Oriented Spanners

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Abstract

Given a point set P in the Euclidean plane and a parameter t, we define an *oriented t-spanner* as an oriented subgraph of the complete bi-directed graph such that for every pair of points, the shortest cycle in G through those points is at most a factor t longer than the shortest oriented cycle in the complete bi-directed graph. We investigate the problem of computing sparse graphs with small oriented dilation.

As we can show that minimising oriented dilation for a given number of edges is NP-hard in the plane, we first consider one-dimensional point sets. While obtaining a 1-spanner in this setting is straightforward, already for five points such a spanner has no plane embedding with the leftmost and rightmost point on the outer face. This leads to restricting to oriented graphs with a one-page book embedding on the one-dimensional point set. For this case we present a dynamic program to compute the graph of minimum oriented dilation that runs in $\mathcal{O}(n^8)$ time for n points, and a greedy algorithm that computes a 5-spanner in $\mathcal{O}(n \log n)$ time.

Expanding these results finally gives us a result for two-dimensional point sets: we prove that for convex point sets the greedy triangulation results in an oriented $\mathcal{O}(1)$-spanner.

1 Introduction

Computing geometric spanners is an extensively studied problem [5, 20]. Directed geometric spanners have also been considered [1]. Given a point set $P \subset \mathbb{R}^d$ and a parameter t, a *directed t-spanner* $G = (P, E)$ is a subgraph of the complete bi-directed geometric graph on P such that for every pair of points p, p', the shortest path in G is at most a factor t longer than the shortest path in the complete graph, that is, $|p - p'|$. The *dilation* of G then is the smallest such t. Formally, $t = \max\left\{ \frac{|d_G(p,p')|}{|p-p'|} \mid p, p' \in P \right\}$, where $d_G(p, p')$ denotes the *shortest path from p to p' in G*.

(Directed) geometric spanners have a wide range of applications, ranging from wireless ad-hoc networks [7, 21] to robot motion planning [12] and the analysis of road networks [2, 14]. In all of these applications one might want to avoid adding the edge (v, u) if the edge (u, v) was included: in ad-hoc networks this may reduce interference, in motion planning it may reduce congestion and simplify collision avoidance, in road networks this corresponds to one-way

roads or tracks, which may be necessary because of space limitations, and in communication networks one could require two neighbouring devices not to exchange data by the same (bi-directional) direct connection, for example, in two-way authentication.

This motivates our study of *oriented graphs* as spanners, i.e. directed spanners $G = (P, E)$ where $(p, p') \in E$ implies $(p', p) \notin E$. With this restriction, if the edge (p, p') is added, the dilation in the other direction is never 1. Even worse, given a set P of three points, where p and p' are very close to each other and p'' is far away from both, any oriented graph will have arbitrarily high dilation for either (p, p') or (p', p) (see Figure 1). Therefore, considering the dilation for an oriented graph as $t = \max\left\{ \frac{|d_G(p,p')|}{|p-p'|} \mid p, p' \in P \right\}$ would not tell us much about the quality of the spanner. To obtain meaningful results, we define *oriented dilation*.
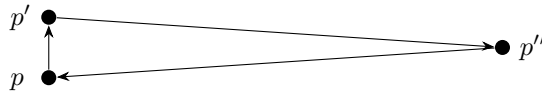
Figure 1 If p and p' are very close to each other and p'' is far away from both, any oriented graph will have arbitrarily high (directed) dilation.

By $C_G(p, p')$ we denote the *shortest oriented cycle* containing the points p and p' in an oriented graph G. The *optimal oriented cycle* $\Delta(p, p')$ for two points $p, p' \in P$ is the shortest oriented cycle containing p and p' in the complete graph on P. Notice, $\Delta(p, p')$ is the triangle $\Delta_{pp'p''}$ with $p'' = \underset{p^* \in P \setminus \{p, p'\}}{\arg\min} \left(|p - p^*| + |p^* - p'| \right)$.

▶ **Definition 1** (oriented dilation). *Given a point set P and an oriented graph G on P, the oriented dilation of two points $p, p' \in P$ is defined as*

$$\mathrm{odil}(p, p') = \frac{|C_G(p, p')|}{|\Delta(p, p')|}.$$

The dilation t of an oriented graph is defined as $t = \max\{\mathrm{odil}(p, p') \mid \forall p, p' \in P\}$.

An oriented graph with dilation at most t is called an *oriented t-spanner*. We frequently contrast our results to known results on undirected geometric spanners, and refer to the known results by using the adjective *undirected*. Our new measure for oriented graphs is similar to the definition of dilation in round trip spanners [9, 10] that has been considered in the setting of (non-geometric) directed graph spanners; but round trip spanners require a starting graph, and using the complete bi-directed geometric graph would not give meaningful results.

In this paper, we initiate the study of oriented spanners. As is common for spanners, our general goal is to obtain *sparse* spanners, i.e. with linear number of edges. The goal can be achieved by bounding the number of edges explicitly or by restricting to a class of sparse graphs like plane graphs [5]. We refer to a spanner as a *minimum (oriented) spanner* if it minimises t under the given restriction.

It is known that computing a minimum undirected spanner with at most $n - 1$ edges, i.e. a minimum dilation tree, is NP-hard [15]. The corresponding question for oriented spanners asks for the *minimum dilation cycle*. We prove this problem to be NP-hard in Section 3.1.

The problem of computing the minimum undirected spanner restricted to the class of plane straight-line graphs is called the *minimum dilation triangulation* problem; its hardness is still open [14, 15], but it is conjectured to be NP-hard [6]. As this undirected problem can be emulated in the oriented setting by suitable vertex gadgets, it is unlikely that finding a *minimum plane straight-line (oriented) spanner* can be done efficiently.

Therefore, in Section 2, we start with one-dimensional point sets. For such a point set P with n points, we can give a minimum spanner with $3n - 6$ edges. However, if we are interested in a one-dimensional result analogous to minimum plane spanners, this spanner is not suitable: it has no plane embedding with leftmost and rightmost point on the outer face. Therefore, we restrict our attention to a graph class that is closer to the plane case for two-dimensional point sets: one-page book embeddings.

We show how to compute a t-spanner which is a one-page plane book embedding for a one-dimensional point set in Section 2.2. We prove that with a greedy algorithm, we can always generate such a t-spanner with $t \leq 5$ in $\mathcal{O}(n \log n)$ time. An optimal one-page plane spanner can be computed in $\mathcal{O}(n^8)$ time (Theorem 11).

As the resulting spanner is outerplanar, this particular class of graphs is also motivated by the problem of finding a minimum plane spanner for points in convex position. Using these results, in Section 3.2, we show that suitably orienting the greedy triangulation leads to oriented $\mathcal{O}(1)$-spanners for two-dimensional point sets in convex position (Theorem 16). For general (non-convex) point sets, there are examples where all orientations of the greedy triangulation have large dilation.

The greedy triangulation fulfils the α-diamond-property [11], and all triangulations with this property are undirected $\mathcal{O}(1)$-spanners. This raises the question whether all triangulations fulfilling this property are also oriented $\mathcal{O}(1)$-spanners for convex point sets. In Section 3.3 we answer this question negatively.

Table 1 Overview of the results of the paper.

Point set	Spanner type	Dilation	Time complexity	Reference
2-dim.	unrestricted	minimum	NP-hard	Theorem 14
2-dim.	unrestricted	≤ 2	$\mathcal{O}(n^2)$	Proposition 13
2-dim.	plane	minimum	Min.Dil. Triangulation	Observation 15
2-dim. convex	plane	$\mathcal{O}(1)$	$\mathcal{O}(n \log n)$	Theorem 16
1-dim.	unrestricted	1	$\mathcal{O}(n)$	Corollary 3
1-dim.	2-page-plane	≤ 2	$\mathcal{O}(n)$	Proposition 4
1-dim.	1-page-plane	minimum	$\mathcal{O}(n^8)$	Theorem 11
1-dim.	1-page-plane	≤ 5	$\mathcal{O}(n \log n)$	Theorem 10

2 One-Dimensional Point Sets

We first focus on points in one dimension. We will always draw points on a horizontal line with the minimum point leftmost, and the maximum point rightmost. We observe that in one dimension only the dilation of a linear number of candidate pairs needs to be checked.

▶ **Lemma* 2.** *Let P be a one-dimensional point set of n points. The oriented dilation t of an oriented graph G on P is*

$$t = \max\{\mathrm{odil}(p_i, p_{i+2}), \mathrm{odil}(p_j, p_{j+3}) \mid 1 \leq i \leq n-2, 1 \leq j \leq n-3\}.$$

This observation directly leads to an oriented 1-spanner with $3n - 6$ edges for every one-dimensional point set.

The proofs of results marked by ∗ are given in a long version of this paper.

▶ **Corollary 3** (oriented 1-spanner). *For every one-dimensional point set P, $G = (P, E)$ with*

$$E = \{(p_i, p_{i+1}), (p_{j+2}, p_j), (p_{k+3}, p_k) \mid 1 \leq i \leq n-1, 1 \leq j \leq n-2, 1 \leq k \leq n-3\}$$

is an oriented 1-spanner for P (see Figure 2).
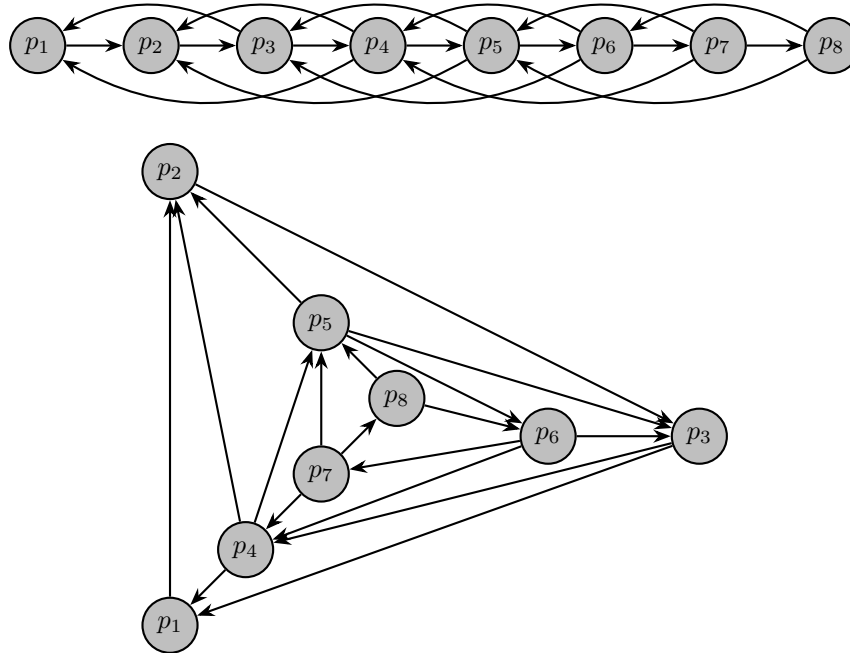
Figure 2 One-dimensional oriented 1-spanner and its plane embedding in the Euclidean space.

In two dimensions, plane (straight-line) spanners are of particular interest. The natural one-dimensional analogue to plane straight-line graphs are one- and two-page book embeddings [3, 8, 13, 22].

A *one-page book embedding* of a graph corresponds to an embedding of the vertices as points on a line with the edges drawn without crossings as circular arcs above the line. In a *two-page book embedding* an edge may be drawn as an arc above or below the line. In such a (one- or two-page) book embedding, for consecutive points on the line, we may draw their edge straight on the line. We call a graph *one-page plane* (respectively *two-page plane*) if it has a one-page (respectively two-page) book embedding.

In particular, one-page plane graphs are outerplanar graphs and correspond to plane straight-line graphs if we embed the points on a (slightly) convex arc instead of on a line. Two-page plane graphs are a subclass of planar graphs, while any planar graph has a four-page book embedding [22].

As we argue next, the 1-spanner of Corollary 3 is not two-page plane (and therefore also not one-page plane). This follows from a stronger statement: the graph has no plane embedding with the first and last point on the outer face. Suppose the graph would have such an embedding. The graph has $3n - 6$ edges, but no edge between the first and the last point for $n > 4$. Thus, we could add this edge while maintaining planarity, which contradicts the fact that a planar graph has at most $3n - 6$ edges. Interestingly, the 1-spanner is planar. We construct a stack triangulation by adding points from left to right. The first three points form a triangle. Then, we inductively add the next point into the triangle formed by the previous three points. See Figure 2.

2.1 Two-Page Plane Spanners

As argued above, the 1-spanner given in Corollary 3 is not two-page plane (and thus not one-page plane). Moreover, by Lemma 2, no two-page plane 1-spanner can exist. However, we can give a two-page plane 2-spanner for every one-dimensional point set:

▶ **Proposition* 4** (two-page plane 2-spanner). *For every one-dimensional point set P, the graph $G = (P, E)$ with $E = \{(p_i, p_{i+1}), (p_{j+2}, p_j) \mid 1 \leq i \leq n-1, 1 \leq j \leq n-2\}$ (see Figure 3) is a two-page plane oriented 2-spanner for P.*
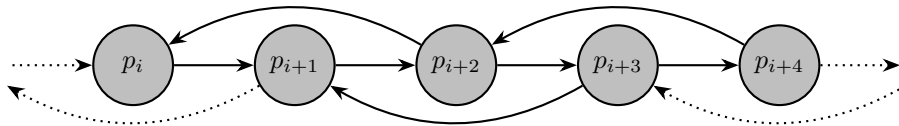
■ **Figure 3** Part of $G = (P, E)$ with $E = \{(p_i, p_{i+1}), (p_{j+2}, p_j) \mid 1 \leq i \leq n-1, 1 \leq j \leq n-2\}$.

2.2 One-Page Plane Spanners

The 2-spanner given by Proposition 4 is two-page plane, but not one-page plane. As noted above, one-page plane graphs on one-dimensional point sets correspond to plane straight-line graphs if we interpret the point set as being convex. We thus place particular focus on one-plane plane graphs, since they are not only of interest in their own right but also aid us in finding oriented plane spanners in two-dimensions.

By *maximal one-page plane*, we mean a one-page plane graph $G = (P, E)$ such that for every edge $e \notin E$, the graph $G' = (P, E \cup \{e\})$ is not one-page plane. We call the edge set $\{(p_i, p_{i+1}) \mid 1 \leq i \leq n-1\}$ the *baseline*. A directed edge (p_j, p_i) with $i < j$ is a *back edge*.

We refer to an oriented one-page plane graph that includes a baseline and all other edges are back edges as a one-page-plane-baseline graph, for short 1-*PPB graph*. Lemma 5 shows that a 1-PPB graph has smaller dilation than an oriented graph with the same edge set but another orientation. Without loss of generality, we consider only 1-PPB graphs instead of one-page plane graphs in general.

▶ **Lemma* 5.** *Let $G = (P, E)$ be a one-page plane oriented t-spanner for a one-dimensional point set P. Let E' be an edge set where the edges are incident to the same vertices as E, but the orientation may be different. If we orientate such that $G' = (P, E')$ is a 1-PPB graph, then the dilation of G' is at most t.*

▶ **Lemma* 6.** *Let P be a one-dimensional point set of n points. The oriented dilation t of a 1-PPB graph for P is*

$$t = \max\{\mathrm{odil}(p_i, p_{i+2}) \mid 1 \leq i \leq n-2\}.$$

Lemma 6 holds for every 1-PPB graph, even if it is not maximal. Proposition 4 shows that Lemma 6 does not hold for non-one-page plane graphs.

Due to one-page planarity, for a point set P with $|P| > 3$, every graph G contains a tuple $p_i, p_{i+2} \in P$ where $C_G(p_i, p_{i+2}) \neq \Delta(p_i, p_{i+2})$. Combining this with Lemma 6, we get:

▶ **Corollary 7.** *There is no one-page plane 1-spanner for any point set P with $|P| > 3$.*

However, there are point sets with a one-page plane almost 1-spanner. Let $G = (P, E)$ be a graph with $|P| = 5$, $E = \{(p_i, p_{i+1}) \mid 1 \leq i \leq n - 1\} \cup \{(p_3, p_1), (p_5, p_3)\}$ and the distances $p_2 - p_1 = p_5 - p_4 = \varepsilon$ and $p_3 - p_2 = p_4 - p_3 = 1$ (see Figure 4). It holds that $t = \mathrm{odil}(p_2, p_4) = \frac{2+2\varepsilon}{2}$. For an arbitrary small ε, G is a one-page plane almost 1-spanner.
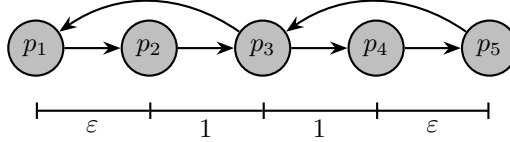

■ **Figure 4** 1-PPB almost 1-spanner.

▶ **Observation*** **8.** *There are one-dimensional point sets where no one-page plane oriented t-spanner exists for $t < 2$.*

We construct a 1-PPB spanner by starting with the base line and greedily adding back edges sorted by length from shortest to longest if they do not cross any of the edges already added. The following leads to a simple, greedy algorithm for constructing this graph: The first edge that we need to add is the shortest edge between two points with exactly one point in between. Imagine deleting the point that was in between. Then again, we need to add the shortest edge with exactly one point in between and so on, until only two points are left. By maintaining the points in a linked list and the relevant distances in a priority queue, this leads to a run time of $\mathcal{O}(n \log n)$.

To bound the dilation of the resulting graph, we need the concept of a *blocker*.

▶ **Definition 9** (blocker). *Let E be the greedily computed back edge set. Because the resulted graph $G = (P, E \cup \{(p_i, p_{i+1}) \mid 1 \leq i \leq n - 1\})$ is maximal, $(p_j, p_i) \notin E$ for $i + 2 \leq j$ implies there is a shorter edge in E which intersects with (p_j, p_i). (The greedy algorithm added this edge first and discarded (p_j, p_i) in a later iteration of the loop.) For the shortest edge intersecting (p_j, p_i), we say it* blocks (p_j, p_i). *The edge can be blocked by (p_k, p_m) with $k > j$ and $i < m < j$ or $(p_m, p_{k'})$ with $k' < i$ and $i < m < j$ (see Figure 5).*
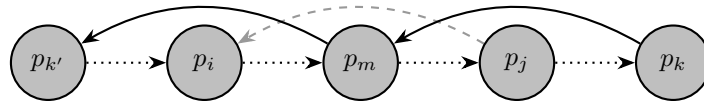
■ **Figure 5** (p_j, p_i) can be blocked by (p_k, p_m) or $(p_m, p_{k'})$ with $i < m < j$, $k > j$ and $k' < i$.

▶ **Theorem*** **10** (one-page plane 5-spanner). *Given a one-dimensional point set P of size n, a one-page plane oriented 5-spanner can be constructed in $\mathcal{O}(n \log n)$ time.*

The full proof is given in the full version, but, since the proof of correctness for the greedy two-dimensional algorithm (see Section 3.2) works similarly, we give a proof sketch here.

Proof sketch. Due to Lemma 6, it is sufficient to bound the dilation $\mathrm{odil}(p_i, p_{i+2})$ of tuples p_i, p_{i+2} with one point in-between. Let (b_1, p_{i+1}) be the blocker of the optimal edge (p_{i+2}, p_i) (blue). There could be a "sequence of blockers" $b_1, \ldots, b_{j+1}$, such that (b_k, p_{i+1}) blocks (b_{k-1}, p_i) for $2 \leq k \leq j + 1$ (green), see Figure 6.

Combining this with a lower bound on the distance of a tuple b_{i-2}, b_i for $3 \leq i \leq j+1$ (violet), we show that there are only two blockers b_j and b_{j+1}, whose the distance to p_{i+1} is larger than $p_{i+1} - p_i$. There are two cases for the shortest oriented cycle $C_G(p_i, p_{i+2})$. For $(b_{j+1}, p_i) \in E$ holds $\mathrm{odil}(p_i, p_{i+2}) = 4$. The worst-case is that $(b_{j+1}, p_i) \notin E$ is blocked by some edge (p_{i+1}, p_l) with $l < i$ (red). Then the dilation is

$$\mathrm{odil}(p_i, p_{i+2}) = \frac{(b_1 - p_l) \cdot 2}{(p_{i+2} - p_i) \cdot 2} \leq \frac{b_1 - p_{i+1} + p_{i+1} - p_l}{p_{i+2} - p_i} \leq \frac{p_{i+2} - p_i + 4 \cdot (p_{i+1} - p_i)}{p_{i+2} - p_i} = 5.$$
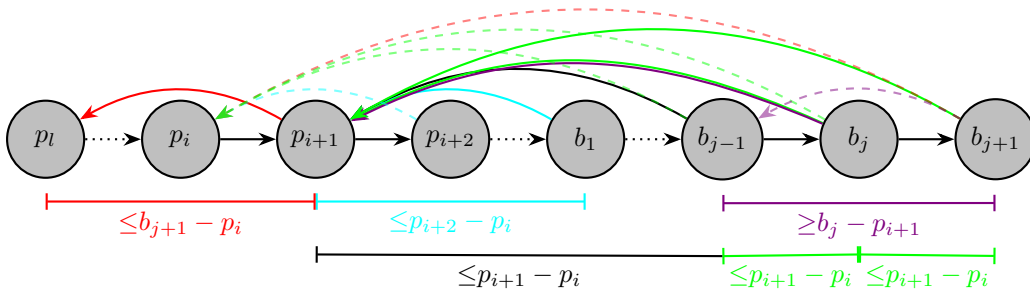

Figure 6 Blockers in the proof of Theorem 10.

Figure 7 shows a point set P and its greedily constructed spanner G. The oriented dilation of G is $t = \frac{5-7\varepsilon}{1+\varepsilon}$. Thus, G is a 5-spanner for P.

However, Figure 8 shows a t-spanner with $t = \frac{2-2\varepsilon}{1+\varepsilon} < 2$ for the same point set. Therefore, the greedy algorithm does not return the minimum spanner for every given one-dimensional point set.
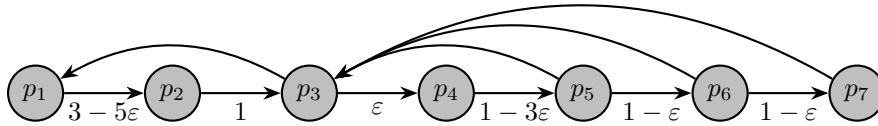

Figure 7 Example of a greedily constructed spanner, its dilation is $t = \mathrm{odil}(p_2, p_4) = \frac{5-7\varepsilon}{1+\varepsilon} < 5$.
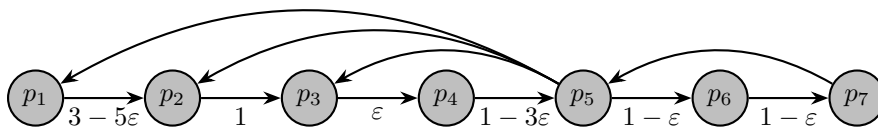
Figure 8 A spanner on the point set of Figure 7 with dilation $t = \mathrm{odil}(p_2, p_4) = \frac{2-2\varepsilon}{1+\varepsilon} < 2$.

The minimum dilation of a 1-PPB spanner for a one-dimensional point set is larger than 1 (Observation 8), unlike non-plane oriented spanners (Corollary 3). Algorithm 1 computes in $\mathcal{O}(n^8)$ time a minimum 1-PPB spanner for a given point set. The dynamic program is based on the following idea. By Lemma 6, it holds that $t = \max\{t', t'', \mathrm{odil}(p_{k-1}, p_{k+1})\}$, where t' is the minimum dilation for $\{p_1, \ldots, p_k\}$ and t'' is the minimum dilation for $\{p_k, \ldots, p_n\}$. Due to one-page-planarity, if $(p_r, p_l) \in E$, it holds that $(p_j, p_i) \notin E$ for $l < i < r < j$ and $i < l < j < r$. We test all candidates for a split point p_k. By adding the edges (p_n, p_k)

and (p_k, p_1), we can compute the optimal edge set for $\{p_1, \ldots, p_k\}$ and $\{p_k, \ldots, p_n\}$ (almost) independently. (We need the parameters l' and r' which represent the only edges needed to consider $\mathrm{odil}(p_{k-1}, p_{k+1})$.)

▶ **Theorem*** **11** (Optimal 1-PPB). *Given a one-dimensional point set P of size n, the minimum one-page plane oriented spanner for P can be calculated in $\mathcal{O}(n^8)$ time.*

■ **Algorithm 1** Minimum One-Page Plane Spanner.

Require: one-dimensional point set $P = \{p_1, \ldots, p_n\}$ (numbered from left to right)
Ensure: minimum one-page plane oriented spanner for P
  Initialise table $\mathrm{oE} = [1, \ldots, n] \times [1, \ldots, n] \times [1, \ldots, n] \times [1, \ldots, n]$
  // $\mathrm{ODIL}(E')$ returns dilation of $G = (P, E' \cup \{(p_i, p_{i+1}) \mid 1 \leq i \leq n-1\})$
  // $\mathrm{oE}(l, l', r', r) =$ back edges set E' of a minimum spanner G for $\{p_l, \ldots, p_r\}$ with
  $(p_{l'}, p_l) \in C_G(p_l, p_{l+1})$ and $(p_r, p_{r'}) \in C_G(p_{r-1}, p_r)$
  Fill table by dynamic program based on the recursion formula:

$$\mathrm{oE}(l, l', r', r) = \begin{cases} \text{``invalid'', if } l' < l+2,\ r' > r-2 \text{ or } l > r \text{ (contradicts definition)} & \text{(i)} \\[6pt] \text{``invalid'', if } r < l+2 \text{ (no oriented spanners for } |P| < 3) & \text{(ii)} \\[6pt] (p_r, p_l), \text{ if } l' = r \text{ and } r' = l \text{ (spanner for } |P| = 3 \text{ is a cycle)} & \text{(iii)} \\[6pt] \text{``invalid'', if } l' < r,\ r' > l \text{ and } l' > r' \text{ (contradicts planarity)} & \text{(iv)} \\[6pt] E' = \mathrm{oE}(l, l', k_r, r-1) \cup \{(p_r, p_l)\}, \text{ if } l' < r \text{ and } r' = l, \text{ choose} \\ l \leq k_r \leq r-3 \text{ s.t. } \mathrm{ODIL}(E') \text{ is minimal} & \text{(v)} \\[6pt] E' = \mathrm{oE}(l+1, k_l, r', r) \cup \{(p_r, p_l)\}, \text{ if } l' = r \text{ and } r' > l, \text{ choose} \\ l+3 \leq k_l \leq r \text{ s.t. } \mathrm{ODIL}(E') \text{ is minimal} & \text{(vi)} \\[6pt] E' = \mathrm{oE}(l, l', k_r, k) \cup \mathrm{oE}(k, k_l, r', r) \cup \{(p_r, p_l)\}, \text{ if } l' < r,\ r' > l \\ \text{and } l' \leq r', \text{ choose } l \leq k_r \leq k-2,\ l+2 \leq k \leq r-2 \text{ and} \\ k+2 \leq k_l \leq r \text{ s.t. } \mathrm{ODIL}(E') \text{ is minimal} & \text{(vii)} \end{cases}$$

  $E' \leftarrow \mathrm{oE}(1, l', r', n)$, choose l' and r' s.t. $\mathrm{ODIL}(E')$ is minimal
  return $G = (P, E' \cup \{(p_i, p_{i+1}) \mid 1 \leq i \leq n-1\})$ // add baseline

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3   Two-Dimensional Point Sets

In the one-dimensional case, we have seen that a 1-spanner exists for every point set, though it is not plane. In two dimensions, the complete bi-directed graph is always a directed 1-spanner. This is not the case for oriented dilation. There we have point sets where no 1-spanner exists, even more, no 1.46-spanner exists:

▶ **Observation*** **12.** *There are two-dimensional point sets for which no oriented t-spanner exists for $t < 2\sqrt{3} - 2 \approx 1.46$.*

  While an oriented complete graph does not lead to a 1-spanner, it does yield a 2-spanner:

▶ **Proposition*** **13.** *For every point set an oriented 2-spanner can be constructed by orienting a complete graph.*

3.1 Hardness

We now have shown that although a 1-spanner does not exist for every two-dimensional point set, we get a 2-spanner via the oriented complete graph. However, this leads to a graph with $\frac{1}{2}n \cdot (n-1)$ edges, so this graph is neither sparse nor plane.

We will see that computing a sparse minimum oriented spanner is NP-hard:

▶ **Theorem* 14.** *Given a two-dimensional set P of n points and the parameters t and m', it is NP-hard to decide if there is an oriented t-spanner $G = (P, E)$ with $|E| \leq m'$.*

For computing a minimum *plane* oriented spanner, i.e. a minimum oriented dilation triangulation, hardness remains open. In the undirected setting this question is a long-standing open problem [6, 14, 15]. To our knowledge, it is not even known whether a PTAS for this problem exists, and for several related open problems there is no FPTAS [15], unless $P = NP$. We show the following relative hardness result.

▶ **Observation 15.** *An FPTAS for minimum plane oriented spanner would imply an FPTAS for minimum dilation triangulation.*
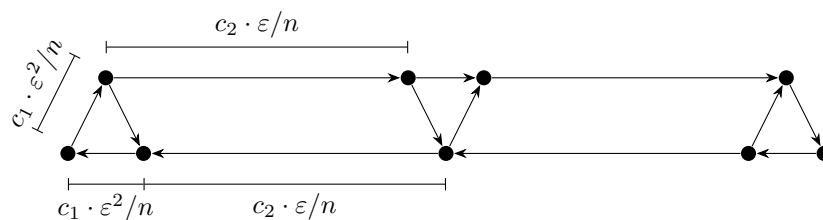
Figure 9 By replacing each point with this construction, we can reduce the minimum dilation triangulation problem to the minimum plane oriented spanner problem.

We sketch a reduction from the minimum dilation triangulation problem. Suppose we are given a set P of n points. The idea is to replace every point with the gadget depicted in Figure 9. The gadget consists of $3 \cdot \lceil \frac{4n}{3} \rceil$ points, positioned along a line in small triangles. Since the triangles are small relative to the distance between the triangles, we obtain an oriented dilation of less than $(1 + c\varepsilon)$ within the gadget by connecting the gadget points as in the figure, where $c > 0$ is a constant that we can choose by suitably picking c_1 and c_2.

The gadget leaves us with room for two edges to every one of n points, one in each direction, without disturbing planarity. Therefore, minimising the oriented dilation in this setting while requiring planarity should pick the edges that correspond to those in the minimum dilation triangulation, except if they are ε-close.

3.2 Greedy Triangulation

Given the relative hardness of computing minimum plane oriented spanners, we next investigate the question of whether plane oriented $\mathcal{O}(1)$-spanners exist. In the undirected setting, prominent examples for plane constant dilation spanners are the Delaunay triangulation and the greedy triangulation. Our main result on oriented spanners in two dimensions is that the greedy triangulation is an $\mathcal{O}(1)$-spanner for convex point sets (i.e. point sets for which the points lay in convex position).

The *greedy triangulation* of a point set is the triangulation obtained by considering all pairs of points by increasing distance, and by adding the straight-line edge if it does not intersect any of the edges already added. The greedy triangulation can be computed in linear time from the Delaunay triangulation [18], and in linear time for a convex point set if the order of the points along the convex hull is given [19].

It is known that the dilation of the undirected greedy triangulation is bounded by a constant. This follows from the fact that the greedy triangulation fulfils the α-diamond-property [11]. The currently best upper bound on the dilation t of such a triangulation is $t \leq \frac{8(\pi-\alpha)^2}{\alpha^2 \sin^2(\alpha/4)}$ with a lower bound on α of $\pi/6$ for the greedy triangulation [4].
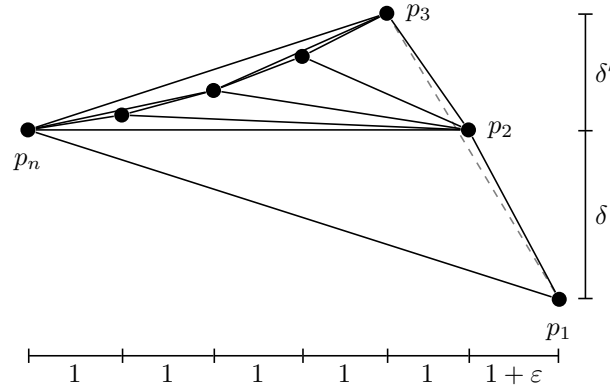
Figure 10 Greedy triangulation for a non-convex point set. It is is also a minimum weight triangulation.

We first observe that restricting to convex point sets is necessary to obtain constant dilation from orienting the greedy triangulation. For this, consider the greedy triangulation $T = (P, E)$ on the following non-convex point set $P = \{p_1, \ldots, p_n\}$. As shown in Figure 10, we place the points $\{p_3, \ldots, p_n\}$ on a very flat parabola. By arbitrarily decreasing the y-distances $\delta, \delta' > 0$, we reduce the construction to an almost one-dimensional problem. The x-distance of each consecutive point pair p_i, p_{i+1} is 1 for $2 \leq i < n$. We place p_1 slightly right of p_2 such that the x-distance is $1 + \varepsilon$ for a small $\varepsilon > 0$. Therefore, the x-distance of p_1 to p_i is ε-larger than x-distance of p_2 to p_{i+1} for $3 \leq i < n$. By this, the greedy triangulation added $(p_2, p_{i+1}) \in E$ and discarded $(p_1, p_i) \notin E$ for $3 \leq i < n$. Further, we place p_1 slightly below p_2 so that, due to planarity, p_1 is only adjacent to p_2 and p_n. Since p_1 has degree 2, for any orientation of T, every shortest oriented cycle containing p_1 contains the subpath p_n, p_1, p_2 or vice versa. From this construction, it follows that $\mathrm{odil}(p_1, p_3) \geq \frac{n+\varepsilon}{2+\varepsilon}$, regardless of the orientation of T. Therefore, every orientation of greedy triangulation has oriented dilation $\Omega(n)$.

In contrast, for convex point sets, the greedy triangulation results in a $\mathcal{O}(1)$-spanner. For this, we use a *consistent* orientation of the edges. This means each face of an oriented triangulation is confined by an oriented cycle. If such an orientation exists for a given triangulation, we call it an *orientable triangulation*. Since the dual graph of a triangulation of a convex point set is a tree, it is orientable.

▶ **Theorem 16.** *By orienting the greedy triangulation of a convex two-dimensional point set P consistently, we get a plane oriented $\mathcal{O}(1)$-spanner for P.*

Proof. Let $T = (P, E)$ be the greedy triangulation of P and $G = (P, \overrightarrow{E})$ its consistent orientation (which is unique up to reversing all edges). Note that T is an undirected graph, whereas G is directed. To improve readability, undirected edges are written with curly

brackets and directed edges with round brackets. Due to α-diamond property, the undirected dilation of any greedy triangulation can be bounded by a constant. Let t_g be the (smallest such) constant.

We distinguish between i) $\{p, p'\}$ is in E or ii) not.

For i), we prove that there is a path from p to p' in $T - \{p, p'\}$ of length in $\mathcal{O}(|\Delta(p, p')|)$, where $\Delta(p, p')$ is a smallest triangle in the complete graph incident to p and p'. By that, we bound the length of the shortest oriented cycle $C_G(p, p')$.

Let $q \in P$ be the third point incident to $\Delta(p, p')$. Let Π be the undirected path from p to q in T with first edge $\{p, q'\}$, w.l.o.g. $q' \neq p'$. (If $q' = p'$, then the proof would be the same with switched roles for p and p'.)

If $\{q', p'\} \in E$, then $\Delta_{pq'p'} \in E$. Regardless of the orientation of $\{p, q'\}$, we can bound $|C_G(p, p')| \leq |\Delta_{pq'p'}| \leq 2 \cdot t_g \cdot |\Delta(p, p')|$. However, $\{q', p'\}$ could be blocked by another edge. Since P is convex and T is planar, this edge is incident to p. Analogous to Theorem 10, we show that there could be a "sequence of blockers" (see Definition 9), but that there is path from p to q of length $\mathcal{O}(|p - q|)$.

Let b_{j+2c} be such that $\{q', b_{j+2c}\} \in E$ and $\{q', b_{j+2c}\}$ blocks $\{q', p'\}$. Let the points $b_1, \ldots, b_{j+2c}$ be ordered such that $\{p, b_i\} \in E$ is the shortest edge that blocks $\{q', b_{i-1}\}$ with $b_0 = p'$ (see Figure 11a).

The following inequalities hold for these blockers:

$$|p - b_i| \leq |q' - b_{i-1}|, \tag{1}$$
$$|p - b_i| \leq |p - b_k| \text{ for } 1 \leq i < k \leq j + 2c, \text{ and} \tag{2}$$
$$|p - b_{i-1}| \leq |b_{i-2} - b_i| \text{ for } 3 \leq i \leq j + 2c. \tag{3}$$

Equation 2 is true, because otherwise $\{p, b_k\}$ would block $\{q', b_{i-1}\}$ instead of $\{p, b_i\}$.

Equation 3 is explained as follows: due to convexity and planarity, $\{p, b_{i+1}\} \in E$ implies $\{b_i, b_{i+2}\} \notin E$. This means the greedy algorithm added $\{p, b_{i+1}\}$ and discarded $\{b_i, b_{i+2}\}$ in later iteration (see Figure 11b).

Let γ be an arbitrary constant. We call an edge *long* if its length is larger than $|q' - p| \cdot \gamma$. Because of Equation 2, once an edge $\{p, b_i\}$ is long, every edge $\{p, b_k\}$ is also long, for $1 \leq i < k \leq j + 2c$. Let b_j be the point such that $\{p, b_j\}$ is the *last short* blocker, i.e.

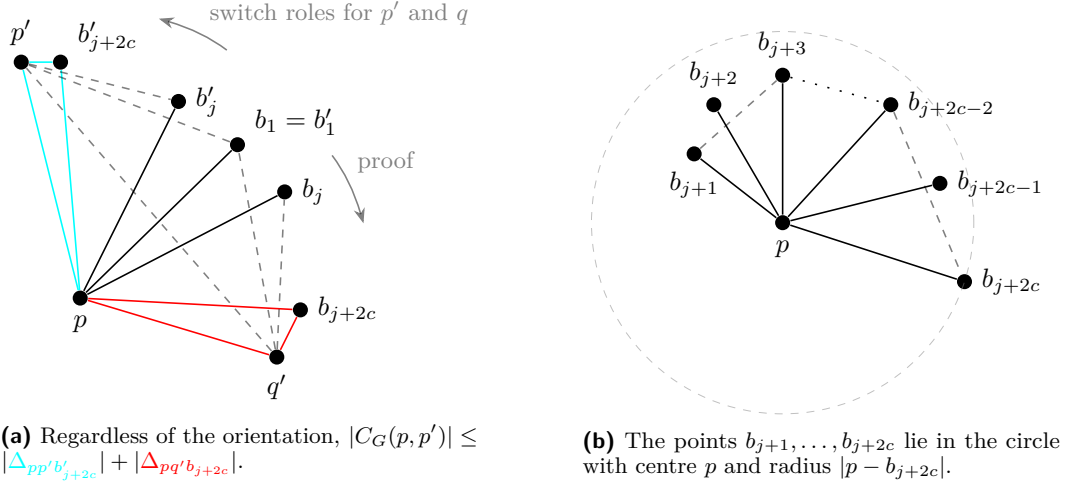$$|p - b_j| \leq |q' - p| \cdot \gamma \text{ and} \tag{4}$$
$$|p - b_k| > |q' - p| \cdot \gamma \text{ for all } j + 1 \leq k \leq j + 2c. \tag{5}$$

By this, the length of the last blocker $\{p, b_{j+2c}\}$ depends on γ and c:

$$
\begin{aligned}
|p - b_{j+2c}| &\overset{\text{eq. 1}}{\leq} |q' - b_{j+2c-1}| \overset{\substack{\text{triangle} \\ \text{inequality}}}{\leq} |q' - p| + |p - b_{j+2c-1}| \\
&\overset{\text{eq. 1}}{\leq} |q' - p| + |q' - b_{j+2c-2}| \overset{\substack{\text{triangle} \\ \text{inequality}}}{\leq} \ldots \leq 2c \cdot |q' - p| + |p - b_j| \\
&\overset{\text{eq. 4}}{\leq} |q' - p| \cdot (2c + \gamma).
\end{aligned}
\tag{6}
$$

Due to convexity, the points $b_{j+1}, \ldots, b_{j+2c}$ must be contained in the circle with centre p and radius $|p - b_{j+2c}|$ (see Figure 11b). Therefore, their pairwise distances are bounded by its circumference. Upper and lower bounding the sum of the pairwise distances of tuples b_{j+i}, b_{j+i+2} for $1 \leq i \leq 2c - 2$, it follows c is bounded by a function dependent on γ:

(a) Regardless of the orientation, $|C_G(p, p')| \leq |\Delta_{pp'b'_{j+2c}}| + |\Delta_{pq'b_{j+2c}}|$.

(b) The points $b_{j+1}, \ldots, b_{j+2c}$ lie in the circle with centre p and radius $|p - b_{j+2c}|$.

Figure 11 Visualisation of the central proof steps of case i) in Theorem 16.

$$2c \cdot \gamma \cdot |q' - p| \overset{\text{eq. 5}}{<} \sum_{i=2}^{2c-1} |p - b_{j+i}| \overset{\text{eq. 3}}{\leq} \sum_{i=1}^{2c-2} |b_{j+i} - b_{j+i+2}| \overset{\text{eq. 6}}{\leq} 2\pi(2c + \gamma) \cdot |q' - p|$$

$$\iff c < \pi \cdot \frac{\gamma - 1}{\gamma - 2\pi}.$$

For $\gamma = \mathcal{O}(1)$, it holds that $c = \mathcal{O}(1)$. Thus, the length of the last blocker $\{p, b_{j+2c}\}$ is bounded by a constant times $|p - q'|$.

This proof can be repeated with switched roles for p' and q. Then, the points $b'_1, \ldots, b'_{j+2c}$ are ordered such that $\{p, b'_i\}$ blocks $\{p', b_{i-1}\}$ with $b_0 = q'$ (Figure 11a).

By definition of the point set $b_1, \ldots, b_{j+2c}$ (respectively $b'_1, \ldots, b'_{j+2c}$), it holds that $\{q', b_{j+2c}\} \in E$ and $\{p', b'_{j+2c}\} \in E$. Regardless of the orientation, the length of the shortest oriented cycle is bounded by $|C_G(p, p')| \leq |\Delta_{pq'b_{j+2c}}| + |\Delta_{pp'b'_{j+2c}}|$. This permits a constant dilation for the case $\{p, p'\} \in E$ as

$$
\begin{aligned}
\operatorname{odil}(p, p') = \frac{|C_G(p, p')|}{|\Delta(p, p')|} &\leq \frac{|\Delta_{pq'b_{j+2c}}| + |\Delta_{pp'b'_{j+2c}}|}{|\Delta_{pp'q}|} \\
&\leq \max\left\{ \frac{|\Delta_{pq'b_{j+2c}}|}{|\Delta_{pp'q}|}, \frac{|\Delta_{pp'b'_{j+2c}}|}{|\Delta_{pp'q}|} \right\} \overset{\text{wlog}}{=} \frac{|p - q'| + |q' - b_{j+2c}| + |b_{j+2c} - p|}{|\Delta_{pp'q}|} \\
&\overset{\text{triangle inequality}}{\leq} \frac{2 \cdot |p - q'| + 2 \cdot |p - b_{j+2c}|}{|\Delta_{pp'q}|} \\
&\overset{\text{eq. 6}}{\leq} \frac{2 \cdot |p - q'| + 2 \cdot |p - q'| \cdot (2c + \gamma)}{|\Delta_{pp'q}|} = \frac{|p - q'| \cdot (4c + 2\gamma + 2)}{|p - p'| + |p' - q| + |q - p|} \\
&\overset{\alpha\text{-diamond-property}}{\leq} \frac{t_g \cdot |p - q| \cdot (4c + 2\gamma + 2)}{|\Delta_{pp'q}|} \leq t_g \cdot (4c + 2\gamma + 2) = \mathcal{O}(1).
\end{aligned}
$$

For ii) $\{p, p'\} \notin E$, due to α-diamond property, there is an undirected path Π from p to p' in T of length $|\Pi| \leq t_g \cdot |p - p'|$. Applying case i) gives factor $\lambda = \mathcal{O}(1)$ for each path edge $\{q', q''\} \in \Pi$. The dilation for the case $\{p, p'\} \notin E$ is bounded by
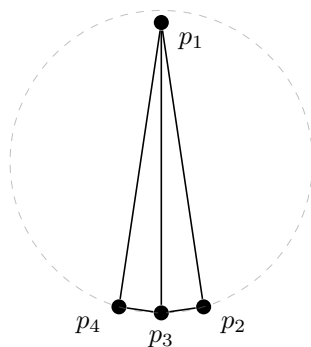
Figure 12 Delaunay triangulation for convex point set. It is also a minimum dilation triangulation.

$$\mathrm{odil}(p, p') = \frac{|C_G(p, p')|}{|\Delta(p, p')|} \leq \frac{\sum_{\{q', q''\} \in \Pi} |C_G(q', q'')|}{|\Delta(p, p')|}$$

$$\overset{\text{case i)}}{\leq} \frac{\lambda \cdot |\Pi|}{|\Delta(p, p')|} \leq \frac{\lambda \cdot t_g \cdot |p - p'|}{|\Delta(p, p')|} = \lambda \cdot t_g = \mathcal{O}(1). \qquad \blacktriangleleft$$

3.3 Other Triangulations

As the greedy triangulation leads to a plane oriented $\mathcal{O}(1)$-spanner for convex point sets, the question arises whether the α-diamond property implies the existence of oriented spanners.

We already have seen that this is not the case for the greedy triangulation on general point sets. Likewise, the minimum weight triangulation has the α-diamond property [17] but does not yield an $\mathcal{O}(1)$-spanner in general. This can be seen by the same example as for the greedy triangulation in Figure 10. By essentially the same argument, the minimum weight triangulation will have the same edges incident to p_1, which results in an oriented dilation of $\Omega(n)$. Whether the minimum weight triangulation is an oriented $\mathcal{O}(1)$-spanner for convex point sets, we leave as an open problem.

For the Delaunay triangulation the situation is even worse. The Delaunay triangulation has the α-diamond property [16] and is the basis for many undirected plane spanner constructions [5]. However, for $n \geq 4$ its oriented dilation can be arbitrary large, no matter how we orient its edges, even for convex point sets.

Figure 12 shows the Delaunay triangulation $T = (P, E)$ of a convex point set of size 4. The shortest cycle $C_G(p_2, p_4)$ contains p_1, for any orientation of T. We can place p_2, p_3, p_4 arbitrarily close to each other without decreasing the radius of the circle through them. By placing p_1 in the circle and sufficiently far away from p_2, p_3 and p_4, the oriented dilation $\mathrm{odil}(p_2, p_4) \geq \frac{|C_G(p_2, p_4)|}{|\Delta_{p_2 p_3 p_4}|}$ can be made arbitrary large. The example can be modified to include more points without changing $\mathrm{odil}(p_2, p_4)$.

Finally, we consider orienting the undirected minimum dilation triangulation. However, essentially the same example as for the Delaunay triangulation shows that there are convex point sets for which any orientation has arbitrarily large oriented dilation. In Figure 12, consider the case in which we have fixed all positions except for the y-coordinates of p_2 and p_4. Let y_0 be the y-coordinate of p_3, and let $y_0 + \varepsilon$ be the y-coordinate of p_2 and p_4. By decreasing $\varepsilon > 0$, we can make sure that the minimum dilation triangulation chooses the same diagonal as the Delaunay triangulation, since the undirected dilation between p_2 and p_4 by the path through p_3 becomes arbitrarily close to 1. However, as in the Delaunay triangulation, the oriented dilation between p_2 and p_4 can be made arbitrarily large.

Thus, the quest for a triangulation of small oriented dilation for non-convex point sets remains open.

4   Conclusion and Outlook

Motivated by applications of geometric spanners, we introduced the concept of oriented geometric spanners. We provided a wide range of extremal and algorithmic results for oriented spanners in one and two dimensions.

Intriguingly, orienting the greedy triangulation yields a plane $\mathcal{O}(1)$-spanner for point sets in convex position, but not for general point sets. Furthermore, other natural triangulations like the Delaunay and the minimum weight triangulation do not lead to plane constant dilation spanners.

This raises the question of whether a plane constant dilation spanner exists and can be computed efficiently. If this is not possible, what is the lowest dilation that we can guarantee? Until now, even showing whether the greedy triangulation yields an $\mathcal{O}(n)$-spanner remains open.

As the concept of oriented spanners is newly introduced, it opens up many new avenues of research, for example:

- We know that the minimum one-page plane oriented spanner achieves a dilation of at most 5 (since this is the bound for the greedy algorithms) and that there are point sets where it has a dilation of 2. What is its worst-case dilation $2 \le t \le 5$? Can we compute it faster?
- We constructed a two-page plane oriented 2-spanner for any one-dimensional point set. Is 2 a tight upper bound on the dilation of a minimum two-page plane spanner? Is there an efficient algorithm to compute such a spanner?
- For two-dimensional point sets, the question of bounding minimum oriented dilation already arises without restricting to plane graphs. What is the worst-case dilation $2\sqrt{3} - 2 \le t \le 2$ of the minimum dilation oriented complete graph?
- Given an undirected geometric graph, can we efficiently compute an orientation minimising the dilation? For which graph classes is this possible?
- While undirected dilation compares the shortest path from p to p' in G with the edge between them in the complete graph K_n, oriented dilation compares the shortest oriented cycle through p to p' in G to the shortest triangle in K_n. An analogous measure in undirected graphs, *cyclic dilation*, would compare the shortest simple cycle in G to the shortest triangle in K_n. Can we efficiently compute sparse graphs, in particular triangulations, of low cyclic dilation? Another measure of interest would be *detour dilation*, which compares the shortest path not using the edge $\{p, p'\}$ in G (i.e. the shortest path in $G - \{p, p'\}$) with the triangle in K_n. Low detour dilation is a necessary condition for low oriented dilation, and actually at the core of our analysis of the greedy triangulation. Is there a triangulation with constant detour dilation? Can it be oriented to obtain constant oriented dilation?
- In the applications mentioned, it is desirable to reduce the number of bi-directional edges but it may not be necessary to avoid them completely. This opens up a whole new set of questions on the trade-off between directed dilation and the number of bi-directional edges. For instance, given a parameter t, compute the directed t-spanner with as few bi-directional edges as possible (and no bound on the number of oriented edges). Or, given a plane graph with certain edges marked as *one-way*, compute the orientation of these edges that minimises the directed dilation (while all other edges are bi-directional). For which families of graphs can this be done efficiently? For which is this problem NP-hard?

References

1   Hugo A. Akitaya, Ahmad Biniaz, and Prosenjit Bose. On the spanning and routing ratios of the directed Θ_6-graph. *Comput. Geom.*, 105-106:101881, 2022. `doi:10.1016/j.comgeo.2022.101881`.

2   Boris Aronov, Kevin Buchin, Maike Buchin, Bart Jansen, Tom De Jong, Marc van Kreveld, Maarten Löffler, Jun Luo, Bettina Speckmann, and Rodrigo Ignacio Silveira. Connect the dot: Computing feed-links for network extension. *J. Spatial Information Science*, 3:3–31, 2011. `doi:10.5311/JOSIS.2011.3.47`.

3   Michael A. Bekos, Martin Gronemann, and Chrysanthi N. Raftopoulou. Two-page book embeddings of 4-planar graphs. *Algorithmica*, 75(1):158–185, 2016. `doi:10.1007/s00453-015-0016-8`.

4   Prosenjit Bose, Aaron Lee, and Michiel H. M. Smid. On generalized diamond spanners. In *Proc. 10th Internat. Sympos. Algorithms and Data Struct.*, volume 4619 of *Lecture Notes in Computer Science*, pages 325–336. Springer-Verlag, 2007. `doi:10.1007/978-3-540-73951-7_29`.

5   Prosenjit Bose and Michiel Smid. On plane geometric spanners: A survey and open problems. *Comput. Geom. Theory Appl.*, 46(7):818–830, 2013. `doi:10.1016/j.comgeo.2013.04.002`.

6   Aléx F. Brandt, Miguel M. Gaiowski, Cid C. de Souza, and Pedro J. de Rezende. Minimum dilation triangulation: Reaching optimality efficiently. In *Proc. 26th Canad. Conf. Comput. Geom. (CCCG)*, 2014.

7   Martin Burkhart, Pascal Von Rickenbach, Rogert Wattenhofer, and Aaron Zollinger. Does topology control reduce interference? In *Proc. 5th ACM Internat. Sympos. Mobile Ad Hoc Networking and Computing*, pages 9–19, 2004. `doi:10.1145/989459.989462`.

8   Fan R. K. Chung, Frank Thomson Leighton, and Arnold L. Rosenberg. Embedding graphs in books: A layout problem with applications to vlsi design. *SIAM J. Algebraic Discrete Methods*, 8(1):33–58, 1987. `doi:10.1137/0608002`.

9   Lenore J. Cowen and Christopher G. Wagner. Compact roundtrip routing for digraphs. In *Proc. 10th Annu. ACM-SIAM Sympos. Discrete Algorithms (SODA)*, pages 885–886. SIAM, 1999. `doi:10.5555/314500.315068`.

10 Lenore J. Cowen and Christopher G. Wagner. Compact roundtrip routing in directed networks. *J. Algorithms*, 50(1):79–95, 2004. `doi:10.1016/j.jalgor.2003.08.001`.

11 Gautam Das and Deborah Joseph. Which triangulations approximate the complete graph? In *Optimal Algorithms*, volume 401 of *Lecture Notes in Computer Science*, pages 168–192. Springer-Verlag, 1989. `doi:10.1007/3-540-51859-2_15`.

12 Andrew Dobson and Kostas E. Bekris. Sparse roadmap spanners for asymptotically near-optimal motion planning. *Internat. J. Robotics Research*, 33(1):18–47, 2014. `doi:10.1177/0278364913498`.

13 Vida Dujmovic and David R. Wood. On linear layouts of graphs. *Discret. Math. Theor. Comput. Sci.*, 6(2):339–358, 2004. `doi:10.46298/dmtcs.317`.

14 David Eppstein. Spanning trees and spanners. In *Handbook of Computational Geometry*, pages 425–461. Elsevier, 2000. `doi:h10.1016/B978-044482537-7/50010-3`.

15 Panos Giannopoulos, Rolf Klein, Christian Knauer, Martin Kutz, and Dániel Marx. Computing geometric minimum-dilation graphs is NP-hard. *Internat. J. Comput. Geom. Appl.*, 20(2):147–173, 2010. `doi:10.1142/S0218195910003244`.

16 Robert L. (Scot) Drysdale III, Scott A. McElfresh, and Jack Snoeyink. On exclusion regions for optimal triangulations. *Discret. Appl. Math.*, 109(1-2):49–65, 2001. `doi:10.1016/S0166-218X(00)00236-5`.

17 J. Mark Keil and Carl A. Gutwin. Classes of graphs which approximate the complete euclidean graph. *Discrete Comput. Geom.*, 7:13–28, 1992. `doi:10.1007/BF02187821`.

18 Christos Levcopoulos and Drago Krznaric. The greedy triangulation can be computed from the Delaunay triangulation in linear time. *Comput. Geom. Theory Appl.*, 14(4):197–220, 1999. `doi:10.1016/S0925-7721(99)00037-1`.

19   Christos Levcopoulos and Andrzej Lingas. Fast algorithms for greedy triangulation. *BIT*, 32(2):280–296, 1992. `doi:10.1007/BF01994882`.

20   Giri Narasimhan and Michiel Smid. *Geometric Spanner Networks*. Cambridge University Press, 2007. `doi:10.1017/CBO9780511546884`.

21   Christian Schindelhauer, Klaus Volbert, and Martin Ziegler. Geometric spanners with applications in wireless networks. *Comput. Geom. Theory Appl.*, 36(3):197–214, 2007. `doi:10.1016/j.comgeo.2006.02.001`.

22   Mihalis Yannakakis. Embedding planar graphs in four pages. *J. Comput. Syst. Sci.*, 38(1):36–67, 1989. `doi:10.1016/0022-0000(89)90032-9`.