Unified Almost Linear Kernels for Generalized Covering and Packing Problems on Nowhere Dense Classes

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Abstract

Let $\mathcal{F}$ be a family of graphs, and let p, r be nonnegative integers. For a graph G and an integer k, the $(p, r, \mathcal{F})$-Covering problem asks whether there is a set $D \subseteq V(G)$ of size at most k such that if the p-th power of G has an induced subgraph isomorphic to a graph in $\mathcal{F}$, then it is at distance at most r from D. The $(p, r, \mathcal{F})$-Packing problem asks whether G^p has k induced subgraphs $H_1, \ldots, H_k$ such that each H_i is isomorphic to a graph in $\mathcal{F}$, and for $i, j \in \{1, \ldots, k\}$, the distance between $V(H_i)$ and $V(H_j)$ in G is larger than r.

We show that for every fixed nonnegative integers p, r and every fixed nonempty finite family $\mathcal{F}$ of connected graphs, $(p, r, \mathcal{F})$-Covering with $p \leqslant 2r + 1$ and $(p, r, \mathcal{F})$-Packing with $p \leqslant 2\lfloor r/2 \rfloor + 1$ admit almost linear kernels on every nowhere dense class of graphs, parameterized by the solution size k. As corollaries, we prove that Distance-r Vertex Cover, Distance-r Matching, $\mathcal{F}$-Free Vertex Deletion, and Induced-$\mathcal{F}$-Packing for any fixed finite family $\mathcal{F}$ of connected graphs admit almost linear kernels on every nowhere dense class of graphs. Our results extend the results for Distance-r Dominating Set by Drange et al. (STACS 2016) and Eickmeyer et al. (ICALP 2017), and for Distance-r Independent Set by Pilipczuk and Siebertz (EJC 2021).

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1 Introduction

The Dominating Set problem is one of the classical NP-hard problems which asks whether a graph G contains a set of at most k vertices whose closed neighborhood contains all the vertices of G. A natural variant of it is the Distance-r Dominating Set problem which asks whether G contains a set of at most k vertices such that every vertex of G is at distance at most r from one of these vertices. Dominating Set has been intensively studied in the context of fixed-parameter algorithms. In a parameterized problem Π, we are given an instance (x, k) where k is a parameter, and the central question is whether the parameterized problem admits an algorithm, called *fixed-parameter* algorithm, that runs in time $f(k) \cdot |x|^c$ for some computable function f and a constant c. We say that Π is *fixed-parameter tractable*, or FPT for short if it admits a fixed-parameter algorithm. It is known that Dominating Set is W[2]-complete parameterized by k [14, 15], meaning that it is not FPT unless an unexpected collapse occurs in the parameterized complexity hierarchy. Thus, it is natural to restrict graph classes and see whether a fixed-parameter algorithm exists. Dominating Set admits a fixed-parameter algorithm on planar graphs [13, 27], and the project of finding larger sparse graph classes on which fixed-parameter algorithms for Dominating Set exist has been studied intensively, see [11, 4, 30, 38, 29, 24, 40].

A *kernelization* algorithm for a parameterized problem takes an instance (x, k) and outputs an equivalent instance (x', k') in time polynomial in $|x| + k$, where $|x'| + k' \leqslant g(k)$ for some computable function g. We call the function g the *size* of the kernel. If g is a polynomial (resp. linear), then such an algorithm is called a polynomial (resp. linear) kernel. It is well known that a parameterized decision problem is FPT if and only if it admits a kernelization; see [16]. Furthermore, with a polynomial kernel, we can compress inputs to instances of polynomial size, which lead to boost up the running time of exact algorithms solving the problem, like the brute-force search algorithm. Therefore, it is natural and applicable to investigate the existence of a polynomial kernel or a linear kernel. In particular, the existence of linear kernels for Dominating Set on sparse graph classes have been investigated.

One of the first results is a linear kernel for Dominating Set on planar graphs due to Alber, Fellows, and Niedermeier [3]. It has been generalized to classes of bounded genus graphs [26], H-minor free graphs [23], and H-topological minor free graphs [24]. Drange et al. [17] extended the previous results to classes of graphs with bounded expansion for Distance-r Dominating Set, and Eickmeyer et al. [20] obtained almost linear kernels for Distance-r Dominating Set on nowhere dense classes of graphs. Classes of graphs with bounded expansion and nowhere dense classes of graphs were introduced by Nešetřil and Ossona de Mendez [37], which are defined in terms of shallow minors and capture most of well-studied sparse graph classes.

Independent Set is another classic NP-hard problem which asks to find a set of k vertices in a given graph whose pairwise distance is more than 1, and Distance-r Independent Set is the problem obtained by replacing 1 with r. It is known that Independent Set is W[1]-complete parameterized by k [15]. The distance variations of Dominating Set and Independent Set are closely related, in a sense that the size of a distance-$2r$ independent set is a lower bound for the minimum size of a distance-r dominating set. Dvořák [18] presented an approximation algorithm for Distance-r Dominating Set, which outputs a set of size bounded by a function of the $2r$-weak coloring number and the maximum size of a distance-$2r$ independent set. Pilipczuk and Siebertz [39] recently presented an almost linear kernel for Distance-r Independent Set on nowhere dense classes of graphs.

For a fixed r, both Distance-r Dominating Set and Distance-r Independent Set can be expressed in first-order logic. Thus, by the meta-theorem of Grohe, Kreutzer, and Siebertz [29], there are almost-linear-time fixed-parameter tractable on every nowhere dense class of graphs. Fabiański, Pilipczuk, Siebertz, and Toruńczyk [22] presented linear-time fixed-parameter algorithms for Distance-r Dominating Set on various graph classes, including powers of nowhere dense classes and map graphs, and a linear-time fixed-parameter algorithm for Distance-r Independent Set on every nowhere dense class of graphs.

A natural question is whether there are linear/polynomial kernels for other problems on classes of graphs with bounded expansion and nowhere dense classes of graphs. Meta-type kernelization results have been studied for graphs on bounded genus [6], H-minor free graphs [25], H-topological minor free graphs [32], classes of graphs with bounded expansion, and nowhere dense classes of graphs [28]. Note that the last result by Gajarsky et al. [28] is to obtain kernelizations parameterized by the size of a modulator to constant tree-depth, and not by the solution size. Currently, limited investigations have been conducted on variations of Distance-r Dominating Set and Distance-r Independent Set. In this paper, we consider generic problems to hit finite graphs by the r-th neighborhood of a set of vertices.

Let $\mathcal{F}$ be a family of graphs, and let p and r be nonnegative integers. For a graph G, let G^p be the graph with vertex set $V(G)$ such that distinct vertices v and w are adjacent in G^p if and only if the distance between v and w in G is at most p, and let $N^r_G[D]$ be the set of all vertices in G at distance at most r from D in G. For a graph G, a set $D \subseteq V(G)$ is a $(p, r, \mathcal{F})$*-cover* of G if there is no set $X \subseteq V(G) \backslash N^r_G[D]$ such that $G^p[X]$ is isomorphic to a graph in $\mathcal{F}$. We denote by the $\gamma^{\mathcal{F}}_{p,r}(G)$ the minimum size of a $(p, r, \mathcal{F})$-cover of G. For a graph G and an integer k, the $(p, r, \mathcal{F})$-Covering problem asks whether $\gamma^{\mathcal{F}}_{p,r}(G) \leqslant k$. Note that Distance-$r$ Dominating Set is equal to $(1, r, \{K_1\})$-Covering.

Our main results are the following. For classes of graphs with bounded expansion, we can obtain linear kernels. Let $\mathbb{N}$ be the set of nonnegative integers and $\mathbb{R}_+$ be the set of positives.

▶ **Theorem 1.1.** *For every nowhere dense class $\mathcal{C}$ of graphs, there is $g_{\text{cov}} : \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N}$ such that for every nonempty family $\mathcal{F}$ of connected graphs with at most d vertices, $p, r \in \mathbb{N}$ with $p \leqslant 2r + 1$, and $\varepsilon > 0$, there is a polynomial-time algorithm that given a graph $G \in \mathcal{C}$ and $k \in \mathbb{N}$, either correctly decides that $\gamma^{\mathcal{F}}_{p,r}(G) > k$, or outputs a graph G' with $|V(G')| \leqslant g_{\text{cov}}(r, d, \varepsilon) \cdot k^{1+\varepsilon}$ such that $\gamma^{\mathcal{F}}_{p,r}(G) \leqslant k$ if and only if $\gamma^{\mathcal{F}}_{p,r}(G') \leqslant k + 1$.*

A $(p, r, \mathcal{F})$*-packing* of G is a family of sets $A_1, \ldots, A_\ell \subseteq V(G)$ such that each $G^p[A_i]$ is isomorphic to a graph in $\mathcal{F}$, and for all $1 \leqslant i < j \leqslant \ell$, the distance between A_i and A_j in G is more than r. We denote by $\alpha^{\mathcal{F}}_{p,r}(G)$ the maximum size of a $(p, r, \mathcal{F})$-packing of G. For a graph G and an integer k, the $(p, r, \mathcal{F})$-Packing problem asks whether $\alpha^{\mathcal{F}}_{p,r}(G) \geqslant k$. Note that Distance-$r$ Independent Set is equal to $(1, r, \{K_1\})$-Packing.

▶ **Theorem 1.2.** *For every nowhere dense class $\mathcal{C}$ of graphs, there is $g_{\text{pack}} : \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N}$ such that for every nonempty family $\mathcal{F}$ of connected graphs with at most d vertices, $p, r \in \mathbb{N}$ with $p \leqslant 2\lfloor r/2 \rfloor + 1$, and $\varepsilon > 0$, there is a polynomial-time algorithm that given a graph $G \in \mathcal{C}$ and $k \in \mathbb{N}$, either correctly decides that $\alpha^{\mathcal{F}}_{p,r}(G) = 0$, or correctly decides that $\alpha^{\mathcal{F}}_{p,r}(G) > k$, or outputs a graph G' with $|V(G')| \leqslant g_{\text{pack}}(r, d, \varepsilon) \cdot k^{1+\varepsilon}$ such that $\alpha^{\mathcal{F}}_{p,r}(G) \geqslant k$ if and only if $\alpha^{\mathcal{F}}_{p,r}(G') \geqslant k + 1$.*

Applications. Our kernels for the covering problems have the following applications. Canales, Hernández, Martins, and Matos [9] introduced a distance-r vertex cover and a distance-r guarding set. For a graph G and a positive integer r, a set $D \subseteq V(G)$ is a *distance-r vertex cover* if $G - N^r_G[D]$ has no edge, and a *distance-r guarding set* if $G - N^{r-1}_G[D]$ has no triangle. For a positive integer k, the Distance-r Vertex Cover problem asks whether a graph G has a distance-r vertex cover of size at most k. Similarly, the Distance-r Guarding Set problem asks whether a graph G has a distance-r guarding set of size at most k. Distance-r Vertex Cover and Distance-r Guarding Set for $r \geqslant 1$ can be formulated as $(1, r, \{K_2\})$-Covering and $(1, r - 1, \{K_3\})$-Covering, respectively. By Theorem 1.1, both problems admit almost linear kernels on every nowhere dense class of graphs, and these kernels also can be translated to fixed-parameter algorithms solving those problems on every nowhere dense class of graphs.

For a family $\mathcal{F}$ of graphs, a graph G, and an integer k, the $\mathcal{F}$-Free Vertex Deletion problem asks whether there is a set S of at most k vertices in G such that $G - S$ has no induced subgraph isomorphic to $\mathcal{F}$. If all graphs in $\mathcal{F}$ have at most d vertices, then this problem is related to the d-Hitting Set problem, and has a kernel of size $\mathcal{O}(k^{d-1})$ [1]. Our results imply that if $\mathcal{F}$ is a finite set of connected graphs, then $\mathcal{F}$-Free Vertex Deletion admits an almost linear kernel on every nowhere dense class of graphs. This can be applied to Cograph Vertex Deletion [36], to Cluster Vertex Deletion [5, 41], and to Claw-Free Vertex Deletion [7].

Our kernels for the packing problems have the following applications. A matching of a graph G is a set M of edges of G such that no two edges in M share an end. For a positive integer r, a *distance-r matching* of G is a matching M of G such that for distinct edges $u_1u_2, v_1v_2 \in M$, $\min\{\mathrm{dist}_G(u_i, v_j) : i, j \in [2]\} \geqslant r$. For an integer k, the Distance-r Matching problem asks whether G has a distance-r matching of size at least k. Distance-1 Matching is nothing but finding a matching of size at least k, so can be solved in polynomial time [19]. Distance-r Matching for $r \geqslant 2$ is equal to $(1, r-1, \{K_2\})$-Packing. Moser and Sikdar [35] presented linear kernels for Distance-2 Matching on planar graphs and graphs of bounded degree, and a cubic kernel for the same problem on graphs of girth at least 6. Later, Kanj, Pelsmajer, Schaefer, and Xia [31] presented a kernel of size $40k$ for Distance-2 Matching on planar graphs. By Theorem 1.2, Distance-r Matching for every $r \geqslant 2$ admits an almost linear kernel on every nowhere dense class of graphs.

We can further generalize the matching problem. For a graph H and an integer k, the H-Matching problem asks whether a graph G has k vertex-disjoint subgraphs isomorphic to H. For every integer $d \geqslant 1$, let P_d be a path on d vertices. Dell and Marx [10] presented a kernel for P_3-Matching with $O(k^{2.5})$ edges, and a unified kernel for P_d-Matching with $O(d^{d^2} d^7 k^3)$ vertices. They also showed that for every integer $d \geqslant 3$ and every $\varepsilon > 0$, under some complexity hypothesis, K_d-Matching does not have kernels with $O(k^{d-1-\varepsilon})$ edges. By taking $\mathcal{F}$ as the set of all graphs on $|V(H)|$ vertices that contain H as a subgraph, we can formulate H-Matching as $(1, 0, \mathcal{F})$-Packing. Generally, we may consider Induced-$\mathcal{F}$-Packing which asks whether a graph has k vertex-disjoint induced subgraphs each isomorphic to some graph in $\mathcal{F}$. By Theorem 1.2, Induced-$\mathcal{F}$-Packing for every fixed finite family $\mathcal{F}$ of connected graphs admits almost linear kernel on every nowhere dense class of graphs.

We may formulate $(p, r, \mathcal{F})$-Covering and $(p, r, \mathcal{F})$-Packing on fixed powers of a given graph. Formally, for a fixed positive integer t, $(pt, rt, \mathcal{F})$-Covering on a graph G is exactly same as $(p, r, \mathcal{F})$-Covering on its t-th power G^t. Therefore, our result provides the existence of almost linear kernels for both problems on t-th powers G^t of graphs from a nowhere dense class of graphs, assuming that the original graph G is given. However, if the power G^t is only given, then we need to find the graph G to apply our kernelization algorithm.

Organization. We organize this paper as follows. In Section 2, we present some terminology from graph theory, especially lemmas on nowhere dense classes of graphs. In Sections 3 and 4, we present almost linear kernels for $(p, r, \mathcal{F})$-Covering and $(p, r, \mathcal{F})$-Packing on every nowhere dense class of graphs, respectively.

2 Preliminaries

In this paper, all graphs are simple and finite and have at least one vertex. For an equivalence relation $\sim$ on a set X, we denote by $\mathrm{index}(\sim)$ the number of equivalence classes of $\sim$ in X. For every integer n, let $[n]$ be the set of positive integers at most n. Throughout this section, we let p, r be nonnegative integers, let G be a graph, and let A, B be subsets of $V(G)$. We follow the notations from the textbook of Diestel [12]. We denote by $\mathrm{dist}_G(v, w)$ the distance between vertices v and w in G and by $\mathrm{dist}_G(A, B)$ be the shortest distance between a vertex in A and a vertex in B. The *p-th power* of G, denoted by G^p, is the graph with vertex set $V(G)$ such that distinct vertices v and w are adjacent in G^p if and only if $\mathrm{dist}_G(v, w) \leqslant p$. For a vertex v of G, let $N_G^r[v]$ be the set of vertices of G which are at distance at most r from v in G, and $N_G^r(v) := N_G^r[v] \backslash \{v\}$. Let $N_G^r[A] := \bigcup_{v \in A} N_G^r[v]$ and $N_G^r(A) := N_G^r[A] \backslash A$.

A set $X \subseteq V(G)$ is a *distance-r independent set* in G if the vertices in X are pairwise at distance larger than r in G. We denote by $\alpha_r(G)$ the maximum size of a distance-r independent in G. A set $D \subseteq V(G)$ is a *distance-r dominating set* of G if every vertex of G lies in $N_G^r[D]$. We denote by $\gamma_r(G)$ the minimum size of a distance-r dominating set of G.

Sparse graphs. A graph H with vertex set $\{v_1, \ldots, v_n\}$ is an *r-shallow minor* of G if there exist pairwise disjoint subsets $V_1, \ldots, V_n$ of $V(G)$ such that each $G[V_i]$ has radius at most r and for all edges $v_i v_j \in E(H)$, $\mathrm{dist}_G(V_i, V_j) = 1$. A class $\mathcal{C}$ of graphs has *bounded expansion* if there is $f : \mathbb{N} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$, $G \in \mathcal{C}$, and an r-shallow minor H of G, $|E(H)|/|V(H)| \leqslant f(r)$. A class $\mathcal{C}$ of graphs is *nowhere dense* if there is $g : \mathbb{N} \to \mathbb{N}$ such that for all $r \in \mathbb{N}$ and $G \in \mathcal{C}$, $K_{g(r)}$ is not an r-shallow minor of G.

For vertices $v \in A$ and $u \in V(G) \backslash A$, a path P from u to v is *A-avoiding* if $V(P) \cap A = \{v\}$. For a vertex $u \in V(G) \backslash A$, the *r-projection* of u on A, denoted by $M_r^G(u, A)$, is the set of all vertices $v \in A$ connected to u by an A-avoiding path of length at most r in G. The *r-projection profile* of u on A is a function $\rho_r^G[u, A] : A \to [r] \cup \{\infty\}$ such that for each vertex $v \in A$, $\rho_r^G[u, A](v)$ is ∞ if there is no A-avoiding path of length at most r from u to v, and otherwise the length of a shortest A-avoiding path from u to v. Let $\mu_r(G, A) := |\{\rho_r^G[u, A] : u \in V(G) \backslash A\}|$. We will use the following lemmas.

▶ **Lemma 2.1** (Eickmeyer et al. [20]). *For every nowhere dense class $\mathcal{C}$ of graphs, there is $f_{\mathrm{proj}} : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N}$ such that for all $r \in \mathbb{N}$, $\varepsilon > 0$, $G \in \mathcal{C}$, and $X \subseteq V(G)$, $\mu_r(G, X) \leqslant f_{\mathrm{proj}}(r, \varepsilon) \cdot |X|^{1+\varepsilon}$.*

For $t \geqslant 0$, a set $X \subseteq V(G)$ is *(r, t)-close* if $|M_r^G(u, X)| \leqslant t$ for every $u \in V(G) \backslash X$.

▶ **Lemma 2.2** (Eickmeyer et al. [20]). *For every nowhere dense class $\mathcal{C}$ of graphs, there exist $f_{\mathrm{cl}} : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N}$ and a polynomial-time algorithm that for all $r \in \mathbb{N}$, $\varepsilon > 0$, $G \in \mathcal{C}$, and $X \subseteq V(G)$, outputs an $(r, f_{\mathrm{cl}}(r, \varepsilon) \cdot |X|^{\varepsilon})$-close set $X_{\mathrm{cl}} \supseteq X$ of size at most $f_{\mathrm{cl}}(r, \varepsilon) \cdot |X|^{1+\varepsilon}$.*

For a set $X \subseteq V(G)$, an *r-path closure* of X is a set $X_{\mathrm{pth}} \supseteq X$ such that for $u, v \in X$, if $\mathrm{dist}_G(u, v) \leqslant r$, then $\mathrm{dist}_{G[X_{\mathrm{pth}}]}(u, v) = \mathrm{dist}_G(u, v)$.

▶ **Lemma 2.3** (Eickmeyer et al. [20]). *For every nowhere dense class $\mathcal{C}$ of graphs, there exist $f_{\mathrm{pth}} : \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N}$ and a polynomial-time algorithm that for all $r \in \mathbb{N}$, $\varepsilon > 0$, $G \in \mathcal{C}$, and $X \subseteq V(G)$, outputs an r-path closure of X having size at most $f_{\mathrm{pth}}(r, \varepsilon) \cdot |X|^{1+\varepsilon}$.*

Drange et al. [17] showed analogues of these three lemmas on classes of graphs with bounded expansion. By substituting Lemmas 2.1, 2.2, and 2.3 with their analogues, we can easily obtain linear kernels for $(p, r, \mathcal{F})$-Covering and $(p, r, \mathcal{F})$-Packing on classes of graphs with bounded expansion. Thus, we mainly focus on constructing almost linear kernels for the problems on nowhere dense classes of graphs.

A class $\mathcal{C}$ of graphs is *uniformly quasi-wide* if there exist $N : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$ such that for all $G \in \mathcal{C}$ and $A \subseteq V(G)$ with $|A| \geqslant N(r, m)$, there exist sets $S \subseteq V(G)$ and $B \subseteq A \backslash S$ such that $|S| \leqslant s(r)$, $|B| \geqslant m$, and B is distance-r independent in $G - S$.

▶ **Theorem 2.4** (Kreutzer, Rabinovich, and Siebertz [33]). *Let $\mathcal{C}$ be a nowhere dense class of graphs. For every $r \geqslant 0$, there are $p(r), s(r)$ such that for all $G \in \mathcal{C}$, $m \in \mathbb{N}$, and $A \subseteq V(G)$ with $|A| \geqslant m^{p(r)}$, there are sets $S \subseteq V(G)$ and $B \subseteq A \backslash S$ such that $|S| \leqslant s(r)$, $|B| \geqslant m$, and B is distance-r independent in $G - S$. Moreover, if K_c is not an r-shallow minor of G, then $s(r) \leqslant c \cdot r$ and one can find desired sets S and B in $O(r \cdot c \cdot |A|^{c+6} \cdot |V(G)|^2)$ time.*

VC-dimension. A *set-system* is a family of subsets of a set, called the *ground set*. Let $\mathcal{S}$ be a set-system with the ground set S. A set $S' \subseteq S$ is *shattered* by $\mathcal{S}$ if $|\{S' \cap T : T \in \mathcal{S}\}| = 2^{|S'|}$. The *Vapnik-Chervonenkis dimension*, or *VC-dimension* for short, of $\mathcal{S}$ is the largest cardinality of a shattered subset of S by $\mathcal{S}$. Observe that if a set $S' \subseteq S$ is shattered by $\mathcal{S}$, then every subset of S' is also shattered by $\mathcal{S}$. In addition, for every $\mathcal{S}' \subseteq \mathcal{S}$, the VC-dimension of $\mathcal{S}'$ is at most that of $\mathcal{S}$.

▶ **Proposition 2.5** (See [34, Proposition 10.3.3]). *Let $F(X_1, \ldots, X_d)$ be a set-theoretic expression using set variables $X_1, \ldots, X_d$ and the operations of union, intersection, and difference. Let $\mathcal{S}$ be a set-system with the ground set S, and $\mathcal{T} := \{F(S_1, \ldots, S_d) : S_1, \ldots, S_d \in \mathcal{S}\}$. If $\mathcal{S}$ has VC-dimension $c < \infty$, then $\mathcal{T}$ has VC-dimension $O(cd \log d)$.*

A *hitting set* of $\mathcal{S}$ is a set $X \subseteq S$ such that for every $T \in \mathcal{S}$, $T \cap X \neq \varnothing$. Let $\tau(\mathcal{S})$ be the minimum size of a hitting set of $\mathcal{S}$.

Brönnimann and Goodrich [8] and Even, Rawitz, and Shahar [21] presented polynomial-time algorithms finding a hitting set X of a nonempty set-system $\mathcal{S}$ having VC-dimension at most c with $|X| = O(c \cdot \tau(\mathcal{S}) \cdot \ln \tau(\mathcal{S}))$.

▶ **Theorem 2.6** ([8, 21]). *There exist a constant C_τ and a polynomial-time algorithm that for every nonempty set-system $\mathcal{S}$ having VC-dimension at most c, outputs a hitting set of $\mathcal{S}$ having size at most $C_\tau \cdot c \cdot \tau(\mathcal{S}) \cdot \ln \tau(\mathcal{S}) + 1$.*

The VC-dimension of G is defined by the VC-dimension of $\{N_G[v] : v \in V(G)\}$.

▶ **Theorem 2.7** (Adler and Adler [2]). *Let $\mathcal{C}$ be a nowhere dense class of graphs and $\phi(x, y)$ be a first-order formula such that for all $G \in \mathcal{C}$ and vertices v and w of G, $G \models \phi(v, w)$ if and only if $G \models \phi(w, v)$. For a graph $G \in \mathcal{C}$, let $G_\phi := (V(G), \{vw : G \models \phi(v, w)\})$. Then there exists a nonnegative integer c depending on $\mathcal{C}$ and ϕ such that every graph in $\{G_\phi : G \in \mathcal{C}\}$ has VC-dimension at most c.*

For every $p \in \mathbb{N}$, the property that the distance between two vertices is at most p can be expressed in a first-order formula, so Theorem 2.7 has the following corollary.

▶ **Corollary 2.8.** *For every nowhere dense class $\mathcal{C}$ of graphs, there exists a function $f_{\mathrm{vc}} : \mathbb{N} \to \mathbb{N}$ such that for all $p \in \mathbb{N}$ and $G \in \mathcal{C}$, G^p has VC-dimension at most $f_{\mathrm{vc}}(p)$.* ◀

3 Kernels for the $(p, r, \mathcal{F})$-Covering problems

Let p, r be nonnegative integers with $p \leqslant 2r + 1$ and let $\mathcal{F}$ be a nonempty finite family of connected graphs. In this section, we present an almost linear kernel for $(p, r, \mathcal{F})$-Covering on every nowhere dense class of graphs. To do this, we divert to an annotated variant of $(p, r, \mathcal{F})$-Covering. For a graph G and a set $A \subseteq V(G)$, a set $D \subseteq V(G)$ is a $(p, r, \mathcal{F})$*-cover* of A in G if there is no set $X \subseteq A \backslash N_G^r[D]$ such that $G^p[X]$ is isomorphic to a graph in $\mathcal{F}$. We denote by $\gamma_{p,r}^{\mathcal{F}}(G, A)$ the minimum size of a $(p, r, \mathcal{F})$-cover of A in G. For a graph G, a set $A \subseteq V(G)$, and an integer k, the Annotated $(p, r, \mathcal{F})$-Covering problem asks whether $\gamma_{p,r}^{\mathcal{F}}(G, A) \leqslant k$.

We first construct an almost linear kernel for Annotated $(p,r,\mathcal{F})$-Covering on every nowhere dense class of graphs. Every instance of $(p,r,\mathcal{F})$-Covering can be seen as an instance of an annotated variant, so we apply the almost linear kernel to the input instance. Afterwards, we construct an equivalent instance of $(p,r,\mathcal{F})$-Covering by attaching a small graph, which will be called a $(p,\mathcal{F})$-critical graph, to the resulting instance obtained from the almost linear kernel.

For a graph G and a set $A \subseteq V(G)$, a $(p,r,\mathcal{F})$-*core* of A in G is a set $Z \subseteq A$ such that every minimum-size $(p,r,\mathcal{F})$-cover of Z in G is a $(p,r,\mathcal{F})$-cover of A in G. Observe that $\gamma^{\mathcal{F}}_{p,r}(G,A) = \gamma^{\mathcal{F}}_{p,r}(G,Z)$ and A is a $(p,r,\mathcal{F})$-core of A in G. We derive an almost linear kernel for Annotated $(p,r,\mathcal{F})$-Covering from Proposition 3.1 saying that we can either confirm that the given instance is a no-instance, or reduce the size of a $(p,r,\mathcal{F})$-core of A in G.

▶ **Proposition 3.1.** *For every nowhere dense class $\mathcal{C}$ of graphs, there is $f_{\mathrm{core}} : \mathbb{N}\times\mathbb{N}\times\mathbb{R}_+ \to \mathbb{N}$ such that for every nonempty family $\mathcal{F}$ of connected graphs with at most d vertices, $p, r \in \mathbb{N}$ with $p \leqslant 2r+1$, and $\varepsilon > 0$, there is a polynomial-time algorithm that given a graph $G \in \mathcal{C}$, $A \subseteq V(G)$, $k \in \mathbb{N}$, and a $(p,r,\mathcal{F})$-core Z of A in G with $|Z| > f_{\mathrm{core}}(r,d,\varepsilon)\cdot k^{1+\varepsilon}$, either correctly decides that $\gamma^{\mathcal{F}}_{p,r}(G,A) > k$, or outputs a vertex $z \in Z$ such that $Z\backslash\{z\}$ is a $(p,r,\mathcal{F})$-core of A in G.*

We will use the following proposition to prove Proposition 3.1.

▶ **Proposition 3.2.** *For every nowhere dense class $\mathcal{C}$ of graphs, there is $f_{\mathrm{apx}} : \mathbb{N}\times\mathbb{N}\times\mathbb{R}_+ \to \mathbb{N}$ such that for every nonempty family $\mathcal{F}$ of connected graphs with at most d vertices, $p, r \in \mathbb{N}$, and $\varepsilon > 0$, there is a polynomial-time algorithm that given a graph $G \in \mathcal{C}$ and $A \subseteq V(G)$, outputs a $(p,r,\mathcal{F})$-cover of A in G having size at most $f_{\mathrm{apx}}(r,d,\varepsilon)\cdot\gamma^{\mathcal{F}}_{p,r}(G,A)^{1+\varepsilon}$.*

Proof. Let $\mathcal{N} := \{N^r_G[v] : v \in V(G)\}$ and $\mathcal{N}_A := \{N^r_G[v] : v \in A\}$. By Corollary 2.8, $\mathcal{N}$ has VC-dimension at most $f_{\mathrm{vc}}(r)$. Since $\mathcal{N}_A \subseteq \mathcal{N}$, $\mathcal{N}_A$ has VC-dimension at most $f_{\mathrm{vc}}(r)$. Let $\mathcal{H}_0 := \{N^r_G[B] : B \subseteq A,\ |B| \leqslant d\}$. Let $\mathcal{H}_1$ be the family of sets $B \subseteq A$ such that $G^p[B]$ is isomorphic to a graph in $\mathcal{F}$, and $\mathcal{H}_2 := \{N^r_G[B] : B \in \mathcal{H}_1\}$. Since $\mathcal{N}$ has VC-dimension at most $f_{\mathrm{vc}}(r)$, by Proposition 2.5, $\mathcal{H}_0$ has VC-dimension at most $O(f_{\mathrm{vc}}(r)\cdot d\log d)$. Since $\mathcal{H}_2 \subseteq \mathcal{H}_0$, $\mathcal{H}_2$ has VC-dimension at most $O(f_{\mathrm{vc}}(r)\cdot d\log d)$.

Let $\gamma := \gamma^{\mathcal{F}}_{p,r}(G,A)$ and δ be the VC-dimension of $\mathcal{H}_2$. Observe that $(p,r,\mathcal{F})$-covers of A in G correspond to hitting sets of $\mathcal{H}_2$, and vice versa. By Theorem 2.6, one can find in polynomial time a hitting set X of $\mathcal{H}_2$ having size at most $C_\tau\cdot\delta\cdot\gamma\cdot\ln\gamma + 1$. Thus, one can choose the function $f_{\mathrm{apx}}(r,d,\varepsilon)$ with $|X| \leqslant f_{\mathrm{apx}}(r,d,\varepsilon)\cdot\gamma^{1+\varepsilon}$. ◀

Proof of Proposition 3.1. The function $f_{\mathrm{core}}(r,d,\varepsilon)$ will be defined later. At the beginning, we assume that $|Z| > f_{\mathrm{core}}(r,d,\varepsilon)\cdot k^{1+C\varepsilon}$ for some constant C, and at the end, we scale ε accordingly. If Z contains a vertex v such that for every set $B \subseteq Z\backslash\{v\}$ with $|B| \leqslant d-1$, $G^p[B\cup\{v\}]$ is isomorphic to no graph in $\mathcal{F}$, then the statement holds by taking v as z. Thus, we may assume that for every $v \in Z$, there is a set $B \subseteq Z\backslash\{v\}$ such that $G^p[B\cup\{v\}]$ is isomorphic to a graph in $\mathcal{F}$.

By Proposition 3.2, one can find in polynomial time a $(p,r,\mathcal{F})$-cover X of Z in G having size at most $f_{\mathrm{apx}}(r,d,\varepsilon)\cdot\gamma^{\mathcal{F}}_{p,r}(G,Z)^{1+\varepsilon}$. If $|X| > f_{\mathrm{apx}}(r,d,\varepsilon)\cdot k^{1+\varepsilon}$, then $\gamma^{\mathcal{F}}_{p,r}(G,A) = \gamma^{\mathcal{F}}_{p,r}(G,Z) > k$. Thus, we may assume that $|X| \leqslant f_{\mathrm{apx}}(r,d,\varepsilon)\cdot k^{1+\varepsilon}$. Let $r' := 2pd+3r$. By Lemma 2.2, one can find in polynomial time an $(r', f_{\mathrm{cl}}(r',\varepsilon)\cdot|X|^{\varepsilon})$-close set $X_{\mathrm{cl}} \supseteq X$ of size at most $f_{\mathrm{cl}}(r',\varepsilon)\cdot|X|^{1+\varepsilon} \leqslant f_{\mathrm{cl}}(r',\varepsilon)\cdot f_{\mathrm{apx}}(r,d,\varepsilon)^{1+\varepsilon}\cdot k^{1+3\varepsilon}$.

Let $\sim$ be an equivalence relation on $Z\backslash X_{\mathrm{cl}}$ such that for vertices $u, v \in Z\backslash X_{\mathrm{cl}}$, $u \sim v$ if and only if $\rho^G_{r'}[u, X_{\mathrm{cl}}] = \rho^G_{r'}[v, X_{\mathrm{cl}}]$. By Lemma 2.1,

$$\mathrm{index}(\sim) \leqslant f_{\mathrm{proj}}(r',\varepsilon)\cdot|X_{\mathrm{cl}}|^{1+\varepsilon} \leqslant f_{\mathrm{proj}}(r',\varepsilon)\cdot f_{\mathrm{cl}}(r',\varepsilon)^{1+\varepsilon}\cdot f_{\mathrm{apx}}(r,d,\varepsilon)^{1+3\varepsilon}\cdot k^{1+7\varepsilon}.$$

Let $p(r')$ and $s := s(r')$ be the constants in Theorem 2.4. Let

$$\xi := 2 \cdot f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{apx}}(r, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + d^2/4 + s + 1 \quad \text{and} \quad m := 2^{2^{d^2/2+sd} \cdot (r+1)^{sd}} \cdot \xi + 1.$$

By setting $C = 7 + 2 \cdot p(r')$, one can choose $f_{\mathrm{core}}(r, d, \varepsilon)$ with $f_{\mathrm{core}}(r, d, \varepsilon) \cdot k^{1+C\varepsilon} \geqslant |X_{\mathrm{cl}}| + \mathrm{index}(\sim) \cdot m^{p(r')}$. Since $|Z| > f_{\mathrm{core}}(r, d, \varepsilon) \cdot k^{1+C\varepsilon}$, we have that $|Z \backslash X_{\mathrm{cl}}| > \mathrm{index}(\sim) \cdot m^{p(r')}$. Thus, by the pigeonhole principle, there is an equivalence class λ of $\sim$ with $|\lambda| > m^{p(r')}$. By Theorem 2.4, one can find in polynomial time sets $S \subseteq V(G)$ and $L \subseteq \lambda \backslash S$ such that $|S| \leqslant s$, $|L| \geqslant m$, and L is distance-r' independent in $G - S$.

We are going to find a desired vertex z from L. To do this, we define the following. For each $i \in [d]$, let $\mathcal{G}_i$ be the set of all graphs whose vertex sets are $[i]$. Note that $|\mathcal{G}_i| = 2^{i(i-1)/2}$ for each $i \in [d]$. Let $\mathcal{H}$ be the set of functions $\rho : S \to [2r+1] \cup \{\infty\}$. Since $|S| \leqslant s$, we have that $|\mathcal{H}| \leqslant (2r+2)^s$. For each $i \in [d]$, let $\mathcal{H}_i$ be the set of all vectors $(h_1, \ldots, h_i, g)$ of length $i + 1$ where $h_j \in \mathcal{H}$ for each $j \in [i]$ and $g \in \mathcal{G}_i$. Let $\overline{\mathcal{H}} := \bigcup_{i=1}^{d} \mathcal{H}_i$. Note that

$$|\overline{\mathcal{H}}| = \sum_{i=1}^{d} |\mathcal{H}_i| = \sum_{i=1}^{d} (|\mathcal{H}|^i \cdot |\mathcal{G}_i|) \leqslant \sum_{i=1}^{d} ((2r+2)^{si} \cdot 2^{i(i-1)/2}) \leqslant 2^{d^2/2+sd} \cdot (r+1)^{sd}.$$

Let $\ell := |\overline{\mathcal{H}}|$. We take an arbitrary ordering $\sigma_1, \ldots, \sigma_\ell$ of $\overline{\mathcal{H}}$. For each $v \in L$, let $\mathcal{A}_v := \varnothing$ and $\mathbf{x}(v)$ be a zero vector of length ℓ. One can enumerate in polynomial time the sets $B \subseteq Z \backslash \{v\}$ of size at most $d - 1$ such that $G^p[B \cup \{v\}]$ is isomorphic to a graph in $\mathcal{F}$. For each such B, we do the following. If there is an index $i \in [\ell]$ such that the i-th entry of $\mathbf{x}(v)$ is 0 and for $\sigma_i = (h^i_1, \ldots, h^i_t, g_i) \in \overline{\mathcal{H}}$, there is an isomorphism $\phi_i : (B \backslash S) \cup \{v\} \to [t]$ between $(G - S)^p[(B \backslash S) \cup \{v\}]$ and g_i where $\phi_i(v) = 1$ and $\rho^G_{2r+1}[\phi_i^{-1}(j), S] = h^i_j$ for each $j \in [t]$, then we put B into $\mathcal{A}_v$ and convert the i-th entry of $\mathbf{x}(v)$ to 1. Otherwise, we do nothing for the chosen B. Since $|B| \leqslant d - 1$, one can check in polynomial time whether B satisfies the conditions. Thus, the resulting $\mathcal{A}_v$ and $\mathbf{x}(v)$ can be computed in polynomial time.

For each $v \in L$, since $Z \backslash \{v\}$ has a subset B such that $G^p[B \cup \{v\}]$ is isomorphic to a graph in $\mathcal{F}$, $\mathcal{A}_v \neq \varnothing$ and $\mathbf{x}(v)$ has a nonzero entry. For each set $B \in \mathcal{A}_v$, let B^* be the vertex set of the component of $(G - S)^p[(B \backslash S) \cup \{v\}]$ having v, and $\mathcal{B}_v := \bigcup_{B \in \mathcal{A}_v} B^*$.

Since $|L| \geqslant m = 2^{2^{d^2/2+sd} \cdot (r+1)^{sd}} \cdot \xi + 1$ and $\ell \leqslant 2^{d^2/2+sd} \cdot (r+1)^{sd}$, by the pigeonhole principle, L has a subset κ_1 such that $|\kappa_1| \geqslant \xi + 1$ and $\mathbf{x}(v) = \mathbf{x}(w)$ for all $v, w \in \kappa_1$. Let z be a vertex in κ_1 such that $\mathrm{dist}_{G-S}(\mathcal{B}_z, X_{\mathrm{cl}}) \geqslant \mathrm{dist}_{G-S}(\mathcal{B}_v, X_{\mathrm{cl}})$ for every $v \in \kappa_1$.

We show that $Z \backslash \{z\}$ is a $(p, r, \mathcal{F})$-core of A in G. To do this, for a minimum-size $(p, r, \mathcal{F})$-cover D of $Z \backslash \{z\}$ in G, we need to show that D is a $(p, r, \mathcal{F})$-cover of A in G. Since Z is a $(p, r, \mathcal{F})$-core of A in G, it suffices to show that D is a $(p, r, \mathcal{F})$-cover of Z in G.

Suppose for contradiction that D is not a $(p, r, \mathcal{F})$-cover of Z in G. Since D is a $(p, r, \mathcal{F})$-cover of $Z \backslash \{z\}$ in G, there is a set $B_z \subseteq Z \backslash (N^r_G[D] \cup \{z\})$ such that $G^p[B_z \cup \{z\}]$ is isomorphic to a graph in $\mathcal{F}$. In particular, there exist a graph $H \in \mathcal{G}_t$ for some $t \leqslant d$ and an isomorphism $\psi_z : (B_z \backslash S) \cup \{z\} \to [t]$ between $(G - S)^p[(B_z \backslash S) \cup \{z\}]$ and H where $\psi_z(z) = 1$. For each $v \in \kappa_1 \backslash \{z\}$, there exist $B_v \in \mathcal{A}_v$ and an isomorphism $\psi_v : (B_v \backslash S) \cup \{v\} \to [t]$ between $(G - S)^p[(B_v \backslash S) \cup \{v\}]$ and H where $\psi_v(v) = 1$ and for each $j \in [t]$, $\rho^G_{2r+1}[\psi_v^{-1}(j), S] = \rho^G_{2r+1}[\psi_z^{-1}(j), S]$. To derive a contradiction, we do the following steps.

(1) Find a set $\kappa_3 \subseteq \kappa_1 \backslash \{z\}$ such that for each $u \in \kappa_3$, $\mathrm{dist}_{G-S}(\mathcal{B}_u, X_{\mathrm{cl}}) > r$ and $G^p[B^*_u \cup (B_z \backslash B^*_z)]$ is isomorphic to $G^p[B_z \cup \{z\}]$.

(2) Show that $|D| \geqslant |\kappa_3|$.

(3) Construct a $(p, r, \mathcal{F})$-cover of $Z \backslash \{z\}$ in G having size less than $|D|$.

Since D is a minimum-size $(p, r, \mathcal{F})$-cover of $Z \backslash \{z\}$ in G, these steps derive a contradiction.

Let κ_1' be the set of vertices $v \in \kappa_1$ with $\mathrm{dist}_{G-S}(\mathcal{B}_v, X_{\mathrm{cl}}) \leqslant r$ and let $\kappa_2 := \kappa_1 \backslash \kappa_1'$. We can show that $|\kappa_1'| \leqslant |M^G_{r'}(z, X_{\mathrm{cl}})|$. Thus, $|\kappa_2| \geqslant \xi + 1 - |M^G_{r'}(z, X_{\mathrm{cl}})| \geqslant f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{apx}}(r, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + d^2/4 + s + 2$. Since κ_2 is nonempty, by the choice of z, κ_2 contains z. Let B^*_z be the vertex set of the component of $(G-S)^p[(B_z \backslash S) \cup \{z\}]$ having z. Note that for vertices $v, w \in \kappa_2$, $\psi_v^{-1} \circ \psi_w$ is an isomorphism between $(G-S)^p[(B_w \backslash S) \cup \{w\}]$ and $(G-S)^p[(B_v \backslash S) \cup \{v\}]$ assigning w to v. Thus, $\psi_v^{-1} \circ \psi_z(B^*_z) = B^*_v$. The following claim shows that the isomorphism is indeed an isomorphism between induced subgraphs of G^p.

▷ Claim 1. For vertices $v, w \in \kappa_2$, $\psi_w^{-1} \circ \psi_v$ is an isomorphism between $G^p[(B_v \backslash S) \cup \{v\}]$ and $G^p[(B_w \backslash S) \cup \{w\}]$.

Proof. It suffices to show that for $i, j \in [t]$, $\psi_v^{-1}(i)$ is adjacent to $\psi_v^{-1}(j)$ in G^p if and only if $\psi_w^{-1}(i)$ is adjacent to $\psi_w^{-1}(j)$ in G^p. Suppose that $\psi_v^{-1}(i)$ is adjacent to $\psi_v^{-1}(j)$ in G^p. Since $\psi_w^{-1} \circ \psi_v$ is an isomorphism between $(G \backslash S)^p[(B_v \backslash S) \cup \{v\}]$ and $(G \backslash S)^p[(B_w \backslash S) \cup \{w\}]$, we may assume that $\psi_v^{-1}(i)$ and $\psi_v^{-1}(j)$ are nonadjacent in $(G \backslash S)^p[(B_v \backslash S) \cup \{v\}]$. Thus, every path of length at most p in G between $\psi_v^{-1}(i)$ and $\psi_v^{-1}(j)$ has a vertex in S.

We take an arbitrary path Q of G between $\psi_v^{-1}(i)$ and $\psi_v^{-1}(j)$ having length at most p. Let q_i and q_j be the vertices in $V(Q) \cap S$ such that each of $\mathrm{dist}_Q(\psi_v^{-1}(i), q_i)$ and $\mathrm{dist}_Q(\psi_v^{-1}(j), q_j)$ is minimum. Such q_i and q_j exist, because Q has a vertex in S. Let Q_i be the subpath of Q between $\psi_v^{-1}(i)$ and q_i, and Q_j be the subpath of Q between $\psi_v^{-1}(j)$ and q_j. Note that both Q_i and Q_j are S-avoiding paths of length at most $p \leqslant 2r+1$.

Since $\{v, w\} \subseteq \kappa_2 \subseteq \kappa_1$, $\rho^G_{2r+1}[\psi_v^{-1}(i), S]$ and $\rho^G_{2r+1}[\psi_w^{-1}(i), S]$ are same, and therefore G has an S-avoiding path Q_i' between $\psi_w^{-1}(i)$ and q_i whose length is at most that of Q_i. Similarly, G has an S-avoiding path Q_j' between $\psi_w^{-1}(j)$ and q_j whose length is at most that of Q_j. By substituting Q_i and Q_j with Q_i' and Q_j' from Q, respectively, we obtain a walk of G between $\psi_w^{-1}(i)$ and $\psi_w^{-1}(j)$ whose length is at most p. Therefore, $\psi_w^{-1}(i)$ is adjacent to $\psi_w^{-1}(j)$ in G^p. ◁

The following claim shows that except for at most $d^2/4$ vertices in κ_2, for every remaining vertex $u \in \kappa_2$, we can build an isomorphic copy of $G^p[B_z \cup \{z\}]$ by substituting B^*_z with B^*_u.

▷ Claim 2. κ_2 has at most $d^2/4$ vertices v such that $G^p[B^*_v \cup (B_z \backslash B^*_z)]$ is not isomorphic to $G^p[B_z \cup \{z\}]$.

Proof. For vertices $u \in \kappa_2 \backslash \{z\}$ and $i \in \psi_z(B^*_z)$, since $\{u, z\} \subseteq \kappa_2 \subseteq \kappa_1$, $\rho^G_{2r+1}[\psi_u^{-1}(i), S]$ and $\rho^G_{2r+1}[\psi_z^{-1}(i), S]$ are same. Therefore, for each $w \in S$, $\psi_u^{-1}(i)$ is adjacent to w in G^p if and only if $\psi_z^{-1}(i)$ is adjacent to w in G^p. By Claim 1, the restriction of $\psi_u^{-1} \circ \psi_z$ on B^*_z is an isomorphism between $G^p[B^*_z]$ and $G^p[B^*_u]$.

We first show that for all vertices $v \in \kappa_2$, $i \in \psi_z(B^*_z)$, and $w \in B_z \backslash (B^*_z \cup S)$, if $\psi_z^{-1}(i)$ is adjacent to w in G^p, then $\psi_v^{-1}(i)$ is adjacent to w in G^p. Suppose that $\psi_z^{-1}(i)$ is adjacent to w in G^p. Let Q' be a path of G between $\psi_z^{-1}(i)$ and w of length at most p. Since $(G-S)^p[B^*_z]$ is a component of $(G-S)^p[(B_z \backslash S) \cup \{z\}]$ having z and $w \notin B^*_z$, $Q' \cap S \neq \varnothing$.

Let q be the the vertex in $V(Q') \cap S$ such that $\mathrm{dist}_{Q'}(\psi_z^{-1}(i), q)$ is minimum. Such q exists, because Q' has a vertex in S. Let Q_1' be the subpath of Q' between $\psi_z^{-1}(i)$ and q. Note that Q_1' is an S-avoiding path of length at most $p \leqslant 2r+1$. Since $\rho^G_{2r+1}[\psi_v^{-1}(i), S] = \rho^G_{2r+1}[\psi_z^{-1}(i), S]$, there is an S-avoiding path Q_2' in G between $\psi_v^{-1}(i)$ and q having length at most that of Q_1'. By substituting Q_1' with Q_2' from Q', we obtain a walk of G between $\psi_v^{-1}(i)$ and w having length at most p. Thus, $\psi_v^{-1}(i)$ is adjacent to w in G^p. So, there is no pair of $i \in \psi_z(B^*_z)$ and $w \in B_z \backslash (B^*_z \cup S)$ so that in G^p, $\psi_z^{-1}(i)$ is adjacent to w and $\psi_u^{-1}(i)$ is nonadjacent to w.

We now show that if there exist vertices $i \in \psi_z(B_z^*)$ and $w \in B_z \backslash (B_z^* \cup S)$ such that $\psi_z^{-1}(i)$ is nonadjacent to w in G^p, then κ_2 contains at most one vertex x such that $\psi_x^{-1}(i)$ is adjacent to w in G^p. To prove the claim, it suffices to show this statement, because $|B_z^*| \cdot |B_z \backslash (B_z^* \cup S)| \leqslant d^2/4$.

Suppose for contradiction that there exist vertices $i \in \psi_z(B_z^*)$, $w \in B_z \backslash (B_z^* \cup S)$, and distinct $x, x' \in \kappa_2$ such that $\psi_z^{-1}(i)$ is nonadjacent to w in G^p and both $\psi_x^{-1}(i)$ and $\psi_{x'}^{-1}(i)$ are adjacent to w in G^p. Then G has paths R and R' of length at most p from w to $\psi_x^{-1}(i)$ and $\psi_{x'}^{-1}(i)$, respectively. We can verify that R or R' has a vertex in S, as otherwise L is not distance-r' independent in $G - S$. By symmetry, we may assume that R has a vertex in S. Let t be the vertex in $V(R) \cap S$ such that $\mathrm{dist}_R(\psi_x^{-1}(i), t)$ is minimum. Let R_0 be the subpath of R between $\psi_x^{-1}(i)$ and t. Note that R_0 is an S-avoiding path of length at most $p \leqslant 2r + 1$. Since $\rho_{2r+1}^G[\psi_x^{-1}(i), S] = \rho_{2r+1}^G[\psi_z^{-1}(i), S]$, G has an S-avoiding path R_0' between $\psi_z^{-1}(i)$ and t having length at most that of R_0. By substituting R_0 with R_0' from R, we obtain a walk of G between $\psi_z^{-1}(i)$ and w having length at most p, contradicting the assumption that $\psi_z^{-1}(i)$ is nonadjacent to w in G^p, and this proves the claim. ◁

Since $|\kappa_2| \geqslant f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{apx}}(r, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + d^2/4 + s + 2$, by Claim 2, $\kappa_2 \backslash \{z\}$ has a subset κ_3 of size at least $f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{apx}}(r, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + s + 1$ such that for each vertex $u \in \kappa_3$, $G^p[B_u^* \cup (B_z \backslash B_z^*)]$ is isomorphic to $G^p[B_z \cup \{z\}]$, which is isomorphic to a graph in $\mathcal{F}$. This is the end of the first step.

We now show that $|D| \geqslant |\kappa_3|$. For each vertex $u \in \kappa_3$, since $B_u^* \cup (B_z \backslash B_z^*) \subseteq Z \backslash \{z\}$ and D is a $(p, r, \mathcal{F})$-cover of $Z \backslash \{z\}$ in G, there exist vertices $x_u \in B_u^* \cup (B_z \backslash B_z^*)$ and $d_u \in D$ with $\mathrm{dist}_G(x_u, d_u) \leqslant r$. Observe that $x_u \in \psi_u^{-1} \circ \psi_z(B_z^*)$, because $B_z \backslash B_z^* \subseteq B_z \subseteq Z \backslash (N_G^r[D] \cup \{z\})$. Let P_u be an arbitrary path in G between x_u and d_u of length at most r.

▷ Claim 3. For each $u \in \kappa_3$, $V(P_u) \cap (S \cup X_{\mathrm{cl}}) = \varnothing$.

Proof. Let u be a vertex in κ_3. Suppose for contradiction that $V(P_u) \cap S \neq \varnothing$. Let q be the vertex in $V(P_u) \cap S$ such that $\mathrm{dist}_{P_u}(x_u, q)$ is minimum. Let P_1 be the subpath of P_u between x_u and q. Note that P_1 is an S-avoiding path of length at most r. Since $\{u, z\} \subseteq \kappa_2 \subseteq \kappa_1$, G has an S-avoiding path P_2 between $\psi_z^{-1} \circ \psi_u(x_u)$ and q having length at most that of P_1. By substituting P_1 with P_2 from P_u, we obtain a walk of G between $\psi_z^{-1} \circ \psi_u(x_u) \in B_z^* \subseteq B_z \cup \{z\}$ and d_u having length at most r, contradicting the assumption that $B_z \cap N_G^r[D] = \varnothing$. Hence, $V(P_u) \cap S = \varnothing$.

Since $u \notin \kappa_1'$ and $B_u \in \mathcal{A}_u$, we have that

$$\mathrm{dist}_{G \backslash S}(x_u, X_{\mathrm{cl}}) \geqslant \mathrm{dist}_{G \backslash S}(B_u^*, X_{\mathrm{cl}}) \geqslant \mathrm{dist}_{G \backslash S}(\mathcal{B}_u, X_{\mathrm{cl}}) > r.$$

Since P_u is a path of $G \backslash S$ having length at most r, $V(P_u) \cap X_{\mathrm{cl}} = \varnothing$. ◁

We now derive $|D| \geqslant |\kappa_3|$ from the following.

▷ Claim 4. For distinct $u, u' \in \kappa_3$, the vertices d_u and $d_{u'}$ are distinct.

As the last step, we now construct a $(p, r, \mathcal{F})$-cover of $Z \backslash \{z\}$ in G having size less than $|D|$. Let $D_{\mathrm{sell}} := \{d_u : u \in \kappa_3\}$, $D_{\mathrm{buy}} := M_{r'}^G(z, X_{\mathrm{cl}}) \cup S$, and $D' := (D \backslash D_{\mathrm{sell}}) \cup D_{\mathrm{buy}}$. By Claim 4,

$$|D_{\mathrm{sell}}| = |\kappa_3| \geqslant f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{apx}}(r, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + s + 1,$$
$$|D_{\mathrm{buy}}| \leqslant |M_{r'}^G(z, X_{\mathrm{cl}})| + |S| \leqslant f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{apx}}(r, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + s.$$

Since $D_{\mathrm{sell}} \subseteq D$, we have that $|D'| < |D|$. For a contradiction, we show the following claim.

▷ Claim 5. D' is a $(p, r, \mathcal{F})$-cover of $Z \backslash \{z\}$ in G.

Proof. Suppose not. Then there is a set $B' \subseteq Z \backslash (N_G^r[D'] \cup \{z\})$ such that $G^p[B']$ is isomorphic to a graph in $\mathcal{F}$. Since D is a $(p, r, \mathcal{F})$-cover of $Z \backslash \{z\}$ in G and $D \backslash D' \subseteq D_{\text{sell}}$, D_{sell} contains a vertex d_u for some $u \in \kappa_3$ with $\text{dist}_G(d_u, B') \leqslant r$.

Since $(G \backslash S)^p[B_u^*]$ is connected and $|B_u^*| \leqslant d$, $G \backslash S$ has a path Q_0 of length at most $p(d-1)$ between u and x_u. More specifically, Q_0 is a concatenation of paths $Q_0^1, \ldots, Q_0^{t_1}$ for $t_1 \leqslant d-1$ such that for each $i \in [t_1]$, the length of Q_0^i is at most p and the ends of Q_0^i are in B_u^*. Since $\text{dist}_G(d_u, B') \leqslant r$, G has a path Q_1 of length at most r between d_u and $w_1 \in B'$. Since X_{cl} is a $(p, r, \mathcal{F})$-cover of Z in G, G has a path Q_2 of length at most r between $w_2 \in B'$ and $x \in X_{\text{cl}}$. Since $G^p[B']$ is isomorphic to a connected graph in $\mathcal{F}$ and $|B'| \leqslant d$, G has a path R of length at most $p(d-1)$ between w_1 and w_2. More specifically, R is a concatenation of paths $R_1, \ldots, R_{t_2}$ for $t_2 \leqslant d-1$ such that for each $i \in [t_2]$, the length of R_i is at most p and the ends of R_i are in B'. By concatenating Q_0, P_u, Q_1, R, and Q_2, we obtain a walk of G between u and x having length at most

$$\begin{aligned}&|E(Q_0)| + |E(P_u)| + |E(Q_1)| + |E(R)| + |E(Q_2)| \\ &\leqslant p(d-1) + r + r + p(d-1) + r = 2p(d-1) + 3r \leqslant r'.\end{aligned}$$

Let P be a path of G between u and x consisting of edges of the walk. Let b be the vertex in $V(P) \cap (S \cup X_{\text{cl}})$ such that $\text{dist}_P(u, b)$ is minimum. Such b exists, because $x \in X_{\text{cl}}$.

We first show that $\text{dist}_G(b, B') \leqslant r$. Note that Q_0 has no vertex in S. Since $u \notin \kappa_1'$, $\text{dist}_{G \backslash S}(\mathcal{B}_u, X_{\text{cl}}) > r$. Since $p \leqslant 2r+1$, for some $j \in [t_1]$, if Q_0^j has a vertex in X_{cl}, then $\text{dist}_{G \backslash S}(\mathcal{B}_u, X_{\text{cl}}) \leqslant \text{dist}_{G \backslash S}(B_u^*, X_{\text{cl}}) \leqslant r$, a contradiction. Therefore, Q_0 has no vertex in X_{cl}. By Claim 3, $V(P_u) \cap (S \cup X_{\text{cl}}) = \varnothing$. These imply that $b \in V(Q_1) \cup V(R) \cup V(Q_2)$. If $b \in V(Q_1) \cup V(Q_2)$, then $\text{dist}_G(b, B') \leqslant r$ clearly. Since $p \leqslant 2r+1$, for some $j \in [t_2]$, if $b \in R_j$, then $\text{dist}_G(b, B') \leqslant r$. Therefore, $\text{dist}_G(b, B') \leqslant r$.

Since $B' \subseteq Z \backslash (N_G^r[D'] \cup \{z\})$, b is not contained in D'. Since $S \subseteq D_{\text{buy}} \subseteq D'$, b is contained in $X_{\text{cl}} \backslash S$. Since the subpath of P between u and b is an X_{cl}-avoiding path of length at most r', b is contained in $M_{r'}^G(u, X_{\text{cl}})$. Since $\{u, z\} \subseteq \kappa_2 \subseteq \lambda$ where λ is an equivalence class of $\sim$, $M_{r'}^G(u, X_{\text{cl}})$ and $M_{r'}^G(z, X_{\text{cl}})$ are same. Therefore, $b \in M_{r'}^G(z, X_{\text{cl}}) \subseteq D'$, a contradiction. ◁

Claim 5 contradicts the assumption that D is a minimum-size $(p, r, \mathcal{F})$-cover of $Z \backslash \{z\}$ in G. Thus, $Z \backslash \{z\}$ is a $(p, r, \mathcal{F})$-core of A in G. We conclude the proof by scaling ε to ε / C. ◀

After recursively applying Proposition 3.1, started from A, we may assume that we are given a small $(p, r, \mathcal{F})$-core Z. By taking a $(2r+1)$-path closure Y of some superset of Z with Lemma 2.3, we can derive an almost linear kernel for Annotated $(p, r, \mathcal{F})$-Covering as follows.

▶ **Theorem 3.3.** *For every nowhere dense class $\mathcal{C}$ of graphs, there is $f_{\text{cov}} : \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N}$ such that for every nonempty family $\mathcal{F}$ of connected graphs with at most d vertices, $p, r \in \mathbb{N}$ with $p \leqslant 2r+1$, and $\varepsilon > 0$, there is a polynomial-time algorithm that given a graph $G \in \mathcal{C}$, $A \subseteq V(G)$, and $k \in \mathbb{N}$, either correctly decides that $\gamma_{p,r}^{\mathcal{F}}(G, A) > k$, or outputs sets $Y \subseteq V(G)$ of size at most $f_{\text{cov}}(r, d, \varepsilon) \cdot k^{1+\varepsilon}$ and $Z \subseteq A \cap Y$ such that $\gamma_{p,r}^{\mathcal{F}}(G[Y], Z) = \gamma_{p,r}^{\mathcal{F}}(G, A)$.*

We now convert the resulting instance of Theorem 3.3 to an equivalent instance of $(p, r, \mathcal{F})$-Covering. To do this, we will use the following definition and lemmas. For an integer $q \geqslant 0$ and a nonempty family $\mathcal{G}$ of graphs, a graph H is $(q, \mathcal{G})$-*critical* if either

- H is a 1-vertex graph and $\mathcal{G}$ contains a 1-vertex graph, or
- H has at least two vertices, H^q has an induced subgraph isomorphic to a graph in $\mathcal{G}$, and for every vertex v of H, $(H-v)^q$ has no induced subgraph isomorphic to a graph in $\mathcal{G}$.

▶ **Lemma 3.4.** *Let $\mathcal{G}$ be a nonempty family of graphs. Let F be a graph in $\mathcal{G}$ and d be the order of F. For every positive integer q, there is a $(q,\mathcal{G})$-critical graph of order at most $d(dq+1)/2$. Moreover, if every graph in $\mathcal{G}$ has order at most d, then one can construct the $(q,\mathcal{G})$-critical graph in time polynomial in d.*

Proof. Let F_0 be the q-subdivision of F. Since F has at most $d(d-1)/2$ edges,

$$|V(F_0)| \leqslant d + \frac{d(d-1)(q-1)}{2} = d \cdot \frac{dq-d-q+3}{2} \leqslant \frac{d(dq+1)}{2}.$$

Let H be a graph which is initially set as F_0. Note that H^q has an induced subgraph isomorphic to $F \in \mathcal{G}$. If $|V(H)| = 1$, then H is $(q,\mathcal{G})$-critical. Otherwise, for each vertex v of H, we check whether $(H\backslash v)^q$ has an induced subgraph isomorphic to a graph in $\mathcal{G}$. If H has no such vertex, then H is $(q,\mathcal{G})$-critical. Otherwise, we set H by $H\backslash v$ and do the above process until either $|V(H)| = 1$ or H has no such a vertex. It is readily seen that the resulting graph is $(q,\mathcal{G})$-critical graph and has at most $d(dq+1)/2$ vertices. Whole these processes work in polynomial time when every graph in $\mathcal{G}$ has at most d vertices. ◀

The following lemma shows that every vertex of a $(q,\mathcal{G})$-critical graph is a $(q,\lfloor q/2\rfloor,\mathcal{G})$-cover of it.

▶ **Lemma 3.5.** *Let $\mathcal{G}$ be a nonempty family of graphs, and q be a positive integer. If H is a $(q,\mathcal{G})$-critical graph and there is a set $B \subseteq V(H)$ such that $H^q[B]$ is isomorphic to a graph in $\mathcal{G}$, then for every $x \in V(H)$, B contains a vertex in $N_H^{\lfloor q/2\rfloor}[x]$.*

Proof. Suppose for contradiction that B contains no vertex in $N_H^{\lfloor q/2\rfloor}[x]$. Since H is $(q,\mathcal{G})$-critical, $(H\backslash x)^q[B]$ is isomorphic to no graph in $\mathcal{G}$. Since $H^q[B]$ is isomorphic to a graph in $\mathcal{G}$, B contains distinct vertices v and w such that v and w are adjacent in H^q and every path of H between v and w having length at most q should contain x. However, since neither v nor w is in $N_H^{\lfloor q/2\rfloor}[x]$, if H has a path P between v and w having x as an internal vertex, then the length of P is at least $2\lfloor q/2\rfloor + 2 > q$, a contradiction. ◀

To prove Theorem 1.1, we construct an equivalent instance of $(p,r,\mathcal{F})$-Covering by attaching a $(p,\mathcal{F})$-critical graph to the resulting instance of Theorem 3.3.

Sketch of the proof of Theorem 1.1. The cases where either $r=0$ or $p=0$ are relatively easy to deal with. Thus, in this sketch, we assume that both r and p are positive. Let d be the maximum order of a graph in $\mathcal{F}$. By Lemma 3.4, one can find in polynomial time a $(p,\mathcal{F})$-critical graph H having at most $d(dp+1)/2$ vertices. Let $p' := \lfloor p/2\rfloor$ and x be a vertex of H. We construct the graph G' as follows: take the disjoint union of $G[Y]$ and H, add a new vertex h, and for each vertex $v \in (Y\backslash Z) \cup N_H^{p'}[x]$, connect h and v by a path P_v of length r. We can show that the resulting graph G' is the desired one by Lemma 3.5. ◀

4 Kernels for the $(p,r,\mathcal{F})$-Packing problems

Let p,r be nonnegative integers with $p \leqslant r+1$ and $\mathcal{F}$ be a nonempty finite family of connected graphs. We present an almost linear kernel for $(p,r,\mathcal{F})$-Packing on every nowhere dense class of graphs. We also divert to the annotated variant of $(p,r,\mathcal{F})$-Packing. Although the proof scheme is similar to that of the kernel for $(p, r, \mathcal{F})$-Covering, we need a more intricate approximation algorithm, and the key lemma and the main steps of its proof are quite different from those of $(p, r, \mathcal{F})$-Covering.

For a graph G and a set $A \subseteq V(G)$, a $(p, r, \mathcal{F})$-*packing* of A in G is a family of subsets of A, say $A_1, \dots, A_\ell$ such that each $G^p[A_i]$ is isomorphic to a graph in $\mathcal{F}$, and for all $1 \leqslant i < j \leqslant \ell$, $\mathrm{dist}_G(A_i, A_j) > r$. We denote by $\alpha^{\mathcal{F}}_{p,r}(G, A)$ the maximum size of a $(p, r, \mathcal{F})$-packing of A in G. For a graph G, a set $A \subseteq V(G)$, and an integer k, Annotated $(p, r, \mathcal{F})$-Packing asks whether $\alpha^{\mathcal{F}}_{p,r}(G, A) \geqslant k$. We first derive an almost linear kernel for this problem.

▶ **Proposition 4.1.** *For every nowhere dense class $\mathcal{C}$ of graphs, there is $f_{\mathrm{rd}} : \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N}$ such that for every nonempty family $\mathcal{F}$ of connected graphs with at most d vertices, $p, r \in \mathbb{N}$ with $p \leqslant 2\lfloor r/2\rfloor + 1$, and $\varepsilon > 0$, there is a polynomial-time algorithm that given a graph $G \in \mathcal{C}$, $A \subseteq V(G)$ with $|A| > f_{\mathrm{rd}}(r, d, \varepsilon) \cdot k^{1+\varepsilon}$, and $k \in \mathbb{N}$, either correctly decides that $\alpha^{\mathcal{F}}_{p,r}(G, A) > k$, or outputs a vertex $z \in A$ such that $\alpha^{\mathcal{F}}_{p,r}(G, A) \geqslant k$ if and only if $\alpha^{\mathcal{F}}_{p,r}(G, A \backslash \{z\}) \geqslant k$.*

▶ **Proposition 4.2.** *For every nowhere dense class $\mathcal{C}$ of graphs, there is $f_{\mathrm{dual}} : \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N}$ such that for every nonempty family $\mathcal{F}$ of connected graphs with at most d vertices, $p, r, r_0 \in \mathbb{N}$ with $\max\{p, r_0\} \leqslant 2r + 1$, and $\varepsilon > 0$, there is a polynomial-time algorithm that given a graph $G \in \mathcal{C}$ and $A \subseteq V(G)$, outputs a $(p, r, \mathcal{F})$-cover of A in G having size at most $f_{\mathrm{dual}}(r, d, \varepsilon) \cdot \alpha^{\mathcal{F}}_{p,r_0}(G, A)^{1+\varepsilon}$.*

Proof of Proposition 4.1. The function $f_{\mathrm{rd}}(r, d, \varepsilon)$ will be defined later. At the beginning, we assume that $|A| > f_{\mathrm{rd}}(r, d, \varepsilon) \cdot k^{1+C\varepsilon}$ for some constant C, and at the end, we scale ε accordingly. We may assume that for every $v \in A$, there is $B \subseteq A \backslash \{v\}$ such that $G^p[B \cup \{v\}]$ is isomorphic to a graph in $\mathcal{F}$. Since $p \leqslant 2\lfloor r/2\rfloor + 1$, by Proposition 4.2, one can find a $(p, \lfloor r/2\rfloor, \mathcal{F})$-cover X of A in G having size at most $f_{\mathrm{dual}}(\lfloor r/2\rfloor, d, \varepsilon) \cdot \alpha^{\mathcal{F}}_{p,r}(G, A)^{1+\varepsilon}$ in polynomial time. If $|X| > f_{\mathrm{dual}}(\lfloor r/2\rfloor, d, \varepsilon) \cdot k^{1+\varepsilon}$, then $\alpha^{\mathcal{F}}_{p,r}(G, A) > k$. Thus, we may assume that $|X| \leqslant f_{\mathrm{dual}}(\lfloor r/2\rfloor, d, \varepsilon) \cdot k^{1+\varepsilon}$. Let $r' := 4pd + 3r$. By Lemma 2.2, one can find an $(r', f_{\mathrm{cl}}(r', \varepsilon) \cdot |X|^{\varepsilon})$-close set $X_{\mathrm{cl}} \supseteq X$ of size at most $f_{\mathrm{cl}}(r', \varepsilon) \cdot |X|^{1+\varepsilon} \leqslant f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{dual}}(\lfloor r/2\rfloor, d, \varepsilon)^{1+\varepsilon} \cdot k^{1+3\varepsilon}$ in polynomial time.

We define an equivalence relation $\sim$ on $A \backslash X_{\mathrm{cl}}$ such that for $u, v \in A \backslash X_{\mathrm{cl}}$, $u \sim v$ if and only if $\rho^G_{r'}[u, X_{\mathrm{cl}}] = \rho^G_{r'}[v, X_{\mathrm{cl}}]$. Then $\mathrm{index}(\sim) \leqslant f_{\mathrm{proj}}(r', \varepsilon) f_{\mathrm{cl}}(r', \varepsilon)^{1+\varepsilon} f_{\mathrm{dual}}(\lfloor r/2\rfloor, d, \varepsilon)^{1+3\varepsilon} k^{1+7\varepsilon}$ by Lemma 2.1. Let $p(r')$ and $s := s(r')$ be the constants in Theorem 2.4. Let

$$\xi := d \cdot (f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{dual}}(\lfloor r/2\rfloor, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + s + d^2/4 + 1) \quad \text{and} \quad m := 2^{2^{d^2/2} \cdot (r+2)^{sd}} \cdot \xi + 1.$$

By setting $C = 7 + 2 \cdot p(r')$, one can choose $f_{\mathrm{rd}}(r, d, \varepsilon)$ with $f_{\mathrm{rd}}(r, d, \varepsilon) \cdot k^{1+C\varepsilon} \geqslant |X_{\mathrm{cl}}| + \mathrm{index}(\sim) \cdot m^{p(r')}$. Since $|A| > f_{\mathrm{rd}}(r, d, \varepsilon) \cdot k^{1+C\varepsilon}$, we have that $|A \backslash X_{\mathrm{cl}}| > \mathrm{index}(\sim) \cdot m^{p(r')}$. Thus, by the pigeonhole principle, there is an equivalence class λ of $\sim$ with $|\lambda| > m^{p(r')}$. By Theorem 2.4, one can find in polynomial time sets $S \subseteq V(G)$ and $L \subseteq \lambda \backslash S$ such that $|S| \leqslant s$, $|L| \geqslant m$, and L is distance-r' independent in $G - S$.

We are going to find a desired vertex z from L. To do this, we define the following. For each $i \in [d]$, let $\mathcal{G}_i$ be the set of all graphs whose vertex sets are $[i]$. Note that $|\mathcal{G}_i| = 2^{i(i-1)/2}$ for each $i \in [d]$. Let $\mathcal{H}'$ be the set of functions $\rho : S \to [r + 1] \cup \{\infty\}$. Since $|S| \leqslant s$, we have that $|\mathcal{H}'| \leqslant (r + 2)^s$. For each $i \in [d]$, let $\mathcal{H}'_i$ be the set of all vectors $(h_1, \dots, h_i, g)$ of length $i + 1$ where $h_j \in \mathcal{H}'$ for each $j \in [i]$ and $g \in \mathcal{G}_i$. Let $\overline{\mathcal{H}'} := \bigcup_{i=1}^{d} \mathcal{H}'_i$. Similar to the proof of Proposition 3.1, we can show that $|\overline{\mathcal{H}'}| \leqslant 2^{d^2/2} \cdot (r + 2)^{sd}$. Let $\ell := |\overline{\mathcal{H}'}|$. We take an arbitrary ordering $\sigma_1, \dots, \sigma_\ell$ of $\overline{\mathcal{H}'}$. For each $v \in L$, let $\mathcal{A}_v := \varnothing$ and $\mathbf{x}(v)$ be a zero vector of length ℓ. One can enumerate in polynomial time the sets $B \subseteq A \backslash \{v\}$ of size at most $d - 1$ such that $G^p[B \cup \{v\}]$ is isomorphic to a graph in $\mathcal{F}$ in polynomial time. For each such B, we do the following. If there is an index $i \in [\ell]$ such that the i-th entry of $\mathbf{x}(v)$ is 0 and for $\sigma_i = (h^i_1, \dots, h^i_t, g_i) \in \overline{\mathcal{H}'}$, there is an isomorphism $\phi_i : (B\backslash S) \cup \{v\} \to [t]$ between $(G-S)^p[(B\backslash S) \cup \{v\}]$ and g_i where $\phi_i(v) = 1$ and for each $j \in [t]$, $\rho^G_r[\phi^{-1}(j), S] = h^i_j$, then we put B into $\mathcal{A}_v$ and convert the i-th entry of $\mathbf{x}(v)$ to 1. Otherwise, we do nothing for the chosen B. Since $|B| \leqslant d-1$, one can check in polynomial time whether B satisfies the conditions. Thus, the resulting $\mathcal{A}_v$ and $\mathbf{x}(v)$ can be computed in polynomial time.

For each $v \in L$, since $A\backslash\{v\}$ has a subset B such that $G^p[B \cup \{v\}]$ is isomorphic to a graph in $\mathcal{F}$, $\mathcal{A}_v \neq \varnothing$ and $\mathbf{x}(v)$ has a nonzero entry. For each $B \in \mathcal{A}_v$, let B^* be the vertex set of the component of $(G-S)^p[(B\backslash S) \cup \{v\}]$ having v. Since $|L| \geqslant m = 2^{2^{d^2/2} \cdot (r+2)^{sd}} \cdot \xi + 1$ and $\ell \leqslant 2^{d^2/2} \cdot (r+2)^{sd}$, by the pigeonhole principle, L has a subset κ_1 such that $|\kappa_1| \geqslant \xi + 1$ and $\mathbf{x}(v) = \mathbf{x}(w)$ for all $v, w \in \kappa_1$. Let z be an arbitrary vertex in κ_1.

We show that $\alpha^{\mathcal{F}}_{p,r}(G, A) \geqslant k$ if and only if $\alpha^{\mathcal{F}}_{p,r}(G, A\backslash\{z\}) \geqslant k$. The backward direction is obvious. Suppose that G has a $(p, r, \mathcal{F})$-packing I of A in G having size at least k. We may assume that z is contained in some $B_z \in I$, because otherwise I is also a $(p, r, \mathcal{F})$-packing of $A\backslash\{z\}$. In particular, there exist a graph $H \in \mathcal{G}_t$ for some $t \leqslant d$ and an isomorphism $\psi_z : (B_z\backslash S) \cup \{z\} \to [t]$ between $(G-S)^p[(B_z\backslash S) \cup \{z\}]$ and H where $\psi_z(z) = 1$. To show that $\alpha^{\mathcal{F}}_{p,r}(G, A\backslash\{z\}) \geqslant k$, it suffices to show that there exist a vertex $z' \in \kappa_1\backslash\{z\}$ and a set $B_{z'} \subseteq A\backslash\{z\}$ such that $z' \in B_{z'}$ and $(I\backslash\{B_z\}) \cup \{B_{z'}\}$ is a $(p, r, \mathcal{F})$-packing of $A\backslash\{z\}$ in G having the same size as I.

Suppose for contradiction that no such z' exists. It means that for each $v \in \kappa_1\backslash\{z\}$, if $A\backslash\{z\}$ has a subset B such that $v \in B$ and $G^p[B]$ is isomorphic to a graph in $\mathcal{F}$, then $I\backslash\{B_z\}$ contains an element B' with $\mathrm{dist}_G(B, B') \leqslant r$, because otherwise we can substitute B_z with B from I. For each $v \in \kappa_1\backslash\{z\}$, there exist $B_v \in \mathcal{A}_v$ and an isomorphism $\psi_v : (B_v\backslash S) \cup \{v\} \to [t]$ between $(G-S)^p[(B_v\backslash S) \cup \{v\}]$ and H where $\psi_v(v) = 1$ and for each $j \in [t]$, $\rho^G_{r+1}[\psi_v^{-1}(j), S] = \rho^G_{r+1}[\psi_z^{-1}(j), S]$. For each $v \in \kappa_1\backslash\{z\}$, let $f(v) := B^*_v \cup (B_z\backslash B^*_z)$.

To derive a contradiction, we do the following steps.

(1) Find a set $\kappa_4 \subseteq \kappa_1\backslash\{z\}$ such that for each $u \in \kappa_4$, $G^p[f(u)]$ is isomorphic to $G^p[B_z \cup \{z\}]$ and I contains an element C_u with $\mathrm{dist}_G(f(u), C_u) \leqslant r$ and $\mathrm{dist}_G(C_u, S) > \lfloor r/2 \rfloor$.

(2) Show that κ_4 contains distinct vertices v and v' with $\mathrm{dist}_{G-S}(v, v') \leqslant r'$.

Since $\kappa_4 \subseteq L$ is distance-r' independent in $G-S$, these steps derive a contradiction.

Let B^*_z be the vertex set of the component of $(G-S)^p[(B_z\backslash S) \cup \{z\}]$ having z. Note that for vertices $v, w \in \kappa_1$, $\psi_v^{-1} \circ \psi_w$ is an isomorphism between $(G-S)^p[(B_w\backslash S) \cup \{w\}]$ and $(G-S)^p[(B_v\backslash S) \cup \{v\}]$ assigning w to v. Thus, $\psi_v^{-1} \circ \psi_z(B^*_z) = B^*_v$.

For the first step, we will use the following three claims. The proofs of Claims 6 and 7 are similar to those of Claims 1 and 2, respectively.

▷ Claim 6. For vertices $v, w \in \kappa_1$, $\psi_w^{-1} \circ \psi_v$ is an isomorphism between $G^p[(B_v\backslash S) \cup \{v\}]$ and $G^p[(B_w\backslash S) \cup \{w\}]$.

▷ Claim 7. κ_1 has at most $d^2/4$ vertices v where $G^p[f(v)]$ is not isomorphic to $G^p[B_z]$.

Proof. For vertices $u \in \kappa_1\backslash\{z\}$ and $i \in \psi_z(B^*_z)$, since $\{u, z\} \subseteq \kappa_1$, $\rho^G_{r+1}[\psi_u^{-1}(i), S]$ and $\rho^G_{r+1}[\psi_z^{-1}(i), S]$ are same. Therefore, for each $w \in S$, $\psi_u^{-1}(i)$ is adjacent to w in G^p if and only if $\psi_z^{-1}(i)$ is adjacent to w in G^p. By Claim 6, the restriction of $\psi_u^{-1} \circ \psi_z$ on B^*_z is an isomorphism between $G^p[B^*_z]$ and $G^p[B^*_u]$.

We first show that for all vertices $v \in \kappa_1$, $i \in \psi_z(B^*_z)$, and $w \in B_z\backslash(B^*_z \cup S)$, if $\psi_z^{-1}(i)$ is adjacent to w in G^p, then $\psi_v^{-1}(i)$ is adjacent to w in G^p. Suppose that $\psi_z^{-1}(i)$ is adjacent to w in G^p. We take an arbitrary path Q' of G between $\psi_z^{-1}(i)$ and w having length at most p. Since $(G\backslash S)^p[B^*_z]$ is a component of $(G\backslash S)^p[(B_z\backslash S) \cup \{z\}]$ having z and $w \notin B^*_z$, Q' must have a vertex in S.

Let q be the the vertex in $V(Q') \cap S$ such that $\mathrm{dist}_{Q'}(\psi_z^{-1}(i), q)$ is minimum. Let Q'_1 be the subpath of Q' between $\psi_z^{-1}(i)$ and q. Note that Q'_1 is an S-avoiding path of length at most $p \leqslant r+1$. Since $\rho^G_{r+1}[\psi_v^{-1}(i), S] = \rho^G_{r+1}[\psi_z^{-1}(i), S]$, G has an S-avoiding path Q'_2 between $\psi_v^{-1}(i)$ and q having length at most that of Q'_1. By substituting Q'_1 with Q'_2 from Q', we obtain a walk of G between $\psi_v^{-1}(i)$ and w having length at most p. Hence, $\psi_v^{-1}(i)$ is adjacent to w in G^p.

Thus, there is no pair of vertices $i \in \psi_z(B_z^*)$ and $w \in B_z \backslash (B_z^* \cup S)$ such that $\psi_z^{-1}(i)$ is adjacent to w in G^p and $\psi_u^{-1}(i)$ is nonadjacent to w in G^p.

We now show that if there exist vertices $i \in \psi_z(B_z^*)$ and $w \in B_z \backslash (B_z^* \cup S)$ such that $\psi_z^{-1}(i)$ is nonadjacent to w in G^p, then κ_1 contains at most one vertex x such that $\psi_x^{-1}(i)$ is adjacent to w in G^p. To prove the claim, it suffices to show this statement, because $|B_z^*| \cdot |B_z \backslash (B_z^* \cup S)| \leqslant d^2/4$.

Suppose for contradiction that there exist $i \in \psi_z(B_z^*)$, $w \in B_z \backslash (B_z^* \cup S)$, and distinct $x, x' \in \kappa_1$ such that $\psi_z^{-1}(i)$ is nonadjacent to w in G^p and both $\psi_x^{-1}(i)$ and $\psi_{x'}^{-1}(i)$ are adjacent to w in G^p. Then G has paths R and R' of length at most p from w to $\psi_x^{-1}(i)$ and $\psi_{x'}^{-1}(i)$, respectively.

We first verify that R or R' has a vertex in S. Suppose not. Since $|B_x^*| \leqslant d$, $G \backslash S$ has a path R_1 of length at most $p(d-1)$ between x and $\psi_x^{-1}(i)$. Similarly, $G \backslash S$ has a path R'_1 of length at most $p(d-1)$ between x' and $\psi_{x'}^{-1}(i)$. Since neither R nor R' has a vertex in S, by concatenating R_1, R, R', and R'_1, we obtain a walk of $G \backslash S$ of length at most $2pd \leqslant r'$ between x and x', contradicting the assumption that L is distance-r' independent in $G \backslash S$. Hence, R or R' has a vertex in S. By symmetry, we may assume that R has a vertex in S.

Let t be the vertex in $V(R) \cap S$ such that $\mathrm{dist}_R(\psi_x^{-1}(i), t)$ is minimum. Let R_0 be the subpath of R between $\psi_x^{-1}(i)$ and t. Note that R_0 is an S-avoiding path of length at most $p \leqslant r+1$. Since $\rho^G_{r+1}[\psi_x^{-1}(i), S] = \rho^G_{r+1}[\psi_z^{-1}(i), S]$, G has an S-avoiding path R'_0 between $\psi_z^{-1}(i)$ and t having length at most that of R_0. By substituting R_0 with R'_0 from R, we obtain a walk of G between $\psi_z^{-1}(i)$ and w having length at most p, contradicting the assumption that $\psi_z^{-1}(i)$ is nonadjacent to w in G^p, and this proves the claim. ◁

Since $|\kappa_1| \geqslant d \cdot (f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{dual}}(\lfloor r/2 \rfloor, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + s + d^2/4 + 1) + 1$, by Claim 7, $\kappa_1 \backslash \{z\}$ has a subset κ_2 of size at least $d \cdot (f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{dual}}(\lfloor r/2 \rfloor, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + s + 1)$ such that for each vertex $u \in \kappa_2$, $G^p[f(u)]$ is isomorphic to $G^p[B_z]$, which is isomorphic to a graph in $\mathcal{F}$.

For each $u \in \kappa_2$, since $f(u) \subseteq A \backslash \{z\}$, by assumption, $I \backslash \{B_z\}$ contains an element C_u with $\mathrm{dist}_G(f(u), C_u) \leqslant r$. We take an arbitrary path P_u of G between $b_u \in f(u)$ and $c_u \in C_u$ having length at most r. Since $\{B_z, C_u\} \subseteq I$ which is a $(p, r, \mathcal{F})$-packing of A in G, $\mathrm{dist}_G(B_z \backslash B_z^*, C_u) \geqslant \mathrm{dist}_G(B_z, C_u) > r$. Thus, $b_u \in f(u) \backslash (B_z \backslash B_z^*) = B_u^*$.

▷ Claim 8. For each $u \in \kappa_2$, $V(P_u) \cap S = \varnothing$.

Proof. Suppose for contradiction that for some $u \in \kappa_2$, $V(P_u) \cap S \neq \varnothing$. Let q be the vertex in $V(P_u) \cap S$ such that $\mathrm{dist}_{P_u}(b_u, q)$ is minimum. Let P_1 be the subpath of P_u between b_u and q. Note that P_1 is an S-avoiding path of length at most r. Since $\{u, z\} \subseteq \kappa_1$, G has an S-avoiding path P_2 between $\psi_z^{-1} \circ \psi_u(b_u)$ and q having length at most that of P_1. By substituting P_1 with P_2 from P_u, we obtain a walk of G between $\psi_z^{-1} \circ \psi_u(b_u) \in B_z$ and c_u having length at most r, contradicting the assumption that $\mathrm{dist}_G(B_z, C_u) > r$. ◁

Since L is distance-r' independent in $G - S$ and $2r \leqslant r'$, by Claim 8, $c_u \neq c_{u'}$ for distinct $u, u' \in \kappa_2$. Since $|\kappa_2| \geqslant d \cdot (f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{dual}}(\lfloor r/2 \rfloor, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + s + 1)$ and every element in I contains at most d vertices, there is a set $\kappa_3 \subseteq \kappa_2$ of size at least $f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{dual}}(\lfloor r/2 \rfloor, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + s + 1$ such that $C_u \neq C_{u'}$ for all distinct $u, u' \in \kappa_3$.

Let κ_3' be the set of vertices $u \in \kappa_3$ with $\mathrm{dist}_G(C_u, S) \leqslant \lfloor r/2 \rfloor$. Since I is a $(p, r, \mathcal{F})$-packing of A in G, for all distinct $u, u' \in \kappa_3$, $\mathrm{dist}_G(C_u, C_{u'}) > r$. Thus, we deduce that $|\kappa_3'| \leqslant |S| \leqslant s$. Let $\kappa_4 := \kappa_3 \backslash \kappa_3'$. Note that $|\kappa_4| \geqslant f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{dual}}(\lfloor r/2 \rfloor, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + 1$.

We now show that κ_4 contains distinct vertices v and v' with $\mathrm{dist}_{G-S}(v, v') \leqslant r'$. For each $u \in \kappa_4$, since $G^p[C_u]$ is isomorphic to a graph in $\mathcal{F}$ and X_{cl} is a $(p, \lfloor r/2 \rfloor, \mathcal{F})$-cover of A in G, G has a path R_u of length at most $\lfloor r/2 \rfloor$ between some $y_u \in C_u$ and $x_u \in X_{\mathrm{cl}}$. Since $u \notin \kappa_3'$, $V(R_u) \cap S = \emptyset$. Since $G^p[C_u]$ is isomorphic to a connected graph in $\mathcal{F}$, G has a path Q_u of length at most $p(d-1)$ between c_u and y_u. More specifically, Q_u is a concatenation of $Q_u^1, \ldots, Q_u^{t'}$ for $t' \leqslant d-1$ such that for each $i \in [t']$, the length of Q_u^i is at most p and the ends of Q_u^i are in C_u. Since $p \leqslant 2\lfloor r/2 \rfloor + 1$, for some $j \in [t']$, if $V(Q_u^j) \cap S \neq \emptyset$, then $\mathrm{dist}_G(C_u, S) \leqslant \lfloor r/2 \rfloor$, contradicting that $u \notin \kappa_3'$. Thus, $V(Q_u) \cap S = \emptyset$. By Claim 8, $V(P_u) \cap S = \emptyset$. Since $(G-S)^p[B_u^*]$ is connected and $|B_u^*| \leqslant d$, $G - S$ has a path O_u of length at most $p(d-1)$ between u and b_u. By concatenating O_u, P_u, Q_u, and R_u, we obtain a walk of $G - S$ between u and x_u having length at most

$$|E(O_u)| + |E(P_u)| + |E(Q_u)| + |E(R_u)| \leqslant p(d-1) + r + p(d-1) + \lfloor r/2 \rfloor \leqslant \lfloor r'/2 \rfloor.$$

Let W_u be a path of $G - S$ between u and x_u consisting of edges of the walk. Let w_u be the vertex in $V(W_u) \cap X_{\mathrm{cl}}$ such that $\mathrm{dist}_{W_u}(u, w_u)$ is minimum. Such w_u exists, because $x_u \in X_{\mathrm{cl}}$. Note that the subpath of W_u between u and w_u is an X_{cl}-avoiding path of length at most $\lfloor r'/2 \rfloor$. Thus, w_u is contained in $M_{r'}^G(u, X_{\mathrm{cl}})$. Since $\{u, z\} \subseteq \kappa_1 \subseteq \lambda$ where λ is an equivalence class of $\sim$, $M_{r'}^G(u, X_{\mathrm{cl}})$ and $M_{r'}^G(z, X_{\mathrm{cl}})$ are same. Therefore, $w_u \in M_{r'}^G(z, X_{\mathrm{cl}})$.

Since $|\kappa_4| \geqslant f_{\mathrm{cl}}(r', \varepsilon) \cdot f_{\mathrm{dual}}(\lfloor r/2 \rfloor, d, \varepsilon)^{\varepsilon} \cdot k^{2\varepsilon} + 1 \geqslant |M_{r'}^G(z, X_{\mathrm{cl}})| + 1$, by the pigeonhole principle, there are distinct $v, v' \in \kappa_4$ with $w_v = w_{v'}$. By concatenating W_v and $W_{v'}$, we obtain a walk of $G - S$ between v and v' having length at most r', contradicting the assumption that L is distance-r' independent in $G - S$. Therefore, there are a vertex $z' \in \kappa_1 \backslash \{z\}$ and a set $B_{z'} \subseteq A \backslash \{z\}$ such that $z' \in B_{z'}$ and $(I \backslash \{B_z\}) \cup \{B_{z'}\}$ is a $(p, r, \mathcal{F})$-packing of $A \backslash \{z\}$ in G having the same size as I. We conclude the proof by scaling ε to ε / C. ◀

By recursively applying Proposition 4.1 and taking an $(r+1)$-path closure of the resulting set Z, we can construct an almost linear kernel for Annotated $(p, r, \mathcal{F})$-Packing as follows.

▶ **Theorem 4.3.** *For every nowhere dense class $\mathcal{C}$ of graphs, there is a function $f_{\mathrm{pck}} : \mathbb{N} \times \mathbb{N} \times \mathbb{R}_+ \to \mathbb{N}$ such that for every nonempty family $\mathcal{F}$ of connected graphs with at most d vertices, $p, r \in \mathbb{N}$ with $p \leqslant 2\lfloor r/2 \rfloor + 1$, and $\varepsilon > 0$, there is a polynomial-time algorithm that given a graph $G \in \mathcal{C}$, $A \subseteq V(G)$, and $k \in \mathbb{N}$, either correctly decides that $\alpha_{p,r}^{\mathcal{F}}(G, A) > k$, or outputs sets $Y \subseteq V(G)$ of size at most $f_{\mathrm{pck}}(r, d, \varepsilon) \cdot k^{1+\varepsilon}$ and $Z \subseteq A \cap Y$ such that $\alpha_{p,r}^{\mathcal{F}}(G, A) \geqslant k$ if and only if $\alpha_{p,r}^{\mathcal{F}}(G[Y], Z) \geqslant k$.*

To prove Theorem 1.2, we first apply the kernel in Theorem 4.3 and attach a $(p, \mathcal{F})$-critical graph to the resulting instance of this kernel. The way is similar to that of the proof of Theorem 1.1, but slightly different.

Sketch of the proof of Theorem 1.2. The cases where either $r \leqslant 1$ or $p = 0$ are relatively easy to deal with. Thus, in this sketch, we assume that $r \geqslant 2$ and $p \geqslant 1$. Let d be the maximum order of a graph in $\mathcal{F}$. By Lemma 3.4, one can find in polynomial time a $(p, \mathcal{F})$-critical graph H having at most $d(dp+1)/2$ vertices. Let $p' := \lfloor p/2 \rfloor$ and x be a vertex of H. We construct a graph G' as follows: take the disjoint union of $G[Y]$ and H, add a new vertex h, for each $v \in Y \backslash Z$, connect h and v by a path of length $\lfloor r/2 \rfloor$, and for each $v \in N_H^{p'}[x]$, connect h and v by a path of length $\lceil r/2 \rceil$. We can show that the resulting graph G' is the desired one by Lemma 3.5. ◀

References

1 Faisal N. Abu-Khzam. A kernelization algorithm for d-hitting set. *J. Comput. System Sci.*, 76(7):524–531, 2010. doi:10.1016/j.jcss.2009.09.002.

2 Hans Adler and Isolde Adler. Interpreting nowhere dense graph classes as a classical notion of model theory. *European J. Combin.*, 36:322–330, 2014. doi:10.1016/j.ejc.2013.06.048.

3 Jochen Alber, Michael R. Fellows, and Rolf Niedermeier. Polynomial-time data reduction for dominating set. *J. ACM*, 51(3):363–384, 2004. doi:10.1145/990308.990309.

4 Noga Alon and Shai Gutner. Linear time algorithms for finding a dominating set of fixed size in degenerated graphs. *Algorithmica*, 54(4):544–556, 2009. doi:10.1007/s00453-008-9204-0.

5 Manuel Aprile, Matthew Drescher, Samuel Fiorini, and Tony Huynh. A tight approximation algorithm for the cluster vertex deletion problem. *Math. Program.*, 197(2):1069–1091, 2023. doi:10.1007/s10107-021-01744-w.

6 Hans L. Bodlaender, Fedor V. Fomin, Daniel Lokshtanov, Eelko Penninkx, Saket Saurabh, and Dimitrios M. Thilikos. (Meta) kernelization. *J. ACM*, 63(5):Art. 44, 69, 2016. doi:10.1145/2973749.

7 Flavia Bonomo-Braberman, Julliano R. Nascimento, Fabiano S. Oliveira, Uéverton S. Souza, and Jayme L. Szwarcfiter. Linear-time algorithms for eliminating claws in graphs. In *Computing and combinatorics*, volume 12273 of *Lecture Notes in Comput. Sci.*, pages 14–26. Springer, Cham, 2020.

8 H. Brönnimann and M. T. Goodrich. Almost optimal set covers in finite VC-dimension. *Discrete Comput. Geom.*, 14(4):463–479, 1995. ACM Symposium on Computational Geometry. doi:10.1007/BF02570718.

9 Santiago Canales, Gregorio Hernández, Mafalda Martins, and Inês Matos. Distance domination, guarding and covering of maximal outerplanar graphs. *Discrete Appl. Math.*, 181:41–49, 2015. doi:10.1016/j.dam.2014.08.040.

10 Holger Dell and Dániel Marx. Kernelization of packing problems. In *23rd Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 68–81. ACM, New York, 2012.

11 Erik D. Demaine, Fedor V. Fomin, Mohammadtaghi Hajiaghayi, and Dimitrios M. Thilikos. Subexponential parameterized algorithms on bounded-genus graphs and H-minor-free graphs. *J. ACM*, 52(6):866–893, 2005. doi:10.1145/1101821.1101823.

12 Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Berlin, fifth edition, 2018. Paperback edition of [MR3644391].

13 Frederic Dorn. Dynamic programming and fast matrix multiplication. In *14th Annual European Symposium, Zurich, Switzerland, September 11-13, 2006*, volume 4168 of *Lecture Notes in Comput. Sci.*, pages 280–291. Springer, Berlin, 2006. doi:10.1007/11841036_27.

14 Rod G. Downey and Michael R. Fellows. Fixed-parameter tractability and completeness. I. Basic results. *SIAM J. Comput.*, 24(4):873–921, 1995. doi:10.1137/S0097539792228228.

15 Rod G. Downey and Michael R. Fellows. Fixed-parameter tractability and completeness. II. On completeness for $W[1]$. *Theoret. Comput. Sci.*, 141(1-2):109–131, 1995. doi:10.1016/0304-3975(94)00097-3.

16 Rodney G. Downey and Michael R. Fellows. *Fundamentals of parameterized complexity*. Texts in Computer Science. Springer, London, 2013. doi:10.1007/978-1-4471-5559-1.

17 Pål Grønås Drange, Markus Dregi, Fedor V. Fomin, Stephan Kreutzer, Daniel Lokshtanov, Marcin Pilipczuk, Michał Pilipczuk, Felix Reidl, Fernando Sánchez Villaamil, Saket Saurabh, Sebastian Siebertz, and Somnath Sikdar. Kernelization and sparseness: the case of dominating set. In *33rd Symposium on Theoretical Aspects of Computer Science*, volume 47 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 31, 14. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2016. doi:10.4230/LIPIcs.STACS.2016.31.

18 Zdeněk Dvořák. Constant-factor approximation of the domination number in sparse graphs. *European J. Combin.*, 34(5):833–840, 2013. doi:10.1016/j.ejc.2012.12.004.

19 Jack Edmonds. Paths, trees, and flowers. *Canadian J. Math.*, 17:449–467, 1965. doi:10.4153/CJM-1965-045-4.

20 Kord Eickmeyer, Archontia C. Giannopoulou, Stephan Kreutzer, O-joung Kwon, Michał Pilipczuk, Roman Rabinovich, and Sebastian Siebertz. Neighborhood complexity and kernelization for nowhere dense classes of graphs. In *44th International Colloquium on Automata, Languages, and Programming*, volume 80 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 63, 14. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2017. doi:10.4230/LIPIcs.ICALP.2017.63.

21 Guy Even, Dror Rawitz, and Shimon Shahar. Hitting sets when the VC-dimension is small. *Inform. Process. Lett.*, 95(2):358–362, 2005. doi:10.1016/j.ipl.2005.03.010.

22 Grzegorz Fabiański, Michał Pilipczuk, Sebastian Siebertz, and Szymon Toruńczyk. Progressive algorithms for domination and independence. In *36th International Symposium on Theoretical Aspects of Computer Science*, volume 126 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 27, 16. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2019.

23 Fedor V. Fomin, Daniel Lokshtanov, Saket Saurabh, and Dimitrios M. Thilikos. Linear kernels for (connected) dominating set on H-minor-free graphs. In *23rd Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 82–92. ACM, New York, 2012.

24 Fedor V. Fomin, Daniel Lokshtanov, Saket Saurabh, and Dimitrios M. Thilikos. Kernels for (connected) dominating set on graphs with excluded topological minors. *ACM Trans. Algorithms*, 14(1):Art. 6, 31, 2018. doi:10.1145/3155298.

25 Fedor V. Fomin, Daniel Lokshtanov, Saket Saurabh, and Dimitrios M. Thilikos. Bidimensionality and kernels. *SIAM J. Comput.*, 49(6):1397–1422, 2020. doi:10.1137/16M1080264.

26 Fedor V. Fomin and Dimitrios M. Thilikos. Fast parameterized algorithms for graphs on surfaces: linear kernel and exponential speed-up. In *Automata, languages and programming*, volume 3142 of *Lecture Notes in Comput. Sci.*, pages 581–592. Springer, Berlin, 2004. doi:10.1007/978-3-540-27836-8_50.

27 Fedor V. Fomin and Dimitrios M. Thilikos. Dominating sets in planar graphs: branch-width and exponential speed-up. *SIAM J. Comput.*, 36(2):281–309, 2006. doi:10.1137/S0097539702419649.

28 Jakub Gajarský, Petr Hliněný, Jan Obdržálek, Sebastian Ordyniak, Felix Reidl, Peter Rossmanith, Fernando Sánchez Villaamil, and Somnath Sikdar. Kernelization using structural parameters on sparse graph classes. *J. Comput. System Sci.*, 84:219–242, 2017. doi:10.1016/j.jcss.2016.09.002.

29 Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. *J. ACM*, 64(3):Art. 17, 32, 2017. doi:10.1145/3051095.

30 Shai Gutner. Polynomial kernels and faster algorithms for the dominating set problem on graphs with an excluded minor. In *Parameterized and exact computation*, volume 5917 of *Lecture Notes in Comput. Sci.*, pages 246–257. Springer, Berlin, 2009. doi:10.1007/978-3-642-11269-0_20.

31 Iyad Kanj, Michael J. Pelsmajer, Marcus Schaefer, and Ge Xia. On the induced matching problem. *J. Comput. System Sci.*, 77(6):1058–1070, 2011. doi:10.1016/j.jcss.2010.09.001.

32 Eun Jung Kim, Alexander Langer, Christophe Paul, Felix Reidl, Peter Rossmanith, Ignasi Sau, and Somnath Sikdar. Linear kernels and single-exponential algorithms via protrusion decompositions. *ACM Trans. Algorithms*, 12(2):Art. 21, 41, 2016. doi:10.1145/2797140.

33 Stephan Kreutzer, Roman Rabinovich, and Sebastian Siebertz. Polynomial kernels and wideness properties of nowhere dense graph classes. *ACM Trans. Algorithms*, 15(2):Art. 24, 19, 2019. doi:10.1145/3274652.

34 Jiří Matoušek. *Lectures on discrete geometry*, volume 212 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002. doi:10.1007/978-1-4613-0039-7.

35 Hannes Moser and Somnath Sikdar. The parameterized complexity of the induced matching problem. *Discrete Appl. Math.*, 157(4):715–727, 2009. doi:10.1016/j.dam.2008.07.011.

36 James Nastos and Yong Gao. Bounded search tree algorithms for parametrized cograph deletion: efficient branching rules by exploiting structures of special graph classes. *Discrete Math. Algorithms Appl.*, 4(1):1250008, 23, 2012. doi:10.1142/S1793830912500085.

37 Jaroslav Nešetřil and Patrice Ossona de Mendez. Grad and classes with bounded expansion. I. Decompositions. *European J. Combin.*, 29(3):760–776, 2008. doi:10.1016/j.ejc.2006.07.013.

38 Geevarghese Philip, Venkatesh Raman, and Somnath Sikdar. Polynomial kernels for dominating set in graphs of bounded degeneracy and beyond. *ACM Trans. Algorithms*, 9(1):Art. 11, 23, 2012. doi:10.1145/2390176.2390187.

39 Michał Pilipczuk and Sebastian Siebertz. Kernelization and approximation of distance-r independent sets on nowhere dense graphs. *European J. Combin.*, 94:103309, 19, 2021. doi:10.1016/j.ejc.2021.103309.

40 J. A. Telle and Y. Villanger. FPT algorithms for domination in sparse graphs and beyond. *Theoret. Comput. Sci.*, 770:62–68, 2019. doi:10.1016/j.tcs.2018.10.030.

41 Dekel Tsur. Faster parameterized algorithm for cluster vertex deletion. *Theory Comput. Syst.*, 65(2):323–343, 2021. doi:10.1007/s00224-020-10005-w.