say anything on the order between the elements of the subsets in the union.

For any non-empty multiset m, we define $S_m \stackrel{\text{def}}{=} (\cap_{x\in m} X_{\not\geq x}) \cap (\cup_{x\in m} X_{<x})$ a subset of X. Thus for any multiset m' in $M^r(X)$, $m' \stackrel{\not\emptyset}{<} m$ iff $m' \in M^r(S_m)$. Therefore:

$$M^r(X)_{<m} = \bigcup_{m_1+m_2=m, m_1\neq\emptyset} \{\ m' + m_2 \ :\ m' \in M^r(S_{m_1})\ \} \ .$$

Observe that $\boldsymbol{h}(S_{m_1}) < \boldsymbol{h}(X)$ by definition of S_{m_1}. Hence by induction hypothesis $\boldsymbol{h}(M^r(S_{m_1})) \leq \omega^{\boldsymbol{h}(S_{m_1})} < \omega^{\boldsymbol{h}(X)}$. Moreover, $\omega^{\boldsymbol{h}(X)}$ is indecomposable. Hence according to Lemma A.2:

$$\boldsymbol{h}(M^r(X)_{<m}) \leq \bigoplus_{m_1+m_2=m, m_1\neq\emptyset} \boldsymbol{h}(M^r(\cup_{x\in m_1} X_{<x})) < \omega^{\boldsymbol{h}(X)} \ .$$

Therefore $\boldsymbol{h}(M^r(X)) \leq \omega^{\boldsymbol{h}(X)}$ according to Equation (Res-h). ◀

B Proof of Theorem 3.4

First we prove intermediary lower and upper bounds on the width of the multiset ordering.

▶ **Lemma B.1.** *Let X be a* wpo*. Then*

$$\boldsymbol{w}(M^r(X)) \geq \sup_{x\in X, n<\omega} \boldsymbol{w}(M^r(X)_{\perp\langle x\rangle}) \cdot n + 1$$

Proof. This proof follows the same structure as the proof of Lemma 2.3: We study the residual of $M^r(X)$ which contains every element incomparable to some multiset of the form $\langle x\rangle \times n$, and slice this residual into a family of quasi-incomparable subsets.

According to Equation (Res-w),

$$\begin{aligned} w(M^r(X)) &= \sup_{m\in M^r(X)} \boldsymbol{w}(M^r(X)_{\perp m}) + 1 \\ &\geq \sup_{x\in X, n<\omega} \boldsymbol{w}(M^r(X)_{\perp \langle x\rangle\times n}) + 1 \ . \end{aligned}$$

For all $k \in [1, n]$, let $M_k = \{\ \langle x\rangle \times (n-k) \cup m \ :\ m \in M^r(X)_{\perp\langle x\rangle}\ \}$. Observe that $M_k \equiv M^r(X)_{\perp\langle x\rangle}$ for any $k \in [1, n]$, and for all $m \in M_k$, $m \perp \langle x\rangle \times n$. We claim that $(M_k)_{k\in[1,n]}$ is a quasi-incomparable family of subsets of $M^r(X)_{\perp(\langle x\rangle\times n)}$: Let $i < n$ and Y a finite subset of $M_1 \cup \cdots \cup M_i$. We define m_Y and M'_{i+1} as

$$\begin{aligned} m_Y &\stackrel{\text{def}}{=} \bigcup_{j\leq i} \bigcup_{m\in(M_j\cap Y)} (m \setminus (\langle x\rangle \times (n-j))) \ , \\ M'_{i+1} &\stackrel{\text{def}}{=} \{\ \langle x\rangle \times (n-i-1) \cup m_Y \cup m \ :\ m \in M^r(X)_{\perp\langle x\rangle}\ \} \ . \end{aligned}$$

Observe that M'_{i+1} is an isomorphic subset of M_{i+1}, and $Y \perp M'_{i+1}$.

Therefore according to Lemma 1.14, $\boldsymbol{w}(M^r(X)_{\perp(\langle x\rangle\times n)}) \geq \boldsymbol{w}(M^r(X)_{\perp\langle x\rangle}) \cdot n$. ◀

▶ **Lemma B.2.** *Let X be a* wpo*. Then*

$$\boldsymbol{w}(M^r(X)) \leq \sup_{x\in X, n<\omega} \boldsymbol{w}(M^r(X)_{\perp\langle x\rangle}) \otimes n + 1$$

Proof. By definition, for any multisets $m, m' \in M^r(X)$, $m \perp m'$ means that $m \neq m'$ and there exists m_1, m'_1, m_2 such that $m = m_1 \cup m_2$, $m' = m'_1 \cup m_2$ and $m_1 \overset{\not\cap}{\perp} m'_1$.

Therefore, the residual $M^r(X)_{\perp m}$ can be partitioned as an augmentation of a disjoint union:

$$M^r(X)_{\perp m} \geq_{\text{aug}} \bigsqcup_{m_1+m_2=m, m_1\neq\emptyset} \{ \, m'_1 + m_2 \; : \; m' \in M^r(X), m'_1 \overset{\not\cap}{\perp} m_1 \, \} \, ,$$

which can be reformulated into

$$M^r(X)_{\perp m} \geq_{\text{aug}} \bigsqcup_{m_1\subseteq m, m_1\neq\emptyset} M^r(X)_{\overset{\not\cap}{\perp} m_1}$$

where $M^r(X)_{\overset{\not\cap}{\perp} m_1}$ is the residual $\{ \, m' \in M^r(X) \; : \; m' \overset{\not\cap}{\perp} m_1 \, \}$.

Let us observe this residual: $m' \overset{\not\cap}{\perp} m_1$ means that m' and m_1 are disjoint and there exists $x \in m_1$ such that for all $y' \in m'$, $x \not\leq y'$, and there exists $x' \in m'$ such that for all $y \in m_1$, $x' \not\leq y$. In particular $x' \not\leq x$. Hence $m' \overset{\not\cap}{\perp} m_1$ implies there exists $x \in m_1$ such that $\langle x \rangle \overset{\not\cap}{\perp} m'$, which is equivalent to $\langle x \rangle \perp m'$. Therefore the support of $M^r(X)_{\overset{\not\cap}{\perp} m_1}$ is included in a union on $x \in m_1$ of residuals $M^r(X)_{\perp\langle x\rangle}$. With an augmentation we get a disjoint union:

$$M^r(X)_{\overset{\not\cap}{\perp} m_1} \leq_{\text{st}} \geq_{\text{aug}} \bigsqcup_{x\in m_1} M^r(X)_{\perp\langle x\rangle} \, .$$

Hence according to Table 1, $M^r(X)_{\perp m} \leq \bigoplus_{m_1\subseteq m, m_1\neq\emptyset} \bigoplus_{x\in m_1} \boldsymbol{w}(M^r(X)_{\perp\langle x\rangle})$.

Let $x \in m$ such that $\boldsymbol{w}(M^r(X)_{\perp\langle x\rangle})$ is maximal. Then $\boldsymbol{w}(M^r(X)_{\perp m}) \leq \boldsymbol{w}(M^r(X)_{\perp\langle x\rangle}) \otimes n$ for some $n < \omega$. Hence according to Equation (Res-w),

$$\boldsymbol{w}(M^r(X)) = \sup_{m\in M^r(X)} \boldsymbol{w}(M^r(X)_{\perp m}) + 1 \leq \sup_{x\in X, n<\omega} \boldsymbol{w}(M^r(X)_{\perp\langle x\rangle}) \otimes n + 1 \, . \qquad ◀$$

The bounds provided in Lemmas B.1 and B.2 actually match. Furthermore, they can be reformulated in such a way that the residual on $M^r(X)$ boils down to a residual on X:

▶ **Lemma B.3.** *For any non-linear* wpo X*,*

$$\boldsymbol{w}(M^r(X)) = \sup\{ \, \boldsymbol{w}(M^r(X_{\not\geq x})) \cdot \omega \; : \; x \in X, X_{\perp x} \neq \emptyset \, \} \, . \tag{W}$$

Proof. For any ordinal α, $\sup_{n<\omega}(\alpha \cdot n + 1) = \sup_{n<\omega}(\alpha \otimes n + 1) = \alpha \cdot \omega$. Hence according to Lemmas B.1 and B.2, $\boldsymbol{w}(M^r(X)) = \sup_{x\in X}(\boldsymbol{w}(M^r(X)_{\perp\langle x\rangle}) \cdot \omega)$.

Let $x \in X$. If $X_{\perp x} = \emptyset$, then $M^r(X)_{\perp\langle x\rangle} = \emptyset$. Otherwise let $y \in X_{\perp x}$. Observe that, for any $m \in M^r(X_{\not\geq x})$, $m \cup \langle y \rangle \perp \langle x \rangle$. Hence

$$\{ \, \langle y \rangle \cup m \; : \; m \in M^r(X_{\not\geq x}) \, \} \leq_{\text{st}} M^r(X)_{\perp\langle x\rangle} \leq_{\text{st}} M^r(X_{\not\geq x}) \, .$$

Therefore $\boldsymbol{w}(M^r(X)_{\perp\langle x\rangle}) = \boldsymbol{w}(M^r(X_{\not\geq x}))$ if $X_{\perp x} \neq \emptyset$, otherwise $\boldsymbol{w}(M^r(X)_{\perp\langle x\rangle}) = 0$. ◀

Proof of Theorem 3.4. If X is linear, $Bad_\perp(X)$ only contains the empty sequence, hence $\boldsymbol{o}_\perp(X) = 0$ and $\boldsymbol{w}(Mulr(X) = 1$. Otherwise, observe that Equation (W) is quite similar to Equation (Res-f) in its structure. Thus $\boldsymbol{w}(M^r(X)) = \omega^{\boldsymbol{o}_\perp(X)}$ follows directly from Equation (W). ◀

C Proof of Theorem 4.5

▶ **Lemma C.1.** *For any* wpo X*, for any maximal linearisation* $\ell : str(X) \to \boldsymbol{o}(str(X))$*, there exists a friendly subset* X' *such that* ℓ *restricted to* X' *verifies the friendly condition, and* $\boldsymbol{o}(X') \geq \delta(\boldsymbol{o}(str(X)))$*.*

Proof. We claim that for any $\beta \leq \boldsymbol{o}(str(X))$, there exists $X_\beta \subseteq \ell^{-1}(\{\ \gamma : \gamma < \beta\ \})$ a friendly subset of X where ℓ restricted to X_β verifies the friendly condition, such that $\boldsymbol{o}(X_\beta) \geq \delta(\beta)$. In this proof, when we say that a subset is friendly, it is always implied that ℓ restricted to this subset witnesses the friendly condition.

We build the subsets $(X_\beta)_{\beta \leq \boldsymbol{o}(str(X))}$ as follows:

- $X_0 = \emptyset$,
- For γ limit, $X_\gamma = \bigcup_{\beta<\gamma} X_\beta$,
- For any β, $X_{\beta+1} = X_\beta \cup \ell^{-1}(\beta)$ if friendly, otherwise $X_{\beta+1} = X_\beta$.

First observe that X_β is friendly for any $\beta \leq \boldsymbol{o}(str(X))$. Indeed, X_0 is friendly, and since for any $\beta < \beta'$, $X_\beta \subseteq X_{\beta'}$, then the union $\bigcup_{\beta<\gamma} X_\beta$ for γ limit is friendly by induction.

Let us prove the claim $\boldsymbol{o}(X_\beta) \geq \delta(\beta)$, by showing that for any $\beta + 2 \leq \boldsymbol{o}(str(X))$, we have $\boldsymbol{o}(X_{\beta+2}) > \boldsymbol{o}(X_\beta)$. Let $x = \ell^{-1}(\beta')$ and $x' = \ell^{-1}(\beta' + 1)$. Assume for the sake of contradiction that $X_{\beta+2} = X_\beta$. This means that neither $X_\beta \cup \{x\}$ nor $X_\beta \cup \{x'\}$ are friendly. Hence there exists $y, y' \in X_\beta$ such that for any $z \in X$, we have $z \perp y \implies z \geq x$ and $z \perp y' \implies z \geq x$. Now because of ℓ we know that $x \ngeq x'$ and $y, y' \ngeq x, x'$. Since $y, y' \in str(X)$, then $X_{\perp y}$ and $X_{\perp y'}$ are both non-empty, so actually $x \perp y$ and $x' \perp y'$. And since $x \ngeq x'$, we know $y' < x$. Therefore $x \perp x'$, hence $y < x'$. Which leads to a contradiction on the relationship between y and y'. ◀

For any friendly subset X', $\boldsymbol{o}(X') \leq \boldsymbol{o}(str(X))$, and there exist X' such that $\boldsymbol{o}(X') \geq \delta(\boldsymbol{o}(str(X)))$. Therefore there exists a friendly subset X' which maximizes $\boldsymbol{o}(X')$.

Proof of Theorem 4.5. We say that a bad sequence $x_1, \ldots x_n$ respects a maximal linearisation ℓ when $\ell(x_1) > \cdots > \ell(x_n)$. Let X' be a friendly subset of X and ℓ a maximal linearisation of X' that verifies the friendly condition. Observe that $Bad(X')$ restricted to sequences that respect ℓ has for rank $\boldsymbol{o}(X')$, and is embedded in $Bad_\perp(X)$. Hence $\boldsymbol{o}_\perp(X) \geq \boldsymbol{o}(X')$.

We prove the upper bound by induction on $\boldsymbol{o}_\perp(X)$. If $\boldsymbol{o}_\perp(X) = 0$ then the only friendly subset of X is the empty set. Now suppose that $\boldsymbol{o}_\perp(X) > 0$. For any $x \in str(X)$, by induction hypothesis on $X_{\ngeq x}$, there exists a friendly subset X' of $X_{\ngeq x}$, with a maximal linearisation ℓ which verifies the friendly condition, such that $\boldsymbol{o}(X') \geq \boldsymbol{o}_\perp(X_{\ngeq x})$. We extend ℓ to the subset $X' \cup \{x\}$ of X, such that $\ell(x) = \boldsymbol{o}(X')$. Now ℓ is a maximal linearisation of $X' \cup \{x\}$ which verifies the friendly condition, therefore $\boldsymbol{o}(X' \cup \{x\})$ is a friendly subset of X and $\boldsymbol{o}(X' \cup \{x\}) > \boldsymbol{o}_\perp(X_{\ngeq x})$. Let X' be a friendly subset of X which maximizes $\boldsymbol{o}(X')$. Then for any $x \in str(X)$, $\boldsymbol{o}_\perp(X_{\ngeq x}) < \boldsymbol{o}(X')$. Therefore $\boldsymbol{o}_\perp(X) \leq \boldsymbol{o}(X')$ according to Equation (Res-f). ◀