Coloring Tournaments with Few Colors: Algorithms and Complexity

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Abstract

A *k-coloring* of a tournament is a partition of its vertices into k acyclic sets. Deciding if a tournament is 2-colorable is NP-hard. A natural problem, akin to that of coloring a 3-colorable graph with few colors, is to color a 2-colorable tournament with few colors. This problem does not seem to have been addressed before, although it is a special case of coloring a 2-colorable 3-uniform hypergraph with few colors, which is a well-studied problem with super-constant lower bounds.

We present an efficient decomposition lemma for tournaments and show that it can be used to design polynomial-time algorithms to color various classes of tournaments with few colors, including an algorithm to color a 2-colorable tournament with ten colors. For the classes of tournaments considered, we complement our upper bounds with strengthened lower bounds, painting a comprehensive picture of the algorithmic and complexity aspects of coloring tournaments.

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1 Introduction

A tournament $T = (V, A)$ is a complete, oriented graph: For each pair of vertices $i, j \in V$, there is either an arc from i to j or an arc from j to i (but not both). A subset of vertices $S \subseteq V$ induces the *subtournament* $T[S]$. If this subtournament contains no directed cycles, then it is said to be *acyclic*. The problem of *coloring a tournament* is that of partitioning the vertices into the minimum number of acyclic sets, sometimes referred to as the *dichromatic number* [32]. Since a tournament contains a directed cycle if and only if it contains a directed triangle, the problem of coloring a tournament is equivalent to partitioning the vertices into the minimum number of sets so that each set does not contain a directed triangle.

Coloring tournaments can be compared to the problem of coloring undirected graphs. For the latter, deciding if a graph is 2-colorable (i.e., bipartite) is easy, but it is NP-hard to decide if a graph is 3-colorable. A widely-studied promise problem is that we are given a graph promised to be 3-colorable and the goal is to color it (in polynomial time) with few colors [34, 5, 23, 24]. For tournaments, it is easy to decide whether or not a tournament is 1-colorable (i.e., transitive), since this is exactly when the tournament is acyclic. However, deciding if a tournament is 2-colorable is already NP-hard [8].

This suggests the following promise problem: Given a tournament promised to be 2-colorable, what is the fewest number of colors with which it can be colored in polynomial time? This question is the starting point for this paper and naturally leads to related problems of determining upper and lower bounds for coloring various classes of tournaments. For

Table 1 Best known lower and upper bounds for various graph coloring problems. All inapproximability results are under the assumption $\mathsf{P} \neq \mathsf{NP}$ except those denoted by *, which are under the d-To-1 Conjecture [26]. The lower bound should be read as, "It is hard to color a 3-colorable graph with 5 colors." The upper bound as, "A 3-colorable graph can be (efficiently) colored with $\tilde{O}(n^{0.19996})$ colors."

Graph Type	Lower Bound	Upper Bound
3-Colorable graphs	5 [6], $O(1)^*$ [18]	$\tilde{O}(n^{0.19996})$[24]
k-Colorable graphs, $k \geq 3$	$2k-1$ [6], $O(1)^*$[18]	$O(n^{1-\frac{3}{k+1}})$ [23]
General graphs	$n^{1-\epsilon}$ [22, 35]	$O(n(\log\log n)^2(\log n)^{-3})$ [19]
3-Uniform 2-colorable hypergraphs	$O(1)$ [11]	$\tilde{O}(n^{\frac{1}{5}})$ [29]

comparison, the complexity landscape of graph coloring is well studied and we have a general understanding of what it looks like. (See Table 1.) In contrast, the problem of coloring tournaments has been studied very little from the algorithmic or complexity perspective. This paper is an effort to address this disparity.

1.1 Previous Work

The problem of coloring a 2-colorable tournament with few colors is a special case of coloring a 2-colorable 3-uniform hypergraph with few colors. Deciding if a 3-uniform hypergraph is 2-colorable is NP-hard [31] and more recently it was proved to be NP-hard to color with any constant number of colors [11]. On the positive side, a 2-colorable 3-uniform hypergraph can be colored in polynomial time with $\tilde{O}(n^{1/5})$ colors [1, 7, 29], a result which uses tools from and is analogous to that of [23] for 3-colorable graphs. Thus, $\tilde{O}(n^{1/5})$ is the best-known upper bound on the number of colors needed to efficiently color a 2-colorable tournament. Deciding if a tournament is 2-colorable is NP-hard [8] and furthermore, deciding if a tournament is k-colorable for any $k \geq 2$ is NP-hard [15]. It is consistent with these results that we can, say, efficiently color a 2-colorable tournament with three colors.

From a structural graph theory perspective, the problem of coloring tournaments has been widely studied due to its connection to the famous Erdős-Hajnal Conjecture [12, 9], which has an equivalent formulation in terms of tournaments [2]. The latter posits that for any tournament H, there is a constant ϵ_H (where $0 < \epsilon_H \leq 1$) such that any H-free tournament on n vertices has a transitive subtournament of size at least $O(n^{\epsilon_H})$. [4] exactly characterize the tournaments for which $\epsilon_H = 1$, which they call *heroes*. Forbidding a hero in a tournament T actually results in T being colorable with a constant number of colors [4], which yields a transitive induced subtournament of linear size. These results are existential and do not provide an efficient algorithm to color an H-free tournament with a constant number of colors, when H is some fixed hero.

1.2 Our Results

We consider some basic algorithmic and computational complexity questions on the subject of coloring tournaments. Our main algorithmic tool, presented in Section 2, is a decomposition lemma which can be used to obtain efficient algorithms for coloring tournaments in various cases when certain conditions are met. On a high level, it bears some resemblance to decompositions previously used to prove bounded dichromatic number in tournaments and in dense digraphs with forbidden subgraphs [4, 20]. To apply our decomposition lemma to 2-colorable tournaments, we use an observation used by [1, 7, 29] which states that

Table 2 Best known polynomial time inapproximability results and approximation algorithms for various tournament coloring problems. Previous results are indicated with a citation. All the results without a citation are established in this paper. Lower bounds are under the assumption $\mathsf{P} \neq \mathsf{NP}$ except those marked with a *, which hold under the d-To-1 Conjecture [26]. The function $g(k)$ denotes the number of colors needed to efficiently color a k-colorable graph, while $f(k)$ is the number of colors needed to efficiently color a k-colorable tournament. The entry indicated by † is a hardness of approximation result.

Tournament Type	Lower Bound	Upper Bound
2-Colorable tournaments	2[8], 3	10
3-Colorable tournaments	5, $O(1)$ *	$\tilde{O}(n^{0.19996})$
k-Colorable tournaments, $k \geq 2$	$2k-1$, $O(1)$ *	$5 \cdot f(k-1) \cdot g(k)$
2-Colorable light tournaments	in P?	5
Light tournaments	in P?	9
General tournaments	$n^{\frac{1}{2}-\epsilon}$ †	$n/\log n$[13]

there is an efficient algorithm to partition a 2-colorable tournament into two tournaments that are each light. A *light tournament* is one in which for each arc uv, the set of vertices $N(uv) = \{w \mid uvw \text{ forms a directed triangle}\}$ is transitive. (Let C_3 denote a directed triangle. A light tournament is H-free where H is the hero $(C_3, 1, 1)$.)

In fact, due to this observation and the fact that [4] showed that light tournaments have constant dichromatic number, it cannot be NP-hard (unless NP= co-NP) to color a 2-colorable tournament with $O(1)$ colors. (This does not however immediately imply that there is an efficient algorithm, since there are many search problems that are believed to be intractable even though their decision variant is easy, e.g., those in the class TFNP.) Although [4] did not provide an efficient algorithm to color a light tournament with a constant number of colors, a careful modification of their techniques indeed results in a polynomial-time algorithm using around 35 colors to color a light tournament.

Like some other lemmas which show that the dichromatic number of a tournament is bounded (i.e., constant) if the out-neighborhoods of vertices have bounded dichromatic number [21], our decomposition lemma also has a local-to-global flavor: If the sets $N(uv)$ can be efficiently colored with few colors for all arcs uv and if there are two vertices s and t such that the out-neighborhood of s and the in-neighborhood of t can be efficiently colored with few colors, then our decomposition lemma yields an efficient algorithm to color the whole tournament with few colors.

We give applications of our algorithmic decomposition lemma in Section 3. Specifically, we show that 2-colorable tournaments can be efficiently colored with ten colors and that light tournaments can be efficiently colored with nine colors. We then use our toolbox to study 3-colorable tournaments. Here we show that the problem of coloring a 3-colorable tournament has a constant-factor reduction to the problem of coloring 3-colorable graphs.

Next, we strengthen the lower bounds by showing in Section 4 that it is NP-hard to color a 2-colorable tournament with three colors. We then give a reduction from coloring graphs to coloring tournaments, which implies, for example, that it is hard to color 3-colorable tournaments with $O(1)$ colors under the d-To-1 Conjecture of Khot [26]. Finally, we show that it is NP-hard to approximate the number of colors required for a general tournament to within a factor of $O(n^{1/2-\epsilon})$ for any $\epsilon > 0$. Our results are summarized in Table 2.

1.3 Notation and Preliminaries

Let $T = (V, A)$ be a tournament with vertex set V and arc set A. Sometimes, we use $V(T)$ to denote its vertex set and $A(T)$ to denote its arc set. For $S \subset V$, we use $T[S]$ to denote the subtournament induced on vertex set S, although we sometimes abuse notation and refer to the subtournament itself as S. We define $uv \in A$ to be an arc directed from u to v. We define $N^+(v)$ to be all $w \in V$ such that arc $vw \in A$ and $N^-(v)$ to be all $w \in V$ such that arc $wv \in A$. We let $N^+[v] = N^+(v) \cup \{v\}$ and $N^-[v] = N^-(v) \cup \{v\}$. For $S \subset V$, we define $N^+(S) = \bigcup_{v \in S} N^+(v)$, and we define $N^-(S), N^+[S], N^-[S]$ analogously. We use $N^{\pm}(S)$ to denote vertices in $V \setminus S$ that have at least one in-neighbor and at least one out-neighbor in S. Sometimes we refer to $N^{\pm}(S)$ of a set as its *mixed neighborhood*.

For $S, U \subset V$ such that $S \cap U = \emptyset$, we use $S \Rightarrow U$ to indicate that all arcs between S and U are directed from S to U. Let C_3 denote a directed triangle; usually, we refer to this simply as a triangle. Define $N(uv) \subset V$ to contain all vertices w such that uvw forms the directed triangle consisting of arcs uv, vw and wu. In other words, $N(uv) = N^-(u) \cap N^+(v)$. For three tournaments T_1, T_2 and T_3, we use $\Delta(T_1, T_2, T_3)$ to denote the tournament resulting from adding all arcs from T_1 to T_2, all arcs from T_2 to T_3 and all arcs from T_3 to T_1.

A tournament $T = (V, A)$ is *k-colorable* if there is a partition of V into k vertex-disjoint sets, $V_1, V_2, \dots, V_k$, such that $T[V_i]$ is transitive for all $i \in \{1, \dots, k\}$. We use $\vec{\chi}(T)$ to denote the *dichromatic number* of T (i.e., the minimum number of transitive subtournaments into which $V(T)$ can be partitioned). Computing the value $\vec{\chi}(T)$ is in general NP-hard [8]. We therefore use $\vec{\chi}_{\mathcal{C}}(T)$ to denote the number of colors by which T can be efficiently colored. Our goal is to find upper and lower bounds on $\vec{\chi}_{\mathcal{C}}(T)$.

We remark that we will always assume that a tournament T which we want to color is strongly connected; if this were not the case, we can color each strongly connected component separately. Therefore, each vertex has an out-neighborhood containing at least one vertex.

2 Efficient Tournament Decomposition for Coloring

We present a decomposition for a tournament that can be computed in polynomial time and yields an efficient method to color a tournament tournaments with few colors in certain cases.

▶ **Definition 1.** *We define a c-*vertex chain *$(v_i)_{0 \le i \le k}$ of a tournament T the following way: Let v_0 and v_k be a pair of vertices such that $\vec{\chi}_{\mathcal{C}}(N^+(v_0) \cup N^-(v_k)) \le c$, and let $(v_i)_{0 \le i \le k}$ be the vertices in the shortest directed path from v_0 to v_k.*

Additionally, we define an *arc chain* $(e_i)_{1 \le i \le k}$ corresponding to a vertex chain, where e_i is the arc from v_{i-1} to v_i. The main idea behind this decomposition is to build zones that can be efficiently colored, and such that all arcs between zones at distance more than four (i.e., *long* arcs) go backwards.

▶ **Definition 2.** *Given a c-vertex chain, a* path decomposition *of a tournament T is defined as:*

- $D_0 = N^+(v_0)$.
- *For* $1 \le i \le k$, $D_i = N(e_i) \setminus (\cup_{0 \le j \le i-1} D_j)$.
- $D_{k+1} = N^-(v_k) \setminus (\cup_{0 \le j \le k} D_j)$.

First we prove that this is indeed a decomposition of T.

▶ **Lemma 3.** *Let $T = (V, A)$ be a tournament and let $(D_0, \dots, D_{k+1})$ be a path decomposition of T. Then $V = \cup_{0 \le i \le k+1} D_i$.*

Proof. We will prove this lemma by contradiction: Suppose there is a vertex $w \in V$ that does not belong to any D_i. Assume that w does not belong to the vertex chain. Since w is neither in D_0 nor in D_{k+1}, then $w \in N^-(v_0)$ and $w \in N^+(v_k)$. Take the smallest integer i such that $w \in N^+(v_i)$. There must be one since $w \in N^+(v_k)$. Notice that $i \geq 1$ since $w \notin N^+(v_0)$, so e_i belongs to the arc chain and $w \in N(e_i)$. Therefore, $w \in D_i$, which is a contradiction.

Now consider the case in which w is in the vertex chain. An arc with both endpoints in the vertex chain that is not in the arc chain is backwards. Thus, $v_i \in N(e_{i+2})$ for all $0 \leq i \leq k-2$. Notice that v_{k-1} can belong to D_{k+1} (if it does not belong to D_j for some $j < k+1$). Finally, $v_k \in N(e_{k-1})$. ◀

We remark that, for the sake of simplicity and to more easily visualize the decomposition, it might be easier to not include the vertices in the vertex chain in the path decomposition. In this case, these vertices can be colored with two extra colors. Since all arcs not in the arc chain with both endpoints in the vertex chain go backwards (with respect to the arc chain; otherwise there would be an even shorter path), we can use two colors so that all forwards arcs (those in the arc chain) are bicolored.

▶ **Lemma 4.** *Let $0 \leq i, j \leq k+1$ and let $j \geq i+5$. For $u \in D_i$ and $w \in D_j$, we have $u \in N^+(w)$.*

Proof. We will prove this by contradiction. Suppose $j \geq i+5$ and $u \in N^-(w)$. Then there is a path of three arcs from v_i to v_{j-1}, namely (v_i, u, w, v_{j-1}). (By definition of the decomposition, $u \in D_i$ implies $u \in N^+(v_i)$ and $w \in D_j$ implies $w \in N^-(v_{j-1})$.) This is not possible since by the definition of the vertex chain as the shortest path, there can be no path between v_i and v_{j-1} with fewer than four arcs (since $(j-1)-i \geq (i+5-1)-i = 4$). ◀

▶ **Lemma 5.** *If T has a c-vertex chain that can be found in polynomial time and if $\vec{\chi}_C(N(e)) \leq c$ for each arc e in the corresponding arc chain, then $\vec{\chi}_C(T) \leq 5c$.*

Proof. Given a c-vertex chain, we construct a path decomposition. We make five palettes of c colors each with labels from 0 to 4. We color each D_i using the color palette with label i mod 5. Let us show that we can do this in polynomial time. First, note that the set of colors used is of size c for every D_i. Then, let us consider D_0: $N^+(v_0)$ can be colored efficiently with c colors by definition of a vertex chain. Similarly, D_{k+1} is a subset of $N^-(v_k)$ and can thus also be efficiently colored with c colors. Finally, for every $1 \leq i \leq k$, D_i is a subset of $N(e_i)$, which can be colored efficiently with c colors by the condition of the lemma.

Our goal is now to prove that this is a proper coloring of T. We will do this by showing that all forward arcs between different D_i are bicolored. By Lemma 4, there are no forwards arcs between D_i and D_j when $j \geq i+5$. Furthermore, by the definition of the coloring, no vertex in D_i and D_j can share a color for $i+1 \leq j \leq i+4$. Thus all forward arcs from D_i to D_j will be bicolored. Since every D_i is properly colored, and all forward arcs between different D_i are bicolored, T is properly colored. ◀

The next lemma has essentially the same proof as Lemma 5.

▶ **Lemma 6.** *If T has a c-vertex chain that can be found in polynomial time and if $\vec{\chi}_C(N(e)) \leq d$ for each arc e in the arc chain and if $c > d$, then $\vec{\chi}_C(T) \leq c + 4d$.*

Proof. We find the path decomposition using the c-vertex chain. We can color the set $S = D_0 \cup D_{k+1}$ with c colors and the remaining sets D_i for $1 \leq i \leq k$ with d colors each. For the last $c-d$ of the colors used for S, we can remove these vertices from S since these

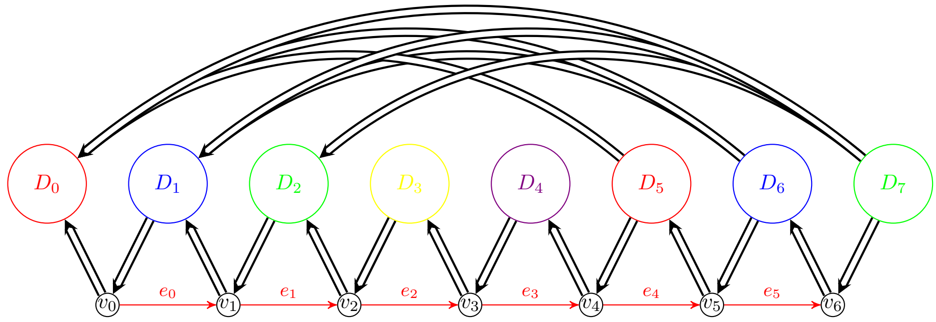

Figure 1 A path decomposition of T. The red arcs (e_i) form a shortest path from v_0 to v_k, thus all the arcs not depicted between the v_i's go backward. All the vertices in a given D_i are colored from the color palette indicated by the color of the D_i. Notice that because there are no long forward arcs between the D_i's, all arcs between D_i's that share a color palette are backwards.

colors will not be used again and call the remaining vertices in S (colored with the first d colors) S'. For the remaining vertices in S, we decompose them into $D_0 := D_0 \cap S'$ and $D_{k+1} := D_{k+1} \cap S'$ Now we have sets $D_0, D_1, \ldots, D_{k+1}$ each colored with d colors. We color these sets using five color palettes of d colors each and use the palette i mod 5 for set D_i. By Lemma 4, this does not create any monochromatic forward arcs. Thus, the total number of colors used is $(c - d) + 5d = c + 4d$. ◀

3 Algorithms for Coloring Tournaments

We consider various special cases of tournaments and show how to use our tools to color them with few colors.

3.1 2-Colorable Tournaments

A tournament $T = (V, A)$ is *2-colorable* if $\vec{\chi}(T) = 2$, and a 2-coloring of tournament T is a partition of V into two vertex sets, V_1 and V_2, such that $T[V_1]$ and $T[V_2]$ are each transitive. In this section, our goal is to prove Theorem 7.

▶ **Theorem 7.** *Let T be a 2-colorable tournament. Then $\vec{\chi}_{\mathcal{C}}(T) \leq 10$.*

We say an arc uv in A is *heavy* if there exist three vertices $a, b, c \in N(uv)$ which form a triangle abc. If a tournament contains no heavy arcs, then it is *light*. We will use the following observation.

▶ **Observation 8.** *Let T be a 2-colorable tournament. Then T can be partitioned into two light subtournaments T_1 and T_2 such that $\vec{\chi}_{\mathcal{C}}(T) \leq \vec{\chi}_{\mathcal{C}}(T_1) + \vec{\chi}_{\mathcal{C}}(T_2)$.*

This observation appears in [1, 7, 29] where it is stated more generally for 2-colorable 3-uniform hypergraphs. We include a proof here for completeness.

▶ **Lemma 9.** *In a 2-coloring of a tournament T, each heavy arc must be 2-colored.*

Proof. If u and v are both, say, blue, then each vertex in $N(uv)$ would be red, forcing a triangle in $N(uv)$ to be all red (i.e., monochromatic), which is not possible in a 2-coloring. ◀

▶ **Corollary 10.** *In a 2-colorable tournament, the heavy arcs form a bipartite graph.*

Now we can prove Observation 8.

Proof of Observation 8. All heavy arcs can be easily detected. By Corollary 10, the set of heavy arcs forms a bipartite graph. The vertex set of this bipartite graph can be colored with two colors (red and blue), such that the tournament induced by each color does not contain a heavy arc. Then we partition the vertices into two sets one containing all the blue vertices and the other containing all the red vertices. The uncolored vertices can go in either set. Since neither of these sets contains any heavy arcs, we can partition the vertices of a 2-colorable tournament into two light subtournaments. ◀

Theorem 7 will follow from Observation 8 and the following theorem.

▶ **Theorem 11.** *Let T be a 2-colorable light tournament. Then $\vec{\chi}_{\mathcal{C}}(T) \leq 5$.*

Our goal it to use Lemma 5 to prove Theorem 11. In other words, we want to show that a 2-colorable light tournament has a 1-vertex chain. We first prove a useful claim.

▶ **Lemma 12.** *Let T be a k-colorable tournament. Then there exist vertices u and w such that $N^+(u) \cup N^-(w)$ is $(k-1)$-colorable.*

Proof. Since $T = (V, A)$ is k-colorable, there exist k transitive sets $X_1, \ldots, X_k$ such that $V = \cup_{i=1}^{k} X_i$. Then take u to be the vertex in X_1 that has only incoming arcs from other vertices in X_1 (i.e., the sink vertex for X_1). Similarly, take w to be the vertex in X_1 that has only outgoing arcs to other vertices in X_1 (i.e., the source vertex for X_1). The out-neighborhood of u and the in-neighborhood of w are both subsets of $V \setminus X_1$, and thus so is their union, which is therefore $(k-1)$-colorable. ◀

Now we are ready to prove that we can find a 1-vertex chain.

▶ **Lemma 13.** *Let T be a 2-colorable, light tournament. Then T contains a 1-vertex chain that can be found in polynomial time.*

Proof. By Lemma 12, there exist u and w such that $N^+(u) \cup N^-(w)$ is transitive. To find them, we can test the transitivity of $N^+(u) \cup N^-(w)$ for every pair of vertices in T. Then we simply need to find a shortest path from u to w, which can be done in polynomial time. Let k denote the length of the path, and define $v_0 = u$, $v_k = w$, and $(v_i)_{1 \leq i \leq k-1}$ the rest of the vertices in the path. ◀

The proof of Theorem 11 follows from Lemma 13, Lemma 5 and the fact that $\vec{\chi}_{\mathcal{C}}(N(e)) \leq 1$ for every arc e in a light tournament.

Certificates of Non-2-Colorability

In Section 3.1, we presented an algorithm to color a 2-colorable tournament with ten colors. Suppose we run this algorithm on an arbitrary tournament T (e.g., one that is *not* 2-colorable). Then our algorithm will either color T with ten colors or it will produce at least one certificate that T is not 2-colorable. A certificate will have the following form: either a) there is an odd cycle of heavy arcs in T, or b) for every ordered pair of vertices (u, v), the subtournament $T[N^+(u) \cup N^-(v)]$ is not transitive. In particular, an 11-chromatic tournament must contain such a certificate.

3.2 3-Colorable Tournaments

Coloring 3-colorable tournaments turns out to be closely related to coloring 3-colorable graphs. This seems surprising since the techniques for 3-colorable graphs were applied to coloring 2-colorable 3-uniform hypergraphs, which are a generalization of 2-colorable tournaments.

We will first show that we can adapt ideas of [34] and [5] to the problem of coloring 3-colorable tournaments by using our algorithm for coloring 2-colorable tournaments with ten colors as a subroutine.

▶ **Lemma 14.** *A 3-colorable tournament can be colored with $O(\sqrt{n})$ colors in polynomial time.*

Proof. Let $T = (V, A)$ be a 3-colorable tournament. Notice that T has at least three vertices each of whose out-neighborhoods is 2-colorable. To see this, consider any proper 3-coloring of T. Each color spans a transitive subtournament and each transitive subtournament has a sink vertex that has outgoing arcs only towards the other two colors.

For any vertex, if its out-neighborhood is 2-colorable, we can color its out-neighborhood with 10 colors by Theorem 7. So we can try to run the algorithm for the out-neighborhood of every vertex, and the algorithm will successfully produce a 10-coloring of the out-neighborhood of at least three vertices.

Therefore, if the minimum outdegree is at least $\sqrt{n}$, we find a transitive set of size at least $\sqrt{n}/10$. On the other hand, if the minimum outdegree is smaller than $\sqrt{n}$, we will make progress another way. In this case, let u be a vertex with outdegree smaller than $\sqrt{n}$. Then, we add u to a set S, and continue the algorithm on the subtournament of T induced on $V \setminus N^+[u]$. We continue this until we find a transitive subtournament of size at least $\sqrt{n}/20$ or until we have removed half the vertices. In the first case, we will have found a transitive set of size $\Omega(\sqrt{n})$, and in the second case, the set S will be transitive, and also of size $\Omega(\sqrt{n})$.

In conclusion, since we can find a transitive set of size $\Omega(\sqrt{n})$ in polynomial time, we can repeat the procedure recursively to find a coloring with $O(\sqrt{n})$ colors in polynomial time (see [5] for example). ◀

We can also use the decomposition of Section 2 to get a coloring with fewer colors based on a reduction to coloring 3-colorable graphs.

▶ **Theorem 15.** *If we can efficiently color a 3-colorable graph G with k colors, then we can efficiently color a 3-colorable tournament with $50k$ colors.*

Proof. Let $T = (V, A)$ be a 3-colorable tournament. For every arc $e \in A$, try coloring $N(e)$ with 10 colors using Theorem 7. If the algorithm fails, the neighborhood of the edge is not 2-colorable, and thus the edge is not monochromatic in any 3-coloring. Let $F \subset E$ denote the set of arcs whose neighborhoods cannot be colored with 10 colors using our algorithm. Ignore the direction of the arcs in F and consider the graph $G = (V, F)$. This graph must be 3-colorable, since no arc in F is monochromatic in any 3-coloring of T.

Now let us show that from a coloring of G with k colors, we can obtain a coloring of T with $50k$ colors. Consider a coloring of the graph $G = (V, F)$ and let V_i be the vertices colored with color i in this coloring. Consider the induced subtournament $T' = T[V_i]$; it has no arc in F and thus the neighborhood of every arc in this tournament can be colored efficiently with 10 colors. Furthermore, by Lemma 12 and Theorem 7, there are vertices u and v in T' such that $N^+_{T'}(u) \cup N^-_{T'}(v)$ is efficiently 10-colorable. So by Lemma 5, we can efficiently color T' with 50 colors. We can do this for the subtournament $T[V_i]$ for each of the i colors used to color G. ◀

Combining this Lemma with approximation algorithm [24], which colors a 3-colorable graph with fewer than $n^{\frac{1}{5}}$ colors, we obtain the same asymptotic bound for 3-colorable tournaments.

▶ **Corollary 16.** *Let T be a 3-colorable tournament on n vertices. Then, $\vec{\chi}_{\mathcal{C}}(T) \leq O(n^{0.19996})$.*

We can extend Theorem 15 to a more general case.

▶ **Lemma 17.** *Let f and g be functions such that we can efficiently color k-colorable graphs (respectively, k-colorable tournaments) with $g(k)$ (respectively, $f(k)$) colors. Then $f(k) \leq 5 \cdot f(k-1) \cdot g(k)$.*

Proof. We use the same reduction as in the proof of Theorem 15, but now F is the set of arcs whose neighborhoods cannot be efficiently $f(k-1)$-colored. Then each V_i in G is colored with $5 \cdot f(k-1)$ colors. So we need a total of $5 \cdot f(k-1) \cdot g(k)$ colors. ◀

3.3 Light Tournaments

Our goal in this section is to prove the following theorem.

▶ **Theorem 18.** *Let T be a light tournament. Then $\vec{\chi}_{\mathcal{C}}(T) \leq 9$.*

We will prove Theorem 18 by showing that every light tournament has a c-vertex chain for some constant c. To do this, we will find one vertex whose in-neighborhood we can color efficiently with a constant number of colors, and another whose out-neighborhood we can color efficiently with a constant number of colors. We will start by establishing some structural claims about light tournaments which are adapted from [4].

Throughout this section $T = (V, A)$ will denote a light tournament. Note that we do not assume that T is necessarily 2-colorable. Recall that a C_3 is a directed triangle.

▶ **Definition 19.** *Define a C_3-chain of length ℓ in T to be a set of ℓ vertex disjoint C_3's, $X = (X_1, X_2, X_3, \ldots, X_\ell)$, such that for each $i \in \{1, \ldots, \ell-1\}$, $X_i \Rightarrow X_{i+1}$.*

A *backwards arc* in a C_3-chain is an arc uv with $u \in X_i$ and $v \in X_j$ for $j < i$.

▶ **Lemma 20.** *A C_3-chain has no backwards arcs.*

This follows from the following claim.

▷ Claim 21. If $X = (X_1, X_2, \ldots, X_\ell)$ is a C_3-chain of length ℓ, then $X_i \Rightarrow X_j$ for $i < j$, where $1 \leq i < j \leq \ell$.

Proof. Notice that there are no arcs from X_{i+1} to X_i, since by definition of a C_3-chain, we have all arcs from X_i to X_{i+1}. Moreover, there is no arc uv from X_{i+2} to X_i since otherwise triangle X_{i+1} would appear in the neighborhood $N(uv)$, meaning that uv is heavy, which is a contradiction. This implies that all arcs go from X_i to X_{i+2} (since T is a tournament). Now suppose $j > i+2$. If there is a back arc uv from $u \in X_j$ to $v \in X_i$, then uv is a heavy arc, because X_{j-1} would be in $N(uv)$ since by induction we have all arcs from X_i to X_{j-1} and from X_{j-1} to X_j. ◁

Let us fix $X = (X_1, X_2, \ldots, X_\ell)$ to be a C_3-chain in T, and let $W = V(T) \setminus V(X)$. Initially, X can be of any length $\ell \geq 1$.

▷ Claim 22. For $w \in W$:
1. If $w \Rightarrow X_i$, then $w \Rightarrow X_j$ for all $j \geq i$.
2. If $X_i \Rightarrow w$, then $X_j \Rightarrow w$ for all $j \leq i$.

Proof. Suppose $w \Rightarrow X_i$ and there is an arc uw with $u \in X_j$ for $j > i$. Then uw is a heavy arc. Similarly, suppose $X_i \Rightarrow w$ and there is an arc wu with $u \in X_j$ for $j < i$, then wu is a heavy arc. ◁

We partition the vertices in W into zones $(Z_0, Z_1, \ldots, Z_\ell)$ using the following criteria. For $w \in W$, if i is the highest index such that $X_i \Rightarrow w$, then w is assigned to zone Z_i. If there is no such X_i, then w is assigned to zone Z_0.

Say a vertex $w \in W$ is *clear* if $w \Rightarrow X_i$ or $X_i \Rightarrow w$ for all X_i in H. Let $C \subseteq W$ be the set of clear vertices.

▷ Claim 23. If C is not transitive, we can extend X.

Proof. If the set $Z_i \cap C$ contains a triangle, then we can extend X by adding a new triangle to the chain between X_i and X_{i+1}.

If there is no i such that $Z_i \cap C$ contains a triangle, then we claim that C is transitive. This follows from the observation that there are no backwards arcs from $Z_j \cap C$ to $Z_i \cap C$ for $i < j$. Indeed, should such an arc uv from $Z_j \cap C$ to $Z_i \cap C$ exist, then $X_{i+1} \subset N(uv)$, so uv would be heavy. ◁

We say that X is a *maximal* C_3*-chain* if C is transitive. Let us also now define the *unclear* vertices U, where $U = W \setminus C$. In a maximal C_3-chain $X = (X_1, \ldots, X_\ell)$, notice that for a vertex $a \in X_1$, we have $N^-(a) \cap U \subseteq N^\pm(X_1)$. (This is because if a vertex $u \in N^-(a)$ has $u \Rightarrow X_i$, then u would be a clear vertex.)

▷ Claim 24. We can efficiently find two directed triangles $X_1 = abc$ and $X_\ell = xyz$ such that the set $S = \{v \mid v \Rightarrow X_1 \text{ or } X_\ell \Rightarrow v\}$ is transitive.

Proof. Find a maximal C_3-chain X and let ℓ be the length of this chain. Let $abc = X_1$ and $xyz = X_\ell$. The set of vertices $\{v \mid v \Rightarrow X_1 \text{ or } X_\ell \Rightarrow v\}$ is a subset of C and is therefore transitive. ◁

▷ Claim 25. Let xyz be a directed triangle. Then $\vec{\chi}_C(N^\pm(\{x, y, z\})) \leq 3$.

Proof. Each vertex $v \in N^\pm(\{x, y, z\})$ belongs to $N(xy), N(yz)$ or $N(zx)$. Since each of these sets is transitive, we conclude that $N^\pm(\{x, y, z\})$ can be colored with three colors. ◁

We can now easily prove Theorem 18, which is a corollary of Lemma 6 and the following lemma.

▶ **Lemma 26.** *Let T be a light tournament. Then T has a 5-vertex chain.*

Proof. Recall that for a vertex $a \in X_1$, we have $N^-(a) \cap U \subseteq N^\pm(X_1)$. If $X_1 = abc$, notice that for $v \in N^-(a) \cap U$, $v \notin N(ca)$. Thus, $N^-(a) \cap U \subseteq N(ab) \cup N(bc)$, which is efficiently 2-colorable. Making an analogous argument for $N^+(z) \cap U$, we conclude that $(N^+(z) \cup N^-(a)) \cap U$ is efficiently 4-colorable. The rest of the vertices in $N^+(z) \cup N^-(a)$ belong to the set S defined in Claim 24 and can be colored with one color. Therefore $\vec{\chi}_C(N^+(z) \cup N^-(a)) \leq 5$, so we can use z and a as the endpoints of a 5-vertex chain. ◀

The approach in this section can be extended to bound the chromatic number of a more general subclass of heroes. See the full version [28] for details.

4 Hardness of Approximate Coloring in Tournaments

In this section, we examine the hardness of approximate coloring of tournaments. [8] showed that deciding if a tournament can be 2-colored is NP-hard. Later, [15] proved that for any k, it is NP-hard to decide if a tournament is k-colorable.

We will first improve upon these NP-hardness results and then show hardness of coloring k-colorable tournaments for $k \geq 3$ with $O(1)$ colors under the d-To-1 conjecture. The d-To-1 conjecture was first introduced by Khot alongside the famous Unique Games conjecture [26], and has since been used to show hardness of coloring 3-colorable graphs with $O(1)$ colors [18].

First notice that the search problem must be at least as hard as its decisional equivalent.

▶ **Observation 27.** *Let $k < \ell$ be any two constants. If we can color k-colorable tournaments with ℓ colors, then we can distinguish k-colorable tournaments from tournaments with chromatic number at least $\ell + 1$.*

This comes immediately from the fact that if we could ℓ-color all k-colorable tournaments, then we could see that they do not have chromatic number $\ell + 1$ or greater. The hardness of distinguishing between chromatic number k and greater or equal to $\ell + 1$ is therefore commonly established as a way of implying the hardness of coloring k-colorable graphs with ℓ colors (see for example [6]).

All proofs of the theorems in this section are provided in the full version [28].

4.1 NP-Hardness of Approximate Coloring of k-Colorable Tournaments

It was shown previously that it is NP-hard to color a 2-colorable tournament with 2 colors [8, 15]. We prove a stronger theorem, that it is NP-hard to 3-color a 2-colorable tournament.

▶ **Theorem 28.** *It is NP-hard, given a tournament T, to distinguish whether $\vec{\chi}(T) = 2$ or $\vec{\chi}(T) \geq 4$.*

The proof of this Theorem relies on a reduction from the problem of coloring 2-colorable tournaments with three colors to the problem of coloring 2-colorable 3-uniform hypergraphs with six colors. This problem is NP-hard, since it was proven that coloring 2-colorable 3-uniform hypergraphs with any constant number of colors is NP-hard [11].

We then use a recursive construction that starts with the tournament obtained in the proof of Theorem 28 to generalize the hardness of approximation to k-colorable tournaments for any constant k.

▶ **Theorem 29.** *It is NP-hard, given a tournament T and a constant k, to distinguish whether $\vec{\chi}(T) = k$ or $\vec{\chi}(T) \geq 2k$.*

4.2 Reduction from Coloring Graphs to Coloring Tournaments

In Section 3.2, we showed that if we can color a 3-colorable graph with k colors, then we can color a 3-colorable tournament with $50k$ colors. We give a reduction in the other direction: We show that the problem of coloring a k-colorable graph with ℓ colors is reducible to the problem of coloring a k-colorable tournament with ℓ colors.

▶ **Theorem 30.** *Given any two constants $k, \ell \geq 3$, if we can efficiently distinguish k-colorable tournaments and tournaments with chromatic number at least ℓ, then we can efficiently distinguish k-colorable graphs and graphs with chromatic number at least ℓ.*

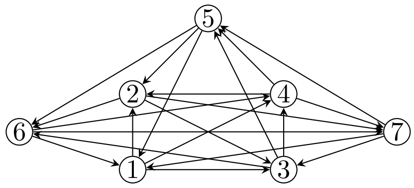

Figure 2 3-chromatic light tournament.

A corollary of this reduction is the hardness of coloring tournaments under the d-To-1 Conjecture of Khot [26]; [18] showed that assuming the d-To-1 Conjecture, it is hard to color 3-colorable graphs with $O(1)$ colors, and using our reduction, we can extend this hardness to tournaments.

▶ **Corollary 31.** *Let $3 \leq k < \ell$ be any two constants. Then if the d-To-1 conjecture is true, we cannot distinguish between tournaments with chromatic number k and tournaments with chromatic number at least ℓ.*

4.3 Hardness of Approximation for General Tournaments

Coloring digon-free digraphs has been shown to be NP-hard to approximate within a factor of $n^{1/2-\epsilon}$ [14]. This proof can easily be extended to the case of tournaments, which provides the following theorem.

▶ **Theorem 32.** *Given any arbitrarily small constant $\epsilon > 0$, it is NP-hard to approximate the chromatic number of tournaments within a factor of $n^{1/2-\epsilon}$.*

5 Conclusion

There are many open questions related to the theorems we have presented since all the rows in Table 2 present gaps between the upper and lower bounds. One example is light tournaments: What is the maximum number of colors required to color a light tournament? From Theorem 18, we know that light tournaments have dichromatic number at most 9. On the other hand, there exist light tournaments that are not 2-colorable. An example of such a tournament is the Paley tournament P_7, one of the four 3-chromatic tournaments on seven vertices [33]. This tournament is represented in Figure 2. We have not found any light tournament with chromatic number at least four. The Paley tournament P_{11} is the unique 4-chromatic tournament on 11 vertices [33]. A light 4-chromatic tournament would have to have at least 13 vertices as [3] proved that any 4-chromatic tournament on 12 vertices must contain an induced copy of P_{11} and P_{11} is not light.

Moreover, notice that if we could show that it is hard to color a 2-colorable tournament with four colors (rather than three as per Theorem 28), this would imply hardness of coloring a 2-colorable light tournament with two colors by Observation 8. Indeed, we have no hardness results for coloring light tournaments. Any upper bound of c on their dichromatic number would imply that it cannot be NP-hard to color them with c colors, because the property of being light is checkable in polynomial time (unlike the property of being, say, 2-colorable).

Another observation is the relation of coloring tournaments and the feedback vertex set (FVS) problem on tournaments. There is an elegant 2-approximation for this problem [30]. Notice that Theorem 7 implies that in a 2-colorable tournament, we can efficiently find a FVS of size at most $9n/10$. In contrast, the algorithm in [30] could just return the whole vertex set if the two transitive sets were of roughly equal size. Finally, we mention that, analogous to a well-studied question for general graphs [10, 27], one can ask what is the largest transitive induced subtournament that one can efficiently find in a 2-colorable tournament? Is it larger than $n/10$?

Finally, we remark that an implication of Theorem 15 is that proving any hardness of coloring 3-colorable tournaments would then provide hardness of coloring 3-colorable graphs with 50 times fewer colors. Since it has taken around 20 years to go from proving NP-hardness of coloring a 3-colorable graph with four colors [25, 16, 17] to NP-hardness of coloring a 3-colorable graph with five colors [6], it would be interesting to see if we can prove hardness of coloring 3-colorable tournaments for a constant larger than five (at least five is shown in Theorem 29), or perhaps show that the two problems are actually equivalent.

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