Efficient Algorithms for Euclidean Steiner Minimal Tree on Near-Convex Terminal Sets

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Abstract

The Euclidean Steiner Minimal Tree problem takes as input a set $\mathcal{P}$ of points in the Euclidean plane and finds the minimum length network interconnecting all the points of $\mathcal{P}$. In this paper, in continuation to the works of [5] and [15], we study Euclidean Steiner Minimal Tree when $\mathcal{P}$ is formed by the vertices of a pair of regular, concentric and parallel n-gons.

We restrict our attention to the cases where the two polygons are not very close to each other. In such cases, we show that Euclidean Steiner Minimal Tree is polynomial-time solvable, and we describe an explicit structure of a Euclidean Steiner minimal tree for $\mathcal{P}$.

We also consider point sets $\mathcal{P}$ of size n where the number of input points not on the convex hull of $\mathcal{P}$ is $f(n) \leq n$. We give an exact algorithm with running time $2^{\mathcal{O}(f(n)\log n)}$ for such input point sets $\mathcal{P}$. Note that when $f(n) = \mathcal{O}(\frac{n}{\log n})$, our algorithm runs in single-exponential time, and when $f(n) = o(n)$ the running time is $2^{o(n \log n)}$ which is better than the known algorithm in [9].

We know that no FPTAS exists for Euclidean Steiner Minimal Tree unless P = NP [6]. On the other hand FPTASes exist for Euclidean Steiner Minimal Tree on convex point sets [14]. In this paper, we show that if the number of input points in $\mathcal{P}$ not belonging to the convex hull of $\mathcal{P}$ is $\mathcal{O}(\log n)$, then an FPTAS exists for Euclidean Steiner Minimal Tree. In contrast, we show that for any $\epsilon \in (0, 1]$, when there are $\Omega(n^{\epsilon})$ points not belonging to the convex hull of the input set, then no FPTAS can exist for Euclidean Steiner Minimal Tree unless P = NP.

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1 Introduction

The Euclidean Steiner Minimal Tree problem asks for a network of minimum total length interconnecting a given finite set $\mathcal{P}$ of n points in the Euclidean plane. Formally, we define the problem as follows, taken from [2]:

> Euclidean Steiner Minimal Tree
> **Input:** A set $\mathcal{P}$ of n points in the Euclidean plane
> **Question:** Find a connected plane graph $\mathcal{T}$ such that $\mathcal{P}$ is a subset of the vertex set $V(\mathcal{T})$, and for the edge set $E(\mathcal{T})$, $\Sigma_{e \in E(\mathcal{T})} \overline{e}$ is minimized over all connected plane graphs with $\mathcal{P}$ as a vertex subset.

Note that the metric being considered is the Euclidean metric, and for any edge $e \in E(\mathcal{T})$, $\overline{e}$ denotes the Euclidean length of the edge. Here, the input set $\mathcal{P}$ of points is often called a set of *terminals*, the points in $\mathcal{S} = V(\mathcal{T}) \setminus \mathcal{P}$ are called *Steiner points*. A solution graph $\mathcal{T}$ is referred to as a *Euclidean Steiner minimal tree*, or simply an SMT.

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The Euclidean Steiner Minimal Tree problem is a classic problem in the field of Computational Geometry. The origin of the problem dates back to Fermat (1601-1665) who proposed the problem of finding a point in the plane such that the sum of its distance from three given points is minimized. This is equivalent to finding the location of the Steiner point when given three terminals as input. Torricelli proposed a geometric solution to this special case of 3 terminal points. The idea was to construct equilateral triangles outside on all three sides of the triangle formed by the terminals, and draw their circumcircles. The three circles meet at a single point, which is our required Steiner point. This point came to be known as the *Torricelli point*. When one of the angles in the triangle is at least 120°, the minimizing point coincides with the obtuse angle vertex of the triangle. In this case, the Torricelli point lies outside the triangle and no longer minimizes the sum of distances from the vertices. However, when vertices of polygons with more than 3 sides are considered as a set of terminals, a solution to the Fermat problem does not in general lead to a solution to the Euclidean Steiner Minimal Tree problem. For a more detailed survey on the history of the problem, please refer to [2, 9]. For convenience, we refer to the Euclidean Steiner Minimal Tree problem as ESMT.

ESMT is NP-hard. In [6], Garey et al. prove a discrete version of the problem (Discrete ESMT) to be strongly NP-complete via a reduction from the Exact Cover by 3-Sets (X3C) problem. Although it is not known if the ESMT problem is in NP, it is at least as hard as any NP-complete problem. So, we do not expect a polynomial time algorithm for it. A recursive method using only Euclidean constructions was given by Melzak in [11] for constructing all the Steiner minimal trees for any set of n points in the plane by constructing full Steiner trees of subsets of the points. Full Steiner trees are interconnecting trees having the maximum number of newly introduced points (Steiner points) where all internal junctions are of degree 3. Hwang improved the running time of Melzak's original exponential algorithm for full Steiner tree construction to linear time in [8]. Using this, we can construct an Euclidean Steiner minimal tree in $2^{\mathcal{O}(n \log n)}$ time for any set of n points. This was the first algorithm for Euclidean Steiner Minimal Tree. The problem is known to be NP-hard even if all the terminals lie on two parallel straight lines, or on a bent line segment where the bend has an angle of less than 120° [13]. Since the above sets of terminals all lie on the boundary of a convex polygon (or, are in convex position), this shows that ESMT is NP-hard when restricted to a set of points that are in weakly convex position.

Although the ESMT problem is NP-hard, there are certain arrangements of points in the plane for which the Euclidean Steiner minimal tree can be computed efficiently, say in polynomial time. One such arrangement is placing the points on the vertices of a regular polygon. This case was solved by Du et al. [5]. Their work gives exact topologies of the Euclidean Steiner minimal trees. Weng et al. [15] generalized the problem by incorporating the centre point of the regular polygon as part of the terminal set, along with the vertices. This case was also found to be polynomial time solvable.

Tractability in the form of approximation algorithms for ESMT has been extensively studied. It was proved in [6] that a fully polynomial time approximation scheme (FPTAS) cannot exist for this problem unless P = NP. However, we do have an FPTAS when the terminals are in convex position [14]. Arora's celebrated polynomial time approximation scheme (PTAS) for the ESMT and other related problems is described in [1]. Around the same time, Rao and Smith gave an efficient polynomial time approximation scheme (EPTAS) in [12]. In recent years, an EPTAS with an improved running time was designed by Kisfaludi-Bak et al. [10].

Our Results

In this paper, we first extend the work of [5] and [15]. We state this problem as ESMT on k-Concentric Parallel Regular n-gons.

▶ **Definition 1** (k-Concentric Parallel Regular n-gons). *k-Concentric Parallel Regular n-gons are k regular n-gons that are concentric and where the corresponding sides of polygons are parallel to each other.*

Please refer to Figure 1(a) for an example of a 2-Concentric Parallel Regular 12-gon. We call k-Concentric Parallel Regular n-gons as k-CPR n-gons for short.

We consider terminal sets where the terminals are placed on the vertices of 2-CPR n-gons. In the case of $k = 2$, the n-gon with the smaller side length will be called the inner n-gon and the other n-gon will be called the outer n-gon. Also, let a be the side length of the inner n-gon, and b be the side length of the outer n-gon. We define $\lambda = \frac{b}{a}$ and refer to it as the *aspect ratio* of the two regular polygons. In Section 3, we derive the exact structures of the SMTs for 2-CPR n-gons when the aspect ratio λ of the two polygons is greater than $\frac{1}{1-4\sin(\pi/n)}$ and $n \geq 13$.

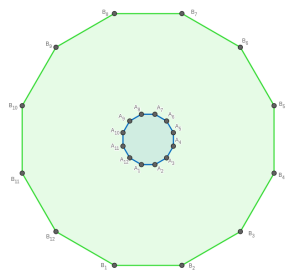
Next, we consider ESMT on an $f(n)$-Almost Convex Point Set.

▶ **Definition 2** ($f(n)$-Almost Convex Point Set). *An $f(n)$-Almost Convex Point Set $\mathcal{P}$ is a set of n points in the Euclidean plane such that there is a partition $\mathcal{P} = \mathcal{P}_1 \uplus \mathcal{P}_2$ where $\mathcal{P}_1$ forms the convex hull of $\mathcal{P}$ and $|\mathcal{P}_2| = f(n)$.*

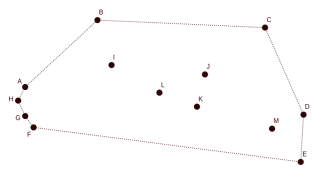
Please refer to Figure 1(b) for an example of a 5-Almost Convex Set of 13 points. In Section 4, we give an exact algorithm for ESMT on $f(n)$-Almost Convex Sets of n terminals. The running time of this algorithm is $2^{\mathcal{O}(f(n)\log n)}$. Thus, when $f(n) = \mathcal{O}(\frac{n}{\log n})$, then our algorithm runs in $2^{\mathcal{O}(n)}$ time, and when $f(n) = o(n)$ then the running time is $2^{o(n \log n)}$. This is an improvement on the best known algorithm for the general case [9].

Next in Section 5, for $f(n) = \mathcal{O}(\log n)$, we give an FPTAS. On the other hand we show that, for all $\epsilon \in (0, 1]$, when $f(n) \in \Omega(n^{\epsilon})$, there cannot exist any FPTAS unless P = NP.

Due to paucity of space, we omit certain proofs that can be found in the full version of the paper.



(a) 2-CPR 12-gons.



(b) $f(n)$-Almost Convex Point Set for $n = 13$, $f(n) = 5$.

Figure 1 Examples for Definition 1 and Definition 2.

2 Preliminaries

Notations. For a given positive integer $k \in \mathbb{N}$, the set of integers $\{1, 2, \ldots, k\}$ is denoted for short as $[k]$. Given a graph G, the vertex set is denoted as $V(G)$ and the edge set as $E(G)$. Given two graphs G_1 and G_2, $G_1 \cup G_2$ denotes the graph G where $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$.

In this paper, a regular n-gon is denoted by $A_1A_2A_3...A_n$ or $B_1B_2B_3...B_n$. For convenience, we define $A_{n+1} := A_1$, $B_{n+1} := B_1$, $A_0 := A_n$ and $B_0 := B_n$. We use the notation $\{A_i\}$ to denote the polygon $A_1A_2A_3 \dots A_n$ and $\{B_i\}$ to denote the polygon $B_1B_2B_3 \dots B_n$. For any regular polygon $A_1A_2A_3...A_n$, the circumcircle of the polygon is denoted as $(A_1A_2A_3...A_n)$.

Given two points P, Q in the Euclidean plane, we denote by $\mathsf{dist}(P, Q)$ the Euclidean distance between P and Q. Given a line segment AB in the Euclidean plane, $\overline{AB} = \mathsf{dist}(A, B)$. For two distinct points A and B, L_{AB} denotes the line containing A and B; and $\overrightarrow{AB}$ denotes the ray originating from A and containing B.

When we refer to a graph $\mathcal{G}$ in the Euclidean plane then $V(\mathcal{G})$ is a set of points in the Euclidean plane, and $E(\mathcal{G})$ is a subset of the family of line segments $\{P_1P_2 | P_1, P_2 \in V(\mathcal{G})\}$. For any tree $\mathcal{T}$ in the Euclidean plane, we denote by the notation $|\mathcal{T}|$ the value of $\Sigma_{e \in E(\mathcal{T})} \overline{e}$. A path in a tree $\mathcal{T}$ is uniquely specified by the sequence of vertices on the path; therefore, P_1, P_2, P_3, $\dots$, P_k (where $P_i \in V(\mathcal{T}), \forall i \in [k]$ and $P_iP_{i+1} \in E(\mathcal{T}), \forall i \in [k-1]$) denotes the path starting from the vertex P_1, going through the vertices P_2, P_3, $\dots$, P_{k-1} and finally ending at P_k. Equivalently, we can specify the same path as *the path from* P_1 *to* P_k, since $\mathcal{T}$ is a tree. Consider the graph T such that $V(T) = \{v_P | P \in V(\mathcal{T})\}$, $E(T) = \{v_{P_1}v_{P_2} | P_1P_2$ is a line segment in $E(\mathcal{T})\}$. Then T is said to be the topology of $\mathcal{T}$ while $\mathcal{T}$ is said to realize the topology T. Given two trees $\mathcal{T}_1$, $\mathcal{T}_2$ in the Euclidean plane, $\mathcal{T}' = \mathcal{T}_1 \cup \mathcal{T}_2$ is the graph where $V(\mathcal{T}') = V(\mathcal{T}_1) \cup V(\mathcal{T}_2)$ and $E(\mathcal{T}') = E(\mathcal{T}_1) \cup E(\mathcal{T}_2)$.

Given any graph G, a Steiner minimal tree or SMT for a terminal set $\mathcal{P} \subseteq V(G)$ is the minimum length connected subgraph G' of G such that $\mathcal{P} \subseteq V(G')$. The Steiner Minimal Tree problem on graphs takes as input a set $\mathcal{P}$ of terminals and aims to find a minimum length SMT for $\mathcal{P}$. For the rest of the paper, we also refer to a Euclidean Steiner minimal tree as an SMT. Given a set of points $\mathcal{P}$ in the Euclidean plane, the convex hull of $\mathcal{P}$ is denoted as $\mathrm{CH}(\mathcal{P})$.

Properties of a Euclidean Steiner minimal tree. A Euclidean Steiner minimal tree (SMT) has certain structural properties as given in [3]. We state them in the following Proposition.

▶ **Proposition 3.** *Consider an SMT on n terminals.*

1. *No two edges of the SMT intersect with each other.*
2. *Each Steiner point has degree exactly 3 and the incident edges meet at $120°$ angles. The terminals have degree at most 3 and the incident edges form angles that are at least $120°$.*
3. *The number of Steiner points is at most $n - 2$, where n is the number of terminals.*

A full Steiner tree (FST) is a Steiner tree (need not be minimal, but may include Steiner points) having exactly $n - 2$ Steiner points, where n is the number of terminals. In an FST, all terminals are leaves and Steiner points are interior nodes. When the length of an FST is minimized, it is called a minimum FST.

All SMTs can be decomposed into FST components such that, in each component a terminal is always a leaf. This decomposition is unique for a given SMT [9]. A topology for an FST is called a full Steiner topology and that of a Steiner tree is called a Steiner topology.

Steiner Hulls. A Steiner hull for a given set of points is defined to be a region which is known to contain an SMT. We get the following propositions from [9].

▶ **Proposition 4.** *For a given set of terminals, every SMT is always contained inside the convex hull of those points. Thus, the convex hull is also a Steiner hull.*

The next two propositions are useful in restricting the structure of SMTs and the location of Steiner points.

▶ **Proposition 5** (The Lune property). *Let* UV *be any edge of an SMT. Let* $L(\mathrm{U},\mathrm{V})$ *be the lune-shaped intersection of circles of radius* $|\mathrm{UV}|$ *centered on* U *and* V*. No other vertex of the SMT can lie in* $L(\mathrm{U},\mathrm{V})$*, except* U *and* V *themselves.*

▶ **Proposition 6** (The Wedge property). *Let* W *be any open wedge-shaped region having angle* $120°$ *or more and containing none of the points from the input terminal set* $\mathcal{P}$*. Then* W *contains no Steiner points from an SMT of* $\mathcal{P}$*.*

Fully Polynomial Time Approximation Scheme (FPTAS). An algorithm is called a fully polynomial time approximation scheme (FPTAS) for a problem if it takes an input instance and a parameter $\epsilon > 0$, and outputs a solution with approximation factor $(1+\epsilon)$ for a minimization problem in time $(1/\epsilon)^{\mathcal{O}(1)} n^{\mathcal{O}(1)}$ where n is the input size.

3 Polynomial cases for Euclidean Steiner Minimal Tree

In this section, we consider the Euclidean Steiner Minimal Tree problem for 2-CPR n-gons. Throughout the section, we denote the inner n-gon as $\{A_i\}$ and the outer n-gon as $\{B_i\}$. First, we consider the configuration of an Euclidean Steiner minimal tree in a subsection of the annular area between $\{A_i\}$ and $\{B_i\}$, which will form an isosceles trapezoid.

Then we prove our result for all 2-CPR n-gons.

3.1 Isosceles Trapezoids and Vertical Forks

In this section, we discuss one particular Steiner topology when the terminal set is formed by the four corners of a given isosceles trapezoid. However, we will limit the discussion to only the isosceles trapezoids such that the angle between the non-parallel sides is of the form $\frac{2\pi}{n}$ where $n \in \mathbb{N}$, $n \geq 4$. The reason is that given 2-CPR n-gons $\{A_i\}$, $\{B_i\}$, for $n \geq 4$ and for any $j \in \{1, \ldots, n-1\}$, the region $A_jA_{j+1}B_jB_{j+1}$ is an isosceles trapezoid such that the angle between the non-parallel sides is of the form $\frac{2\pi}{n}$.

(a) Isosceles trapezoid with $\angle AOB = \frac{2\pi}{8}$.

(b) The Vertical Fork, $\mathcal{T}_{vf}$.

Figure 2 Isosceles Trapezoids and Vertical Forks.

Let $ABQP$ be an isosceles trapezoid with AB, PQ as the parallel sides, and AP, BQ as the non-parallel sides. Assume without loss of generality that AB is shorter than PQ. Let $\overline{AB} = 1$ and $\overline{PQ} = \lambda$, where $\lambda \geq \frac{\sqrt{3}+\tan\frac{\pi}{n}}{\sqrt{3}-\tan\frac{\pi}{n}}$. For brevity, we say $\lambda_v = \frac{\sqrt{3}+\tan\frac{\pi}{n}}{\sqrt{3}-\tan\frac{\pi}{n}}$. Let L_{PA} and L_{QB} be the lines containing the line segments PA and QB respectively. Also let O be the point of intersection of L_{PA} and L_{QB}. Further, let M and N be the midpoints of AB and PQ respectively (as in Figure 2(a)). As mentioned earlier, $\angle AOB = \frac{2\pi}{n}$ where $n \in \mathbb{N}$, $n \geq 4$.

Now, we explore the following Steiner topology of the terminal set $\{A, B, P, Q\}$:

1. A and B are connected to a Steiner point S_1.
2. P and Q are connected to another Steiner point S_2.
3. S_1 and S_2 are directly connected (Please see Figure 2(b)).

We call such a topology a *vertical fork topology* and the Steiner tree realising such a topology, the *vertical fork*. Note that in a vertical fork topology the only unknowns are the locations of the two Steiner points S_1, S_2. Therefore, we have the vertical fork topology as T_{vf}, with $E(T_{vf}) = \{AS_1, BS_1, S_1S_2, S_2P, S_2Q\}$. We show the existence of a vertical fork and calculate its total length in the following lemma.

▶ **Lemma 7.** *A vertical fork $\mathcal{T}_{vf}$ can be constructed for any $n \geq 4$ and for any $\lambda \geq \lambda_v$, where*

$$\lambda_v = \frac{\sqrt{3} + \tan\frac{\pi}{n}}{\sqrt{3} - \tan\frac{\pi}{n}}$$

such that the length of the vertical fork

$$|\mathcal{T}_{vf}| = \frac{(\lambda - 1)}{2\tan\frac{\pi}{n}} + \frac{\sqrt{3}(\lambda + 1)}{2}$$

3.2 Euclidean Steiner Minimal Tree and Large Polygons with Large Aspect Ratios

In this section, we consider the Euclidean Steiner Minimal Tree problem when the terminal set is formed by the vertices of 2-CPR n-gons, namely $\{A_i\}$ and $\{B_i\}$. As mentioned earlier, $\{A_i\}$ is the inner polygon and $\{B_i\}$ is the outer polygon of this set of 2-CPR n-gons. In particular, we consider the case when $n \geq 13$; for $n \leq 12$ these are constant sized input instances and can be solved using any brute-force technique. We also require that the aspect ratio λ has a lower bound λ_1, i.e. we do not want the two polygons to have sides of very similar length. The exact value of λ_1 will be clear during the description of the algorithm. Intuitively, when λ is *very large*, the SMT should look similar to what was derived in [15]. We call the topology of such an SMT a *singly connected topology*, as in Figure 3.

▶ **Definition 8.** *A Steiner topology of $\{A_i\} \cup \{B_i\}$ is a **singly connected topology**, if it has the following structure:*

- *A vertical gadget i.e. five edges $\{A_jS_a, A_{j+1}S_a, S_aS_b, S_bB_j, S_bB_{j+1}\}$ for some $1 \leq j \leq n$, where S_a and S_b are newly introduced Steiner points contained in the isosceles trapezoid $\{A_j, A_{j+1}, B_j, B_{j+1}\}$.*
- *All $(n-2)$ polygon edges of $\{A_i\}$ excluding the edge A_jA_{j+1}*
- *All $(n-2)$ polygon edges of $\{B_i\}$ excluding the edge B_jB_{j+1}*

For the rest of this section, we consider the SMTs for a large enough aspect ratio, λ and show that there is an SMT that must be a realisation of a *singly connected topology*. We refer to an SMT for the terminal set defined by the vertices of $\{A_i\}$ and $\{B_i\}$ as the SMT for $\{A_i\} \cup \{B_i\}$.

Without loss of generality, we consider the edge length of any side A_iA_{i+1} in $\{A_i\}$ to be 1 and that of any side B_iB_{i+1} of $\{B_i\}$ to be λ.

We define the notion of a path in an SMT for the vertices of $\{A_i\}$ and $\{B_i\}$ where the starting point is in $\{A_i\}$ and the ending point is in $\{B_i\}$.

▶ **Definition 9.** *An **A-B path** is a path in a Steiner tree of $\{A_i\} \cup \{B_i\}$ which starts from a vertex in $\{A_i\}$ and ends at a vertex in $\{B_i\}$ with all intermediate nodes (if any) being Steiner points.*

(a) SMT of 2-CPR 13-gons. **(b)** SMT of 2-CPR 20-gons.

Figure 3 *Singly connected topology* of 2-CPR n-gons $n = 13$ and $n = 20$.

The following Definition and Figure 4 is useful for the design of our algorithm.

▶ **Definition 10.** *A* ***counter-clockwise path*** *is a path $P_1, P_2, ...P_m$ in a Steiner tree such that for all $i \in \{2, \ldots, m-1\}, \angle P_{i-1}P_iP_{i+1} = \frac{2\pi}{3}$ in the counter-clockwise direction. Similarly, a* ***clockwise path*** *is a path $P_1, P_2, ...P_m$ in a Steiner tree such that for all $i \in \{2, \ldots, m-1\}, \angle P_{i-1}P_iP_{i+1} = \frac{4\pi}{3}$ in the counter-clockwise direction.*

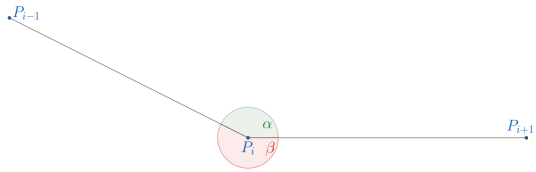


Figure 4 $\angle P_{i-1}P_iP_{i+1} = \alpha$ in the **counter-clockwise direction** and $\angle P_{i-1}P_iP_{i+1} = \beta$ in the **z**, as in Definition 10.

Now, we consider any Steiner point S in any SMT. Let P and Q be two neighbours of S. We now prove that there is no point of the SMT inside the triangle PSQ.

▶ **Observation 11.** *Let S be a Steiner point in any SMT for $\{A_i\} \cup \{B_i\}$, with neighbours P and Q. Then, no point of the SMT lies inside the triangle PSQ.*

Next, we show that in an SMT for $\{A_i\} \cup \{B_i\}$ there cannot be any Steiner point, in the interior of the polygon $\{A_i\}$, that is a direct neighbour of some point B_k in the polygon $\{B_i\}$.

▶ **Observation 12.** *For any SMT for $\{A_i\} \cup \{B_i\}$, there cannot exist a Steiner point S lying in the interior of the polygon $\{A_i\}$ such that SB_k is an edge in an SMT for some $B_k \in \{B_i\}$.*

Proof. For the sake of contradiction, we assume that for some SMT for $\{A_i\} \cup \{B_i\}$ there exists a Steiner point S lying in the interior of the polygon $\{A_i\}$ such that SB_k is an edge in the SMT for some $B_k \in \{B_i\}$. Let A_mA_{m+1} be the edge such that SB_k intersects A_mA_{m+1}. Without loss of generality, assume that A_m is closer to B_k than A_{m+1}. Therefore $\angle B_kA_mS > \angle B_kA_mA_{m+1} \geq \frac{\pi}{2} + \frac{\pi}{n} > \frac{\pi}{2}$. This means that B_kS is the longest edge in the triangle B_kSA_m. Therefore we can remove the edge B_kS from the SMT and replace it with either B_kA_m or SA_m to get another tree connecting the terminal set with a shorter total length than what we started with, which is a contradiction. ◀

We further analyze SMTs for $\{A_i\} \cup \{B_i\}$.

▶ **Observation 13.** *Let $\mathcal{V} = \{A_j, A_{j+1}, \ldots, A_k\}$ be the interval of consecutive vertices of $\{A_i\}$ lying between A_j and A_k (which includes A_{j+1}) such that A_j is distinct from A_{k+1}. Let U be any point on the line segment A_kA_{k+1}. Then an SMT of $\mathcal{V} \cup \{U\}$ is $\mathcal{T}$, with $E(\mathcal{T}) = \{A_jA_{j+1}, A_{j+1}A_{j+2}, \ldots, A_{k-1}A_k\} \cup \{A_kU\}$.*

We proceed by showing that in any SMT for $\{A_i\} \cup \{B_i\}$ there exists at least one A-B path which is also a counter-clockwise path. Symmetrically, we also show that for any SMT for $\{A_i\} \cup \{B_i\}$ there exists another clockwise A-B path which consists of only clockwise turns. We can intuitively see that this is true because, if all clockwise paths starting at a vertex in $\{A_i\}$ also ended in a vertex in $\{A_i\}$, there would be enough paths to form a cycle, which is not possible in a tree.

▶ **Lemma 14.** *In any SMT for $\{A_i\} \cup \{B_i\}$, there exists an A-B path which is also a clockwise path and there exists an A-B path which is also a counter-clockwise path.*

Our next step is to bound the number of "connections" that connect the inner polygon $\{A_i\}$ and the outer polygon $\{B_i\}$ for a large aspect ratio, λ. As λ increases, the area of the annular region between the two polygons increases as well. Therefore, an increase in the number of connections would lead to a longer total length of the SMT considered. Consequently, we will prove that after a certain positive constant λ_1, for $\lambda > \lambda_1$ any SMT for $\{A_i\} \cup \{B_i\}$ will have a single "connection" between the two polygons. Moreover, [15] gives us an evidence that as $\lambda \to \infty$, there will indeed be a single connection connecting the outer polygon and the inner polygon for $n \geq 12$. We can formalize this notion of existence of a single "connection" with the following lemma.

▶ **Lemma 15.** *For any SMT for $\{A_i\} \cup \{B_i\}$ with $n \geq 13$ and $\lambda > \lambda_1$, the number of edges needed to be removed in order to disconnect $\{A_i\}$ and $\{B_i\}$ is 1, where*

$$\lambda_1 = \frac{1}{1 - 4\sin\frac{\pi}{n}}$$

We now proceed to further investigate the connectivity of $\{A_i\}$ and $\{B_i\}$.

▶ **Lemma 16.** *Consider an SMT for $\{A_i\} \cup \{B_i\}$ for $n \geq 13$ and $\lambda \geq \lambda_1$. There must exist $j \in [n]$ and a Steiner point S_1, such that terminals A_j, A_{j+1} form a path A_j, S_1, A_{j+1} in the SMT and each A-B path passes through S_1.*

Proof Sketch. From Lemma 14, we know that there exists one clockwise A-B path and one counterclockwise A-B path in any SMT of $\{A_i\} \cup \{B_i\}$. Let a clockwise A-B path start from A_r and a counter-clockwise A-B path start from A_l. Further following from Lemma 15, as there is one edge common to all A-B paths, the clockwise A-B path from A_r and the counter-clockwise A-B path from A_l must share a common edge S_1S_2. Therefore, each A-B path must pass through S_1 and S_2. Without loss of generality we assume that point S_1 is closer to the polygon $\{A_i\}$ than S_2. This means S_1 is either a Stiener point or a terminal vertex of $\{A_i\}$.

▷ Claim 17. S_1 is not a vertex in $\{A_i\}$.

Therefore, S_1 must be a Steiner point. Let P and Q be the neighbours of S_1 other than S_2, such that $\angle PS_1S_2$ is a clockwise turn while $\angle QS_1S_2$ is a counter-clockwise turn. This means that the clockwise A-B path from A_r passes through P and the counter-clockwise A-B path from A_l passes through Q. We prove the following for P and Q.

▷ Claim 18. P and Q are consecutive vertices of $\{A_i\}$ in any SMT of $\{A_i\} \cup \{B_i\}$.

Therefore, P and Q are consecutive vertices A_j, A_{j+1} of the polygon $\{A_i\}$, for some $j \in [n]$ such that A_j, S_1, A_{j+1} is a path in the SMT, where S_1 is a Steiner point lying on all A-B paths. ◀

Our next step is to investigate some more structural properties of an SMT for $\{A_i\} \cup \{B_i\}$. From [5], we may guess that there would be a lot of polygon edges of both $\{A_i\}$ and $\{B_i\}$ in an SMT. We prove the following Lemma, stating that there is an SMT of $\{A_i\} \cup \{B_i\}$ which contains $(n-2)$ polygon edges of $\{A_i\}$.

▶ **Lemma 19.** *For an SMT for $\{A_i\} \cup \{B_i\}$ with aspect ratio λ, $\lambda > \lambda_1 = \frac{1}{1-4\sin\frac{\pi}{n}}$, let S_1 be the Steiner point such that all A-B paths pass through S_1. Let A_j and A_{j+1} be vertices of $\{A_i\}$ which are connected to S_1. Then, there exists an SMT of $\{A_i\} \cup \{B_i\}$ having $n-2$ polygon edges of $\{A_i\}$ other than A_jA_{j+1}.*

With these set of results in hand, we can now show that there exists an SMT of $\{A_i\} \cup \{B_i\}$ following a *singly connected topology*. To show this, we start with any SMT for $\{A_i\} \cup \{B_i\}$, $\mathcal{T}_0$, that satisfies all the results derived so far and transform it into a Steiner tree of *singly connected topology* having total length not longer than the initial Steiner tree $\mathcal{T}_0$.

▶ **Theorem 20.** *There exists an SMT for $\{A_i\} \cup \{B_i\}$ following a singly connected topology for $n \geq 13$ and $\lambda \geq \lambda_1$, where*

$$\lambda_1 = \frac{1}{1-4\sin\frac{\pi}{n}}$$

Proof. Let $\mathcal{T}_0$ be any SMT of $\{A_i\} \cup \{B_i\}$ which satisfies the properties of Lemma 19. Further, from Lemma 16, there is a Steiner point S_1 which lies on all A-B paths, and there are two consecutive vertices A_j, A_{j+1} such that A_j, S_1, A_{j+1} is a path in $\mathcal{T}_0$. As $\mathcal{T}_0$ satisfies the property of Lemma 19, $\mathcal{T}_0$ has $n-2$ polygon edges of $\{A_i\}$ excluding the edge A_jA_{j+1}.

Let H be the point in the interior of the polygon $\{A_i\}$ such that HA_jA_{j+1} form an equilateral triangle. As $n > 6$, the common centre O of $\{A_i\}$ and $\{B_i\}$ does not lie inside the triangle HA_jA_{j+1}. Now, we modify $\mathcal{T}_0$ as follows:

1. Remove edges A_jS_1, S_1A_{j+1} and add edge S_1H to get the forest $\mathcal{T}_1$. We know from [9] that S_1, S_2 and H are collinear and this transformation does not change the total length. Therefore $|\mathcal{T}_0| = |\mathcal{T}_1|$. Here, $|\mathcal{T}_1|$ denotes the sum of the lengths of edges present in $\mathcal{T}_1$.
2. Add edge HO and remove all polygon edges of $\{A_i\}$ to get $\mathcal{T}_2$. Therefore $|\mathcal{T}_2| = |\mathcal{T}_1| + \overline{HO} - (n-2) = |\mathcal{T}_0| + \overline{HO} - (n-2)$. We observe that $\mathcal{T}_2$ is a tree connecting the points in $\{B_i\} \cup \{O\}$.
3. Let S_0 be the Torricelli point of the triangle OB_jB_{j+1}. Let $\mathcal{T}_3$ be the Steiner tree of $\{B_i\} \cup \{O\}$ with edges S_0O, S_0B_j, S_0B_{j+1} and other points in $\{B_i\}$ connected through $(n-2)$ polygon edges of the polygon $\{B_i\}$. From [15], we know that $\mathcal{T}_3$ is the SMT of $\{B_i\} \cup \{O\}$. Therefore $|\mathcal{T}_3| \leq |\mathcal{T}_2| = |\mathcal{T}_0| + \overline{HO} - (n-2)$. Further we know that H lies on the edge OS_0 (as O, S_0 and H lie on the perpendicular bisector of B_j and B_{j+1}).
4. Remove edge S_0O and add edge S_0H to get $\mathcal{T}_4$. As H lies on the edge OS_0, we have $|\mathcal{T}_4| = |\mathcal{T}_3| - \overline{OH} \leq |\mathcal{T}_0| - (n-2)$.
5. Let S_3 be the intersection of the circumcircle of triangle A_jHA_{j+1} (from Lemma 7 the intersection exists as $\lambda_1 \geq \lambda_v$ for $n \geq 13$). Remove the edge S_3H and add the edges S_3A_j and S_3A_{j+1} to get $\mathcal{T}_5$. Again, from [9] we know that this transformation does not change the total length. Hence $|\mathcal{T}_5| = |\mathcal{T}_5| \leq |\mathcal{T}_0| - (n-2)$. Moreover, as $\lambda > \lambda_v$, we observe that $\{A_j, B_j, A_{j+1}, B_{j+1}, S_3, S_0\}$ form the vertices of the vertical gadget and points O, H, S_3, S_0 appear in that order on the perpendicular bisector of B_j and B_{j+1}.

6. Add back the $(n-2)$ polygon edges of $\{A_i\}$ which were removed in the second step to get $\mathcal{T}_6$. Therefore $|\mathcal{T}_4| = |\mathcal{T}_5| + (n-2) \leq |\mathcal{T}_0|$. We further observe that $\mathcal{T}_6$ is a Steiner tree connecting the points $\{A_i\} \cup \{B_i\}$ with a singly connected topology.

Therefore we started with an arbitrary SMT $\mathcal{T}_0$ and transformed it into a Steiner tree $\mathcal{T}_6$ with a singly connected topology (where $\{A_j, B_j, A_{j+1}, B_{j+1}, S_3, S_0\}$ form the vertices of the vertical gadget) which has a total length not worse than $\mathcal{T}_0$. Hence $\mathcal{T}_6$ must be an SMT of $\{A_i\} \cup \{B_i\}$. This proves the theorem. ◀

▶ Remark 21. Theorem 20 determines the exact structure of the SMT for $\{A_i\} \cup \{B_i\}$. Further from Section 3.1 we determine the exact method to construct the two additional Steiner points in $\mathcal{O}(1)$ steps - note that this construction time is independent of the integer n or the real number λ. Therefore, SMT for $\{A_i\} \cup \{B_i\}$ for $n \geq 13$ and $\lambda \geq \lambda_1$ is solvable in polynomial time.

Interestingly, λ_1 converges to 1 very quickly with increasing n:

n	13	20	40	100	500
λ_1	23.3987	2.6719	1.4574	1.1437	1.0258

4 Euclidean Steiner Minimal Tree on $f(n)$-Almost Convex Point Sets

In this section, we design an exact algorithm for Euclidean Steiner Minimal Tree on $f(n)$-Almost Convex Point Sets running in time $2^{\mathcal{O}(f(n)\log n)}$. Note that $f(n) \leq n$ is always true. Therefore, we are given as input a set $\mathcal{P}$ of n points in the Euclidean Plane such that $\mathcal{P}$ can be partitioned as $\mathcal{P} = \mathcal{P}_1 \uplus \mathcal{P}_2$, where $\mathcal{P}_1$ is the convex hull of $\mathcal{P}$ and $|\mathcal{P}_2| = f(n)$.

We know that the SMT of $\mathcal{P}$ can be decomposed uniquely into one or more full Steiner subtrees, such that two full Steiner subtrees share at most one node [9]. In the following lemma, we further characterize one full Steiner subtree, which we refer to as a *leaf full Steiner subtree*.

▶ **Lemma 22.** *Let $\mathbb{F}$ be the full Steiner decomposition of an SMT of $\mathcal{P}$. Then there exists a full Steiner subtree $\mathcal{F} \in \mathbb{F}$ such that $\mathcal{F}$ has at most one common node with at most one other full Steiner subtree in $\mathbb{F}$.*

Using the above Lemma, along with the bounds on the number of FSTs from [9], we can obtain the following theorem (details can be found in the full version).

▶ **Theorem 23.** *An SMT $\mathcal{T}_{\mathcal{P}}$ of a k-Almost Convex Set $\mathcal{P}$ of terminals can be computed in $\mathcal{O}(n^k \cdot 5^n)$ time.*

The above theorem gives us several improvements in special classes of inputs.

▶ **Corollary 24.** *Let $\mathcal{P}$ be a $f(n)$-Almost Convex Point Set. Then, then there is an algorithm $\mathcal{A}$ for Euclidean Steiner Minimal Tree such that, $\mathcal{A}$ runs in $2^{\mathcal{O}(n+f(n)\log n)}$ time. In particular,*

1. *When $f(n) = \mathcal{O}(n)$, $\mathcal{A}$ runs in $2^{\mathcal{O}(n\log n)}$ time.*
2. *When $f(n) = \Omega(\frac{n}{\log n})$ and $f(n) = o(n)$, $\mathcal{A}$ runs in $2^{o(n\log n)}$.*
3. *When $f(n) = \mathcal{O}(\frac{n}{\log n})$, $\mathcal{A}$ runs in $2^{\mathcal{O}(n)}$ time.*

Therefore, for $f(n) = o(n)$, our algorithm for Euclidean Steiner Minimal Tree does better on $f(n)$-Almost Convex Points Sets than the current best known algorithm [8].

5 Approximation Algorithms for Euclidean Steiner Minimal Tree

The Euclidean Steiner Minimal Tree problem is NP-hard as shown by Garey et al. in [6]. Garey et al. also prove that there cannot be an FPTAS (fully polynomial time approximation scheme) for this problem unless $P = NP$. At the same time, the case when all the terminals lie on the boundary of a convex region admits an FPTAS as given in [14]. In this section, we aim to conduct a more fine-grained analysis for the problem by considering $f(n)$-Almost Convex Point Sets of n terminals and studying the existence of FPTASes for different functions $f(n)$.

First, we present an FPTAS for Euclidean Steiner Minimal Tree on $f(n)$-Almost Convex Sets of n terminals, when $f(n) = \mathcal{O}(\log n)$. The FPTAS follows the strategy of [14] and uses a variant of the Dreyfus-Wagnus Steiner tree algorithm [4] as a subroutine.

▶ **Theorem 25.** *There exists an FPTAS for Euclidean Steiner Minimal Tree on an $f(n)$-Almost Convex Set of n terminals, where $f(n) = \mathcal{O}(\log n)$.*

On the other hand, we prove in the next section that no FPTAS exists for the case when $f(n) = \Omega(n^\epsilon)$, where $\epsilon \in (0, 1]$.

5.1 Hardness of Approximation for Euclidean Steiner Minimal Tree on Cases of Almost Convex Sets

In this section, we consider the Euclidean Steiner Minimal Tree problem on $f(n)$-Almost Convex Sets of n terminal points, where $f(n) = \Omega(n^\epsilon)$ for some $\epsilon \in (0, 1]$. We show that this problem cannot have an FPTAS. The proof strategy is similar to that in [6]. First, we give a reduction for the problem Exact Cover by 3-Sets (defined below) to our problem to show that our problem is NP-hard. Next, we consider a discrete version of our problem and reduce our problem to the discrete version. The discrete version is in NP. Therefore, this chain of reductions imply that the discrete version of our problem is Strongly NP-complete and therefore cannot have an FPTAS, following from [6]. Similar to the arguments in [6], this also implies that our problem cannot have an FPTAS.

Before we describe our reductions, we take a look at the NP-hardness reduction of the Euclidean Steiner Minimal Tree problem from the Exact Cover by 3-Sets (X3C) problem in [6]. In the X3C problem, we are given a universe of elements $U = \{1, 2, \ldots, 3n\}$ and a family $\mathbb{F}$ of 3-element subsets $F_1, F_2, \ldots, F_t$ of the $3n$ elements. The objective is to decide if there exists a subcollection $\mathbb{F}' \subseteq \mathbb{F}$ such that: (i) the elements of $\mathbb{F}'$ are disjoint, and (ii) $\bigcup_{F' \in \mathbb{F}'} F' = U$. The X3C problem is NP-complete [7].

In [6], various gadgets are constructed, i.e. particular arrangements of a set of points. These are then arranged on the plane in a way corresponding to the given X3C instance. Figure 5 shows the reduced ESMT instance obtained for $U = \{1, 2, 3, 4, 5, 6\}$ and $\mathbb{F} = \{\{1, 2, 4\}, \{2, 3, 6\}, \{3, 5, 6\}\}$ (taken from [6]). The squares, hexagons (crossovers), shaded circles (terminators) and lines (rows) all represent specific arrangements of a subset of points. Let $X(\mathbb{F})$ denote the reduced instance. The number of points in $X(\mathbb{F})$ is bounded by a polynomial in n and t. Let this polynomial be $\mathcal{O}(t^\gamma)$, as we can assume $t \geq n$ since otherwise it trivially becomes a NO instance. Here γ is some constant.

We restate Theorem 1 in [6].

▶ **Proposition 26.** *Let $\mathcal{S}^*$ denote an SMT of $X(\mathbb{F})$, the instance obtained by reducing the X3C instance $(n, \mathbb{F})$, and $|\mathcal{S}^*|$ denote its length. If $\mathbb{F}$ has an exact cover, then $|\mathcal{S}^*| \leq f(n, t, \hat{C})$, otherwise $|\mathcal{S}^*| \geq f(n, t, \hat{C}) + \frac{1}{200nt}$, where $t = |\mathbb{F}|$, $\hat{C}$ is the number of crossovers, i.e. hexagonal gadgets, and f is a positive real-valued function of $n, t, \hat{C}$.*

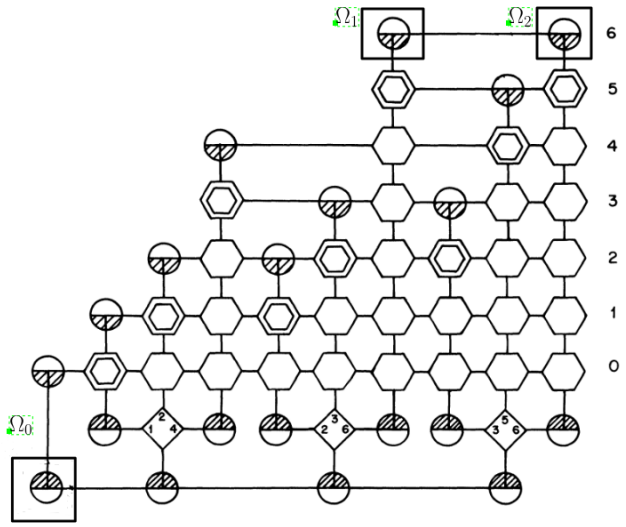


Figure 5 Reduced instance of ESMT from X3C (taken from [6]).

We extend this construction to prove NP-hardness for instances of Euclidean Steiner Minimal Tree where the terminal set $\mathcal{P}$ has $\Omega(n^\epsilon)$ points inside $\mathrm{CH}(\mathcal{P})$. Here, $\epsilon \in (0, 1]$ and n is the number of terminals.

Let us call the *length* of a gadget to be the maximum horizontal distance between any two points in that gadget. Similarly, we define the *breadth* of a gadget to be the maximum vertical distance between any two points in that gadget.

(a) The Upward Terminator symbol.

(b) The Downward Terminator symbol.

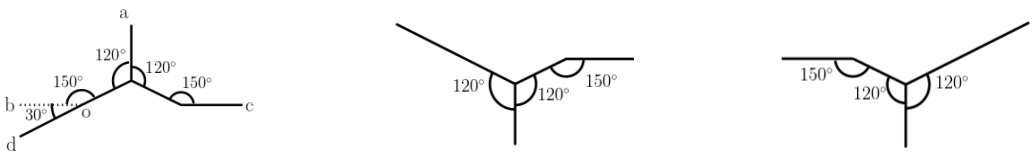
(c) The Downward Terminator symbol.

Figure 6 The Terminator gadget symbol and arrangement of points.

The *terminator* gadget used is shown in Figure 6. The straight lines represent a row of at least 1000 points separated at distances of 1/10 or 1/11. The angles between them are as shown. The upward terminator has the point A above the other points in the terminator, whereas the downward terminator has the point A below the other points. Firstly, we adjust the number of points in the long rows, such that the length and breadth of the terminators is same as that of the hexagonal gadgets (crossovers). We can fix this length and breadth to be some constants, such that the number of points in each gadget is also bounded by some constant. In our construction, we modify the terminators Ω_0, Ω_1, and Ω_2 as shown in Figure 5 enclosed in squares. Ω_1 is the terminator corresponding to the first occurrence of the element $3n \in U$ in some set in $\mathbb{F}$ and Ω_2 is the terminator corresponding to the last occurrence of $3n$ in some set in $\mathbb{F}$ (if there are more than one occurrences of $3n$). If there are no occurrences of $3n$, then it is trivially a no-instance. The modified gadgets are shown in Figure 7. All the other gadgets remain unaltered.

We define a Conic Set of points.

▶ **Definition 27.** *A Conic Set is a set of points consisting of a point T, called the tip of the cone, and the remaining points denoted by $\mathcal{S}$. Let $\mathcal{C}$ be the circle with T as centre and radius r. All the points in $\mathcal{S}$ lie on $\mathcal{C}$, such that the angle at the tip formed by the two extreme points $L, R \in \mathcal{S}$, i.e. $\angle LTR = 30°$ in the anticlockwise direction. So, we have $\overline{TL} = \overline{TR} = r$. The distance between any two consecutive points in $\mathcal{S}$ is the same, say d. Let the number of points in $\mathcal{S}$ be n. We denote the Conic Set as $\mathrm{Cone}(T, r, n)$ and $\mathcal{S}$ as $\mathrm{Circ}(T, r, n)$. We call TL as the left slope of the Conic Set and TR as the right slope of the Conic Set.*



(a) Ω'_0. **(b)** Ω'_1. **(c)** Ω'_2.

Figure 7 The modified terminator gadgets.

We use the Conic Set in the reduction for our problem (please see Figure 8). Now, we give a sketch of the reduction of an X3C instance $(n, \mathbb{F})$ to an instance $X'(\mathbb{F}, \epsilon)$ of ESMT.

Algorithm $\mathcal{A}$ for construction of ESMT instance $X'(\mathbb{F}, \epsilon)$ from X3C instance $(n, \mathbb{F})$:

- Reduce the input X3C instance to the points configuration $X(\mathbb{F})$ according to the reduction given in [6].
- Let $DQCP$ be the smallest axis-parallel rectangle bounding $X(\mathbb{F})$ after certain modifications of gadgets described in [6] (details in the full version).
- Take $\alpha = \frac{1}{\epsilon}$. Define $r = ct^{\alpha} = \mathcal{O}(t^{\alpha})$ and $n' = c't^{\gamma\alpha} = \mathcal{O}(t^{\gamma\alpha})$, where $t = |\mathbb{F}|$ and c and c' are constants. Add the $\mathrm{Cone}(D, r, n')$, such that D is the tip of the Conic Set, and the left slope DE makes an angle of 120° with DP. The right slope DF also makes an angle of 120° with DQ.

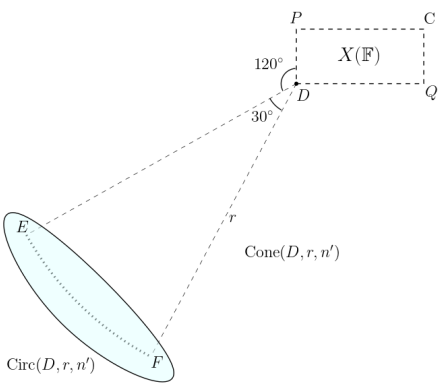


Figure 8 The reduced instance $X'(\mathbb{F}, \epsilon)$.

Now we state a few properties of the constructed instance $X'(\mathbb{F}, \epsilon)$ (detailed proofs can be found in the full version).

▶ **Lemma 28.** *All the points in* $\mathrm{Circ}(D, r, n')$ *(according to Definition 27) lie on the convex hull of the reduced ESMT instance* $X'(\mathbb{F}, \epsilon)$ *constructed by Algorithm* $\mathcal{A}$*, where* $\epsilon \in (0, 1]$.

Let us denote the convex hull of $X'(\mathbb{F}, \epsilon)$ by $\mathrm{CH}(X'(\mathbb{F}, \epsilon))$ and that of the points lying inside or on the bounding rectangle PDQC, i.e. $X'(\mathbb{F}, \epsilon) \setminus \mathrm{Circ}(D, r, n')$ by $\mathrm{CH}(X'(\mathbb{F}, \epsilon) \setminus \mathrm{Circ}(D, r, n'))$.

▶ **Lemma 29.** *The reduced ESMT instance* $X'(\mathbb{F}, \epsilon)$ *constructed by Algorithm* $\mathcal{A}$ *has* $\Omega(N^\epsilon)$ *points inside the convex hull, where* $\epsilon \in (0, 1]$ *and* N *is the total number of terminals in* $X'(\mathbb{F}, \epsilon)$.

We further state structural properties of SMTs of the reduced instance $X'(\mathbb{F}, \epsilon)$ when considering the modified gadgets Ω'_0, Ω'_1, and Ω'_2.

▶ **Lemma 30.** *Consider an SMT* $\mathcal{S}^*$ *of the ESMT instance* $X(\mathbb{F})$ *obtained via reduction from the X3C instance* $(n, \mathbb{F})$ *as per [6]. Consider a tree* $\mathcal{S}'^*$ *on the terminal set of* $X'(\mathbb{F}, \epsilon)$ *obtained from* $\mathcal{S}^*$ *as follows: Consider the modified terminator gadgets* Ω'_i, $i \in \{0, 1, 2\}$ *as in Algorithm* $\mathcal{A}$*. For each* $i \in \{0, 1, 2\}$*, the edge* B_iO_i *is excluded from* $\mathcal{S}^*$ *and the edge* D_iO_i *is included to form* $\mathcal{S}'^*$*.* $\mathcal{S}'^*$ *is an SMT for the terminal set of* $X'(\mathbb{F}, \epsilon)$.

Now we focus on the structure of the SMT of $X'(\mathcal{F}, \epsilon)$. The SMT is basically the union of the SMT $\mathcal{S}'^*$ of the points in the bounding rectangle $PDQC$ as stated in Lemma 30 and the SMT of the set of points $\mathrm{Cone}(D, r, n')$.

$\mathrm{CH}(X'(\mathbb{F}, \epsilon) \setminus \mathrm{Circ}(D, r, n'))$ is enclosed by the bounding rectangle $PDQC$ and D must lie on $\mathrm{CH}(X'(\mathbb{F}, \epsilon) \setminus \mathrm{Circ}(D, r, n'))$. We label the vertices of $\mathrm{CH}(X'(\mathbb{F}, \epsilon) \setminus \mathrm{Circ}(D, r, n'))$ as $D, P_1, P_2, \ldots, P_k$ in the counter-clockwise order. Let $\mathrm{CH}(X'(\mathbb{F}, \epsilon))$ be the convex hull of all the points. By Lemma 28, all the points in $\mathrm{Circ}(D, r, n')$ lie on $\mathrm{CH}(X'(\mathbb{F}, \epsilon))$. Let EP_i and FP_j be edges in $\mathrm{CH}(X'(\mathbb{F}, \epsilon))$, such that $P_i, P_j \notin \mathrm{Circ}(D, r, n')$.

The SMT of $X'(\mathbb{F}, \epsilon)$ clearly lies inside its convex hull, $\mathrm{CH}(X'(\mathbb{F}, \epsilon))$. We show that the Steiner hull can be further restricted to the bounding rectangle $PDQC$ and the convex polygon formed by the points in $\mathrm{Cone}(D, r, n')$. For this we use Theorem 1.5 in [9], as stated below.

▶ **Proposition 31** ([9])**.** *Let* H *be a Steiner hull of* N*. By sequentially removing wedges* abc *from the remaining region, where* a, b, c *are terminals but* $\triangle$abc *contains no other terminal,* a *and* c *are on the boundary and* $\angle\mathrm{abc} \geq 120°$*, a Steiner hull* H' *invariant to the sequence of removal is obtained.*

▶ **Lemma 32.** *The region comprising of the bounding rectangle* $PDQC$ *according to Algorithm* $\mathcal{A}$ *and the convex polygon formed by the set of points* $\mathrm{Cone}(D, r, n')$ *is a Steiner hull of* $X'(\mathbb{F}, \epsilon)$.

Given the nature of the above Steiner hull, we show that we can treat $X(\mathbb{F})$ and $\mathrm{Cone}(D, r, n')$ separately.

▶ **Lemma 33.** *There is an SMT of* $X'(\mathbb{F}, \epsilon)$ *that is the union of an SMT of* $X(\mathbb{F})$ *and an SMT of the points in* $\mathrm{Cone}(D, r, n')$*, with* D *being common to both of them.*

We can identify a structure for an SMT of the points in $\mathrm{Cone}(D, r, n')$ using [15].

▶ **Lemma 34.** *There is an SMT of the points in* $\mathrm{Cone}(D, r, n')$ *that is as shown in Figure 9. In the SMT,* D *is connected to the two middle points in* $\mathrm{Circ}(D, r, n')$ *via a Steiner point* S^t*. The other points in* $\mathrm{Circ}(D, r, n')$ *are connected along the circumference.*

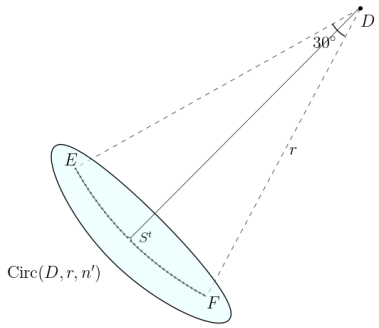


Figure 9 SMT of Cone(D, r, n').

Finally, we prove the NP-hardness of Euclidean Steiner Minimal Tree on $f(n)$-Almost Convex Sets of n terminals, when $f(n) = \Omega(n^\epsilon)$ for some $\epsilon \in (0, 1]$.

▶ **Theorem 35.** *Let $\mathcal{S}^*_{\mathbb{F},\epsilon}$ denote an SMT of $X'(\mathbb{F}, \epsilon)$ and $|\mathcal{S}^*_{\mathbb{F},\epsilon}|$ denote its length. If $\mathbb{F}$ has an exact cover, then $|\mathcal{S}^*_{\mathbb{F},\epsilon}| \leq f(n, t, \hat{C}) + |\mathcal{T}_1|$, otherwise $|\mathcal{S}^*_{\mathbb{F},\epsilon}| \geq f(n, t, \hat{C}) + \frac{1}{200nt} + |\mathcal{T}_1|$, where $\hat{C}$ is the number of crossovers, i.e. hexagonal gadgets, and f is a positive real-valued function of $n, t, \hat{C}$ as stated in Proposition 26.*

Since it is not known if the ESMT problem is in NP, Garey et al. [6] show the NP-completeness of a related problem called the Discrete Euclidean Steiner Minimal Tree (DESMT) problem, which is in NP. We define the DESMT problem as given in [6]. The DESMT problem takes as input a set $\mathcal{X}$ of integer-coordinate points in the plane and a positive integer L, and asks if there exists a set $\mathcal{Y} \supseteq \mathcal{X}$ of integer-coordinate points such that some spanning tree $\mathcal{T}$ for $\mathcal{Y}$ satisfies $|\mathcal{T}|_d \leq L$, where $|\mathcal{T}|_d = \Sigma_{e \in E(\mathcal{T})} \lceil \overline{e} \rceil$, i.e. we round up the length of each edge to the least integer not less than it.

In order to show that DESMT is NP-hard, the same reduction as that of the ESMT problem can be used, followed by scaling and rounding the coordinates of the points. Theorem 4 of [6] proves that the DESMT problem is NP-Complete. Moreover, since it is Strongly NP-Complete, the DESMT problem does not admit any FPTAS. Finally in Theorem 5 of [6], Garey et al. show that as a consequence, the ESMT problem does not have any FPTAS as well.

Now we show that the DESMT problem is NP-hard even on $f(n)$-Almost Convex Sets of n terminals, when $f(n) = \Omega(n^\epsilon)$ and where $\epsilon \in (0, 1]$.

In Section 7 of [6], the reduced instance $X(\mathbb{F})$ of ESMT is converted into an instance $X_d(\mathbb{F})$ of DESMT. The conversion is as follows:
$X_d(\mathbb{F}) = \{(\lceil 12M \cdot 200nt \cdot x_1 \rceil, \lceil 12M \cdot 200nt \cdot x_2 \rceil) : x = (x_1, x_2) \in X(\mathbb{F})\}$, where $M = |X(\mathbb{F})|$.

We apply a similar conversion to the reduced ESMT instance $X'(\mathbb{F}, \epsilon)$, to convert it into a DESMT instance of an $\Omega(n^\epsilon)$-Almost Convex Set. The conversion goes as follows:
$X'_d(\mathbb{F}, \epsilon) = \{(\lceil 12N \cdot 200nt \cdot x_1 \rceil, \lceil 12N \cdot 200nt \cdot x_2 \rceil) : x = (x_1, x_2) \in X'(\mathbb{F}, \epsilon)\}$, where $N = |X'(\mathbb{F}, \epsilon)|$.

The next two lemmas establish the validity of $X'_d(\mathbb{F}, \epsilon)$ as an instance of DESMT and the upper bounds on the size of the constructed instance. Note that the reduction from X3C followed by the conversion can be done in polynomial time.

▶ **Lemma 36.** *The instance $X'_d(\mathbb{F}, \epsilon)$ constructed above is a valid DESMT instance.*

▶ **Lemma 37.** *The reduced DESMT instance $X'_d(\mathbb{F}, \epsilon)$ has N distinct points, where $N = |X'(\mathbb{F}, \epsilon)|$.*

Now we present the following lemma for the constructed DESMT instance $X'_d(\mathbb{F}, \epsilon)$ analogous to Lemma 29 for the ESMT instance $X'(\mathbb{F}, \epsilon)$.

▶ **Lemma 38.** *The reduced DESMT instance $X'_d(\mathbb{F}, \epsilon)$ constructed is an $\Omega(N^\epsilon)$-Almost Convex Set, where $N = |X'_d(\mathbb{F}, \epsilon)|$.*

We get the following theorem from Lemmas 36–38.

▶ **Theorem 39.** *The instance $X'_d(\mathbb{F}, \epsilon)$ constructed is a valid DESMT instance on an $\Omega(N^\epsilon)$-Almost Convex Set, where $|X'_d(\mathbb{F}, \epsilon)| = |X'(\mathbb{F}, \epsilon)| = N$.*

Following Theorems 3 and 4 in [6], we get that the DESMT problem is NP-Complete for $\Omega(N^\epsilon)$-Almost Convex Sets, where N is the total number of terminals. Since we get the reduced instance $X'_d(\mathbb{F}, \epsilon)$ from the X3C instance $(n, \mathbb{F})$, the DESMT problem is strongly NP-complete for $\Omega(N^\epsilon)$-Almost Convex Sets, and does not admit any FPTAS.

Using Theorem 5 of [6], we get that if the ESMT problem has an FPTAS, then the X3C problem can be solved in polynomial time. The Theorem also applies for our case of $\Omega(N^\epsilon)$-Almost Convex Sets. Therefore, we get the following theorem,

▶ **Theorem 40.** *There does not exist any FPTAS for the ESMT problem on $f(n)$-Almost Convex Sets of n terminals, where $f(n) = \Omega(n^\epsilon)$ and $\epsilon \in (0, 1]$, unless $P = NP$.*

6 Conclusion

In this paper, we first study ESMT on vertices of 2-CPR n-gons and design a polynomial time algorithm. It remains open to design a polynomial time algorithm for ESMT on k-CPR n-gons, or show NP-hardness for the problem. Next, we study the problem on $f(n)$-Almost Convex Sets of n terminals. For this NP-hard problem, we obtain an algorithm that runs in $2^{\mathcal{O}(f(n)\log n)}$ time. We also design an FPTAS when $f(n) = \mathcal{O}(\log n)$. On the other hand, we show that there cannot be an FPTAS if $f(n) = \Omega(n^\epsilon)$ for any $\epsilon \in (0, 1]$, unless P = NP. The question of existence of FPTASes when $f(n)$ is a polylogarithmic function remains open.

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