Real Equation Systems with Alternating Fixed-Points

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Abstract

We introduce the notion of a Real Equation System (RES), which lifts Boolean Equation Systems (BESs) to the domain of extended real numbers. Our RESs allow arbitrary nesting of least and greatest fixed-point operators. We show that each RES can be rewritten into an equivalent RES in normal form. These normal forms provide the basis for a complete procedure to solve RESs. This employs the elimination of the fixed-point variable at the left side of an equation from its right-hand side, combined with a technique often referred to as Gauß-elimination. We illustrate how this framework can be used to verify quantitative modal formulas with alternating fixed-point operators interpreted over probabilistic labelled transition systems.

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1 Introduction

The modal mu-calculus is a logic that allows to formulate and verify a very wide range of properties on behaviour, far more expressive than virtually any other behavioural logic around [3, 2]. For instance, CTL and LTL can be mapped to it, but the reverse is not possible. By allowing data parameters in the fixed point variables in modal formulas, this can even be done linearly, without loss of computational effectiveness [5]. Using alternating fixed-points, the modal mu-calculus can intrinsically express various forms of fairness, which in other logics can often only be achieved by adding special fairness operators.

An effective way to evaluate a modal property on a labelled transition system is by translating both to a single Boolean Equation System (BES) with alternating fixed-points [20, 22]. Exactly if the initial boolean variable of the obtained BES has the solution true, the property is valid for the labelled transition system. A BES with alternating fixed-points is equivalent to a parity game [21, 2]. There are many algorithms to solve BESs and parity games [26, 4, 17, 25]. Although, it is a long standing open problem whether a polynomial algorithm exists to solve BESs [4, 17], the existing algorithms work remarkably well in practical contexts.

For a while now, it has been argued that modal logics can become even more effective if they provide quantitative answers [15, 16], such as durations, probabilities and expected values. In this paper we lift boolean equation systems to real numbers to form a framework for the evaluation of quantitative modal formulas, and call the result *Real Equation Systems* (*RESs*), i.e., fixed-point equation systems over the domain of the extended reals, $\mathbb{R} \cup \{-\infty, \infty\}$.

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Conjunction and disjunction are interpreted as minimum and maximum, and new operators such as addition and multiplication with positive constants are added. A typical example of a real equation system is the following

$$\mu X = (\tfrac{1}{2}X + 1) \vee (\tfrac{1}{5}Y + 3),$$
$$\nu Y = ((\tfrac{1}{10}Y - 10) \vee (2X + 5)) \wedge 17.$$

Based on Tarski's fixed-point theorem, this real equation system has a unique solution. Using the method provided in this paper we can determine this solution using algebraic manipulation. In the case above, see Section 4, the second fixed-point equation can be simplified to $\nu Y = -\frac{100}{9} \vee ((2X + 5) \wedge 17)$. It is sound to substitute this in the first equation, which becomes $\mu X = (\frac{1}{2}X + 1) \vee \frac{7}{9} \vee ((\frac{2}{5}X + 4) \wedge \frac{32}{5})$. This equation can be solved for X yielding $X = \frac{32}{5}$, from which it directly follows that $Y = 17$.

Concretely, this paper has the following results. We define real equation systems with alternating fixed-points. The base syntax for expressions is equal to that of [7] with constants, minimum, maximum, addition and multiplication with positive real constants. We add four additional operators, namely two conditional operators, and two tests for infinity, which turn out to be required to algebraically solve arbitrary real equation systems.

We provide algebraic laws that allow to transform any expression to *conjunctive/disjunctive normal form*. Based on this normal form we provide rules that allow to eliminate each variable bound in the left-hand side of an equation from the right-hand side of that equation. This enables "Gauß-elimination", developed for BESs, using which any real equation system can be solved.

We provide a quantitative modal logic, and define how a quantitative formula and a (probabilistic) labelled transition system ((p)LTS) can be transformed into a RES. The solution of the initial variable of this equation system is equal to the evaluation of the quantitative formula on the labelled transition system. We also briefly touch upon the embedding of BESs into RESs.

The approach in this paper follows the tradition of boolean equation systems [19, 20, 21]. By allowing data parameters in the fixed-point variables we obtain Parameterised Boolean Equation Systems (PBESs) which is a very expressive framework that forms the workhorse for model checking [22, 13, 11]. In this paper we do not address such parametric extensions, as they are pretty straightforward, but in combination with parameterised quantitative modal logic, it will certainly provide a very versatile framework for quantitative model checking.

There are a number of extensions of the boolean equation framework to the setting of reals but these typically limit themselves to only single fixed-points. In [7] the minimal integer solutions for a set of equations with only minimal fixed-points is determined. In [8] a polynomial algorithm is provided to find the minimal solution for a set of real equation systems. In [1] convex lattice equation systems are introduced, also restricted to a single fixed-point. In that paper a proof system is given to show that all models of the equations are consistent, meaning that the evaluation of a quantitative modal formula is limited by some upper-bound.

In [24], the Łukasiewicz μ-calculus is studied, which resembles RESs restricted to the interval $[0, 1]$. This logic does allow minimal and maximal fixed-points. They provide two algorithmic ways of computing the solutions for formulas in their logic, *viz.* an indirect method that builds formulas in the first-order theory of linear arithmetic and exploits quantifier elimination, and a method that uses iteration to refine successive approximations of conditioned linear expressions. Embedding our logic in the Łukasiewicz μ-calculus can be done by mapping the extended reals onto the interval $[0, 1]$ using an appropriate sigmoid function. But such a mapping does not map our addition and constant multiplication to available counterparts in the Łukasiewicz μ-calculus, which prevents using algorithms for Łukasiewicz μ-terms [18, 24] to our setting. However, as the Łukasiewicz μ-calculus is directly encodable into the RES framework, all our results are directly applicable to the Łukasiewicz μ-calculus.

2 Expressions and normal forms

We work in the setting of *extended real numbers*, i.e., $\mathbb{R}\cup\{\infty, -\infty\}$, denoted by $\hat{\mathbb{R}}$. We assume the normal total ordering $\leq$ on $\hat{\mathbb{R}}$ where $-\infty \leq x$ and $x \leq \infty$ for all $x \in \hat{\mathbb{R}}$. Throughout this text we employ a set $\mathcal{X}$ of variables and *valuations* $\eta : \mathcal{X} \to \hat{\mathbb{R}}$ that map variables to extended reals. We write $\eta(X)$ to apply η to X, and $\eta[X := r]$ to adapt valuations by:

$$\eta[X := r](Y) = \begin{cases} r & \text{if } X = Y, \\ \eta(Y) & \text{otherwise.} \end{cases}$$

We consider expressions over the set $\mathcal{X}$ of variables with the following syntax.

$$e \;::=\; X \mid d \mid c{\cdot}e \mid e + e \mid e \wedge e \mid e \vee e \mid e \Rightarrow e \diamond e \mid e \to e \diamond e \mid eq_{\infty}(e) \mid eq_{-\infty}(e)$$

where $X \in \mathcal{X}$, $d \in \hat{\mathbb{R}}$ is a constant, $c \in \mathbb{R}_{>0}$ a positive constant, $+$ represents addition, $\wedge$ stands for minimum, $\vee$ for maximum, $\_ \Rightarrow \_ \diamond \_$ and $\_ \to \_ \diamond \_$ are conditional operators, and eq_{∞} and $eq_{-\infty}$ are auxiliary functions to check for $\pm\infty$. The conditional operators and the checks for infinity occur naturally while solving fixed-point equations and therefore, we made them part of the syntax. We apply valuations to expressions, as in $\eta(e)$, where η distributes over all operators in the expression.

The interpretation of these operators on the domain $\hat{\mathbb{R}}$ is largely obvious. A variable X gets a value by a valuation. Multiplying expressions with a constant c is standard, and yields $\pm\infty$ if applied on $\pm\infty$. The conditional operators, addition and infinity operators are defined below where $e, e_1, e_2, e_3 \in \hat{\mathbb{R}}$.

$$e_1 + e_2 = \begin{cases} e_1 + e_2 & \text{if } e_1, e_2 \in \mathbb{R}\text{, i.e., apply normal addition,} \\ \infty & \text{if } e_1 = \infty \text{ or } e_2 = \infty, \\ -\infty & \text{if } e_i{=}-\infty \text{ and } e_{3-i} \neq \infty \text{ for } i = 1, 2. \end{cases}$$

$$e_1 \Rightarrow e_2 \diamond e_3 = \begin{cases} e_2 \wedge e_3 & \text{if } e_1 \leq 0, \\ e_3 & \text{if } e_1 > 0. \end{cases} \qquad e_1 \to e_2 \diamond e_3 = \begin{cases} e_2 & \text{if } e_1 < 0, \\ e_2 \vee e_3 & \text{if } e_1 \geq 0. \end{cases}$$

$$eq_{\infty}(e) = \begin{cases} \infty & \text{if } e = \infty, \\ -\infty & \text{if } e \neq \infty. \end{cases} \qquad eq_{-\infty}(e) = \begin{cases} \infty & \text{if } e \neq -\infty, \\ -\infty & \text{if } e = -\infty. \end{cases}$$

Note that all defined operators are monotonic on $\hat{\mathbb{R}}$. We have the identity $eq_{\infty}(e) = e + -\infty$, and so, we do not treat eq_{∞} as a primary operator. We write $e[X := e']$ for the expression representing the syntactic substitution of e' for X in e. We write $\mathsf{occ}(e)$ for the set of variables from $\mathcal{X}$ occurring in e. Table 1 contains many useful algebraic laws for our operators.

The addition operator $+$ has as property that $-\infty + \infty = \infty + -\infty = \infty$. One may require the other natural addition operator $\hat{+}$, as used in [8], satisfying that $-\infty\hat{+}\infty = \infty\hat{+} - \infty = -\infty$. It can be defined as follows:

$$e_1 \hat{+} e_2 \;=\; eq_{-\infty}(e_1) \Rightarrow -\infty \diamond (eq_{-\infty}(e_2) \Rightarrow -\infty \diamond (e_1 + e_2)).$$

We can extend the syntax with unary negation $-e$ with its standard meaning, and, provided no variable occurs in the scope of its definition within an odd number of negations, negation can be eliminated using standard simplification rules. Therefore, we do not consider it as a primary part of our syntax. At the end of Table 1 we list several identities involving negation. Note that operators $+$ and $\hat{+}$ are each other's dual with regard to negation.

We introduce normal forms, crucial to solve real equation systems, where the sum, conjunction and disjunction over empty domains of variables equal 0, ∞ and $-\infty$, respectively.

▶ **Definition 1.** *Let $\mathcal{X}$ be a set of variables. An expression e is in simple conjunctive normal form iff it has the shape*

$$\bigwedge_{i\in I}\bigvee_{j\in J_i}((\sum_{X\in\mathcal{X}_{ij}} c_{ij}^X{\cdot}X)+(\sum_{X\in\mathcal{X}'_{ij}} eq_{-\infty}(X))+d_{ij})$$

and it is in simple disjunctive normal form iff it has the shape

$$\bigvee_{i\in I}\bigwedge_{j\in J_i}((\sum_{X\in\mathcal{X}_{ij}} c_{ij}^X{\cdot}X)+(\sum_{X\in\mathcal{X}'_{ij}} eq_{-\infty}(X))+d_{ij})$$

where $\mathcal{X}_{ij}\subseteq\mathcal{X}$ and $\mathcal{X}'_{ij}\subseteq\mathcal{X}$ are finite sets of variables, $c_{ij}^X\in\mathbb{R}_{>0}$, and $d_{ij}\in\hat{\mathbb{R}}$.

An expression e is in conjunctive, resp. disjunctive normal form iff

1. *e is in simple conjunctive, resp. disjunctive normal form, or*
2. *e has the shape $e_1\Rightarrow e_2\diamond e_3$ or $e_1\rightarrow e_2\diamond e_3$ where e_1 is in simple conjunctive, resp. disjunctive normal form and e_2 and e_3 are conjunctive resp. disjunctive normal forms.*

▶ **Lemma 2.** *Each expression e not containing the conditional operators $e_1\Rightarrow e_2\diamond e_3$ or $e_1\rightarrow e_2\diamond e_3$ can be rewritten to a simple conjunctive or disjunctive normal form using the equations in Table 1.*

▶ **Lemma 3.** *Expression of the forms $e_1\Rightarrow e_2\diamond e_3$ and $e_1\rightarrow e_2\diamond e_3$ can be rewritten to equivalent expressions where the first argument of such a conditional operator is a simple conjunctive or disjunctive normal form using the equations in Table 1.*

▶ **Theorem 4.** *Each expression e can be rewritten to both a conjunctive and a disjunctive normal form using the equations in Table 1.*

3 Real equation systems and Gauß-elimination

In this section we introduce Real Equation Systems (RESs) as sequences of fixed-point equations, introduce a natural equivalence between RESs, and provide a generic solution method, known as Gauß-elimination [20].

▶ **Definition 5.** *Let $\mathcal{X}$ be a set of variables. A Real Equation System (RES) $\mathcal{E}$ is a finite sequence of (fixed-point) equations*

$$\sigma_1 X_1{=}e_1,\ldots,\sigma_n X_n{=}e_n$$

where σ_i is either the minimal fixed-point operator μ or the maximal fixed-point operator ν, $X_i\in\mathcal{X}$ are variables and e_i are expressions. We write $\mathsf{bnd}(\mathcal{E})$ for the set of variables occurring in the left-hand side, i.e., $\mathsf{bnd}(\mathcal{E})=\{X_1,\ldots,X_n\}$.

Table 1 Algebraic laws.

$\mathrm{I}_\vee$	$e \vee e = e$	$\mathrm{I}_\wedge$	$e \wedge e = e$
D^+_+	$(e_1 + e_2) + e_3 = e_1 + (e_2 + e_3)$	C+	$e_1 + e_2 = e_2 + e_1$
$\mathrm{D}^\vee_\vee$	$(e_1 \vee_2) \vee e_3 = e_1 \vee (e_2 \vee e_3)$	$\mathrm{C}\vee$	$e_1 \vee e_2 = e_2 \vee e_1$
$\mathrm{D}^\wedge_\wedge$	$(e_1 \wedge e_2) \wedge e_3 = e_1 \wedge (e_2 \wedge e_3)$	$\mathrm{C}\wedge$	$e_1 \wedge e_2 = e_2 \wedge e_1$
$\mathrm{D}^\Rightarrow_\Rightarrow$	$(e_1 \Rightarrow e_2 \diamond e_3) \Rightarrow f_1 \diamond f_2 = ((e_1 \vee e_2) \wedge e_3) \Rightarrow f_1 \diamond f_2$		
$\mathrm{D}^\Rightarrow_\rightarrow$	$(e_1 \Rightarrow e_2 \diamond e_3) \rightarrow f_1 \diamond f_2 = e_1 \rightarrow (e_2 \Rightarrow f_1 \diamond f_2) \diamond (e_2 \vee e_3 \Rightarrow f_1 \diamond f_2)$		
$\mathrm{D}^c_\Rightarrow$	$c{\cdot}(e_1 \Rightarrow e_2 \diamond e_3) = e_1 \Rightarrow c{\cdot}e_2 \diamond c{\cdot}e_3$		
$\mathrm{D}^+_\Rightarrow$	$(e_1 \Rightarrow e_2 \diamond e_3) + f = e_1 \Rightarrow (e_2 + f) \diamond (e_3 + f)$		
$\mathrm{D}^\wedge_\Rightarrow$	$(e_1 \Rightarrow e_2 \diamond e_3) \wedge f = e_1 \Rightarrow (e_2 \wedge f) \diamond (e_3 \wedge f)$		
$\mathrm{D}^\vee_\Rightarrow$	$(e_1 \Rightarrow e_2 \diamond e_3) \vee f = e_1 \Rightarrow (e_2 \vee f) \diamond (e_3 \vee f)$		
$\mathrm{D}^\rightarrow_\rightarrow$	$(e_1 \rightarrow e_2 \diamond e_3) \rightarrow f_1 \diamond f_2 = (e_2 \vee (e_1 \wedge e_3)) \rightarrow f_1 \diamond f_2$		
$\mathrm{D}^\Rightarrow_\rightarrow$	$(e_1 \rightarrow e_2 \diamond e_3) \Rightarrow f_1 \diamond f_2 = e_1 \Rightarrow (e_2 \wedge e_3 \rightarrow f_1 \diamond f_2) \diamond (e_3 \rightarrow f_1 \diamond f_2)$		
$\mathrm{D}^c_\rightarrow$	$c{\cdot}(e_1 \rightarrow e_2 \diamond e_3) = e_1 \rightarrow c{\cdot}e_2 \diamond c{\cdot}e_3$		
D+	$(e_1 \rightarrow e_2 \diamond e_3) + f = e_1 \rightarrow (e_2 + f) \diamond (e_3 + f)$		
$\mathrm{D}^\wedge_\rightarrow$	$(e_1 \rightarrow e_2 \diamond e_3) \wedge f = e_1 \rightarrow (e_2 \wedge f) \diamond (e_3 \wedge f)$		
$\mathrm{D}^\vee_\rightarrow$	$(e_1 \rightarrow e_2 \diamond e_3) \vee f = e_1 \rightarrow (e_2 \vee f) \diamond (e_3 \vee f)$		
$\mathrm{D}^+_\wedge$	$e_1 + (e_2 \wedge e_3) = (e_1 + e_2) \wedge (e_1 + e_3)$	$\mathrm{D}^+_\vee$	$e_1 + (e_2 \vee e_3) = (e_1 + e_2) \vee (e_1 + e_3)$
D^c_+	$c{\cdot}(e_1 + e_2) = c{\cdot}e_1 + c{\cdot}e_2$		
$\mathrm{D}^c_\wedge$	$c{\cdot}(e_1 \wedge e_2) = c{\cdot}e_1 \wedge c{\cdot}e_2$	$\mathrm{D}^c_\vee$	$c{\cdot}(e_1 \vee e_2) = c{\cdot}e_1 \vee c{\cdot}e_2$
$\mathrm{D}^\wedge_\vee$	$e_1 \wedge (e_2 \vee e_3) = (e_1 \wedge e_2) \vee (e_1 \wedge e_3)$	$\mathrm{D}^\vee_\wedge$	$e_1 \vee (e_2 \wedge e_3) = (e_1 \vee e_2) \wedge (e_1 \vee e_3)$
D^∞_∞	$eq_\infty(eq_\infty(e)) = eq_\infty(e)$	$\mathrm{D}^{-\infty}_\infty$	$eq_{-\infty}(eq_\infty(e)) = eq_\infty(e)$
$\mathrm{D}^\infty_{-\infty}$	$eq_\infty(eq_{-\infty}(e)) = eq_{-\infty}(e)$	$\mathrm{D}^{-\infty}_{-\infty}$	$eq_{-\infty}(eq_{-\infty}(e)) = eq_{-\infty}(e)$
D^∞_c	$eq_\infty(c{\cdot}e) = eq_\infty(e)$	$\mathrm{D}^{-\infty}_c$	$eq_{-\infty}(c{\cdot}x) = eq_{-\infty}(x)$
D^∞_+	$eq_\infty(e_1 + e_2) = eq_\infty(e_1) + eq_\infty(e_2) = eq_\infty(e_1) \vee eq_\infty(e_2)$		
$\mathrm{D}^{-\infty}_+$	$eq_{-\infty}(e_1 + e_2) = (eq_{-\infty}(e_1) \vee eq_\infty(e_2)) \wedge (eq_\infty(e_1) \vee eq_{-\infty}(e_2))$		
$\mathrm{D}^\infty_\vee$	$eq_\infty(e_1 \vee e_2) = eq_\infty(e_1) \vee eq_\infty(e_2)$	$\mathrm{D}^{-\infty}_\vee$	$eq_{-\infty}(e_1 \vee e_2) = eq_{-\infty}(e_1) \vee eq_{-\infty}(e_2)$
$\mathrm{D}^\infty_\wedge$	$eq_\infty(e_1 \wedge e_2) = eq_\infty(e_1) \wedge eq_\infty(e_2)$	$\mathrm{D}^{-\infty}_\wedge$	$eq_{-\infty}(e_1 \wedge e_2) = eq_{-\infty}(e_1) \wedge eq_{-\infty}(e_2)$
$\mathrm{E}^\wedge_\infty$	$eq_\infty(e) \wedge eq_{-\infty}(e) = eq_\infty(e)$	$\mathrm{E}^\vee_{-\infty}$	$eq_\infty(e) \vee eq_{-\infty}(e) = eq_{-\infty}(e)$
$\mathrm{D}^\infty_\Rightarrow$	$eq_\infty(e_1 \Rightarrow e_2 \diamond e_3) = e_1 \Rightarrow eq_\infty(e_2) \diamond eq_\infty(e_3)$		
$\mathrm{D}^{-\infty}_\Rightarrow$	$eq_{-\infty}(e_1 \Rightarrow e_2 \diamond e_3) = e_1 \Rightarrow eq_{-\infty}(e_2) \diamond eq_{-\infty}(e_3)$		
$\mathrm{D}^\infty_\rightarrow$	$eq_\infty(e_1 \rightarrow e_2 \diamond e_3) = e_1 \rightarrow eq_\infty(e_2) \diamond eq_\infty(e_3)$		
$\mathrm{D}^{-\infty}_\rightarrow$	$eq_{-\infty}(e_1 \rightarrow e_2 \diamond e_3) = e_1 \rightarrow eq_{-\infty}(e_2) \diamond eq_{-\infty}(e_3)$		
D^-_c	$-c{\cdot}e = c{\cdot} -e$		
D^-_+	$-(e_1 + e_2) = -e_1 \hat{+} -e_2$	$\mathrm{D}^-_{\hat{+}}$	$-(e_1 \hat{+} e_2) = -e_1 + -e_2$
$\mathrm{D}^-_\vee$	$-(e_1 \vee e_2) = -e_1 \wedge -e_2$	$\mathrm{D}^-_\wedge$	$-(e_1 \wedge e_2) = -e_1 \vee -e_2$
$\mathrm{D}^-_\Rightarrow$	$-(e_1 \Rightarrow e_2 \diamond e_3) = -e_1 \rightarrow -e_3 \diamond -e_2$	$\mathrm{D}^-_\rightarrow$	$-(e_1 \rightarrow e_2 \diamond e_3) = -e_1 \Rightarrow -e_3 \diamond -e_2$
D^-_∞	$-eq_\infty(e) = eq_{-\infty}(-e)$	$\mathrm{D}^-_{-\infty}$	$-eq_{-\infty}(e) = eq_\infty(-e)$

The empty sequence of equations is denoted by ε.

The semantics of a real equation system is a valuation giving the solutions of all variables, based on an initial valuation η giving the solution for all variables not bound in $\mathcal{E}$.

▶ **Definition 6.** *Let $\mathcal{X}$ be a set of variables and $\mathcal{E}$ be a real equation system over $\mathcal{X}$. The solution $[\![\mathcal{E}]\!]\eta : \mathcal{X} \to \hat{\mathbb{R}}$ yields an extended real number for all $X \in \mathcal{X}$, given a valuation $\eta : \mathcal{X} \to \hat{\mathbb{R}}$ of $\mathcal{E}$. It is inductively defined as follows:*

$$[\![\varepsilon]\!]\eta = \eta,$$
$$[\![\sigma X{=}e, \mathcal{E}]\!]\eta = [\![\mathcal{E}]\!](\eta[X := \sigma(X, \mathcal{E}, \eta, e)])$$

where $\sigma(X, \mathcal{E}, \eta, e)$ is defined as

$$\mu(X, \mathcal{E}, \eta, e) = \bigwedge\{r \in \hat{\mathbb{R}} \mid r \geq [\![\mathcal{E}]\!](\eta[X := r])(e)\} \text{ and}$$
$$\nu(X, \mathcal{E}, \eta, e) = \bigvee\{r \in \hat{\mathbb{R}} \mid [\![\mathcal{E}]\!](\eta[X := r])(e) \geq r\}.$$

It is equivalent to write $=$ instead of $\geq$ in the above sets. This makes the fixed-points easier to understand. Note that if the real equation system is closed, i.e., all variables in the right-hand sides occur in $\mathsf{bnd}(\mathcal{E})$, the value $[\![\mathcal{E}]\!]\eta(X)$ is independent of η for all $X \in \mathsf{bnd}(\mathcal{E})$.

Following [14], we introduce the notion of equivalency between equation systems. We use the symbol $\equiv$ to distinguish this equivalence from "$=$" used in equation systems.

▶ **Definition 7.** *Let $\mathcal{E}, \mathcal{E}'$ be real equation systems. We say that $\mathcal{E} \equiv \mathcal{E}'$ iff $[\![\mathcal{E}, \mathcal{F}]\!]\eta = [\![\mathcal{E}', \mathcal{F}]\!]\eta$ for all valuations η and real equation systems $\mathcal{F}$ with $\mathsf{bnd}(\mathcal{F}) \cap (\mathsf{bnd}(\mathcal{E}) \cup \mathsf{bnd}(\mathcal{E}')) = \emptyset$.*

In [14] it was observed that defining $\mathcal{E} \equiv \mathcal{E}'$ as $[\![\mathcal{E}]\!]\eta = [\![\mathcal{E}']\!]\eta$ for all η is not desirable, as the resulting equivalence is not a congruence. With this alternative notion, we find that $\mu X{=}Y$ and $\nu X{=}Y$ are equivalent. But $\mu X{=}Y, \nu Y{=}X$ and $\nu X{=}Y, \nu Y{=}X$ are not as the first one has solution $X = Y = -\infty$ and the second one has $X = Y = \infty$.

However, if the fixed-point symbol is the same, it is not necessary to take surrounding equations into account. This is a pretty useful lemma which makes the proofs in this paper much easier, and of which we are not aware that it occurs elsewhere in the literature.

▶ **Lemma 8.** *Let X be a variable, e and f be expressions and σ either the minimal or the maximal fixed-point symbol. If for any valuation η it holds that $[\![\sigma X = e]\!]\eta = [\![\sigma X = f]\!]\eta$ then $\sigma X = e \equiv \sigma X = f$.*

The proof of the main Theorem 11 is quite involved and heavily uses the following two lemmas, which we only give for the minimal fixed-point. The formulations for the maximal fixed-point are dual.

▶ **Lemma 9.** *Let $X \in \mathcal{X}$ be a variable and e, f be expressions. It holds that $\mu X = e \equiv \mu X = f$ if for every valuation η:*

1. *for the smallest $r \in \hat{\mathbb{R}}$ such that $r = \eta[X := r](e)$ it holds that there is an $r' \in \hat{\mathbb{R}}$ satisfying that $r' \leq r$ and $r' \geq \eta[X := r'](f)$, and, vice versa,*
2. *for the smallest $r \in \hat{\mathbb{R}}$ such that $r = \eta[X := r](f)$ it holds that there is an $r' \in \hat{\mathbb{R}}$ satisfying that $r' \leq r$ and $r' \geq \eta[X := r'](e)$.*

▶ **Lemma 10.** *If $\mu X = e \equiv \mu X = f$, then for any valuation η it holds that*

1. *for any $r \in \hat{\mathbb{R}}$ such that $r \geq \eta[X := r](e)$, there is an $r' \in \hat{\mathbb{R}}$ such that $r' \leq r$ and $r' = \eta[X := r'](f)$, and, vice versa,*
2. *for any $r \in \hat{\mathbb{R}}$ such that $r \geq \eta[X := r](f)$, there is an $r' \in \hat{\mathbb{R}}$ such that $r' \leq r$ and $r' = \eta[X := r'](e)$.*

Table 2 Properties of the equivalence $\equiv$ on RESs.

E1 $\dfrac{\mathcal{E} \equiv \mathcal{E}'}{\mathcal{F}, \mathcal{E} \ \equiv\ \mathcal{F}, \mathcal{E}'}$. E2 $\dfrac{\mathcal{E} \equiv \mathcal{E}'}{\mathcal{E}, \mathcal{F} \ \equiv\ \mathcal{E}', \mathcal{F}}$.

E3 $\sigma X{=}e, \mathcal{E}, \sigma' Y{=}e' \ \equiv\ \sigma X{=}e[Y := e'], \mathcal{E}, \sigma' Y{=}e'$ if $X, Y \notin \mathsf{bnd}(\mathcal{E})$.

E4 $\sigma X{=}e, \mathcal{E} \ \equiv\ \mathcal{E}, \sigma X{=}e$ if $\mathsf{occ}(e) = \emptyset$ and $X \notin \mathsf{bnd}(\mathcal{E})$.

E5 $\sigma X{=}e, \sigma Y{=}e' \ \equiv\ \sigma Y{=}e', \sigma X{=}e$.

E6 $\dfrac{\mu X = e_1 \ \equiv\ \mu X = f_1 \text{ and } \mu X = e_2 \ \equiv\ \mu X = f_2}{\mu X = e_1 \wedge e_2 \ \equiv\ \mu X = f_1 \wedge f_2}$.

E7 $\dfrac{\nu X = e_1 \ \equiv\ \nu X = f_1 \text{ and } \nu X = e_2 \ \equiv\ \nu X = f_2}{\nu X = e_1 \vee e_2 \ \equiv\ \nu X = f_1 \vee f_2}$.

The notion of equivalence of Definition 7 is an equivalence relation on RESs and it satisfies the properties E1-E7 in Table 2. E1-E5 are proven for boolean equation systems in [14] and the proofs carry over to our setting. In the table, σ and σ' stand for the fixed-point symbols μ and ν. The equivalences E3 and E4 above give a method to solve arbitrary equation systems, provided a single equation can be solved. Here, solving a single equation $\sigma X{=}e$ means replacing it by an equivalent equation $\sigma X{=}e'$ where X does not occur in e', which is the topic of the next section. This method is known as Gauß-elimination as it resembles the well-known Gauß-elimination procedure for sets of linear equations [20].

The idea behind Gauß-elimination for a real equation system $\mathcal{E}$ is as follows. First, the last equation $\sigma_n X_n{=}e_n$ of $\mathcal{E}$ is solved for X_n. Assume the solution is $\sigma_n X_n{=}e'_n$, where X_n does not occur in e'_n. Using E3 the expression e'_n is substituted for all occurrences X_n in right-hand sides of $\mathcal{E}$ removing all occurrences of X_n except in the left hand side of the last equation. Subsequently, this process is repeated for the one but last equation of $\mathcal{E}$ up to the first equation. Now the first equation has the shape $X_1{=}e_1$ where no variable X_1 up till X_n occurs in e_1. Using E4 this equation can be moved to the end of $\mathcal{E}$, and by applying E3 all occurrences of X_1 are removed from the right-hand sides of $\mathcal{E}$. This is then repeated for X_2, which now also does not contain $X_1, \ldots, X_n$, until all variables $X_1, \ldots, X_n$ have been removed from all right-hand sides of $\mathcal{E}$.

A concrete, but simple example is the following. Consider the real equation system

$$\mu X{=}Y, \ \ \nu Y{=}(X+1) \wedge Y.$$

We can derive:

$$\mu X{=}Y,\ \nu Y{=}(X+1) \wedge Y \ \overset{(\dagger)}{\equiv}\ \mu X{=}Y,\ \nu Y{=}X+1 \ \overset{\text{E3}}{\equiv}\ \mu X{=}X+1,\ \nu Y{=}X+1 \ \overset{(\ddagger)}{\equiv}$$
$$\mu X{=}-\infty,\ \nu Y{=}X+1 \ \overset{\text{E4}}{\equiv}\ \nu Y{=}X+1,\ \mu X{=}-\infty,\ \overset{\text{E3}}{\equiv}\ \nu Y{=}-\infty,\ \mu X{=}-\infty.$$

Solving the equation $\nu Y = (X+1) \wedge Y$ at ($\dagger$) above, and $\mu X{=}X+1$ at ($\ddagger$) can be done with simple fixed-point iteration. In $\nu Y = (X+1) \wedge Y$ fixed-pointed iteration starts with $Y = \infty$. This yields in the first iteration $Y = X+1$, and this iteration is stable, and hence it is the maximal fixed-point solution. For $\mu X{=}X+1$, the initial approximation $X = -\infty$ is also a solution, and hence the minimal solution. Unfortunately, fixed-point iteration does not terminate always. For instance, $\mu X{=}(X+1) \vee 0$ has minimal solution $X = \infty$, which can only be obtained via an infinite number of iteration steps.

4 Solving single equations

In this section we show that it is possible to solve each fixed-point equation $\sigma X = e$ in a finite number of steps. First assume that e does not contain conditional operators. If we have a minimal fixed-point equation $\mu X{=}e$, we know via Theorem 4 that we can rewrite e to simple conjunctive normal form. We want to explicitly expose occurrences of the variable X in the normal form of e and do this by denoting the normal form of e as shown in (1). Here, all expressions containing variables different from X are moved to f_{ij} or m_i.

$$\bigwedge_{i\in I}(\bigvee_{j\in J_i}(c_{ij}{\cdot}X + c'_{ij}{\cdot}\mathit{eq}_{-\infty}(X) + f_{ij}) \vee m_i). \tag{1}$$

The expressions f_{ij} and m_i do not contain X. Subexpressions $c_{ij}{\cdot}X$ are optional, i.e., abusing notation, we allow c_{ij} to be 0 if this sub-term is not present. Likewise, $\mathit{eq}_{-\infty}(X)$ is optional and therefore, c'_{ij} is either 0 or 1, where 0 means that the expression is not present. Constants c_{ij} and c'_{ij} cannot both be 0, as in that case the conjunct does not contain X and is hence part of m_i.

We define the solution of $\mu X{=}e$, in which e is assumed to be of shape (1), as $\mu X = \mathit{Sol}^{\mu}_{X=e}$ where:

$$\begin{aligned}\mathit{Sol}^{\mu}_{X=e} = \bigwedge_{i\in I}((&\mathit{eq}_{\infty}(\bigvee_{j\in J_i} f_{ij})) \\ &\Rightarrow (\mathit{eq}_{-\infty}(m_i) \Rightarrow -\infty \diamond ((\bigvee_{j\in J_i|c_{ij}\geq 1} f_{ij} + (c_{ij}-1){\cdot}U_i) \vee \bigvee_{j\in J_i|c'_{ij}=1}\infty \Rightarrow U_i \diamond \infty)) \\ &\diamond \quad \infty)\end{aligned} \tag{2}$$

where $U_i = m_i \vee \bigvee_{j\in J_i|c_{ij}<1} \frac{1}{1-c_{ij}}{\cdot}f_{ij}$.

Note that we use the notation $\bigvee_{j\in J_i|\mathit{cond}}$ where *cond* is a condition. This means that the disjunction is only taken over elements j that satisfy the condition. Also observe that we use expressions such as $\frac{1}{1-c_{ij}}{\cdot}f_{ij}$. This is an ordinary multiplication with $\frac{1}{1-c_{ij}}$ as positive constant. It is worth noting that if only rational numbers are used in the equations, the solutions to the variables are restricted to $-\infty$, ∞ and rationals.

It can be understood that (2) is a solution of (1) as follows. First observe that due to property E6 the solution of a minimal fixed-point distributes over the initial conjunction $\bigwedge_{i\in I}$ of clauses. This means that we can fix some $i \in I$ and only concentrate on understanding how one single clause $\bigvee_{j\in J_i}(c_{ij}{\cdot}X + c'_{ij}{\cdot}\mathit{eq}_{-\infty}(X) + f_{ij}) \vee m_i$ must be solved. If f_{ij} is equal to ∞ for some $j \in J_i$, the solution must be infinite. This is ensured by the outermost conditional operator in (2). Now, assuming that no f_{ij} is equal to ∞, we inspect m_i. If m_i equals $-\infty$, then the minimal solution for the given $i \in I$ is also $-\infty$. This explains the nested conditional operator in (2).

Next consider the innermost conditional operator of (2) and additionally assume $m_i > -\infty$. If there is some c'_{ij} that is equal to 1, then the minimal solution is at least m_i due to the disjunct m_i that appears in the clause. But then it must also be at least $1{\cdot}\mathit{eq}_{-\infty}(m_i) = \infty$. Hence, in this case the solution is ∞, which is ensured by the expression in the condition of the innermost conditional $\bigvee_{j\in J_i|c'_{ij}=1}\infty$. Otherwise, all c'_{ij} equal 0, and both the right-hand side of (1) and the solution (2) can be simplified to

$$\bigvee_{j\in J_i}(c_{ij}{\cdot}X + f_{ij}) \vee m_i \qquad \text{and} \qquad (\bigvee_{j\in J_i|c_{ij}\geq 1} f_{ij} + (c_{ij}-1){\cdot}U_i) \Rightarrow U_i \diamond \infty.$$

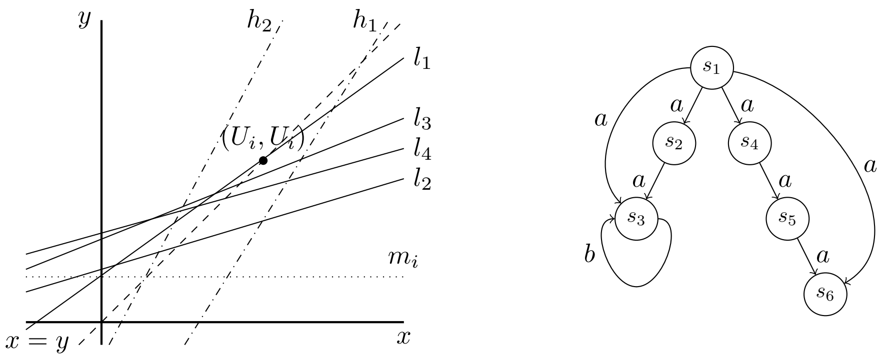

Figure 1 Solving a simple minimal fixed-point equation/An LTS with an infinite sequence of b's.

This resulting situation is best explained using Figure 1 (left). The simple conjunctive normal form consists of a number of disjunctions of the shape $c_{ij}{\cdot}X + f_{ij}$. These characterise lines of which we are interested in their intersection with the line $x = y$. In Figure 1 such lines are drawn as $l_1, \ldots, l_4$, and h_1 and h_2. Due to the disjunction, we are interested in the maximal intersection point. If we first concentrate on those lines with $c_{ij} < 1$, then we see that (U_i, U_i) is the maximal intersection point of these lines above m_i. This intersection point is the solution for the equation unless there is a steep line, with $c_{ij} \geq 1$ which at $x = U_i$ lies above (U_i, U_i). In the figure there is such a line, *viz.* h_2. In such a case the fixed-point lies at the intersection of h_2 with the line $x = y$ for $x > U_i$. As this point does not exist in $\mathbb{R}$, the solution is ∞. The expression $\bigvee_{j\in J_i | c_{ij}\geq 1} f_{ij} + (c_{ij} - 1){\cdot}U_i$ in (2) takes care of this situation. Steep lines, like h_1 which lie below (U_i, U_i) at $x = U_i$ can be ignored, as they do not force the minimal fixed-point U_i to become larger.

In case of a maximal fixed-point equation, $\nu X{=}e$ where e is a simple disjunctive normal form, it is useful to again expose the occurrences of X. We can denote the normal form of e in the following way:

$$\bigvee_{i\in I}\Big(\bigwedge_{j\in J_i}(c_{ij}{\cdot}X + c'_{ij}{\cdot}eq_{-\infty}(X) + f_{ij}) \wedge m_i\Big) \tag{3}$$

where $c_{ij}{\cdot}X$ and $eq_{-\infty}(X)$ are optional, i.e., c_{ij} can be 0, and c'_{ij} is either 0 or 1, where 0 means that the expression is not present. One of c_{ij} and c'_{ij} is not equal to 0. Again, the expressions f_{ij} and m_i do not contain X.

The solution of $\nu X{=}e$, where e is of the shape (3), is $\nu X = Sol^{\nu}_{X=e}$ with

$$\begin{aligned} Sol^{\nu}_{X=e} = \bigvee_{i\in I}\big(&eq_{\infty}(m_i)\\ &\Rightarrow \big(\bigwedge_{j\in J_i | c_{ij}\geq 1 \wedge c'_{ij}=0}(f_{ij} + (c_{ij}-1)){\cdot}U_i\big) \to -\infty \diamond U_i\\ &\diamond \quad \infty\big) \end{aligned} \tag{4}$$

where $U_i = m_i \wedge \bigwedge_{j\in J_i | c_{ij}<1 \wedge c'_{ij}=0} \dfrac{1}{1-c_{ij}}{\cdot}f_{ij}$.

The two fixed-point solutions are not syntactically dual which is due to the fact that simple conjunctive and disjunctive normal forms are not each other's dual, because of the presence of $+$ and $eq_{-\infty}$. We refrain from sketching the intuition underlying the solution to the maximal fixed-point as it is similar to that of the minimal fixed-point.

A full normal form can contain the conditional operators $e_1 \Rightarrow e_2 \diamond e_3$ and $e_1 \rightarrow e_2 \diamond e_3$. Suppose we have an equation $\sigma X = e_1 \Rightarrow e_2 \diamond e_3$ with σ either μ or ν. For the minimal fixed-point the right-hand side of the solution is $Sol^{\mu}_{X=e_1 \Rightarrow e_2 \diamond e_3} = (e_1[X := Sol^{\mu}_{X=e_2} \wedge Sol^{\mu}_{X=e_3}]) \Rightarrow Sol^{\mu}_{X=e_2} \diamond Sol^{\mu}_{X=e_3}$. For the maximal fixed-point we find the right-hand side $Sol^{\nu}_{X=e_1 \Rightarrow e_2 \diamond e_3} = (e_1[X := Sol^{\nu}_{X=e_3}]) \Rightarrow Sol^{\nu}_{X=e_2 \wedge e_3} \diamond Sol^{\nu}_{X=e_3}$.

In case of the other conditional operator $\sigma X = e_1 \rightarrow e_2 \diamond e_3$ we obtain for the right side of the minimal fixed-point $Sol^{\mu}_{X=e_1 \rightarrow e_2 \diamond e_3} = (e_1[X := Sol^{\mu}_{X=e_2}]) \rightarrow Sol^{\mu}_{X=e_2} \diamond Sol^{\mu}_{X=e_2 \vee e_3}$, and for the right side of the maximal fixed-point $Sol^{\nu}_{X=e_1 \rightarrow e_2 \diamond e_3} = (e_1[X := Sol^{\nu}_{X=e_2} \vee Sol^{\nu}_{X=e_3}]) \rightarrow Sol^{\nu}_{X=e_2} \diamond Sol^{\nu}_{X=e_3}$.

The following theorem summarises that these solutions solve fixed-point equations.

▶ **Theorem 11.** *For any fixed-point symbol σ, variable $X \in \mathcal{X}$ and expression e, it holds that*

$$\sigma X = e \;\equiv\; \sigma X = Sol^{\sigma}_{X=e}$$

and $X \notin \mathsf{occ}(Sol^{\sigma}_{X=e})$, where $Sol^{\sigma}_{X=e}$ is defined above.

Proof. By Theorem 4 we can assume that e is in normal form. The proof follows induction on the number of conditional operators. It is straightforward to see that, by construction, X does not occur in $Sol^{\sigma}_{X=e}$.

We only consider the case with a minimal fixed-point where e is a conjunctive normal form. Using property E6 it is possible to solve all conjuncts separately. So, without loss of generality, we assume that e has the shape

$$e = \bigvee_{j \in J} (c_j{\cdot}X + c'_j{\cdot}eq_{-\infty}(X) + f_j) \vee m \tag{5}$$

where $c_j \geq 0$ and $c'_j \in \{0, 1\}$ are constants such that c_j and c'_j are not both 0, and f_j and m are expressions in which X does not occur. We show that the right-hand side of equation (2) without the initial conjunction provides the required term $Sol^{\mu}_{X=e}$ in this theorem. Concretely,

$$\begin{aligned} Sol^{\mu}_{X=e} = {} & (eq_{\infty}(\bigvee_{j \in J} f_j)) \\ & \Rightarrow (eq_{-\infty}(m) \Rightarrow -\infty \diamond (((\bigvee_{j \in J | c_j \geq 1} f_j + (c_j - 1){\cdot}U) \vee \bigvee_{j \in J | c'_j = 1} \infty) \Rightarrow U \diamond \infty)) \\ & \diamond \quad \infty \end{aligned} \tag{6}$$

where $U = m \vee \bigvee_{j \in J | c_j < 1} \frac{1}{1 - c_j}{\cdot}f_j$.

Using Lemma 9 we must prove case 1 and 2 for a valuation η. We start with case 1. So, consider the smallest $r = \eta[X := r](e)$. We define $r' = \eta(Sol^{\mu}_{X=e})$ automatically satisfying the first proof obligation of Lemma 9, where it should be noted that X does not occur in $Sol^{\mu}_{X=e}$. Hence, we only need to show that $r' \leq r$. We distinguish a number of cases.

- Suppose there is some f_j such that $\eta[X := r](f_j) = \infty$. In that case both $r = \infty$ and $r' = \infty$. So, clearly, $r' \leq r$. Below we can now assume that there is no $j \in J$ such that $\eta[X := r](f_j) = \infty$.
- Now assume $\eta(m) = -\infty$. By the previous case we know that $f_j \neq \infty$. In that case $r' = \eta(Sol^{\mu}_{X=e}) = -\infty$, as $\eta(eq_{-\infty}(m)) = -\infty \leq 0$, and hence, $r' \leq r$. Below we assume that $\eta(m) \neq -\infty$.
- If there is at least one $j \in J$ such that $c'_j = 1$, then $r = \eta[X := r](e) = \infty$. The reason for this is that $r > -\infty$, as r at least has the value $\eta(m)$. But then $r = \infty$ as $\eta[X := r](c'_j{\cdot}eq_{-\infty}(X)) = \infty$. Clearly, $r' \leq r$. So, below we can assume that $c'_j = 0$ for all $j \in J$.

- With the assumptions above, we can write e more compactly.

$$e = \bigvee_{j\in J}(c_j{\cdot}X + f_j) \vee m.$$

We know that r is the smallest value satisfying

$$r = \eta[X := r](e) = \eta[X := r](\bigvee_{j\in J}(c_j{\cdot}X + f_j) \vee m).$$

Consider $r_1 = \eta(m \vee \bigvee_{j\in J|c_j<1}(\frac{f_j}{1-c_j}))$.

  - First assume that there is no $j \in J$ with $c_j \geq 1$ such that $r_1 < \eta[X := r_1](c_j{\cdot}X + f_j)$. We show that r_1 is the solution, i.e., $r_1 = r$.
    Consider the case where $\eta(m) \geq \frac{\eta(f_j)}{1-c_j}$ for all $j \in J$ with $c_j < 1$. So, $r_1 = \eta(m)$. In this case $\eta(m)$ is a solution as (i) for those $j \in J$ for which $c_j < 1$, it holds that $\eta(m) \geq c_j{\cdot}\eta(m) + \eta(f_j)$, and (ii) by the assumption of this item for those $j \in J$ such that $c_j \geq 1$, also $\eta(m) < c_j{\cdot}\eta(m) + \eta(f_j)$. It is obvious that $\eta(m)$ must be the smallest solution.
    Now consider the case where $\eta(m) < \frac{\eta(f_j)}{1-c_j}$ for some $j \in J$. In this case $r_1 = \bigvee_{j\in J|c_j<1}(\frac{\eta(f_j)}{1-c_j}) = \frac{\eta(f_{j'})}{1-c_{j'}}$ for some $j' \in J$, where j' is the index of the largest solution. It is straightforward to check that $\frac{\eta(f_{j'})}{1-c_{j'}}$ is a solution. It is also the smallest solution, which can be seen as follows. Suppose there were a smaller solution $r_2 < \frac{\eta(f_{j'})}{1-c_{j'}}$. Hence, $r_2 = \eta(m) \wedge \bigwedge_{j\in J}(c_j{\cdot}r_2 + \eta(f_j)) \geq c_{j'}{\cdot}r_2 + \eta(f_{j'})$. From this it follows that $r_2 \geq \frac{\eta(f_{j'})}{1-c_{j'}}$ contradicting that it is a smaller solution.
    It follows that $r_1 = r$ is the smallest solution. Furthermore, $r' = \eta(Sol^{\mu}_{X=e}) = \eta(U) = \eta(m \vee \bigvee_{j\in J|c_j<1}\frac{f_j}{1-c_j}) = r_1 = r$. Obviously, $r' \leq r$.
  - Now assume that there is a $j \in J$ with $c_j \geq 1$ such that $r_1 < \eta[X := r_1](c_j{\cdot}X + f_j)$. We show that $r = \infty$. Using the argumentation of the previous item, the smallest solution r is at least r_1. But clearly, r_1 is larger than the non-infinite solution of $X = \eta[X := r_1](c_j{\cdot}X + f_j)$ as by the assumption $r_1 > \frac{\eta(f_j)}{1-c_j}$. Note that if $c_j > 1$, this solution exists, and if $c_j = 1$ there is only a finite solution if $f_j = 0$, but in this latter case the assumption of this item is invalid. Hence, the only remaining minimal solution is $r = \infty$. Clearly, for any choice of r' it holds that $r' < r$.

Now we concentrate on case 2 for the minimal fixed-point of Lemma 9. We know that $r = \eta(Sol^{\mu}_{X=e})$ is the minimal solution for $\eta(Sol^{\mu}_{X=e})$ and we must show that there is an $r' \leq r$ such that $r' \geq \eta[X := r'](e)$. We take $r' = r$ leaving us with the obligation to show that $r \geq \eta[X := r](e)$.

We distinguish the following cases.

- Assume that there is some f_j such that $\eta(f_j) = \infty$. In that case $r = \infty$, which satisfies $\infty \geq \eta[X := \infty](e)$. Below we assume that $\eta(f_j) < \infty$ for all $j \in J$.
- Now assume that $\eta(m) = -\infty$. Note that for any $j \in J$ it is the case that $c_j \neq 0$ or $c'_j \neq 0$. In this case, $r = -\infty$ is the solution as $\eta[X := -\infty](e) = -\infty$ and this implies our proof obligation. So, in the steps below we assume that $\eta(m) > -\infty$.
- With the conditions above, if there is at least one $j \in J$ such that $c'_j = 1$, then $r = \infty$ is the fixed-point satisfying our proof obligation. Below we assume that for all $j \in J$ it holds that $c'_j = 0$.
- As all c'_j can be assumed to be 0, we can simplify the equation for X to:

$$\mu X = \bigvee_{j\in J}(c_j{\cdot}X + f_j) \vee m.$$

We find $\eta(U) = \eta(m \vee \bigvee_{j\in J|c_j<1} \frac{f_j}{1-c_j})$. If there is no $j \in J$ with $c_j \geq 1$ such that $\eta(f_j - (1-c_j)\cdot U) > 0$ we find that $r = \eta(Sol^{\mu}_{X=e}) = \eta(U)$. We show that $r \geq \eta[X := r](e)$. If $\eta(m) \geq \bigvee_{j\in J|c_j<1} \frac{\eta(f_j)}{1-c_j}$ then $r = \eta(m)$. For a $j \in J$ with $c_j < 1$ we find that $c_j\cdot\eta(m) + \eta(f_j) \leq \eta(m)$ as $\eta(m) \geq \frac{\eta(f_j)}{1-c_j}$. For a $j \in J$ with $c_j \geq 1$, we find by the condition above that $\eta(f_j + c_j\cdot U) \leq \eta(U)$, or in other words $\eta(f_j + c_j\cdot m) \leq \eta(m)$. So, $r = \eta(m) = \eta[X := r](e)$ as we had to show.
Otherwise, there is some $j' \in J$ with $c_{j'} < 1$ such that $\frac{\eta(f_{j'})}{1-c_{j'}} = \bigvee_{j\in J|c_j<1} \frac{\eta(f_j)}{1-c_j}$. In this case $r = \frac{\eta(f_{j'})}{1-c_{j'}}$. From the conditions, we can see that $r = \eta[X := r](e)$ as we had to show.

- Now assume that there is a $j \in J$ with $c_j \geq 1$ such that $\eta(f_j - (1-c_j)\cdot U) > 0$. In this case $r = \eta(Sol^{\mu}_{X=e}) = \infty$, clearly satisfying our proof obligation.

This finishes our proof for a minimal fixed-point equation. ◀

5 Relation to boolean equation systems

A boolean equation system (BES) is a restricted form of a real equation system where solutions can only be *true* or *false* [20]. Concretely, the syntax for expressions is

$$e \ ::= \ X \mid \mathit{true} \mid \mathit{false} \mid e \vee e \mid e \wedge e$$

where X is taken from some set $\mathcal{X}$ of variables [20]. A boolean equation system is a sequence of fixed-point equations $\sigma_1 X_1 = e_1, \ldots, \sigma_n X_n = e_n$ where σ_i are fixed-point operators, X_i are variables from $\mathcal{X}$ ranging over *true* and *false*, and e_i are boolean expressions.

We do not spell out the semantics of boolean equation systems, as it is similar to that of RESs. However, we believe that it is useful to indicate the relation with real equation systems.

The simplest embedding is where a given BES is literally transformed to a RES and *true* and *false* are interpreted as ∞ and $-\infty$. We consider a minimal fixed-point equation. The right-hand side can be rewritten to a simple conjunctive normal form. We write this in the shape of equation (1). So, $c_{ij} = 1$, $c'_{ij} = 0$, f_{ij} is absent and m_i does not contain X and can only be interpreted as $\pm\infty$. Exactly if J_i is not empty, X is present in conjunct i.

$$\mu X = \bigwedge_{i\in I}((\bigvee_{j\in J_i} X) \vee m_i).$$

The solution is given by equation (2), which can be simplified to:

$$\bigwedge_{i\in I}(eq_{-\infty}(m_i) \Rightarrow -\infty \diamond ((\bigvee_{j\in J_i} 0) \Rightarrow m_i \diamond \infty)) = \bigwedge_{i\in I} m_i = \bigwedge_{i\in I}((\bigvee_{j\in J_i} -\infty) \vee m_i).$$

The latter exactly coincides with the Gauß-elimination rule for BESs that says that in an equation $\mu X = e$, any occurrence of X in e can safely be replaced by *false*. For the maximal fixed-point operator, dual reasoning applies. As Gauß-elimination is a complete way to solve a BES with *true* and *false*, and exactly the same reduction works with the corresponding RES with ∞ and $-\infty$, this confirms that this interpretation works.

An alternative interpretation is given by taking two arbitrary constants c_{true} and c_{false} with as only constraint that $c_{true} > c_{false}$. A boolean equation system $\sigma_1 X_1 = e_1, \ldots, \sigma_n X_n = e_n$ is translated into $\sigma_1 X_1 = c_{false} \vee (c_{true} \wedge e_1), \ldots, \sigma_n X_n = c_{false} \vee (c_{true} \wedge e_n)$ of which the validity can be established in the same way as above.

6 Quantitative modal formulas and their translation to RESs

We can write quantitative modal formulas that yield a value instead of true and false. In the next section we provide examples of what can be expressed. Our formulas have the syntax

$$\phi \ ::= \ X \mid d \mid c{\cdot}\phi \mid \phi+\phi \mid \phi\vee\phi \mid \phi\wedge\phi \mid \langle a\rangle\phi \mid [a]\phi \mid \mu X.\phi \mid \nu X.\phi.$$

Here $d \in \hat{\mathbb{R}}$ and $c \in \mathbb{R}$ with $c > 0$ are constants, $X \in \mathcal{X}$ is a variable, and $a \in \mathcal{A}$ is an action from some set of actions $\mathcal{A}$. Although there are many similar logics around, we have not encountered this exact form before.

We evaluate these modal formulas on probabilistic LTSs. For a finite set of states S, we use distributions $d : S \to [0,1]$ where $d(s)$ is the probability to end up in state s. Distributions satisfy that $\sum_{s\in S} d(s) = 1$. The set of all distributions over S is denoted by $\mathcal{D}(S)$.

▶ **Definition 12.** *A probabilistic labelled transition system (pLTS) is a four-tuple $M = (S, \mathcal{A}, \to, d_0)$ where S is a finite set of states, $\mathcal{A}$ is a finite set of actions, the relation $\to \subseteq S\times\mathcal{A}\times\mathcal{D}(S)$ represents the transition relation, and $d_0 \in \mathcal{D}(S)$ is the initial distribution.*

We leave out the definition of the interpretation of quantitative modal formulas on probabilistic LTSs, as it is standard. Instead, we define the real equation system that is generated given a modal formula ϕ and a probabilistic labelled transition system $M = (S, \mathcal{A}, \to, d_0)$, following the translations in [20, 14, 21, 11]. The function $Eq(\phi)$ generates the required sequence of RES equations for ϕ and $rhs(s, \phi)$ yields the expression for the right-hand side of such an equation representing the value of ϕ in state s.

$$\begin{array}{ll}
Eq(X) = \epsilon, & rhs(s, X) = X_s,\\
Eq(d) = \epsilon, & rhs(s, d) = d,\\
Eq(c{\cdot}\phi) = Eq(\phi), & rhs(s, c{\cdot}\phi) = c{\cdot}rhs(s, \phi),\\
Eq(\phi_1+\phi_2) = Eq(\phi_1), Eq(\phi_2), & rhs(s, \phi_1+\phi_2) = rhs(s, \phi_1) + rhs(s, \phi_2),\\
Eq(\phi_1\vee\phi_2) = Eq(\phi_1), Eq(\phi_2), & rhs(s, \phi_1\vee\phi_2) = rhs(s, \phi_1) \vee rhs(s, \phi_2),\\
Eq(\phi_1\wedge\phi_2) = Eq(\phi_1), Eq(\phi_2), & rhs(s, \phi_1\wedge\phi_2) = rhs(s, \phi_1) \wedge rhs(s, \phi_2),\\
Eq(\langle a\rangle\phi) = Eq(\phi), & rhs(s, \langle a\rangle\phi) = \bigvee_{\{d\in\mathcal{D}(S)\mid s \xrightarrow{a} d\}} \sum_{s'\in S} d(s'){\cdot}rhs(s', \phi),\\
Eq([a]\phi) = Eq(\phi), & rhs(s, [a]\phi) = \bigwedge_{\{d\in\mathcal{D}(S)\mid s \xrightarrow{a} d\}} \sum_{s'\in S} d(s'){\cdot}rhs(s', \phi),\\
Eq(\mu X.\phi) = \langle \mu X_s = rhs(s,\phi) \mid s \in S\rangle, Eq(\phi), & rhs(s, \mu X.\phi) = X_s,\\
Eq(\nu X.\phi) = \langle \nu X_s = rhs(s,\phi) \mid s \in S\rangle, Eq(\phi). & rhs(s, \nu X.\phi) = X_s.
\end{array}$$

We use the notation $\langle \sigma X_s = e_s \mid s \in S\rangle$ for the sequence of all equations $\sigma X_s = e_s$ for all states $s \in S$.

The evaluation of a modal formula ϕ in M with initial distribution d_0 is the solution in $\hat{\mathbb{R}}$ of variable X_{init} in the RES $\mu X_{init} = (\sum_{s\in S} d_0(s){\cdot}rhs(s, \phi))$, $Eq(\phi)$. The use of the minimal fixed-point for the initial variable is of no consequence as X_{init} does not occur elsewhere in the equation system. A maximal fixed-point could also be used.

7 Applications

7.1 The longest a-sequence to a b-loop

We are interested in the longest sequence of actions a to reach a state where an infinite sequence of actions b can be done. The modal formula that expresses this is the following:

$$\mu X.(1 + \langle a\rangle X) \vee (0 \wedge \nu Y.\langle b\rangle Y).$$

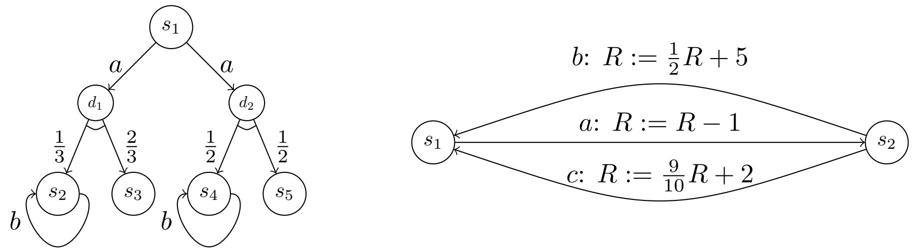

Figure 2 A probabilistic LTS with a loop/An LTS with rewards.

The last part with the maximal fixed-point $0 \wedge \nu Y.\langle b\rangle Y$ when evaluated in a state equals $-\infty$ if no infinite sequence of b's is possible. Otherwise, it evaluates to 0. The first part $1 + \langle a\rangle X$ yields 1 plus the maximum values of the evaluation of X in all states reachable by an action a. If no infinite b-sequence can be reached from such a state, this value is $-\infty$, and otherwise it represents the maximal number of steps to reach such an infinite b-sequence.

We evaluate this formula in the labelled transition system given at the right in Figure 1. This leads to the following real equation system where X_i and Y_i correspond to the value of X, resp. Y in state s_i. The solution of the equation system is written behind each equation.

$$\begin{array}{llcll}
\mu X_1 = (1 + (X_2 \vee X_3 \vee X_4 \vee X_6)) \vee (0 \wedge Y_1) & 2 & & \nu Y_1 = -\infty & -\infty \\
\mu X_2 = (1 + X_3) \vee (0 \wedge Y_2) & 1 & & \nu Y_2 = -\infty & -\infty \\
\mu X_3 = (1 + -\infty) \vee (0 \wedge Y_3) & 0 & & \nu Y_3 = Y_3 & \infty \\
\mu X_4 = (1 + X_5) \vee (0 \wedge Y_4) & -\infty & & \nu Y_4 = -\infty & -\infty \\
\mu X_5 = (1 + X_6) \vee (0 \wedge Y_5) & -\infty & & \nu Y_5 = -\infty & -\infty \\
\mu X_6 = (1 + -\infty) \vee (0 \wedge Y_6) & -\infty & & \nu Y_6 = -\infty & -\infty
\end{array}$$

We find that the longest sequence of actions a is 2, which matches our expectation.

7.2 The probability to reach a loop

We are interested in the probability to reach a b-loop. We apply it to the LTS at the left in Figure 2. Due to the non-determinism there are more paths to such loops, and we are interested in the path with the highest probability. This is expressed by the modal formula

$$\mu X.\langle a\rangle X \vee \langle b\rangle X \vee ((\nu Y.\langle b\rangle Y \vee 0) \wedge 1).$$

As we want a probability, we use $\_ \wedge 1$ and $\_ \vee 0$ to enforce that the solution is in $[0, 1]$. The formula $\nu Y.\langle b\rangle Y \vee 0$ yields ∞ if an infinite sequence of actions b is possible and 0 otherwise.

The translation of this formula on the labelled transition system in Figure 2 yields the following real equation system.

$$\begin{array}{lcllcl}
\mu X_1 = (\frac{1}{3}\cdot X_2 + \frac{2}{3}\cdot X_3) \vee (\frac{1}{2}\cdot X_4 + \frac{1}{2}\cdot X_5) \vee (Y_1 \wedge 1) & & & \nu Y_1 = -\infty \vee 0 & = & 0, \\
\qquad\qquad = \frac{1}{3} \vee \frac{1}{2} \vee 0 = \frac{1}{2}, & & & & & \\
\mu X_2 = X_2 \vee (Y_2 \wedge 1) & = & X_2 \vee 1 = 1, & \nu Y_2 = Y_2 & = & \infty, \\
\mu X_3 = -\infty \vee (Y_3 \wedge 1) & = & -\infty \vee 0 = 0, & \nu Y_3 = -\infty \vee 0 & = & 0, \\
\mu X_4 = X_4 \vee (Y_4 \wedge 1) & = & X_4 \vee 1 = 1, & \nu Y_4 = Y_4 & = & \infty, \\
\mu X_5 = -\infty \vee (Y_5 \wedge 1) & = & -\infty \vee 0 = 0, & \nu Y_5 = -\infty \vee 0 & = & 0.
\end{array}$$

This shows that the maximal probability to reach a b-loop is $\frac{1}{2}$.

7.3 Determining the reward of process behaviour

In Figure 2 at the right a labelled transition system is drawn, where a reward R is changed when a transition takes place. The transition labelled with action a costs one unit, b yields $\frac{1}{2}R + 5$ units, and the transition c adapts the reward by $\frac{9}{10}R + 2$. We want to know what the maximal stable reward is. This is expressed by the following formula:

$$\mu R.\langle a\rangle(R-1) \vee \langle b\rangle(\tfrac{1}{2}\cdot R+5) \vee \langle c\rangle(\tfrac{9}{10}\cdot R+2) \vee 0.$$

Note that we express this as the minimal reward larger than 0, which is the maximum of all individual rewards. Translating this to a real equation system yields

$$\mu R_1 = (R_2 - 1) \vee -\infty \vee -\infty \vee 0, \qquad \mu R_2 = -\infty \vee (\tfrac{1}{2}\cdot R_1 + 5) \vee (\tfrac{9}{10}R_1 + 2) \vee 0.$$

We solve this using Gauß-elimination. This means that the second equation is substituted in the first, which, after some straightforward simplifications, gives us

$$\mu R_1 = (\tfrac{1}{2}\cdot R_1 + 4) \vee (\tfrac{9}{10}\cdot R_1 + 1) \vee 0.$$

We solve this equation using the technique of Section 4, leading to:

$$R_1 = \frac{4}{1-\frac{1}{2}} \vee \frac{1}{1-\frac{9}{10}} \vee 0 = 10.$$

8 Conclusions and outlook

We introduce real equation systems (RESs) as the pendant of Boolean Equation Systems with solutions in the domain of the reals extended with $\pm\infty$. By a number of examples we show how this can be used to evaluate a wide range of quantitative properties of process behaviour.

We provide a complete method to solve RESs using an extension of what is called "Gauß-elimination" [21] to solve boolean equation systems. It shows that any RES can be solved by carrying out a finite number of substitutions. As solving RESs generalises solving BESs, and Gauß-elimination on BESs is exponential, our Gauß-elimination technique can also lead to exponential growth of intermediate terms. A prototype implementation shows that depending on the nature of the system being analysed, this may or may not be an issue. For instance, analysing the Game of the Goose [12] or The Ant on a Grid [6], are practically undoable with the method proposed here, while the Lost Boarding Pass Problem [10] is easily solved, even for planes with 100,000 passengers.

We believe that the next step is to come up with algorithms that are more efficient in practice than Gauß-elimination. This is motivated by the situation with BESs where for instance the recursive algorithm [23, 26] turns out to be practically far more efficient than Gauß-elimination [9].

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