Nash Equilibria of Two-Player Matrix Games Repeated Until Collision

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Abstract

We introduce and initiate the study of a natural class of repeated two-player matrix games, called *Repeated-Until-Collision* (RUC) games. In each round, both players simultaneously pick an action from a common action set $\{1, 2, \ldots, n\}$. Depending on their chosen actions, they derive payoffs given by $n \times n$ matrices A and B, respectively. If their actions *collide* (i.e., they pick the same action), the game ends, otherwise, it proceeds to the next round. Both players want to maximize their total payoff until the game ends. RUC games can be interpreted as pursuit-evasion games or repeated hide-and-seek games. They also generalize *hand cricket*, a popular game among children in India.

We show that under mild assumptions on the payoff matrices, every RUC game admits a Nash equilibrium (NE). Moreover, we show the existence of a *stationary* NE, where each player chooses their action according to a probability distribution over the action set that does not change across rounds. Remarkably, we show that all NE are effectively the same as the stationary NE, thus showing that RUC games admit an *almost* unique NE. Lastly, we also show how to compute (approximate) NE for RUC games.

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1 Introduction

Two-player matrix games, or *bimatrix* games, are among the most well-studied classes of games in non-cooperative game theory [27]. A bimatrix game involves two players with a finite set of actions $\{1, 2, \ldots, n\}$, and can be represented by two $n \times n$ payoff matrices A and B for the two players. A play of the game can be thought of as one player choosing a row i and the other player choosing a column j simultaneously. The "row" player gets a payoff of $A_{i,j}$ and the "column" player gets a payoff of $B_{i,j}$. To avoid being predictable, players can choose their actions according to a *randomized* or mixed strategy which is a probability distribution over their action set. Naturally, each player is interested in maximizing their (expected) payoff.

Arguably the most popular solution concept in game theory is that of Nash equilibrium [24]. A Nash equilibrium (NE) for a bimatrix game is a pair of mixed strategies where no player has any incentive to unilaterally deviate and change her strategy. In a celebrated result, Nash showed that each bimatrix game admits a Nash equilibrium. Given that two-player games are ubiquitous and have widespread applications in networks and communications [8], financial markets [1], robotics [18], etc., the field of algorithmic game theory has extensively studied the existence and computation of NE in bimatrix games and their generalizations.

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A repeated two-player game is one such generalization, which involves repetitions of a base bimatrix game [20, 33, 6]. In a repeated game, a player can choose their moves based on the moves played by the players in the previous rounds, while recognizing that their current move will impact the choice of moves of their opponent in the future, leading to added complexity. While finitely repeated games are well-understood using backward-induction [6], the equilibria of infinitely repeated games can differ from that of the base game [20].

In this paper, we introduce and initiate the study of a natural class of repeated bimatrix games, called *Repeated-Until-Collision* (RUC) games. For $n \times n$ non-negative matrices A and B, an RUC game (A, B) has two players, called *max player* and *min player*, where A is called the max player's score matrix and B is called the min player's cost matrix. The game consists of multiple rounds. Suppose the max player and min player pick actions i and j, respectively, in a round. Then the max player earns a score of $A_{i,j}$ and the min player incurs a cost of $B_{i,j}$. If $i = j$, the game ends (we call this event a *collision*). Otherwise, the game proceeds to the next round. We assume that the max player and min player may use randomized strategies. The max player wants to maximize her expected total score and the min player wants to minimize her expected total cost. Below we discuss a few applications of RUC games, thus underscoring their importance.

Applications. RUC games can be interpreted as variants of pursuit-evasion games [17, 7]. As a simplified example, consider a game between a drug dealer (max player) and the law enforcement (min player). Each day, the drug dealer chooses one of n locations for a drop, while law enforcement chooses a location for a random check. If their locations coincide, the game ends as the drug dealer is caught. Until then, the drug dealer wants to maximize his revenue from the sale of drugs, while law enforcement wants to minimize the cost to society by illicit drug use. Similar examples can be found in reinforcement learning or robot motion planning, where an agent, e.g., a spy robot, is trying to learn an unknown environment. The agent gets a reward from exploring one of n locations, while its adversary (security systems, nature, etc.) actively tries to minimize how much the agent discovers before catching the infiltrating agent.

RUC games generalize *hand cricket* [13, 3], a popular game played by children in India. Hand cricket is a contest between a "batter" (max player) and a "bowler" (min player). In a round, the batter and bowler simultaneously choose actions i and j from $\{1, \ldots, n\}$. This gives the batter a score of i, while the bowler suffers a cost of i. The game ends if $i = j$, i.e., the batter is declared "out". Hand cricket involves the batter trying to maximize her total score and the bowler trying to minimize it. Note that this is an example of a *zero-sum* RUC game, where the payoff to the max player equals the cost borne by the min player.

We also note that RUC games can be thought of as repeated hide-and-seek games [31, 30] between two players, the Hider and the Seeker. In a hide-and-seek game, there are n locations containing varying rewards. The Hider tries to collect as much reward as possible, before getting caught by the Seeker, who aims to minimize the reward lost to the Hider. Like hand cricket, repeated hide-and-seek is also a zero-sum RUC game.

Nash equilibria in RUC games. Our paper addresses the following natural questions about RUC games:

Do Repeated-Until-Collision games admit Nash equilibria? If so, are they unique? Can they be computed efficiently?

At first glance, it is not even clear if an NE should exist for general RUC games. Intuitively, the min player would like a collision to happen soon to prevent accumulating a large cost, while the max player would like to delay a collision. Additionally, if some row of A has large numbers, the max player would want to pick the action corresponding to that row more frequently. These two approaches are at odds with each other, e.g., if the max player picks a single action very frequently, the min player can pick the same action and cause a collision very soon. Hence, the players must pick distributions of actions that balance the per-round score (or cost) and the duration of the game. Whether distributions in which neither player has an incentive to deviate unilaterally (i.e., Nash equilibrium) exist is unclear.

1.1 Our Results

As a warm-up, we begin by looking at a simpler version of RUC games called *stationary Repeated-Until-Collision* (SRUC) games, where the players are restricted to use only *stationary strategies.* A stationary strategy is one where in each round, the player samples an action from the same distribution. Formally, given a vector $\mathrm{x} \in \Delta_n$ (where Δ_n is the n-dimensional standard simplex), the stationary strategy x is to pick action i in each round independently randomly with probability x_i for each $i \in \{1, \ldots, n\}$. Intuitively, stationary strategies make sense for RUC games, since the game has a recursive structure: if a collision doesn't happen in the first round, the remaining game is identical to the original. Moreover, stationary strategies are a natural choice for players constrained on computational resources, since they are independent of the game history.

Under mild assumptions on matrices A and B, we show that a Nash equilibrium always exists for the SRUC game (A, B), and is unique.

▶ **Definition 1.** *For any matrix $A \in \mathbb{R}^{n\times n}_{\geq 0}$, let* $\mathrm{graph}(A) := (V, E)$ *be a directed graph where $V := \{1, \ldots, n\}$ and $E := \{(i, j) : i \in V, j \in V \setminus \{i\}, A_{i,j} > 0\}$. Then A is called* irreducible *iff* $\mathrm{graph}(A)$ *is strongly connected.*

▶ **Theorem 2.** *Let A and B be irreducible matrices. Then a Nash equilibrium exists for the SRUC game (A, B). Furthermore, the Nash equilibrium is unique iff* $\mathrm{graph}(A)$ *is a subgraph of* $\mathrm{graph}(B)$.

We prove Theorem 2 in Section 3. Our existence result uses the Perron-Frobenius theorem [21], a central result from matrix theory, and shows that the NE strategies can be computed from the leading eigenvectors of the payoff matrices.

In Appendix E of the full version [23], using more involved techniques, we also show the existence of NE for SRUC games with reducible payoff matrices. Furthermore, if either A or B is reducible, Nash equilibria need not be unique.

In Section 4, we switch back from SRUC games to RUC games, i.e., we allow players to play non-stationary strategies. This setting is significantly more challenging to analyze due to its large and complicated strategy space. Since strategies can be history-dependent, a deterministic strategy can be viewed as a function assigning an action to every possible state of the game, where the state is given by the players' actions in past rounds. Such functions have an infinite domain since the game's history can grow arbitrarily large, which implies that the set of deterministic strategies is uncountably infinite. So, randomized strategies, defined as probability distributions over deterministic strategies, are tough to analyze formally since these probability distributions have a large and unusual sample space.

Although RUC games allow non-stationary strategies, it is not obvious if players benefit from this extra freedom. If one player uses a stationary strategy, it is a priori unclear if the other player can gain by deviating unilaterally to a non-stationary strategy. Interestingly, we show that every RUC game admits a Nash equilibrium with stationary strategies.

▶ **Theorem 3.** *Let A and B be irreducible matrices. Then there exists a pair of stationary strategies* $(\mathrm{x}^*, \mathrm{y}^*)$ *that is a Nash equilibrium for both the RUC game* (A, B) *and the SRUC game* (A, B).

Furthermore, we show that for zero-sum RUC games (i.e., when $A = B$), the Nash equilibrium $(\mathrm{x}^*, \mathrm{y}^*)$ is *unique up to equivalence*, i.e., it is impossible to distinguish between different Nash equilibria just by observing the players' actions.

Having studied the existence of Nash equilibria in RUC games, we now turn to computation. Since the eigenvectors could be irrational, we consider *approximate* Nash equilibria. In an ε-approximate NE, neither player can improve their payoff by a factor of $(1 + \varepsilon)$ through unilateral deviations. We show that fine-enough approximations to the leading eigenvectors of the payoff matrices can be used to compute approximate NE of an RUC game.

▶ **Theorem 4.** *Let* $(\mathrm{x}^*, \mathrm{y}^*)$ *be a Nash equilibrium for the RUC game* (A, B) *with full support. For* $\varepsilon \in [0, 1)$, *let* $\widehat{\mathrm{x}}, \widehat{\mathrm{y}} \in \Delta_n$ *be such that* $|\mathrm{x}_i^* - \widehat{\mathrm{x}}_i| \leq \varepsilon \mathrm{x}_i^*$ *and* $|\mathrm{y}_i^* - \widehat{\mathrm{y}}_i| \leq \varepsilon \mathrm{y}_i^*$, *for all* $i \in [1..n]$. *Then* $(\widehat{\mathrm{x}}, \widehat{\mathrm{y}})$ *is a* $\frac{4\varepsilon}{(1-\varepsilon)^2}$*-approximate Nash equilibrium.*

Estimating the leading eigenvalue and eigenvector of a matrix can be done via the power method [22], an iterative method with a linear convergence rate.

1.2 Related Work

Among the earliest results on bimatrix games are von Neumann's Minimax Theorem in 1928 on zero-sum games [25], and Nash's fundamental result proving the existence of Nash equilibria in bimatrix games [24]. Subsequently the field of algorithmic game theory [26] has devoted considerable attention to the computation of NE in games, with a series of works showing PPAD-completeness [28] even for computing an approximate NE [28, 12, 9]. PPAD-hardness was then shown for many important subclasses, including constant-rank games [19], sparse games [10], win-lose games [11], etc. On the other hand, approximation schemes for finding approximate NE are known for classes like low rank games [16, 2] (FPTAS), and when $(A + B)$ is sparse [5] (PTAS).

Repeated games are well-understood in terms of "Folk Theorems", which indicate that several models of repeated games admit many Nash equilibria [14]. There are many models of repeated games, differing on the horizon for which the game is played (finite or infinite), aggregate utility to the players (arithmetic mean of payoffs in each round, or a sum of payoffs with a discount $\delta \in (0, 1)$), and the kind of equilibrium in consideration (Nash or subgame perfect Nash equilibrium). Finite games with arithmetic mean of payoffs as the utility admit a Nash equilibrium via backward induction [6]. Infinite games where the utility is the limit of the arithmetic mean of the payoffs have been shown to admit Nash equilibria assuming certain kinds of punishments used to deter players from deviating [4, 29]. Infinite games with the utility being a discounted sum of payoffs per round have also been shown to admit equilibria under different conditions on the base game and punishments [15, 14]. In contrast to these works, in our model of RUC games, the utility of a player is the (undiscounted) sum of per-round payoffs. Furthermore, while an RUC game can allow for infinitely many rounds, under reasonable conditions on the payoff matrices (discussed in Section 3), the play terminates in finite time with probability 1 due to collisions. We also do not assume any external model of punishment to prevent agents from deviating.

Finally, we note a superficial similarity of RUC games with stochastic games [32]. A stochastic game is a repeated game with an underlying state space. In each round, players simultaneously choose actions from some action set, based on which they get payoffs and the state of the game changes stochastically according to some transition matrix. The utility to a player is typically assumed to be the discounted sum of the payoffs per round. Stochastic games are known to admit Nash equilibria [32]. A key point of difference between stochastic games and RUC games is that literature on stochastic games only considers stationary strategies, whereas strategies in our RUC games do not have this restriction.

2 Preliminaries

Notation. We introduce some relevant notation.

1. For any boolean proposition P, let $\mathbf{1}(P)$ be 1 if P is true and 0 otherwise.
2. For $n \in \mathbb{Z}_{\geq 0}$, let $[1..n] := \{1, 2, \ldots, n\}$.
3. For any matrix $A \in \mathbb{R}^{m \times n}$, let $A_{i,j}$ or $A[i, j]$ denote the entry in the i^{th} row and j^{th} column. For any vector $\mathrm{v} \in \mathbb{R}^n$, let v_i denote the i^{th} entry of the vector.
4. For $n \in \mathbb{Z}_{\geq 1}$, let $\Delta_n := \{\mathrm{x} \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n \mathrm{x}_i = 1\}$.
5. For any vector $\mathrm{v} \in \mathbb{R}^n$, let $\mathrm{support}(\mathrm{v}) := \{i \in [1..n] : \mathrm{v}_i \neq 0\}$. v is said to have *full support* if $\mathrm{support}(\mathrm{v}) = [1..n]$.
6. For a vector $\mathrm{v} \in \mathbb{R}^n$, define $\|\mathrm{v}\|_1 := \sum_{i=1}^n |\mathrm{v}_i|$.
7. For any $i \in \mathbb{Z}_{\geq 1}$, let $\mathrm{e}^{(i)}$ be a vector such that $\mathrm{e}^{(i)}_i = 1$ and $\mathrm{e}^{(i)}_j = 0$ for all $j \neq i$.

Two-Player Games. We study two-player games between a *max player*, who is interested in maximizing her payoff, and a *min player*, who is interested in minimizing her payoff. We let $\mathcal{X}$ and $\mathcal{Y}$ denote the strategy space of the max and min players, respectively. When the max and min players use strategies $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, respectively, the max player gets a score $f_1(x, y)$ and the min player incurs a cost of $f_2(x, y)$, where these payoffs are given by functions $f_1, f_2 : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{\infty\}$. When $f_1 = f_2$, the game is said to be a *zero-sum* game.

Nash equilibrium. A Nash equilibrium (NE) is pair of strategies where no player can improve her payoff by unilaterally changing her strategy.

▶ **Definition 5.** *Let $f_1, f_2 : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{\infty\}$ be the players' payoff functions. The pair $(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ is called a* Nash equilibrium *if no player can improve their payoff by switching to a different strategy. Formally,*

1. *(Max player cannot improve) $\forall x \in \mathcal{X}$, $f_1(x, y^*) \leq f_1(x^*, y^*)$.*
2. *(Min player cannot improve) $\forall y \in \mathcal{Y}$, $f_2(x^*, y) \geq f_2(x^*, y^*)$.*

While NE always exist for bimatrix games, they need not exist for general games. Moreover, NE may not be unique, i.e., multiple distinct NE may exist. We now define approximate NE.

▶ **Definition 6.** *Let $f_1, f_2 : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{\infty\}$ be the players' payoff functions. The pair $(x^*, y^*) \in \mathcal{X} \times \mathcal{Y}$ is called an* ε-approximate Nash equilibrium *if no player can improve their payoff by a factor of $(1 + \varepsilon)$ by switching to a different strategy. Formally,*

1. *(Max player cannot improve) $\forall x \in \mathcal{X}$, $f_1(x, y^*) \leq (1 + \varepsilon) \cdot f_1(x^*, y^*)$.*
2. *(Min player cannot improve) $\forall y \in \mathcal{Y}$, $f_2(x^*, y) \geq (1 + \varepsilon)^{-1} \cdot f_2(x^*, y^*)$.*

Zero-Sum Games. For a two-player zero-sum game, the following theorem shows that all Nash equilibria give the same payoffs to the agents, so the agents don't prefer any one of them over the other.

▶ **Proposition 7.** *Let (f, f) be a zero-sum game, where $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \cup \{\infty\}$. If (f, f) admits multiple Nash equilibria, they have the same payoff. Formally, if (x_1, y_1) and (x_2, y_2) are Nash equilibria, then $f(x_1, y_1) = f(x_2, y_2) = f(x_1, y_2) = f(x_2, y_1)$. Moreover, (x_1, y_2) and (x_2, y_1) are also Nash equilibria.*

Proof. Since (x_1, y_1) is a Nash equilibrium, the max player doesn't gain by switching to x_2 and min player doesn't gain by switching to y_2. Hence, $f(x_2, y_1) \le f(x_1, y_1) \le f(x_1, y_2)$. Since (x_2, y_2) is a Nash equilibrium, the max player doesn't gain by switching to x_1 and the min player doesn't gain by switching to y_1. Hence, $f(x_1, y_2) \le f(x_2, y_2) \le f(x_2, y_1)$. Combining these inequalities gives us $f(x_1, y_1) = f(x_2, y_2) = f(x_1, y_2) = f(x_2, y_1)$.

For any x and y, we get $f(x, y_2) \le f(x_2, y_2) = f(x_1, y_2) = f(x_1, y_1) \le f(x_1, y)$, and $f(x, y_1) \le f(x_1, y_1) = f(x_2, y_1) = f(x_2, y_2) \le f(x_2, y)$. Hence, (x_1, y_2) and (x_2, y_1) are also Nash equilibria. ◀

3 Nash Equilibria in Stationary RUC games

In this section, we study Nash equilibria for SRUC games, i.e., an RUC game (A, B) where agents are *forced* to use only stationary strategies. Let $e^{(A)}(\mathrm{x}, \mathrm{y})$ and $e^{(B)}(\mathrm{x}, \mathrm{y})$ be the max player's expected total score and the min player's expected total cost, when the max and min players play stationary strategies x and y, respectively.

▶ **Lemma 8.** *Let (A, B) represent an SRUC game. Let x and y be the max player's and min player's stationary strategies, respectively. Then for $C \in \{A, B\}$, we have*

$$e^{(C)}(\mathrm{x}, \mathrm{y}) = \begin{cases} \dfrac{\mathrm{x}^T C \mathrm{y}}{\mathrm{x}^T \mathrm{y}} & \text{if } \mathrm{x}^T \mathrm{y} > 0 \\ \infty & \text{if } \mathrm{x}^T \mathrm{y} = 0 \text{ and } \mathrm{x}^T C \mathrm{y} > 0 \\ 0 & \text{if } \mathrm{x}^T \mathrm{y} = \mathrm{x}^T C \mathrm{y} = 0 \end{cases}.$$

Proof sketch. (See Appendix A of the full version [23] for the full proof.)

The max player's expected per-round score is $\mathrm{x}^T A \mathrm{y}$, and the probability of collision in a round is $\mathrm{x}^T \mathrm{y}$. If $\mathrm{x}^T \mathrm{y} = 0$, a collision never happens, and so her total score is ∞ or 0 depending on whether $\mathrm{x}^T A \mathrm{y} > 0$. If $\mathrm{x}^T \mathrm{y} > 0$, then we can find $\mu := e^{(A)}(\mathrm{x}, \mathrm{y})$ by solving the equation $\mu = \mathrm{x}^T A \mathrm{y} + (1 - \mathrm{x}^T \mathrm{y})\mu$, which gives $\mu = \mathrm{x}^T A \mathrm{y} / \mathrm{x}^T \mathrm{y}$. The min player's expected total cost can be found analogously. ◀

From now on, instead of looking at SRUC games as multi-round games, we will treat them like single round games where the strategy space is Δ_n for both players, and the payoff functions are $e^{(A)}$ and $e^{(B)}$ for the max player and min player, respectively.

3.1 Existence of Nash Equilibrium

This section shows the existence of a Nash equilibrium in SRUC games. We first recall the definitions of an eigenvalue and eigenvector of a matrix.

▶ **Definition 9.** *For a square matrix $A \in \mathbb{R}^n$, complex number λ, and complex vector $\mathrm{v} \in \mathbb{C}^n - \{0\}$, (λ, v) is called an* eigenpair *of A (and λ is called an* eigenvalue *of A and v is called an* eigenvector *of A) if $A\mathrm{v} = \lambda v$.*

▶ **Lemma 10.** *For a square matrix A, λ is an eigenvalue of A iff λ is an eigenvalue of A^T.*

Proof. λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$, and $\det(A - \lambda I) = \det(A^T - \lambda I)$. ◀

The following lemma establishes a sufficient condition for a strategy pair to be a Nash equilibrium of an SRUC game.

▶ **Lemma 11.** *Consider the SRUC game (A, B). Let (α, y^*) be an eigenpair for A such that $\|\mathrm{y}^*\|_1 = 1$ and $\mathrm{y}_i^* > 0$ for all $i \in [1..n]$. Let (β, x^*) be an eigenpair for B^T such that $\|\mathrm{x}^*\|_1 = 1$ and $\mathrm{x}_i^* > 0$ for all $i \in [1..n]$. Then $(\mathrm{x}^*, \mathrm{y}^*)$ is a Nash equilibrium. Moreover, for any $\mathrm{x}, \mathrm{y} \in \Delta_n$, we have $e^{(A)}(\mathrm{x}, \mathrm{y}^*) = \alpha$ and $e^{(B)}(\mathrm{x}^*, \mathrm{y}) = \beta$.*

Proof. Let $\mathrm{x}, \mathrm{y} \in \Delta_n$. Since x^* and y^* have full support, ${\mathrm{x}^*}^T\mathrm{y} > 0$ and $\mathrm{x}^T\mathrm{y}^* > 0$. Also, by Lemma 8,

$$e^{(A)}(\mathrm{x}, \mathrm{y}^*) = \frac{\mathrm{x}^T A \mathrm{y}^*}{\mathrm{x}^T \mathrm{y}^*} = \frac{\mathrm{x}^T(\alpha \mathrm{y}^*)}{\mathrm{x}^T \mathrm{y}^*} = \alpha. \qquad e^{(B)}(\mathrm{x}^*, \mathrm{y}) = \frac{{\mathrm{x}^*}^T B \mathrm{y}}{{\mathrm{x}^*}^T \mathrm{y}} = \frac{(\beta \mathrm{x}^*)^T \mathrm{y}}{{\mathrm{x}^*}^T \mathrm{y}} = \beta.$$

Hence, $(\mathrm{x}^*, \mathrm{y}^*)$ is a Nash equilibrium. ◀

We now show that such strategy pairs exist due to the Perron-Frobenius theorem stated below, thus proving the existence of Nash equilibrium for SRUC games in Theorem 14.

▶ **Theorem 12** (Perron-Frobenius [21]). *Let $A \in \mathbb{R}_{\geq 0}^{n\times n}$ be an irreducible matrix. Then*

1. *There exists a (unique) positive eigenvalue ρ of A, called the* Perron root *of A, such that for any other (possibly complex) eigenvalue λ of A, $|\lambda| \leq \rho$.*
2. *There exist unique vectors u and v such that $A^T\mathrm{u} = \rho\mathrm{u}$, $A\mathrm{v} = \rho\mathrm{v}$, and $\sum_{i=1}^n \mathrm{u}_i = \sum_{j=1}^n \mathrm{v}_j = 1$. u and v are called the left and right* Perron vectors *of A, respectively.*
3. *$\mathrm{u}_i > 0$ and $\mathrm{v}_i > 0$ for all $i \in [1..n]$.*
4. $\displaystyle\min_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} \leq \rho \leq \max_{i=1}^{n} \sum_{j=1}^{n} A_{i,j},$ *and* $\displaystyle\min_{j=1}^{n} \sum_{i=1}^{n} A_{i,j} \leq \rho \leq \max_{j=1}^{n} \sum_{i=1}^{n} A_{i,j}.$

▶ **Definition 13.** *For irreducible matrices A and B, let* $\mathrm{perronSolution}(A, B)$ *be the tuple $(\rho_A, \rho_B, \mathrm{x}^*, \mathrm{y}^*)$, where ρ_A is the Perron root of A, y^* is the right Perron vector of A, ρ_B is the Perron root of B, and x^* is the left Perron vector of B.*

▶ **Theorem 14.** *For the SRUC game (A, B), where A and B are irreducible matrices, let $(\rho_A, \rho_B, \mathrm{x}^*, \mathrm{y}^*) = \mathrm{perronSolution}(A, B)$. Then $(\mathrm{x}^*, \mathrm{y}^*)$ is a Nash equilibrium. Moreover, for any $\mathrm{x}, \mathrm{y} \in \Delta_n$, we have $e^{(A)}(\mathrm{x}, \mathrm{y}^*) = \rho_A$ and $e^{(B)}(\mathrm{x}^*, \mathrm{y}) = \rho_B$.*

Proof. Follows from Theorem 12 and Lemma 11. ◀

3.2 Uniqueness of Nash Equilibrium

In this section, we show that an SRUC game has a unique NE under mild assumptions on the payoff matrices. We begin with two lemmas which together show that any NE of an SRUC game has both players using strategies with full support, provided the payoff matrices A and B are irreducible and $\mathrm{graph}(A)$ is a subgraph of $\mathrm{graph}(B)$.

▶ **Lemma 15.** *Let $(\mathrm{x}^*, \mathrm{y}^*)$ be a Nash equilibrium for the SRUC game (A, B), where A is irreducible and* $\mathrm{graph}(A)$ *is a subgraph of* $\mathrm{graph}(B)$*. Then* $\mathrm{support}(\mathrm{y}^*) = [1..n]$.

Proof. The key idea is that if y^* doesn't have full support, then the max player is incentivized to play a strategy outside $\mathrm{support}(\mathrm{y}^*)$ to get an infinite score. Such a strategy would impose an infinite cost on the min player. Finally, it's possible for the min player to have a bounded cost by playing a full-support strategy.

For any $C \in \{A, B\}$, let $([1..n], E_C) := \text{graph}(C)$. Let $\overline{E}_C := [1..n] \times [1..n] \setminus E_C$. For any $\mathrm{x}, \mathrm{y} \in \Delta_n$ such that $\mathrm{x}^T\mathrm{y} = 0$, we get $\mathrm{x}^T C\mathrm{y} = 0 \iff \text{support}(\mathrm{x}) \times \text{support}(\mathrm{y}) \subseteq \overline{E}_C$. Since $E_A \subseteq E_B$, we get $\mathrm{x}^T A\mathrm{y} > 0 \implies \mathrm{x}^T B\mathrm{y} > 0$.

Let $S_y := \text{support}(\mathrm{y}^*)$. Suppose $S_y \neq [1..n]$. Then $\exists (i, j) \in E_A$ from $[1..n] \setminus S_y$ to S_y, since A is irreducible. Hence, $(\mathrm{e}^{(i)})^T A\mathrm{y}^* = \sum_{k \in S_y} A_{i,k}\mathrm{y}^*_k \geq A_{i,j}\mathrm{y}^*_j > 0$. Also, $(\mathrm{e}^{(i)})^T\mathrm{y}^* = 0$, since $i \notin S_y$. Hence, $e^{(A)}(\mathrm{e}^{(i)}, \mathrm{y}^*) = \infty$.

Since $(\mathrm{x}^*, \mathrm{y}^*)$ is a Nash equilibrium, $\infty = e^{(A)}(\mathrm{e}^{(i)}, \mathrm{y}^*) \leq e^{(A)}(\mathrm{x}^*, \mathrm{y}^*)$. This is only possible if ${\mathrm{x}^*}^T\mathrm{y}^* = 0$ and ${\mathrm{x}^*}^T A\mathrm{y}^* > 0$. This means ${\mathrm{x}^*}^T B\mathrm{y}^* > 0$, and so $e^{(B)}(\mathrm{x}^*, \mathrm{y}^*) = \infty$.

Let $\widehat{\mathrm{y}}_j = 1/n$ for all $j \in [1..n]$. Then ${\mathrm{x}^*}^T\widehat{\mathrm{y}} > 0$, so $e^{(B)}(\mathrm{x}^*, \widehat{\mathrm{y}})$ is finite. Since $(\mathrm{x}^*, \mathrm{y}^*)$ is a Nash equilibrium, we get $\infty = e^{(B)}(\mathrm{x}^*, \mathrm{y}^*) \leq e^{(B)}(\mathrm{x}^*, \widehat{\mathrm{y}}) \neq \infty$. This is a contradiction, so $\text{support}(\mathrm{y}^*) = [1..n]$. ◀

▶ **Lemma 16.** *Let* $(\mathrm{x}^*, \mathrm{y}^*)$ *be a Nash equilibrium for SRUC game* (A, B)*, where* B *is irreducible and* $\text{support}(\mathrm{y}^*) = [1..n]$*. Then* $\text{support}(\mathrm{x}^*) = [1..n]$*.*

Proof. The key idea is that if x^* doesn't have full support, then the min player can reduce her cost by not playing actions outside $\text{support}(\mathrm{x}^*)$.

Let $S_x := \text{support}(\mathrm{x}^*)$. Let $\alpha := \sum_{j \in S_x} \mathrm{y}^*_j$. Then $1 - \alpha = \sum_{j \in [1..n] \setminus S_x} \mathrm{y}^*_j$. Suppose $S_x \neq [1..n]$. Then $0 < \alpha < 1$, since $\text{support}(\mathrm{y}^*) = [1..n]$. Define vectors $\widehat{\mathrm{y}}$ and $\widetilde{\mathrm{y}}$ as follows:

$$\widehat{\mathrm{y}}_j = \begin{cases} \mathrm{y}^*_j/\alpha & \text{if } j \in S_x \\ 0 & \text{if } j \notin S_x \end{cases}, \qquad \widetilde{\mathrm{y}}_j = \begin{cases} 0 & \text{if } j \in S_x \\ \mathrm{y}^*_j/(1-\alpha) & \text{if } j \notin S_x \end{cases}.$$

Then $\mathrm{y}^* = \alpha\widehat{\mathrm{y}} + (1 - \alpha)\widetilde{\mathrm{y}}$ and ${\mathrm{x}^*}^T\mathrm{y}^* = \alpha{\mathrm{x}^*}^T\widehat{\mathrm{y}}$. Hence,

$$e^{(B)}(\mathrm{x}^*, \mathrm{y}^*) = \frac{{\mathrm{x}^*}^T B\mathrm{y}^*}{{\mathrm{x}^*}^T\mathrm{y}^*} = \frac{\alpha{\mathrm{x}^*}^T B\widehat{\mathrm{y}} + (1-\alpha){\mathrm{x}^*}^T B\widetilde{\mathrm{y}}}{\alpha{\mathrm{x}^*}^T\widehat{\mathrm{y}}} = e^{(B)}(\mathrm{x}^*, \widehat{\mathrm{y}}) + \frac{1-\alpha}{\alpha}\frac{{\mathrm{x}^*}^T B\widetilde{\mathrm{y}}}{{\mathrm{x}^*}^T\widehat{\mathrm{y}}}.$$

Since $(\mathrm{x}^*, \mathrm{y}^*)$ is a Nash equilibrium, $e^{(B)}(\mathrm{x}^*, \mathrm{y}^*) \leq e^{(B)}(\mathrm{x}^*, \widehat{\mathrm{y}})$. Hence, ${\mathrm{x}^*}^T B\widetilde{\mathrm{y}} = 0$. However,

$${\mathrm{x}^*}^T B\widetilde{\mathrm{y}} = \sum_{i \in S_x} \sum_{j \in [1..n] \setminus S_x} \mathrm{x}^*_i B_{i,j}\widetilde{\mathrm{y}}_j.$$

Since B is irreducible, there is an edge (i, j) in $\text{graph}(B)$ from S_x to $[1..n] \setminus S_x$. Therefore, ${\mathrm{x}^*}^T B\widetilde{\mathrm{y}} \geq B_{i,j}\mathrm{x}^*_i\widetilde{\mathrm{y}}_j > 0$, which is a contradiction. Hence, $\text{support}(\mathrm{x}^*) = [1..n]$. ◀

We now state a simple result about the ratio of sums (Lemma 17), and use it to prove that full support Nash equilibria must be eigenvectors (Lemma 18).

▶ **Lemma 17.** *Let* $\mathrm{u} \in \mathbb{R}^n$ *and* $\mathrm{v} \in \mathbb{R}^n_{>0}$*. Let*

$$\alpha := \min_{i=1}^{n} \frac{\mathrm{u}_i}{\mathrm{v}_i} \qquad \beta := \max_{i=1}^{n} \frac{\mathrm{u}_i}{\mathrm{v}_i} \qquad z := \frac{\sum_{i=1}^n \mathrm{u}_i}{\sum_{i=1}^n \mathrm{v}_i}.$$

Then either $z = \alpha = \beta$ *or* $\alpha < z < \beta$*.*

Proof. If $\alpha = \beta$, then $\mathrm{u}_i/\mathrm{v}_i = \alpha$ for all $i \in [1..n]$, and hence $z = \alpha$. Now let $\alpha < \beta$. Then $\alpha\mathrm{v}_i \leq \mathrm{u}_i \leq \beta\mathrm{v}_i$ for all $i \in [1..n]$. Pick p and q such that $\mathrm{u}_p/\mathrm{v}_p = \alpha$ and $\mathrm{u}_q/\mathrm{v}_q = \beta$. Then

$$\sum_{i=1}^n \mathrm{u}_i \leq \alpha\mathrm{v}_p + \sum_{i \in [1..n] \setminus \{p\}} \beta\mathrm{v}_i < \beta\sum_{i=1}^n \mathrm{v}_i, \qquad \sum_{i=1}^n \mathrm{u}_i \geq \sum_{i \in [1..n] \setminus \{q\}} \alpha\mathrm{v}_i + \beta\mathrm{v}_q > \alpha\sum_{i=1}^n \mathrm{v}_i.$$

Hence, $\alpha < z < \beta$. ◀

▶ **Lemma 18.** *Let* $(\mathrm{x}^*, \mathrm{y}^*)$ *be a Nash equilibrium for SRUC game* (A, B) *such that* $\mathrm{support}(\mathrm{x}^*) = \mathrm{support}(\mathrm{y}^*) = [1..n]$*. Then* $(e^{(B)}(\mathrm{x}^*, \mathrm{y}^*), \mathrm{x}^*)$ *is an eigenpair of* B^T *and* $(e^{(A)}(\mathrm{x}^*, \mathrm{y}^*), \mathrm{y}^*)$ *is an eigenpair of* A*.*

Proof. For $i \in [1..n]$, let $\mathrm{w}_i := \mathrm{x}_i^* \mathrm{y}_i^*$. Then $\mathrm{w}_i > 0$ for all $i \in [1..n]$.

For $j \in [1..n]$, let $\mathrm{u}_j := (B^T \mathrm{x}^*)_j \mathrm{y}_j^*$. Let

$$\alpha_x := \min_{j=1}^{n} \frac{\mathrm{u}_j}{\mathrm{w}_j} = \min_{j=1}^{n} \frac{(B^T \mathrm{x}^*)_j}{\mathrm{x}_j^*} \qquad \beta_x := \max_{j=1}^{n} \frac{\mathrm{u}_j}{\mathrm{w}_j} = \max_{j=1}^{n} \frac{(B^T \mathrm{x}^*)_j}{\mathrm{x}_j^*}$$

Let p and q be indices such that $\mathrm{u}_p/\mathrm{w}_p = \alpha_x$ and $\mathrm{u}_q/\mathrm{w}_q = \beta_x$. Then $e^{(B)}(\mathrm{x}^*, \mathrm{e}^{(p)}) = \alpha_x$, $e^{(B)}(\mathrm{x}^*, \mathrm{e}^{(q)}) = \beta_x$, and $e^{(B)}(\mathrm{x}^*, \mathrm{y}^*) = (\sum_{j=1}^n \mathrm{u}_j)/(\sum_{j=1}^n \mathrm{w}_j)$. Suppose $\alpha_x \neq \beta_x$. Then by Lemma 17, $e^{(B)}(\mathrm{x}^*, \mathrm{e}^{(p)}) < e^{(B)}(\mathrm{x}^*, \mathrm{y}^*) < e^{(B)}(\mathrm{x}^*, \mathrm{e}^{(q)})$. This contradicts the fact that $(\mathrm{x}^*, \mathrm{y}^*)$ is a Nash equilibrium. Hence, $\alpha_x = \beta_x = e^{(B)}(\mathrm{x}^*, \mathrm{y}^*)$ and $(B^T \mathrm{x}^*)_j / \mathrm{x}_j^* = \alpha_x$ for all $j \in [1..n]$. Hence, (α_x, x^*) is an eigenpair of B^T.

For $i \in [1..n]$, let $\mathrm{v}_i := \mathrm{x}_i^* (A\mathrm{y}^*)_i$. Let

$$\alpha_y := \min_{i=1}^{n} \frac{\mathrm{v}_i}{\mathrm{w}_i} = \min_{i=1}^{n} \frac{(A\mathrm{y}^*)_i}{\mathrm{y}_i^*} \qquad \beta_y := \max_{i=1}^{n} \frac{\mathrm{v}_i}{\mathrm{w}_i} = \max_{i=1}^{n} \frac{(A\mathrm{y}^*)_i}{\mathrm{y}_i^*}$$

We can similarly show that $\alpha_y = \beta_y = e^{(A)}(\mathrm{x}^*, \mathrm{y}^*)$ and (α_y, y^*) is an eigenpair of A. ◀

Finally, we show that non-negative eigenvectors are essentially Perron vectors (Lemma 19), and use this result to establish uniqueness of Nash equilibrium (Theorem 20).

▶ **Lemma 19.** *Let* $A \in \mathbb{R}_{\geq 0}^{n \times n}$ *be an irreducible matrix. Let* ρ *be the Perron root of* A*, and* u *and* v *be the left and right Perron vectors of* A*, respectively. Then both of these hold:*

1. *If* (λ, y) *is an eigenpair of* A *such that* $\mathrm{y}_i \geq 0\ \forall i$ *and* $\|\mathrm{y}\|_1 = 1$*, then* $\lambda = \rho$ *and* $\mathrm{y} = \mathrm{v}$*.*
2. *If* (λ, x) *is an eigenpair of* A^T *such that* $\mathrm{x}_i \geq 0\ \forall i$ *and* $\|\mathrm{x}\|_1 = 1$*, then* $\lambda = \rho$ *and* $\mathrm{x} = \mathrm{u}$*.*

Proof. $\rho \mathrm{u}^T \mathrm{y} = (A^T \mathrm{u})^T \mathrm{y} = \mathrm{u}^T (A\mathrm{y}) = \lambda \mathrm{u}^T \mathrm{y}$. Since $\mathrm{support}(\mathrm{u}) = [1..n]$, we get $\mathrm{u}^T \mathrm{y} > 0$. Hence, $\lambda = \rho$. By Theorem 12, v is the unique eigenvector corresponding to eigenvalue ρ such that $\|\mathrm{v}\|_1 = 1$. Hence, $\mathrm{y} = \mathrm{v}$. We get part 2 of the lemma by applying part 1 on A^T. ◀

▶ **Theorem 20.** *Let* (A, B) *be an SRUC game, where* A *and* B *are irreducible and* $\mathrm{graph}(A)$ *is a subgraph of* $\mathrm{graph}(B)$*. Let* $(\rho_A, \rho_B, \mathrm{x}^*, \mathrm{y}^*) = \mathrm{perronSolution}(A, B)$*. Then* $(\mathrm{x}^*, \mathrm{y}^*)$ *is the unique Nash equilibrium for this SRUC game.*

Proof. Let $(\widehat{\mathrm{x}}, \widehat{\mathrm{y}})$ be any Nash equilibrium for SRUC game (A, B). From Lemmas 15 and 16, we get $\mathrm{support}(\widehat{\mathrm{x}}) = \mathrm{support}(\widehat{\mathrm{y}}) = [1..n]$.

Let $\sigma_C := e^{(C)}(\widehat{\mathrm{x}}, \widehat{\mathrm{y}})$ for $C \in \{A, B\}$. By Lemma 18, $(\sigma_B, \widehat{\mathrm{x}})$ is an eigenpair of B^T and $(\sigma_A, \widehat{\mathrm{y}})$ is an eigenpair of A. By Lemma 19, we get $\sigma_B = \rho_B$, $\widehat{\mathrm{x}} = \mathrm{x}^*$, $\sigma_A = \rho_A$, $\widehat{\mathrm{y}} = \mathrm{y}^*$. ◀

Theorem 20 shows uniqueness of NE of an SRUC game (A, B) under two conditions: (i) A and B are irreducible, and (ii) $\mathrm{graph}(A) \subseteq \mathrm{graph}(B)$. The following two lemmas show that if either of these conditions is relaxed, we can no longer guarantee uniqueness of NE.

▶ **Lemma 21.** *Let* (A, B) *be an SRUC game. Suppose* $\exists (i, j)$ *such that* $i \neq j$*,* $A_{i,j} > 0$*, and* $B_{i,j} = 0$*. Then* $(\mathrm{e}^{(i)}, \mathrm{e}^{(j)})$ *is a Nash equilibrium. Moreover,* $e^{(A)}(\mathrm{e}^{(i)}, \mathrm{e}^{(j)}) = \infty$ *and* $e^{(B)}(\mathrm{e}^{(i)}, \mathrm{e}^{(j)}) = 0$*.*

Proof. ${\mathrm{e}^{(i)}}^T A \mathrm{e}^{(j)} = A[i, j] > 0$, ${\mathrm{e}^{(i)}}^T B \mathrm{e}^{(j)} = B[i, j] = 0$, and ${\mathrm{e}^{(i)}}^T \mathrm{e}^{(j)} = 0$. ◀

The NE in Lemma 21 is deterministic but the NE in Theorem 14 is randomized, implying that the game has at least two NE. Hence, the condition $\mathrm{graph}(A) \subseteq \mathrm{graph}(B)$ is necessary for uniqueness of NE if A and B are irreducible. The next lemma shows that if either A or B is reducible, then uniqueness of NE is not guaranteed.

▶ **Lemma 22.** *Let* (A, A) *be an SRUC game where* $A := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. *Then* (x, y) *is a Nash equilibrium iff* $\mathrm{support}(\mathrm{x}) = \{1, 2\}$ *and* $\mathrm{y} = [1, 0]$.

Proof. Observe that A is reducible. For any $\mathrm{x}, \mathrm{y} \in \Delta_2$, if $\mathrm{x}_2 > 0$ and $\mathrm{y}_2 > 0$, then $e^{(A)}(\mathrm{x}, \mathrm{y}) = 1 + \mathrm{x}_2\mathrm{y}_2/(\mathrm{x}_1\mathrm{y}_1 + \mathrm{x}_2\mathrm{y}_2) > 1$. When $\mathrm{x}^T\mathrm{y} = 0$, we get $e^{(A)}(\mathrm{x}, \mathrm{y}) = 0$. Otherwise, $e^{(A)}(\mathrm{x}, \mathrm{y}) = 1$. This is summarized by the following table:

$x \setminus y$	$[1, 0]$	mix	$[0, 1]$
$[1, 0]$	1	1	0
mix	1	> 1	> 1
$[0, 1]$	0	> 1	> 1

This table shows that (x, y) is a NE iff $\mathrm{support}(\mathrm{x}) = \{1, 2\}$ and $\mathrm{y} = [1, 0]$. ◀

The above example also shows that unlike the guarantee of Theorem 14, Nash equilibrium strategies may not have full support.

3.3 Approximate Nash Equilibrium

It may not be possible to compute Nash equilibria for SRUC games exactly, e.g., if eigenvectors are irrational. Hence, we would like to compute approximate Nash equilibria. We will use the following lemma.

▶ **Lemma 23.** *Let* $\mathrm{x}^*, \widehat{\mathrm{x}} \in \Delta_n$ *and* $\delta \in [0, 1)$ *such that* $\mathrm{support}(\mathrm{x}^*) = [1..n]$ *and* $|\widehat{\mathrm{x}}_i - \mathrm{x}^*_i| \le \delta \mathrm{x}^*_i$ *for all* $i \in [1..n]$. *Then* $\mathrm{support}(\widehat{\mathrm{x}}) = [1..n]$ *and for any matrix* $C \in \mathbb{R}^{n \times n}_{\ge 0}$ *and any* $j \in [1..n]$,

$$\frac{(C^T\widehat{\mathrm{x}})_j}{\widehat{\mathrm{x}}_j} \in \left[\frac{1-\delta}{1+\delta}\frac{(C^T\mathrm{x}^*)_j}{\mathrm{x}^*_j}, \frac{1+\delta}{1-\delta}\frac{(C^T\mathrm{x}^*)_j}{\mathrm{x}^*_j}\right].$$

Proof. For any $i \in [1..n]$, we have $\widehat{\mathrm{x}}_i \in [(1-\delta)\mathrm{x}^*_i, (1+\delta)\mathrm{x}^*_i]$. Hence, $\mathrm{support}(\widehat{\mathrm{x}}) = [1..n]$.

$$\begin{aligned}
&|(C^T\widehat{\mathrm{x}})_j - (C^T\mathrm{x}^*)_j| \le |(C^T(\widehat{\mathrm{x}} - \mathrm{x}^*))_j| \le \left|\sum_{i=1}^n C[i,j](\widehat{\mathrm{x}}_i - \mathrm{x}^*_i)\right| \\
&\le \sum_{i=1}^n C[i,j]|\widehat{\mathrm{x}}_i - \mathrm{x}^*_i| \le \delta \sum_{i=1}^n C[i,j]\mathrm{x}^*_i = \delta(C^T\mathrm{x}^*)_j. \\
&\implies (C^T\widehat{\mathrm{x}})_j \in [(1-\delta)(C^T\mathrm{x}^*)_j, (1+\delta)(C^T\mathrm{x}^*)_j] \\
&\implies \frac{(C^T\widehat{\mathrm{x}})_j}{\widehat{\mathrm{x}}_j} \in \left[\frac{1-\delta}{1+\delta}\frac{(C^T\mathrm{x}^*)_j}{\mathrm{x}^*_j}, \frac{1+\delta}{1-\delta}\frac{(C^T\mathrm{x}^*)_j}{\mathrm{x}^*_j}\right].
\end{aligned}$$
◀

We now show that a close-enough approximation to a NE is an approximate NE of an SRUC game. This result allows us to compute approximate NE by using methods for approximately estimating the leading eigenvector of a matrix, such as power iteration [22].

▶ **Theorem 24.** *Let* $(\mathrm{x}^*, \mathrm{y}^*)$ *be a Nash equilibrium for the SRUC game* (A, B) *such that* $\mathrm{support}(\mathrm{x}^*) = \mathrm{support}(\mathrm{y}^*) = [1..n]$. *For* $\delta \in [0, 1)$, *let* $\widehat{\mathrm{x}}, \widehat{\mathrm{y}} \in \Delta_n$ *be such that for all* $i \in [1..n]$, $|\widehat{\mathrm{x}}_i - \mathrm{x}^*_i| \le \delta\mathrm{x}^*_i$ *and* $|\widehat{\mathrm{y}}_i - \mathrm{y}^*_i| \le \delta\mathrm{y}^*_i$. *Then* $(\widehat{\mathrm{x}}, \widehat{\mathrm{y}})$ *is a* $\frac{4\delta}{(1-\delta)^2}$*-approximate Nash equilibrium.*

Proof. By Lemma 18, (ρ_B, x^*) is an eigenpair of B^T. Hence, by Lemma 23, for all $j \in [1..n]$,

$$\frac{(B^T\widehat{\mathrm{x}})_j}{\widehat{\mathrm{x}}_j} \in \left[\frac{1-\delta}{1+\delta}\rho_B, \frac{1+\delta}{1-\delta}\rho_B\right].$$

For any $\mathrm{y} \in \Delta_n$, using Lemma 17, we get

$$e^{(B)}(\widehat{\mathrm{x}}, \mathrm{y}) = \frac{\sum_{j=1}^n (B^T\widehat{\mathrm{x}})_j \mathrm{y}_j}{\sum_{j=1}^n \widehat{\mathrm{x}}_j \mathrm{y}_j} \in \left[\min_{j=1}^n \frac{(B^T\widehat{\mathrm{x}})_j}{\widehat{\mathrm{x}}_j}, \max_{j=1}^n \frac{(B^T\widehat{\mathrm{x}})_j}{\widehat{\mathrm{x}}_j}\right] = \left[\frac{1-\delta}{1+\delta}\rho_B, \frac{1+\delta}{1-\delta}\rho_B\right].$$

In particular, for $\mathrm{y} = \widehat{\mathrm{y}}$, we get $e^{(B)}(\widehat{\mathrm{x}}, \widehat{\mathrm{y}}) \le \frac{1+\delta}{1-\delta}\rho_B$. Let $\varepsilon := \frac{4\delta}{(1-\delta)^2}$. Thus, for any $\mathrm{y} \in \Delta_n$,

$$e^{(B)}(\widehat{\mathrm{x}}, \mathrm{y}) \ge \frac{1-\delta}{1+\delta}\rho_B \ge \left(\frac{1-\delta}{1+\delta}\right)^2 e^{(B)}(\widehat{\mathrm{x}}, \widehat{\mathrm{y}}) = \frac{e^{(B)}(\widehat{\mathrm{x}}, \widehat{\mathrm{y}})}{1+\varepsilon}.$$

We can analogously show that for any $\mathrm{x} \in \Delta_n$,

$$e^{(A)}(\mathrm{x}, \widehat{\mathrm{y}}) \le \left(\frac{1+\delta}{1-\delta}\right)^2 e^{(A)}(\widehat{\mathrm{x}}, \widehat{\mathrm{y}}) = (1+\varepsilon) \cdot e^{(A)}(\widehat{\mathrm{x}}, \widehat{\mathrm{y}}).$$

Hence, $(\widehat{\mathrm{x}}, \widehat{\mathrm{y}})$ is an ε-approximate NE. ◀

4 Nash Equilibria in General RUC games

In this section, we study RUC games when players are allowed to use non-stationary strategies. As discussed in the Introduction, the strategy space can be complicated in general RUC games. We first describe a framework for defining strategies in a RUC game.

4.1 Strategy Space of RUC games

A deterministic (possibly non-stationary) strategy in an RUC game is a function f that takes as input a list of actions played by the opponent so far and outputs the next action for the player. E.g., if the max player is using a deterministic strategy f, then in the k^{th} round, the max player will play the action $f(J)$, where $J := [j_1, j_2, \dots, j_{k-1}]$ is the list of actions played by the min player in the first $k-1$ rounds.

A randomized strategy is a distribution over deterministic strategies. Formally, let $[1..n]^*$ be the set of all finite lists where each element is in $[1..n]$. Let Ω be the set of all deterministic strategies, i.e., the set of all functions from $[1..n]^*$ to $[1..n]$. Then a randomized strategy is given by a probability space $(\Omega, \mathcal{E}, P)$. Recall that in a probability space, $\mathcal{E} \subseteq 2^\Omega$ is the set of events and $P : \mathcal{E} \to [0,1]$ is a probability measure. If a randomized strategy f is sampled from this probability space, then for any set $F \in \mathcal{E}$ of deterministic strategies, we say $\Pr(f \in F) := P(F)$.

For any $\mathrm{x} \in \Delta_n$, and any random variable $X \in [1..n]$, we write $X \sim \mathrm{x}$ to say that $\Pr(X = i) = \mathrm{x}_i$ for all $i \in [1..n]$. The length of a list L, denoted by $|L|$, is the number of elements in L. Unless specified otherwise, assume all lists are finite.

▶ **Definition 25** (stationary strategy)**.** *Let $\mathrm{x} \in \Delta_n$. Let $[I_0, I_1, \dots]$ be an infinite sequence of independent random variables where for each $t \in \mathbb{Z}_{\ge 0}$, $I_t \in [1..n]$ and $I_t \sim \mathrm{x}$. Then* $\mathrm{stat}(\mathrm{x})$ *is a strategy f where $f(J) := I_{|J|}$.* $\mathrm{stat}(\mathrm{x})$ *is called the* stationary strategy *for parameter* x.

With a little abuse of notation, we will sometimes write x instead of $\mathrm{stat}(\mathrm{x})$.

Residual Strategies. For any two lists L_1 and L_2, let $L_1 + L_2$ denote their concatenation. Let $\emptyset$ denote the empty list. Let $[x]$ denote a list of length 1 containing the element x.

Given a list $K \in [1..n]^*$ and a strategy f, define the function f^K as $f^K(K') := f(K+K')$. Intuitively, if a player is using strategy f, then after the opponent has played actions K, f^K is the strategy for the remaining game. f^K is called the K-*residual* strategy of f. Due to the recursive nature of RUC games, residual strategies are helpful in their analysis.

Let $I := [i_1, \ldots, i_k]$ and $J := [j_1, \ldots, j_k]$ be lists. Let f be a randomized strategy. We want to define $f^{(I,J)}$ as the strategy f^J conditioned on the player responding with actions I when the opponent plays actions J. Formally, let $\text{isResp}(f, I, J)$ be the event that a player using strategy f responds with actions I to opponent's actions J, i.e., $\forall t \in [1..k], f([j_1, \ldots, j_{t-1}]) = i_t$. Call (I, J) a *feasible history* for f if $\Pr(\text{isResp}(f, I, J)) > 0$. For any feasible history (I, J) of f, define $f^{(I,J)}$, called the (I, J)-*residual strategy* of f, as a strategy having distribution

$$\Pr(f^{(I,J)} \in F) := \Pr(f^J \in F \mid \text{isResp}(f, I, J)).$$

Intuitively, stationary strategies should remain unchanged when conditioned on past actions. We prove this formally.

▶ **Lemma 26.** $\text{stat}(\mathrm{x})^{(I,J)}$ *has the same distribution as* $\text{stat}(\mathrm{x})$.

Proof. (See Appendix C.1 of the full version [23].) ◀

Expected Score. Consider the RUC game (A, B), where A and B are $n \times n$ matrices. Assume that the max player's and min player's strategies are independent. We now formally define the *expected score (or cost) of a pair of strategies*, and obtain a recursive expression for it to make analysis easier.

▶ **Definition 27** (score). *Let* $S^{(A,r)}(f, g)$ *and* $S^{(B,r)}(f, g)$ *be the max player's total score and min player's total cost, respectively, in the first* r *rounds of the RUC game* (A, B) *when the max player uses strategy* f *and the min player uses strategy* g*. (Note that when* f *and* g *are randomized strategies,* $S^{(A,r)}(f, g)$ *and* $S^{(B,r)}(f, g)$ *are random variables.) For* $C \in \{A, B\}$*, let* $e^{(C,r)}(f, g) := \mathbb{E}(S^{(C,r)}(f, g))$ *and* $e^{(C,\infty)}(f, g) := \lim_{r\to\infty} e^{(C,r)}(f, g)$*.*

▶ **Observation 28.** *The sequence* $[S^{(C,r)}(f, g)]_{r=0}^{\infty}$ *is monotonically increasing, so the sequence* $[e^{(C,r)}(f, g)]_{r=0}^{\infty}$ *is also monotonically increasing. By the monotone convergence theorem,* $e^{(C,\infty)}(f, g) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$*, i.e., the sequence* $[e^{(C,r)}(f, g)]_{r=0}^{\infty}$ *either has a non-negative limit or is unbounded.*

▶ **Observation 29.** *Let* f *and* g *be strategies of the max and min players, respectively. Let* $u := f(\emptyset)$ *and* $v := g(\emptyset)$*. Then for* $C \in \{A, B\}$*,*

$$S^{(C,r)}(f, g) = \begin{cases} C[u, v] + \mathbf{1}(u \neq v) S^{(C,r-1)}(f^{[v]}, g^{[u]}) & \textit{if } r > 0 \\ 0 & \textit{otherwise} \end{cases}.$$

▶ **Lemma 30.** *Let* f *and* g *be independent strategies of the max and min players, respectively. Let* $f(\emptyset) \sim \mathrm{x}$*,* $g(\emptyset) \sim \mathrm{y}$*,* $S_x := \text{support}(\mathrm{x})$*, and* $S_y := \text{support}(\mathrm{y})$*. Then for* $C \in \{A, B\}$*,*

$$e^{(C,r)}(f, g) = \begin{cases} \mathrm{x}^T C \mathrm{y} + \sum_{i \in S_x} \sum_{j \in S_y} \mathrm{x}_i \mathrm{y}_j \mathbf{1}(i \neq j) e^{(C,r-1)}(f^{([i],[j])}, g^{([j],[i])}) & \textit{if } r > 0 \\ 0 & \textit{otherwise} \end{cases},$$

$$e^{(C,\infty)}(f, g) = \mathrm{x}^T C \mathrm{y} + \sum_{i \in S_x} \sum_{j \in S_y} \mathrm{x}_i \mathrm{y}_j \mathbf{1}(i \neq j) e^{(C,\infty)}(f^{([i],[j])}, g^{([j],[i])}).$$

Proof sketch. Follows from Observation 29. See Appendix C.1 of the full version [23] for the full proof. ◀

4.2 Existence of Nash Equilibrium

To see whether a pair of stationary strategies can give us a Nash equilibrium, we first investigate (in the next two lemmas) the upper and lower bounds on a player's payoff when she is free to play any strategy (even non-stationary ones) and her opponent uses a stationary strategy with full support.

▶ **Lemma 31.** *Let (A, B) be an RUC game. Let $\widehat{\mathrm{x}} \in \Delta_n$ such that* $\mathrm{support}(\widehat{\mathrm{x}}) = [1..n]$. *Let $\alpha := \min_{i=1}^{n} (B^T\widehat{\mathrm{x}})_i/\widehat{\mathrm{x}}_i$ and $\beta := \max_{i=1}^{n} (B^T\widehat{\mathrm{x}})_i/\widehat{\mathrm{x}}_i$. Then for any strategy g, we have $\alpha \leq e^{(B,\infty)}(\widehat{\mathrm{x}}, g) \leq \beta$.*

Proof. By the definition of α and β, and by Lemma 17, we get that for any $\mathrm{y} \in \Delta_n$, we have $\alpha\widehat{\mathrm{x}}^T\mathrm{y} \leq \widehat{\mathrm{x}}^T B\mathrm{y} \leq \beta\widehat{\mathrm{x}}^T\mathrm{y}$.

Define the predicate $P(r) : \forall g, e^{(B,r)}(\widehat{\mathrm{x}}, g) \leq \beta$. We will prove $P(r)$ by induction, and that would imply $P(\infty)$. The base case holds, since $e^{(B,0)}(\widehat{\mathrm{x}}, g) = 0 \leq \beta$. Now fix $r \geq 1$, g, and let $g(\emptyset) \sim \mathrm{y}$ and $S_y := \mathrm{support}(\mathrm{y})$. Then by Lemma 30,

$$\begin{aligned} e^{(B,r)}(\widehat{\mathrm{x}}, g) &= \widehat{\mathrm{x}}^T B\mathrm{y} + \sum_{i=1}^{n}\sum_{j\in S_y} \widehat{\mathrm{x}}_i\mathrm{y}_j\mathbf{1}(i \neq j)e^{(B,r-1)}(\widehat{\mathrm{x}}, g^{([j],[i])}) \\ &\leq \beta\widehat{\mathrm{x}}^T\mathrm{y} + \sum_{i=1}^{n}\sum_{j=1}^{n}\widehat{\mathrm{x}}_i\mathrm{y}_j\mathbf{1}(i \neq j)\beta \;= \beta. \qquad \text{(by inductive hypothesis)} \end{aligned}$$

This proves $P(r)$, and hence, also proves $P(\infty)$.

Next, we need to prove that $e^{(B,\infty)}(\widehat{\mathrm{x}}, g) \geq \alpha$ for all g. Let $\mu := \inf_g e^{(B,\infty)}(\widehat{\mathrm{x}}, g)$. Pick any $\varepsilon > 0$. Let $\widehat{g}$ be a strategy such that $e^{(B,\infty)}(\widehat{\mathrm{x}}, \widehat{g}) \leq \mu+\varepsilon$. Let $\widehat{g}(\emptyset) \sim \widehat{\mathrm{y}}$ and $S_y := \mathrm{support}(\widehat{\mathrm{y}})$. Let $\gamma := \min_{i=1}^{n} \widehat{\mathrm{x}}_i$. Then $\widehat{\mathrm{x}}^T\widehat{\mathrm{y}} \geq \gamma$, and by Lemma 30,

$$\begin{aligned} \mu + \varepsilon \geq e^{(B,\infty)}(\widehat{\mathrm{x}}, \widehat{g}) &= \widehat{\mathrm{x}}^T B\widehat{\mathrm{y}} + \sum_{i=1}^{n}\sum_{j\in S_y} \widehat{\mathrm{x}}_i\widehat{\mathrm{y}}_j\mathbf{1}(i \neq j)e^{(B,\infty)}(\widehat{\mathrm{x}}, \widehat{g}^{([j],[i])}) \\ &\geq \alpha\widehat{\mathrm{x}}^T\widehat{\mathrm{y}} + \sum_{i=1}^{n}\sum_{j=1}^{n}\widehat{\mathrm{x}}_i\widehat{\mathrm{y}}_j\mathbf{1}(i \neq j)\mu \;= \mu + \widehat{\mathrm{x}}^T\widehat{\mathrm{y}}(\alpha - \mu). \end{aligned}$$

Since $\mathrm{support}(\widehat{\mathrm{x}}) = [1..n]$, we get $\gamma > 0$ and $\alpha - \mu \leq \varepsilon/\widehat{\mathrm{x}}^T\widehat{\mathrm{y}} \leq \varepsilon/\gamma$. Since this is true for all $\varepsilon > 0$, we get $\alpha \leq \mu$. Hence, $\alpha \leq e^{(B,\infty)}(\widehat{\mathrm{x}}, g) \leq \beta$ for every strategy g. ◀

▶ **Lemma 32.** *Let (A, B) be an RUC game. Let $\widehat{\mathrm{y}} \in \Delta_n$ such that* $\mathrm{support}(\widehat{\mathrm{y}}) = [1..n]$. *Let $\alpha := \min_{i=1}^{n}(A\widehat{\mathrm{y}})_i/\widehat{\mathrm{y}}_i$ and $\beta := \max_{i=1}^{n}(A\widehat{\mathrm{y}})_i/\widehat{\mathrm{y}}_i$. Then for any strategy f, we have $\alpha \leq e^{(A,\infty)}(f, \widehat{\mathrm{y}}) \leq \beta$.*

Proof. (Similar to the proof of Lemma 31.) ◀

Next, we identify sufficient conditions (as in Lemma 11 in Section 3) to make the upper and lower bounds in Lemmas 31 and 32 coincide, which gives us a Nash equilibrium.

▶ **Lemma 33.** *Consider the RUC game (A, B). Let (α, y^*) be an eigenpair for A such that $\|\mathrm{y}^*\|_1 = 1$ and $\mathrm{y}^*_i > 0$ for all $i \in [1..n]$. Let (β, x^*) be an eigenpair for B^T such that $\|\mathrm{x}^*\|_1 = 1$ and $\mathrm{x}^*_i > 0$ for all $i \in [1..n]$. Then $(\mathrm{x}^*, \mathrm{y}^*)$ is a Nash equilibrium. Moreover, for any strategies f and g, we have $e^{(A,\infty)}(f, \mathrm{y}^*) = \alpha$ and $e^{(B,\infty)}(\mathrm{x}^*, g) = \beta$.*

Proof. Follows from Lemmas 31 and 32. ◀

Next, we show that conditions of Lemma 33 can be satisfied using the Perron-Frobenius theorem (Theorem 12), so a Nash equilibrium given by stationary strategies always exists.

▶ **Theorem 34.** *Let (A, B) be an RUC game where A and B are irreducible (c.f. Definition 1). Let $(\rho_A, \rho_B, \mathrm{x}^*, \mathrm{y}^*) = \mathrm{perronSolution}(A, B)$. Then $(\mathrm{x}^*, \mathrm{y}^*)$ is a Nash equilibrium. Moreover, for any strategies f and g, we have $e^{(A,\infty)}(f, \mathrm{y}^*) = \rho_A$ and $e^{(B,\infty)}(\mathrm{x}^*, g) = \rho_B$.*

Proof. Follows from Lemma 33. ◀

Note that for any stationary strategies x and y, if (x, y) is a Nash equilibrium for the RUC game (A, B), then (x, y) is also a Nash equilibrium for the SRUC game (A, B). This is because if a player cannot improve her payoff by switching to a different strategy, then she also cannot improve her payoff by switching to a different stationary strategy. Hence, Theorem 14 is a corollary of Theorem 34.

Since computing Nash equilibrium exactly may be hard, we consider approximate Nash equilibria. To do this, we generalize the corresponding result about SRUC games (Theorem 24 in Section 3) to RUC games.

▶ **Theorem 35** (approximate Nash equilibrium)**.** *Let $(\mathrm{stat}(\mathrm{x}^*), \mathrm{stat}(\mathrm{y}^*))$ be a Nash equilibrium for the RUC game (A, B) such that $\mathrm{support}(\mathrm{x}^*) = \mathrm{support}(\mathrm{y}^*) = [1..n]$. For $\delta \in [0, 1)$, let $\widehat{\mathrm{x}}, \widehat{\mathrm{y}} \in \Delta_n$ such that for all $i \in [1..n]$, $|\widehat{\mathrm{x}}_i - \mathrm{x}^*_i| \le \delta \mathrm{x}^*_i$ and $|\widehat{\mathrm{y}}_i - \mathrm{y}^*_i| \le \delta \mathrm{y}^*_i$. Then $(\widehat{\mathrm{x}}, \widehat{\mathrm{y}})$ is a $\frac{4\delta}{(1-\delta)^2}$-approximate Nash equilibrium for the RUC game (A, B).*

Proof sketch. The proof is similar to Theorem 24. It follows from Lemmas 18, 23, 31, and 32. See Appendix C.2 of the full version [23] for the full proof. ◀

4.3 Uniqueness of Nash Equilibrium up to Equivalence

Before we investigate the uniqueness of Nash equilibrium for RUC games, we first describe a phenomenon where two different randomized strategies can behave similarly.

▶ **Definition 36** (collisions)**.** *For lists $I := [i_1, \dots, i_k]$ and $J := [j_1, \dots, j_k]$, let $\mathrm{collisions}(I, J)$ be the number of collisions if the max player and min player play actions I and J, respectively. Formally, $\mathrm{collisions}(I, J) := \sum_{t=1}^{k} \mathbf{1}(i_t = j_t)$.*

The pair (I, J) is called collision-consistent *if $i_t = j_t$ for all $t < k$.*

▶ **Definition 37.** *Two randomized strategies f_1 and f_2 are said to be* equivalent *if for every collision-consistent pair (I, J), we have $\Pr(\mathrm{isResp}(f_1, I, J)) = \Pr(\mathrm{isResp}(f_2, I, J))$.*

▶ **Example 38.** Let $n = 3$. For any $u, v \in \{1, 2\}$, let $f_{u,v}$ be the deterministic strategy where $f_{u,v}(J) = u$ if $J = [2]$, $f_{u,v}(J) = v$ if $J = [3]$, and $f_{u,v}(J) = 1$ otherwise.

For any $p \in [0, 1/2]$, let h_p be a randomized strategy where $\Pr(h_p = f_{1,1}) = \Pr(h_p = f_{2,2}) = p$ and $\Pr(h_p = f_{1,2}) = \Pr(h_p = f_{2,1}) = 1/2 - p$. Then for any lists I and J of the same length, $\Pr(\mathrm{isResp}(h_p, I, J))$ is not a function of p. Hence, h_0 and $h_{1/2}$ are equivalent.

We now show that given a pair of strategies, replacing each strategy by an equivalent strategy makes no difference to anyone's payoff.

▶ **Lemma 39.** *Let f_1 and f_2 be equivalent strategies of the max player and g_1 and g_2 be equivalent strategies of the min player. Then for the RUC game (A, B), we have $e^{(C,r)}(f_1, g_1) = e^{(C,r)}(f_2, g_2)$ for all $C \in \{A, B\}$ and $r \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$.*

Proof sketch. For any $r \in \mathbb{Z}_{\geq 0}$, the expected score in the first r rounds of a pair of strategies only depends on the distribution of collision-consistent pairs (I, J) where $|I| = |J| \leq r$. Hence, $e^{(C,r)}(f_1, g_1) = e^{(C,r)}(f_2, g_2)$. Take the limit $r \to \infty$ to get $e^{(C,\infty)}(f_1, g_1) = e^{(C,\infty)}(f_2, g_2)$. ◀

A corollary to Lemma 39 is that if (f_1, g_1) is a Nash equilibrium, then so is (f_2, g_2).

Next, we give a useful characterization of equivalence to stationary strategies.

▶ **Lemma 40.** *Let* $\mathrm{x} \in \Delta_n$. *A randomized strategy* f *is equivalent to* $\mathrm{stat}(\mathrm{x})$ *iff* $f^{(I,J)}(\emptyset) \sim \mathrm{x}$ *for every feasible history* (I, J) *of* f *where* $\mathrm{collisions}(I, J) = 0$.

Proof sketch. (See Appendix C.1 of the full version [23] for the full proof.)

$\Longleftarrow$: For each collision-consistent pair (I, J), where $I := [i_1, \ldots, i_k]$, we show that $\Pr(\mathrm{isResp}(f, I, J)) = \Pr(\mathrm{isResp}(\mathrm{stat}(\mathrm{x}), I, J)) = \prod_{t=1}^{k} \mathrm{x}_{i_t}$.

$\Longrightarrow$: $\forall i, j \in [1..n]$, $\Pr(f^{(I,J)}(\emptyset) = i) = \Pr(\mathrm{isResp}(f, I + [i], J + [j])) / \Pr(\mathrm{isResp}(f, I, J))$. Replace f by $\mathrm{stat}(\mathrm{x})$ (since they're equivalent) and simplify to get $\Pr(f^{(I,J)}(\emptyset) = i) = \mathrm{x}_i$. ◀

Indeed, for any stationary strategy, it's possible to construct a different equivalent strategy. This rules out uniqueness of Nash equilibrium. However, we can still hope to get uniqueness *up to equivalence*. We found this to be a very difficult problem, and could only resolve it for RUC games of the form (A, A) (i.e., zero-sum RUC games), where A is irreducible. For such games, we show that all Nash equilibria are equivalent.

▶ **Theorem 41.** *Let* ρ *be the Perron root of an irreducible matrix* A *and* x^* *and* y^* *be the left and right Perron vectors of* A*, respectively. Then for any Nash equilibrium* (f^*, g^*) *of the RUC game* (A, A)*,* f^* *is equivalent to* $\mathrm{stat}(\mathrm{x}^*)$ *and* g^* *is equivalent to* $\mathrm{stat}(\mathrm{y}^*)$.

Proof sketch. (See Appendix C.3 of the full version [23] for the full proof.)

We first use Proposition 7 to show that (f^*, y^*) and (x^*, g^*) are also NE. If $f^*(\emptyset)$ is not distributed as x^*, then in the NE (f^*, y^*), we show that the min player can decrease her cost by changing her first action's distribution. This contradicts the fact that (f^*, y^*) is an NE, and hence, proves that $f^*(\emptyset) \sim \mathrm{x}^*$. We similarly prove that $g^*(\emptyset) \sim \mathrm{y}^*$.

Next, we show that for any (I, J) such that $\mathrm{collisions}(I, J) = 0$, $({f^*}^{(I,J)}, {g^*}^{(J,I)})$ is also an NE, otherwise the min player can gain by deviating in the NE (f^*, y^*), or the max player can gain by deviating in the NE (x^*, g^*).

Combining the above results tells us that ${f^*}^{(I,J)}(\emptyset) \sim \mathrm{x}^*$ and ${g^*}^{(J,I)}(\emptyset) \sim \mathrm{y}^*$ for all (I, J) such that $\mathrm{collisions}(I, J) = 0$, which fulfills the condition of equivalence in Lemma 40. ◀

5 Discussion

In this work, we initiated the study of two-player RUC games: games that are repeated until collision. RUC games are related to other well-known repeated games, like pursuit-evasion games, hide-and-seek games, and stochastic games. They also generalize the popular game of hand cricket, and in Appendix B of the full version [23], we discuss its popular variants. Our main result showed the existence of Nash equilibria in RUC games when the players' payoff matrices are irreducible. We studied two other interesting properties: *stationarity* and *uniqueness*. We proved there always exists a Nash equilibrium where players use stationary strategies, and for zero-sum RUC games, all Nash equilibria are essentially equivalent, that is, they cannot be distinguished by observing the players' actions.

In Appendix D of the full version [23], we explore a variant of RUC games where instead of ending the game on the first collision, we end it on the w^{th} collision, for some $w \in \mathbb{Z}_{\geq 0}$.

Our work opens the way for several interesting questions. We can naturally generalize the definition of a collision to include a larger set of colliding actions, encoded via a collision matrix, and investigate the existence of Nash equilibria. Lastly, our result showing uniqueness up to equivalence of Nash equilibria applies only to zero-sum RUC games; showing (non-)uniqueness for general RUC games is another open question.

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