


Sparse Graphs of Twin-Width 2 Have Bounded Tree-Width

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Abstract

Twin-width is a structural width parameter introduced by Bonnet, Kim, Thomassé and Watrigant [FOCS 2020]. Very briefly, its essence is a gradual reduction (a contraction sequence) of the given graph down to a single vertex while maintaining limited difference of neighbourhoods of the vertices, and it can be seen as widely generalizing several other traditional structural parameters. Having such a sequence at hand allows to solve many otherwise hard problems efficiently. Our paper focuses on a comparison of twin-width to the more traditional tree-width on sparse graphs. Namely, we prove that if a graph G of twin-width at most 2 contains no $K_{t,t}$ subgraph for some integer t , then the tree-width of G is bounded by a polynomial function of t . As a consequence, for any sparse graph class \mathcal{C} we obtain a polynomial time algorithm which for any input graph $G \in \mathcal{C}$ either outputs a contraction sequence of width at most c (where c depends only on \mathcal{C}), or correctly outputs that G has twin-width more than 2. On the other hand, we present an easy example of a graph class of twin-width 3 with unbounded tree-width, showing that our result cannot be extended to higher values of twin-width.

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1 Introduction

Twin-width is a relatively new structural width measure of graphs and relational structures introduced in 2020 by Bonnet, Kim, Thomassé and Watrigant [13]. Informally, the twin-width of a graph measures how diverse the neighbourhoods of the graph vertices are. For instance, *cographs* – the graphs which can be built from singleton vertices by repeated operations of a disjoint union and taking the complement – are exactly the graphs of twin-width 0, which means that the graph can be brought down to a single vertex by successively identifying twin vertices. (Two vertices x and y are called *twins* in a graph G if they have the same neighbours in $V(G) \setminus \{x, y\}$.) Hence the name, *twin-width*, for the parameter.



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Importance of this new concept is clearly witnessed by numerous recent papers on the topic, such as the follow-up series [7–11, 14] and more related research papers which consider twin-width from algorithmic [5, 25], combinatorial [2, 3, 23], structural [4, 15] and logical [4, 19] perspective.

In particular, twin-width, as a structural width parameter, has naturally algorithmic applications in the parameterized complexity area. Among the most important ones we mention that the first order (FO) model checking problem – that is, deciding whether a fixed first-order sentence holds in an input graph – can be solved in linear FPT-time [14]. This and other algorithmic applications assume that a contraction sequence of bounded width is given alongside with the input graph, since in general we do not know how to construct such a sequence efficiently. Consequently, finding an efficient algorithm for computing twin-width of input graph G together with its twin-width decomposition (known as a *contraction sequence* of G) is a central problem in the area. Currently, very little is known about this problem. It is known that one can check whether a graph has twin-width 0 and 1 and compute the corresponding contraction-sequence in polynomial time [12]. On the other hand, in general the problem of deciding the exact value of twin-width is NP-hard, and in particular, deciding whether a graph has twin-width 4 is NP-hard [5]. This means that even for fixed k , the best one can hope for is an approximation algorithm which for given input graph G either correctly outputs that G has twin-width more than k or produces a contraction sequence of G of width at most k' , where k' is some fixed number with $k < k'$. However, to the best of our knowledge, no such efficient algorithm is currently known, even for graph classes of twin-width 2.

The importance and popularity of twin-width largely stems from the fact that it generalizes many well-known graph-theoretic concepts. For example, graph classes of bounded clique-width, planar graphs and more generally graph classes defined by excluding a fixed minor, posets of bounded width and various subclasses of geometric graphs have been shown to have bounded twin-width [14, 22]. In many cases, the proof of having bounded twin-width is constructive, meaning that for such class there exists an integer d and a polynomial time algorithm which takes a graph G as input and outputs a contraction sequence of G of width at most d .

Apart from establishing that some well-known graph classes have bounded twin-width, there are also results relating twin-width to various graph classes from the other direction: one considers graph classes of bounded twin-width which are restricted in some sense, and proves that such classes fall within some well-studied framework. Prime examples of this approach are results stating that $K_{t,t}$ -free graph classes of bounded twin-width have bounded expansion [8, 18] or that stable graph classes of bounded twin-width have structurally bounded expansion [20]. Results of this form allow us to use well-established structural and algorithmic tools for (structurally) bounded expansion to restricted classes of bounded twin-width.

Our research is motivated by the following conjecture, which also fits into this line of research and about which we learned from É. Bonnet.

► **Conjecture 1.1.** *Let \mathcal{C} be a class of graphs of twin-width at most 3 such that there exists t such that no $G \in \mathcal{C}$ contains $K_{t,t}$ as a subgraph. Then \mathcal{C} has bounded tree-width.*

For graph classes of twin-width at most one, we can easily argue that Conjecture 1.1 is true as follows. It is known [11] that if a graph class \mathcal{C} is such that every $G \in \mathcal{C}$ has a contraction sequence in which red components (see Section 2) have bounded size, then \mathcal{C} has bounded clique-width. Since in a contraction sequence of width one each red component has

size one, this implies that graph classes of twin-width one have bounded clique-width. In combination with the fact that graph classes of bounded clique-width which exclude $K_{t,t}$ as a subgraph have bounded tree-width, this proves the result.

However, for twin-width 2 or 3 we cannot assume that red components of trigraphs occurring in contraction sequences have bounded size, and so to attack Conjecture 1.1 one has to employ a more fine-tuned approach by directly analyzing contraction sequences.

Our contribution

We confirm Conjecture 1.1 for the case of graph classes of twin-width 2 and disprove the conjecture for graph classes of twin-width 3. Namely on the positive side we prove the following.

► **Theorem 1.2.** *Let t be an integer and let \mathcal{C} be a class of graphs of twin-width at most 2 such that no $G \in \mathcal{C}$ contains $K_{t,t}$ as a subgraph. Then the tree-width of \mathcal{C} is bounded by a polynomial function of t , namely at most $\mathcal{O}(t^{20})$.*

Theorem 1.2 allows us to apply the existing tools for bounded tree-width to sparse classes of twin-width 2, making many hard problems efficiently solvable on such classes. Moreover, Theorem 1.2 also leads to the following algorithmic corollary.

► **Corollary 1.3.** *Let \mathcal{C} be a class of graphs such that there exists t such that no $G \in \mathcal{C}$ contains $K_{t,t}$ as a subgraph. Then there exists a constant c depending on \mathcal{C} , and a polynomial-time algorithm which for every $G \in \mathcal{C}$ either outputs a contraction sequence of G of width at most c , or correctly outputs that G has twin-width more than 2.*

On the negative side, we exhibit an example of a graph class \mathcal{C} of twin-width 3 such that no $G \in \mathcal{C}$ contains a $K_{2,2} \simeq C_4$ subgraph, and that \mathcal{C} has unbounded tree-width.

2 Related Definitions and Tools

We use standard graph-theoretic terminology and notation. All graphs considered in this paper are finite and simple, i.e., without loops and multiple edges. For the sake of completeness, we include the following folklore definition and tool.

► **Definition 2.1** (Tree-width [26]). *A tree-decomposition of a graph G is a pair (X, T) where T is a tree, whose vertices we call nodes, and $X = \{X_i \mid i \in V(T)\}$ is a collection of subsets of $V(G)$ such that*

1. $\bigcup_{i \in V(T)} X_i = V(G)$,
2. for each edge $vw \in E(G)$, there is $i \in V(T)$ such that $v, w \in X_i$ and,
3. for each $v \in V(G)$ the set of nodes $\{i \mid v \in X_i\}$ forms a subtree of T .

The width of tree-decomposition $(\{X_i \mid i \in V(T)\}, T)$ is equal to $\max_{i \in V(T)} \{|X_i| - 1\}$. The tree-width of a graph G is the minimum width over all tree-decompositions of G .

For a graph G and $A, B \subseteq V(G)$, an $A - B$ path is a path with one endvertex in A and the other endvertex in B . We will use Menger's theorem (see for example [17]).

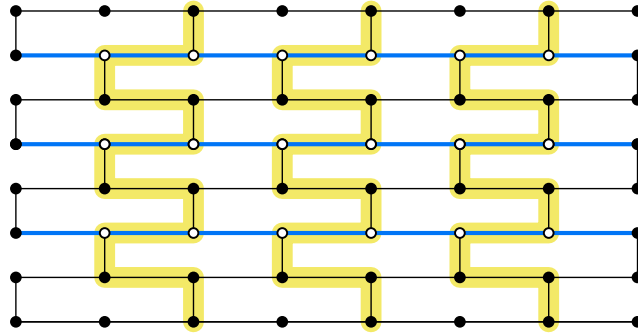
► **Theorem 2.2** (Menger's theorem). *Let G be a graph and $A, B \subseteq V(G)$. The minimum size of a set $S \subseteq V(G)$ such that there is no $A - B$ path in $G - S$ is equal to the maximum number of vertex-disjoint $A - B$ paths in G .*

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For an integer N , the $N \times N$ wall (or *hexagonal grid*) is the graph consisting of N disjoint paths P_1, \dots, P_N , each P_i with vertices v_1^i, \dots, v_N^i in this order in P_i , together with the edges given by the following rule: if both $i, j \in \{1, \dots, N\}$ have the same parity and $i < N$, then v_j^i is adjacent to v_j^{i+1} .

For an integer N , a graph H is called an $N \times N$ *cubic mesh* if the maximum degree of H is 3 and the following holds; we can write $H = Q_1 \cup Q_2$, where each Q_i , $i = 1, 2$, is formed as a vertex-disjoint union of N paths, the ends of paths of Q_i are disjoint from $V(Q_{3-i})$ for $i = 1, 2$, and every component-path of Q_1 intersects every component-path of Q_2 in precisely one (common) subpath. These paths of Q_1 (resp. of Q_2) are called the *rows* (resp. *columns*) of the mesh H , and the vertices of H of degree 3 are called the *branching vertices* of H . The intersection of any row and any column of H must be a subpath of nonzero length since $\Delta(H) = 3$, and it contributes two branching vertices.

► **Observation 2.3.** *Any subdivision of the classical $(2N + 2) \times (2N + 2)$ wall (hexagonal grid) contains an $N \times N$ cubic mesh as a subgraph. See Figure 1.*



■ **Figure 1** Example of an 8×8 wall. We obtain a 3×3 cubic mesh by considering the yellow paths as columns and the blue horizontal paths as rows. Hollow vertices are the branching vertices of this cubic mesh.

The notion of *twin-width* can in general be considered over arbitrary binary relational structures of a finite signature, but here we will define it and deal with it for only finite *simple graphs*. It is based on the following concept.

A *trigraph* is a simple graph G in which some edges are marked as *red*, and with respect to the red edges only, we naturally speak about *red neighbours* and *red degree* in G (while the terms neighbour and degree without “red” refer to all edges of G inclusive of red ones, and edges which are not red are also called *black*). For a pair of (possibly not adjacent) vertices $x_1, x_2 \in V(G)$, we define a *contraction* of the pair x_1, x_2 as the operation creating a trigraph G' which is the same as G except that x_1, x_2 are replaced with a new vertex x_0 (said to *stem* from x_1, x_2) such that:

- the (full) neighbourhood of x_0 in G' (i.e., including the red neighbours), denoted by $N_{G'}(x_0)$, equals the union of the neighbourhoods $N_G(x_1)$ of x_1 and $N_G(x_2)$ of x_2 in G except x_1, x_2 themselves, that is, $N_{G'}(x_0) = (N_G(x_1) \cup N_G(x_2)) \setminus \{x_1, x_2\}$, and
- the red neighbours of x_0 , denoted here by $N_{G'}^r(x_0)$, inherit all red neighbours of x_1 and of x_2 and add those in $N_G(x_1) \Delta N_G(x_2)$, that is, $N_{G'}^r(x_0) = (N_G^r(x_1) \cup N_G^r(x_2) \cup (N_G(x_1) \Delta N_G(x_2))) \setminus \{x_1, x_2\}$, where Δ denotes the symmetric set difference.

► **Definition 2.4** (Contraction sequence and twin-width [13]). A contraction sequence of a trigraph G is a sequence of successive contractions turning G into a single vertex, and its width d is the maximum red degree of any vertex in any trigraph of the sequence. We say that G has or admits a d -contraction sequence if there is a contraction sequence of G of width at most d . The twin-width of a trigraph G is the minimum width over all possible contraction sequences of G .

To define a contraction sequence and the twin-width of an ordinary graph G , we consider G as a trigraph with no red edges. In a summary, a graph has twin-width at most d , if and only if it admits a d -contraction sequence.

For our purpose, the following “inverted” view of a contraction sequence will be useful.

► **Definition 2.5** (Uncontraction sequence). A partitioned graph is a graph G associated with an unordered vertex partition $\mathcal{P} = (P_1, \dots, P_m)$ of $V(G)$. The partitioned trigraph of (G, \mathcal{P}) is a trigraph H on the vertex set $V(H) = \mathcal{P}$ such that $\{P_1, P_2\} \in E(H)$ if and only if G contains an edge from P_1 to P_2 , and this edge $\{P_1, P_2\}$ is red if and only if not all pairs of $P_1 \times P_2$ are edges of G . An uncontraction sequence of an n -vertex graph G is a sequence of partitioned graphs (G, \mathcal{P}^i) for $i = 1, \dots, n$, where $\mathcal{P}^1 = \{V(G)\}$ and \mathcal{P}^n is the partition of $V(G)$ into singletons, and for $1 < i \leq n$ the partition \mathcal{P}^i is obtained from \mathcal{P}^{i-1} by splitting an arbitrary one part of \mathcal{P}^{i-1} into two.

It is easy to observe that if G_0, G_1, \dots, G_{n-1} is a contraction sequence of the n -vertex graph $G = G_0$, then for the trigraph G_{n-k} (on k vertices) we may form the corresponding vertex partition $\mathcal{P}^k = (P_1^k, \dots, P_k^k)$ of $V(G)$ by setting P_i^k to be the set of vertices of G contracted into $w_i \in V(G_{n-k})$, $i = 1, \dots, k$. The partitioned trigraph of (G, \mathcal{P}^k) is hence isomorphic to the trigraph G_{n-k} of the former contraction sequence, and these two possible approaches to twin-width in Definition 2.4 and Definition 2.5 exactly coincide.

3 Proof of Theorem 1.2

Our proof proceeds by contradiction. Thus, we assume that there is a $K_{t,t}$ -free graph G of large tree-width which has twin-width at most 2. We then proceed in two steps:

- **Step I:** Since G has large tree-width, it has to contain a subdivision of a large wall, and hence a large cubic mesh, as a subgraph. Using this and the assumption that G has an uncontraction sequence of width at most 2, we show that there has to be a point in time during the uncontraction sequence such that there are four parts X_1, X_2, X_3, X_4 which form a red path in the corresponding trigraph and there are many disjoint paths of G fully contained in $X_1 \cup X_2 \cup X_3 \cup X_4$ with one endpoint in X_1 and the other in X_4 .
- **Step II:** Using a carefully chosen invariant we show that the structure found in Step I can be maintained indefinitely during the subsequent steps of the uncontraction sequence (still of width at most 2). This yields a contradiction, since the final partition of the uncontraction sequence consists of singletons and there are no red edges.

So, we actually prove the following alternative formulation of Theorem 1.2.

► **Theorem 3.1.** If G is a simple graph not containing a $K_{t,t}$ subgraph, but containing a subgraph (not necessarily induced) $H \subseteq G$ which is an $N \times N$ cubic mesh for $N = 16 \cdot (13t)^2$, then G is of twin-width at least 3.

3.1 Proving Step I

We start with a trivial claim which is crucial in our arguments:

► **Observation 3.2.** *Assume a partitioned graph (G, \mathcal{P}) such that G has no $K_{t,t}$ subgraph, and $X_1, X_2 \in \mathcal{P}$ such that $|X_1|, |X_2| \geq t$. Then the partitioned trigraph of (G, \mathcal{P}) cannot contain a black edge $\{X_1, X_2\}$. In other words, whenever G has an edge from X_1 to X_2 , there is a red edge $\{X_1, X_2\}$ in the partitioned trigraph of (G, \mathcal{P}) .*

The first step in the proof of Theorem 1.2 is precisely formulated and proved next.

► **Lemma 3.3.** *Let t be an integer. Assume that G is a simple graph not containing a $K_{t,t}$ subgraph, but containing a subgraph (not necessarily induced) $H \subseteq G$ which is an $N \times N$ cubic mesh, where $N = 16 \cdot (13t)^2$. If there is an uncontraction sequence (G, \mathcal{P}^i) for $i = 1, \dots, |V(G)|$ of width at most 2, then there exists $m \in \{1, \dots, |V(G)|\}$ such that the following holds:*

- *There are parts $X_1, X_2, X_3, X_4 \in \mathcal{P}^m$ with $|X_i| \geq t$ for each $i \in \{1, \dots, 4\}$ which induce a red path in this order in the partitioned trigraph of (G, \mathcal{P}^m) , and there is no edge between X_1 and X_4 .*
- *The set $X := X_1 \cup X_2 \cup X_3 \cup X_4$ induces in G a subgraph containing $s \geq 4t$ pairwise vertex-disjoint paths from X_1 to X_4 .*

Proof. Set $k := 13t$. Let $m \in \{1, \dots, |V(G)|\}$ be the least index such that every part of \mathcal{P}^m contains less than $4k^2$ branching vertices of H . Then there is a part $Z \in \mathcal{P}^m$ such that Z contains at least $2k^2$ branching vertices of H (one of the two having resulted by the last splitting before \mathcal{P}^m). If Z intersected less than k rows and less than k columns of H , by the condition on intersecting rows and columns we would have less than $2k^2$ branching vertices in Z altogether. Hence, up to symmetry, we may assume that Z hits at least k rows of H . Moreover, since $N = 16 \cdot (13t)^2 = 16 \cdot k^2$, we have that Z contains branching vertices of less than $4k^2 = N/4$ rows of H .

Let $L_2, L_1, R_1, R_2 \in \mathcal{P}^m$ denote the (at most) four parts that are connected to Z by a red path of length ≤ 2 in the partitioned trigraph of (G, \mathcal{P}^m) . That is, we have a red path on, (L_2, L_1, Z, R_1, R_2) in this order (for simplicity, we silently ignore if some of the parts do not exist). Again, each of L_1, L_2, R_1, R_2 contains branching vertices of less than $4k^2 = N/4$ rows of H . Altogether, every row of H hit by Z has at least $2N - 5 \cdot N/4 = 3 \cdot N/4 > 0$ branching vertices not contained in $\bigcup \mathcal{Z}$ where $\mathcal{Z} = \{L_2, L_1, Z, R_1, R_2\}$. We have thus got, as subpaths of the rows hit by Z , a collection $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ of k vertex-disjoint paths in G which connect Z to parts in $\mathcal{P}^m \setminus \mathcal{Z}$. Let the paths in \mathcal{Q} be chosen as inclusion-minimal and for any $Q_i \in \mathcal{Q}$ let a_i and b_i be the endpoints of Q_i .

We now proceed assuming that all L_1, L_2, R_1, R_2 have size at least t , the case when one of $\{L_1, L_2\}$ or one of $\{R_1, R_2\}$ is smaller than t is addressed below. Every path $Q \in \mathcal{Q}$, by its minimality, connects a vertex of Z to a vertex of $Y \in \mathcal{P}^m \setminus \mathcal{Z}$ where Y is a neighbour (red or black) of the set \mathcal{Z} in the partitioned trigraph of (G, \mathcal{P}^m) . For any $X \in \{L_2, L_1, Z, R_1, R_2\}$, since $|X| \geq t$, the union of the parts adjacent to X by black edges in the partitioned trigraph is of cardinality less than t , or we have a $K_{t,t}$ subgraph. Together, at most $5t$ of the paths of \mathcal{Q} in G may end in parts $Y \in \mathcal{P}^m \setminus \mathcal{Z}$ which are not red neighbours of \mathcal{Z} in the partitioned trigraph. Since we have at most two red neighbours of parts of \mathcal{Z} among the parts of $\mathcal{P}^m \setminus \mathcal{Z}$ (namely, the “outside” red neighbours of L_2 and of R_2), up to symmetry, the (unique) red neighbour $L_3 \in \mathcal{P}^m \setminus \mathcal{Z}$ of L_2 contains ends of at least $(k - 5t)/2 = 4t$ of the paths in \mathcal{Q} . In particular, L_3 has to exist, and L_3 is not adjacent to Z in the partitioned trigraph by Observation 3.2. We thus conclude by choosing $X_1 = L_3, X_2 = L_2, X_3 = L_1$ and $X_4 = Z$.

Next, we address the case when exactly one of $\{L_1, L_2\}$ or $\{R_1, R_2\}$ contains W such that $|W| < t$. Without loss of generality assume that $W \in \{R_1, R_2\}$ and so both L_1 and L_2 have size at least t . If $W = R_1$, set $\mathcal{Z}' := \{L_2, L_1, Z\}$, otherwise set $\mathcal{Z}' := \{L_2, L_1, Z, R_1\}$. Every path $Q \in \mathcal{Q}$, by its minimality, connects a vertex of Z to a vertex of $Y \in \mathcal{P}^m \setminus \mathcal{Z}'$ where Y is a neighbour (red or black) of the set \mathcal{Z}' in the partitioned trigraph of (G, \mathcal{P}^m) . For any X in \mathcal{Z} , since $|X| \geq t$, the union of the parts adjacent to X by black edges in the partitioned trigraph is of cardinality less than t , or we have a $K_{t,t}$ subgraph. Together, at most $4t$ of the paths of \mathcal{Q} in G may end in parts $Y \in \mathcal{P}^m \setminus \mathcal{Z}'$ which are not red neighbours of \mathcal{Z}' in the partitioned trigraph, and at most t paths of \mathcal{Q} can go through W . Ignoring all these at most $5t$ paths, all the remaining $|\mathcal{Q}| - 5t > 4t$ paths have to end in the red neighbor of L_2 ; call this neighbor L_3 . We thus again conclude by choosing $X_1 = L_3$, $X_2 = L_2$, $X_3 = L_1$ and $X_4 = Z$.

Finally, we consider the case when there is $W_L \in \{L_1, L_2\}$ with $|W_L| < t$ and $W_R \in \{R_1, R_2\}$ with $|W_R| < t$. We will show that this leads to a contradiction. We first fix the choice of W_L and W_R more precisely. If both L_1 and L_2 have size less than t , then we choose L_1 as W_L and similarly, if both R_1 and R_2 have size less than t , we choose R_1 as W_R . Then in the path $L_2L_1ZR_1R_2$ all parts between W_L and W_R have size at least t . Since each part X between W_L and W_R has size at least t , the union of the parts adjacent to X by black edges has size less than t , as otherwise we have a $K_{t,t}$ as a subgraph. Thus, the union of W_L , W_R , and all black neighbors of (at most 3) parts between W_L and W_R has size at most $5t$. This means that there are at most $5t$ vertices which separate $\{a_1, \dots, a_k\}$ from $\{b_1, \dots, b_k\}$ (recall that for any $i \in [k]$ we denote by a_i and b_i the endpoints of Q_i). Since there are $k > 5t$ disjoint paths between $A := \{a_1, \dots, a_k\}$ and $B := \{b_1, \dots, b_k\}$, this is a contradiction to Menger's theorem. \blacktriangleleft

► **Observation 3.4.** *From Observation 3.2 it follows that each of the s paths from X_1 to X_4 claimed in Lemma 3.3 must also intersect X_2 and X_3 (possibly many times back and forth).*

3.2 Proving Step II

For our convenience, we introduce the following notation. Given a graph G , an uncontraction sequence (G, \mathcal{P}^i) for $i = 1, \dots, |V(G)|$, an integer $j \in \{1, \dots, |V(G)|\}$, and $X \in \mathcal{P}^j$, we define $N_j^b(X)$ as the set of parts of \mathcal{P}^j that have a black edge to X in the partitioned trigraph of (G, \mathcal{P}^j) ; we call this set *the black neighborhood of X in (G, \mathcal{P}^j)* . Also, we set $\|N_j^b(X)\| := \sum_{Y \in N_j^b(X)} |Y|$ to be the total number of vertices of G in any part of the black neighborhood of X in (G, \mathcal{P}^j) .

► **Lemma 3.5.** *Let G be an arbitrary simple graph not containing a $K_{t,t}$ as a subgraph, and let (G, \mathcal{P}^i) for $i = 1, \dots, |V(G)|$ be an uncontraction sequence for G . Suppose that, for some $m \in \{1, \dots, |V(G)|\}$, there are parts $X_1, X_2, X_3, X_4 \in \mathcal{P}^m$ with $|X_i| \geq t$ for each $i \in \{1, \dots, 4\}$ such that:*

- X_1, X_2, X_3, X_4 induce a red path in this order in the partitioned trigraph of (G, \mathcal{P}^m) ,
- there is no edge between X_1 and X_4 , and
- the set $X_1 \cup X_2 \cup X_3 \cup X_4$ induces in G a subgraph containing $s \geq 4t$ pairwise vertex-disjoint paths from X_1 to X_4 .

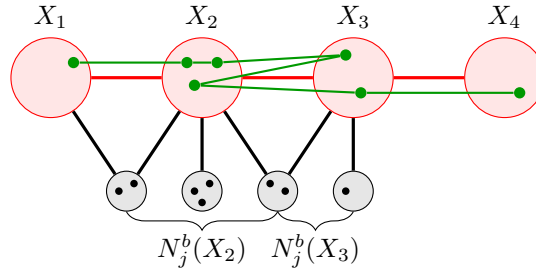
Then the width of this uncontraction sequence is greater than 2.

Proof. For a contradiction, suppose that the width of the considered uncontraction sequence of G is at most 2. We are going to formulate an invariant which is true for (G, \mathcal{P}^m) , and which will remain true at every subsequent step of the uncontraction sequence. Since this invariant, at the same time, will preclude the finest partition into singletons, the assumed sequence of width ≤ 2 cannot exist.

Invariant. At step $j \geq m$ of the uncontraction sequence, in the graph (G, \mathcal{P}^j) and its partitioned trigraph, the following holds. There are 4 parts $X_1, X_2, X_3, X_4 \in \mathcal{P}^j$, each of size at least t , forming a red path in this order in the partitioned trigraph of (G, \mathcal{P}^j) , and parts X_1 and X_4 are not adjacent. Denote by s_j the maximum number of vertex-disjoint paths in $G[X_1 \cup X_2 \cup X_3 \cup X_4]$, starting in X_1 and ending in X_4 . Then $s_j + \|N_j^b(X_2)\| + \|N_j^b(X_3)\| \geq 4t$.

Note that in the base case we have $s_m \geq 4t$, so the invariant is trivially satisfied for $j = m$. Also, without loss of generality we can assume that all the vertex-disjoint paths in $G[X_1 \cup X_2 \cup X_3 \cup X_4]$ are inclusion-wise minimal; so in particular, each path contains exactly one (starting) vertex in X_1 and exactly one (ending) vertex in X_4 . See also an informal illustration in Figure 2.

Now suppose the invariant holds for some $j \in \{m, \dots, |V(G)| - 1\}$ and let us prove that it is also preserved after the j -th uncontraction. First observe that for each $i \in \{1, 2, 3, 4\}$, G contains a bipartite clique with sides X_i and $\bigcup N_j^b(X_i)$. Since $|X_i| \geq t$ and G does not contain $K_{t,t}$ as a subgraph, we have $\|N_j^b(X_i)\| \leq t - 1$. Therefore, $s_j \geq 4t - (t - 1) - (t - 1) \geq 2t$. Each of the s_j disjoint paths in $G[X_1 \cup X_2 \cup X_3 \cup X_4]$ must intersect each of the sets X_1, X_2, X_3, X_4 (Observation 3.4), so actually $|X_i| \geq s_j \geq 2t$ for each $i \in \{1, 2, 3, 4\}$.



■ **Figure 2** Illustration of 4 big parts X_1, X_2, X_3, X_4 forming red path (red), an example of one of s_j paths (green), and parts in black neighbourhoods $N_j^b(X_2), N_j^b(X_3)$ (black).

We now analyze all the possible cases for an uncontraction step. Denote by $x \in \mathcal{P}^j$ the part that is split into new parts $y, z \in \mathcal{P}^{j+1}$. Also select some $s_j \geq 2t$ inclusionwise-minimal pairwise vertex-disjoint paths P_1, \dots, P_{s_j} from X_1 to X_4 in $G[X_1 \cup X_2 \cup X_3 \cup X_4]$. By minimality, Observation 3.2 and Observation 3.4, each path has its first vertex in X_1 , its second vertex in X_2 , its penultimate vertex in X_3 , and its last vertex in X_4 . Also notice that there cannot be any edges between X_1 and X_3 , or any edges between X_2 and X_4 (otherwise, Observation 3.2 would apply and the red degree of X_2 or X_3 would be too large). Recall also there are no edges between X_1 and X_4 .

If $x \notin \{X_1, X_2, X_3, X_4\}$, the black neighbourhood of X_2 and X_3 in the partitioned trigraph can only increase; also, $s_{j+1} = s_j$ as the vertex-disjoint paths in $G[X_1 \cup X_2 \cup X_3 \cup X_4]$ in (G, \mathcal{P}^j) are preserved in the uncontraction step. Hence, the invariant is satisfied.

Suppose $x = X_1$ or $x = X_4$. Without loss of generality, $x = X_1$ (or suppose the red path is actually X_4, X_3, X_2, X_1), as illustrated in Figure 3a. Since at the j -th step, there were at least $s_j \geq 2t$ inclusion-wise minimal pairwise vertex-disjoint paths from X_1 to X_4 in $G[X_1 \cup X_2 \cup X_3 \cup X_4]$, at least t of these start in y or at least t of these start in z ; again without loss of generality, assume that y contains at least t starts of the paths (so naturally $|y| \geq t$). As each of those paths have its second vertex in X_2 , we know that yX_2 is an edge in the partitioned trigraph of (G, \mathcal{P}^{j+1}) ; and it must be a red edge, because both y and X_2 have size at least t and Observation 3.2 applies. We now claim that the red path y, X_2, X_3, X_4 preserves the invariant in (G, \mathcal{P}^{j+1}) . Since $y, z \subseteq X_1$, both y and z are non-adjacent to X_3

and X_4 . Observe also that z cannot be red-adjacent to X_2 in (G, \mathcal{P}^{j+1}) as in this case, the red degree of X_2 would be at least 3. However, z can be either black-adjacent or non-adjacent to X_2 . If z is non-adjacent to X_2 , then z cannot contain the start of any of the s_j paths P_i as the second vertex of each such path is in X_2 . Hence we do not lose any path P_i by removing z from $X_1 \cup X_2 \cup X_3 \cup X_4$, so $s_{j+1} = s_j$ and the invariant still holds. If z is black-adjacent to X_2 , then by a counting argument z contains at most $|z|$ starts of the considered paths P_i ; hence, $s_{j+1} \geq s_j - |z|$. However, the number of vertices of G in the black neighborhood of X_2 also increases by $|z|$, because z was split out of X_1 which was a red neighbour of X_2 . Hence the inequality from the invariant is satisfied:

$$s_{j+1} + \|N_{j+1}^b(X_2)\| + \|N_{j+1}^b(X_3)\| \geq (s_j - |z|) + (\|N_j^b(X_2)\| + |z|) + \|N_j^b(X_3)\| \geq 4t.$$

It is also easy to verify the remaining conditions of the invariant.

It remains to consider $x = X_2$ or $x = X_3$. Without loss of generality, $x = X_2$ (or suppose the red path X_4, \dots, X_1). We first prove that at least one of y, z is red-adjacent to X_1 in (G, \mathcal{P}^{j+1}) . Assume otherwise; then y is either non-adjacent to X_1 (and then $|N_G(X_1) \cap y| = 0$) or black-adjacent to X_1 (and then $|y| \leq t - 1$ by Observation 3.2, so $|N_G(X_1) \cap y| \leq t - 1$). Analogously, $|N_G(X_1) \cap z| \leq t - 1$. Since $X_2 = y \cup z$, we conclude that $|N_G(X_1) \cap X_2| \leq 2t - 2$. But each of $s_j \geq 2t$ vertex-disjoint paths P_1, \dots, P_{s_j} have two consecutive vertices in X_1 and X_2 , so $|N_G(X_1) \cap X_2| \geq s_j - a$ contradiction. Hence at least one of y, z is red-adjacent to X_1 . Repeating the same argument with X_3 instead of X_1 implies that at least one of y, z is red-adjacent to X_3 . Additionally, X_3 is already red-adjacent to X_4 , so exactly one of y, z is red-adjacent to X_3 . Without loss of generality, assume that y is red-adjacent to X_3 . We now consider two separate cases, depending on whether y is red-adjacent to X_1 .

First assume that y is red-adjacent to X_1 , as illustrated in Figure 3b. We claim that the red path X_1, y, X_3, X_4 preserves the invariant in (G, \mathcal{P}^{j+1}) . Observe that z cannot be red-adjacent to y due to the red degree condition of y in (G, \mathcal{P}^{j+1}) . If z is non-adjacent to y and X_3 , no inclusion-wise minimal path from X_1 to X_4 in $G[X_1 \cup X_2 \cup X_3 \cup X_4]$ can be routed through z as all neighbors of z in $X_1 \cup X_2 \cup X_3 \cup X_4$ are in X_1 . Then $s_{j+1} = s_j$ and it is easy to verify the remaining parts of the invariant. Next suppose z is black-adjacent to X_3 ; then by Observation 3.2 we have $|z| \leq t - 1$, hence $|y| = |X_2| - |z| \geq t$. Also observe that at most $|z|$ paths P_i intersect z , so $s_{j+1} \geq s_j - |z|$. Therefore, the inequality from the invariant is preserved:

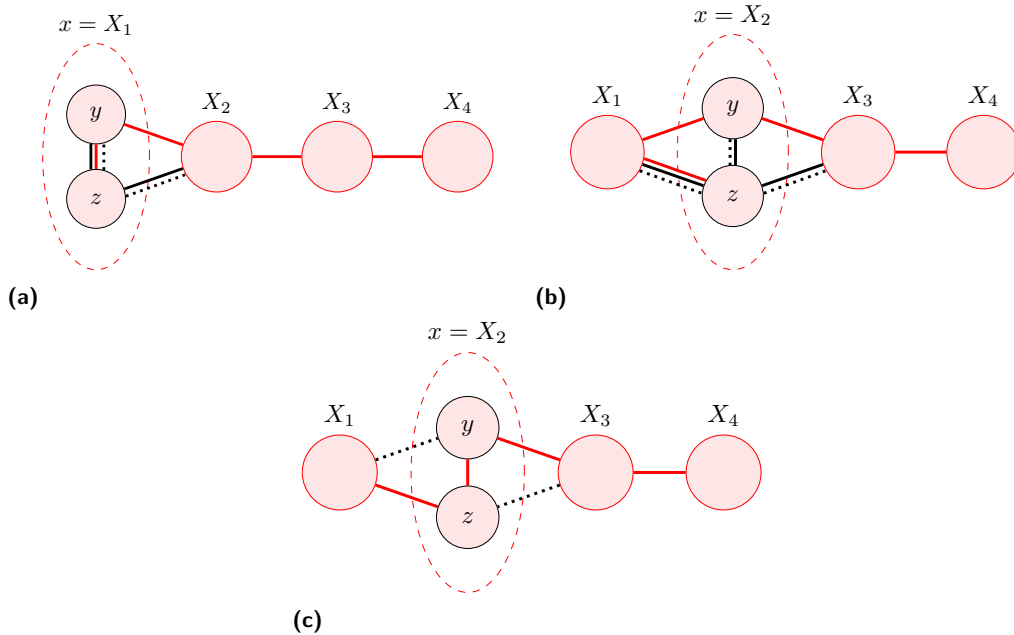
$$s_{j+1} + \|N_{j+1}^b(X_2)\| + \|N_{j+1}^b(X_3)\| \geq (s_j - |z|) + \|N_j^b(X_2)\| + (\|N_j^b(X_3) + |z|\|) \geq 4t,$$

and the remaining conditions of the invariant are easy to verify. Finally suppose z is non-adjacent to X_3 and black-adjacent to y . Then each path P_i must intersect y , hence $|y| \geq s_j > t$. Moreover, $s_{j+1} \geq s_j - |z|$ and again $\|N_{j+1}^b(X_2)\| \geq \|N_j^b(X_2)\| + |z|$, so the inequality from the invariant is preserved; once again, the remaining parts of the invariant follow.

It remains to consider the case where y is not red-adjacent to X_1 (i.e., either non-adjacent or black-adjacent to X_1), as illustrated in Figure 3c. We claim that z, y, X_3, X_4 is a red path preserving the invariant in (G, \mathcal{P}^{j+1}) . We first prove that $|y|, |z| \geq t$. First assume that $|y| \leq t - 1$. Then $|z| = |X_2| - |y| \geq 2t - (t - 1) \geq t$, so by Observation 3.2, z is not black-adjacent to X_3 ; and it is not red-adjacent to X_3 as X_3 is already red-adjacent to y and X_4 . Thus z is non-adjacent to X_3 , so each path P_i must intersect y (this is because $X_2 = y \cup z$ and each path P_i contains an edge between X_2 and X_3). But then $|y| \geq s_t > t - 1$ - a contradiction. Similarly, if $|z| \leq t - 1$, then $|y| = |X_2| - |z| \geq t$, so by Observation 3.2, y is not black-adjacent to X_1 ; and it is not red-adjacent to X_1 by our assumption. So y is

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non-adjacent to X_1 and each path P_i must intersect z – a contradiction. Hence, $|y|, |z| \geq t$. Applying Observation 3.2 three times, we find that y is non-adjacent to X_1 ; z is non-adjacent to X_3 ; and y is not black-adjacent to z . Since y must be adjacent to z (otherwise X_1 and X_4 would be in separate connected components of $G[X_1 \cup X_2 \cup X_3 \cup X_4]$), we get that y is red-adjacent to z . Hence the subgraph of the partitioned trigraph of (G, \mathcal{P}^{j+1}) induced by $\{X_1, z, y, X_3, X_4\}$ contains red edges X_1z , zy , yX_3 and X_3X_4 and no black edges. Therefore, the second vertex of each path P_i is actually in z ; so we can remove a prefix from each path P_i so that each path starts in z , finishes in X_4 and is contained in $z \cup y \cup X_3 \cup X_4$. This witnesses that there exist s_j vertex-disjoint paths from z to X_4 in $G[z \cup y \cup X_3 \cup X_4]$, so $s_{j+1} \geq s_j$. Moreover, $\|N_{j+1}^b(y)\| = \|N_j^b(X_2)\|$ (as the black neighborhoods of y and z in (G, \mathcal{P}^{j+1}) are equal to the black neighborhood of X_2 in (G, \mathcal{P}^j)), and $\|N_{j+1}^b(X_3)\| = \|N_j^b(X_3)\|$ (as the black neighborhood of X_3 did not change during the uncontraction). We conclude that $s_{j+1} + \|N_{j+1}^b(y)\| + \|N_{j+1}^b(X_3)\| \geq 4t$, as required, and all the satisfaction of the remaining parts of the invariant is clear.



■ **Figure 3** Illustration of possible uncontractions from the proof of Lemma 3.5. Multiple edge connections between parts show all possible edges, dotted edge means non-adjacency.

3.3 Concluding the Main Proof

Observe that the basic assumptions of Theorem 3.1 are the same as those of Lemma 3.3, and the assumptions of Lemma 3.5 follow from Lemma 3.3. Hence, as detailed in the formal proof below, Lemma 3.3 and Lemma 3.5 together imply Theorem 3.1.

To extend this to a proof of Theorem 1.2, we use the following tool – an excluded-grid theorem for tree-width in the currently strongest published formulation (modulo polylog factors which we have “rounded up” for simplicity):

► **Theorem 3.6** (Chuzhoy and Tan [16]). *There is a function $f(n) \in \mathcal{O}(n^{10})$ such that, for every positive integer N , if a graph G is of tree-width at least $f(N)$, then G contains a subdivision of the $N \times N$ wall as a subgraph.*

Proof of Theorem 1.2. Let f be the function of Theorem 3.6, and choose $f_1(t) = f(2 \cdot 16 \cdot (13t)^2 + 2) \in \mathcal{O}(t^{20})$. If our graph G has tree-width at least $f_1(t)$, then G contains a subdivision of the $(2 \cdot 16 \cdot (13t)^2 + 2) \times (2 \cdot 16 \cdot (13t)^2 + 2)$ wall by Theorem 3.6. Consequently, by Observation 2.3, G contains a subgraph which is a $(16 \cdot (13t)^2) \times (16 \cdot (13t)^2)$ cubic mesh.

We now invoke Lemma 3.3, assuming G admits an uncontraction sequence of width at most 2; so, we obtain the parts X_1, X_2, X_3, X_4 as claimed by Lemma 3.3 and required by Lemma 3.5. By the subsequent invocation of Lemma 3.5, we immediately arrive at a contradiction to having the uncontraction sequence of width at most 2. Consequently, the tree-width less than $f_1(t) = f(2 \cdot 16 \cdot (13t)^2 + 2) \in \mathcal{O}(t^{20})$. ◀

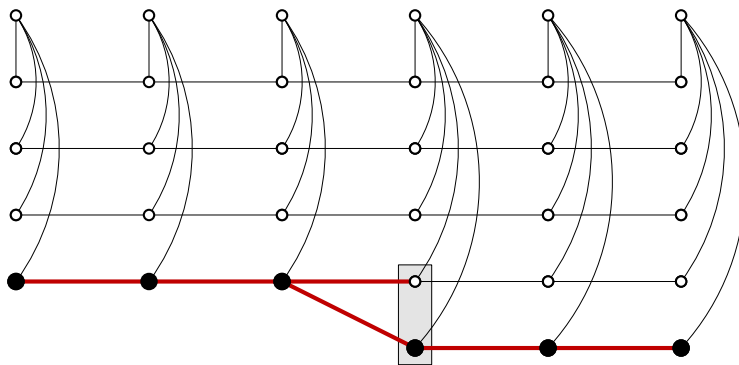
Proof of Corollary 1.3. Let t be an integer and \mathcal{C} a graph class such that no $G \in \mathcal{C}$ contains $K_{t,t}$ as a subgraph. Let f_1 be the polynomial function from (the proof of) Theorem 1.2 and set $k := f_1(t)$, which is a constant depending on \mathcal{C} . Let $G \in \mathcal{C}$ be the input graph. We first use the linear time algorithm of Bodlaender [6] to test whether G has tree-width at most k . If the answer is no, then we know by Theorem 1.2 that G cannot have twin-width at most 2. Otherwise, G has tree-width at most k , and we easily turn the decomposition into a branch-decomposition of width at most $k + 1$. This directly gives a boolean-width decomposition of width at most $k + 1$ [1] (with virtually the same decomposition tree). Finally, by the result of [14], from a boolean-width decomposition of G of width $k + 1$ one can obtain a contraction sequence of width at most $2^{k+2} - 1$ and an easy inspection of the proof shows that this can be done in polynomial time.

Thus, in the end (with respect to the fixed class \mathcal{C}) we compute in polynomial time a contraction sequence of G of width at most $2^{\text{poly}(t)}$. If the class \mathcal{C} and implicit t were to be considered as parameters, the discussed algorithm would have an FPT runtime. ◀

3.4 Case of Twin-width 3

Lastly, we show that in the class of graphs of twin-width 3, no upper bound on the tree-width is possible even if we exclude $K_{2,2}$.

► **Lemma 3.7.** *For every positive integer N , there exists an $(N^2 + N)$ -vertex graph with no $K_{2,2}$ subgraphs whose twin-width is at most 3 and tree-width is at least N .*



■ **Figure 4** Illustration of the $K_{2,2}$ -free graph of twin-width 3 and tree-width $\theta(\sqrt{n})$. It represents a partitioned trigraph of a contraction sequence of width 3: hollow vertices are singletons and the gray rectangle reveals the next contraction.

Proof. Let N be a positive integer. We take G the graph obtained from the disjoint union of N paths P_1, \dots, P_N of length N , and for each $i \in [N]$, we add a new vertex adjacent to the i -th vertex of each path P_1, \dots, P_N . See Figure 4 for an illustration. Obviously G has $N^2 + N$ vertices and no $K_{2,2}$ as subgraph.

Observe that by contracting each path P_1, \dots, P_N , we obtain the complete bipartite graph $K_{N,N}$. As G admits $K_{N,N}$ as a minor, we deduce that its tree-width is at least N .

We claim that the twin-width of G is at most 3. We can assume without loss of generality that the edges of P_1 are red. We contract P_1 and P_2 by iteratively contracting their i -th vertices as illustrated in Figure 4. The maximum red degree of each encountered trigraph is 3 and we end up with a trigraph isomorphic to $G - P_2$. We can repeat this operation to end up with a trigraph isomorphic to $G - (P_2 \cup \dots \cup P_N)$. Then we can contract each pendant vertex with its unique neighbor to obtain a red path whose twin-width is obviously at most 2. We conclude that the twin-width of G is at most 3. ◀

4 Conclusions

We have shown that sparse classes of graphs of twin-width 2 have bounded tree-width. This means that the existing rich machinery of structural and algorithmic tools for tree-width is applicable to such graph classes.

One might wonder how one can relax the requirement on \mathcal{C} being $K_{t,t}$ -free to relate graphs of twin-width 2 to other well-studied graph notions. One could even try to drop any restrictions altogether and conjecture that any class of twin-width at most 2 has bounded clique-width. This cannot be true, since the class of unit interval graphs has twin-width 2 and unbounded clique-width [21]. However, we conjecture the following.

► **Conjecture 4.1.** *Let \mathcal{C} be a stable class of graphs of twin-width at most 2. Then \mathcal{C} has bounded clique-width.*

Here being stable means that there exists k such that no $G \in \mathcal{C}$ contains a half-graph of order k as a semi-induced subgraph. We refer to [20, 24] for details about stability and a discussion on how it relates to tree-width, clique-width and twin-width.

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