A Class of Rational Trace Relations Closed Under Composition

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Abstract

Rational relations on words form a well-studied and often applied notion. While the definition in trace monoids is immediate, they have not been studied in this more general context. A possible reason is that they do not share the main useful properties of rational relations on words. To overcome this unfortunate limitation, this paper proposes a restricted class of rational relations, investigates its properties, and applies the findings to systems equipped with a pushdown that does not hold a word but a trace.

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1 Introduction

Rational relations form a classical and well-studied concept (cf. [15, 14, 4, 17]) that embraces homomorphisms, inverse homomorphisms as well as substitutions. Rational relations appear in the study of automatic structures [1, 18, 32], rational Kripke frames [3], graph databases [2], the representation of infinite graphs and automata [19, 27, 35, 28, 8, 7], and natural language processing [20]. One particular application of rational relations can be found in the theory of pushdown systems: the reachability relation is prefix recognizable [10, 16] and therefore a rational relation which implies that forwards and backwards reachability preserve the regularity of a set of configurations ([6] provides an alternative proof for the backwards reachability).

Also the second theme of this paper has a long and diverse research history starting with Cartier and Foata's work in combinatorics [9] and Mazurkiewicz's ideas about the semantics of concurrent systems [26] that he modelled as equivalence classes of words, called traces today. Much of the work in computer science has concentrated on recognizable sets of traces, on model checking and synthesis problems, and on combinatorics, see [11] for a comprehensive presentation of the theory of traces; many of these results have been extended to more general concurrent systems like concurrent automata (cf., e.g. [13]), message passing automata [25], and other abstract models of distributed automata (e.g. [12, 5]).

Recently, Köcher and the current author considered a generalization of pushdown systems where the stack's contents is not a word, but a trace [23]; these systems were called *cooperating multi-pushdown systems* or cPDS. Our main results state that the forwards reachability relation preserves the rationality and the backwards reachability the recognizability of sets of configurations; these proofs are fundamentally different since the former analyses and simplifies runs of the cPDS while the latter generalises the construction by Bouajjani et al. [6]. While the reachability relation of a classical pushdown system is prefix recognizable, we also observed that this is not the case for cPDS. In addition, Köcher [22] inferred from the main result that the reachability relation is a rational trace relation, but it was not clear whether this rationality could be used to prove the preservation results as in the word case.

In this paper, we first observe that rational trace relations do not compose which leads to the definition of *left-closed rational* (called *lc-rational* from now on) trace relations. The main body of this paper studies their properties, in particular Section 4 shows that the class of lc-rational trace relations is closed under composition, preserves rationality under right- and recognizability under left-application, and is closed under certain concatenations. In Section 5, these properties are used to show that the reachability relation of a cPDS is lc-rational. This allows a uniform proof of the preservation results from [23]. Section 6 returns to the theory of lc-rational relations and characterises them in the spirit of Nivat's theorem for rational word relations; this section also demonstrates that the lc-rationality of a rational relation is not even semi-decidable. In the final Section 7, we return to the consideration of rational trace relations: we characterise them as compositions of the inverse of an lc-rational relation with an lc-rational relation, we show that their left- and right-application transforms recognizable sets into rational ones, and finally provide the class of *closed-rational trace relations* that enjoys all nice properties of rational word relations: closure under composition, preservation of rationality and recognizability under left- and right-application, invertability, and a Nivat-style characterisation.

2 Preliminaries

A *dependence alphabet* is a pair (Σ, D) where Σ is some finite alphabet and $D \subseteq \Sigma \times \Sigma$ is a symmetric and irreflexive relation on Σ, called the *dependence relation. We fix a dependence alphabet* (Σ, D) *throughout this paper.* The *independence relation* I is the irreflexive part of its complement, i.e., $I = \{(a,b) \in \Sigma^2 \mid (a,b) \notin D, a \neq b\}$. For two words $u, v \in \Sigma^*$, we write $(u,v) \in I$ or $u\ I\ v$ if, $(a,b) \in I$ for any two letters a and b occurring in u and v, resp..

Let $\sim\ \subseteq \Sigma^* \times \Sigma^*$ denote the least congruence on the free monoid Σ^* satisfying $ab \sim ba$ for all $(a,b) \in I$. We therefore have $u \sim v$ iff u can be transformed into v by exchanging the order of consecutive independent letters. Then the quotient $\mathbb{M}(\Sigma, D) = \Sigma^*/{\sim}$ is a monoid with multiplication $[u] \cdot [v] = [uv]$ where $[u]$ denotes the equivalence class of the word u with respect to the congruence $\sim$. Usually, we denote the *trace monoid* $\mathbb{M}(\Sigma, D)$ by $\mathbb{M}$. The mapping $\eta\colon \Sigma^* \to \mathbb{M}\colon u \mapsto [u]$ is a monoid homomorphism (thus, $[u]$ and $\eta(u)$ are synonyms).

Let M be a monoid. A set $L \subseteq M$ is *rational* if it can be constructed from finite subsets of M using the operations union, multiplication, and Kleene star (i.e., generated submonoid). Rational subsets of a finitely generated free monoid are called *regular languages*; by Kleene's theorem, they coincide with the languages accepted by deterministic finite automata [21].

Note that also $M \times M$ is a monoid (with componentwise multiplication). Rational subsets of this monoid are called *rational relations on* M. For $M = \Sigma^*$, we will speak of rational word relations, for $M = \mathbb{M}$, of rational trace relations.

Let $\varphi\colon M \to N$ be a monoid homomorphism into the monoid N. Then we have the following (cf. [31, Prop. 6.2] and [34, Prop. II.1.17]):

(i) If $K \subseteq M$ is rational, then $\varphi(K) \subseteq N$ is rational.

(ii) If $L \subseteq N$ is rational and φ is surjective, then there exists a rational set $K \subseteq M$ with $\varphi(K) = L$.

In particular, assuming φ to be surjective, we have that $L \subseteq N$ is rational if, and only if, it is the φ-image of some rational subset of M. In this paper, we will need the following two special cases of this characterisation of rational subsets of N.

(i) The homomorphism $\eta\colon \Sigma^* \to \mathbb{M}$ is surjective. Hence a set $\mathcal{L} \subseteq \mathbb{M}$ of traces is rational if, and only if, there exists a regular language $L \subseteq \Sigma^*$ with $\eta(L) = \mathcal{L}$.

(ii) The mapping $\eta_2\colon \Sigma^* \times \Sigma^* \to \mathbb{M} \times \mathbb{M}\colon (u,v) \mapsto ([u],[v])$ is a surjective homomorphism. Hence a trace relation $\mathcal{R} \subseteq \mathbb{M} \times \mathbb{M}$ is rational if, and only if, there exists a rational word relation $R \subseteq \Sigma^* \times \Sigma^*$ with $\eta_2(R) = \mathcal{R}$.
From now on, we will abuse notation and write $\eta(R)$ for $\eta_2(R)$.

Let, again, M be a monoid. A set $L \subseteq M$ is *recognizable* if there exists a homomorphism $\psi\colon M \to S$ into some finite monoid S such that $L = \psi^{-1}\psi(L)$.[1] If M is a finitely generated free monoid, then L is recognizable iff L is regular (cf. [34, Thm. II.2.1]).

Let $\varphi\colon M \to N$ be a surjective homomorphism into a monoid N, and $L \subseteq N$. Then L is recognizable if, and only if, $\varphi^{-1}(L) \subseteq M$ is recognizable (cf. [31, Prop. 6.3] for the implication "⇒" and [34, Cor. II.2.2 and II.2.12] for the equivalence). In this paper, we will need the following special case. The homomorphism $\eta\colon \Sigma^* \to \mathbb{M}$ is surjective. Therefore, a set $\mathcal{L} \subseteq \mathbb{M}$ of traces is recognizable if, and only if, the language $\eta^{-1}(\mathcal{L}) \subseteq \Sigma^*$ is regular.

Let M be a monoid, $L \subseteq M$ a subset of M, and $R \subseteq M \times M$ a binary relation on M. Then we set

$$L^R = \{y \in M \mid \exists x \in L\colon (x,y) \in R\} \text{ and } {}^R L = \{x \in M \mid \exists y \in L\colon (x,y) \in R\}\,.$$

If R is (the graph of) a function $f\colon M \to M$, then L^R is the image of L under f, i.e., $L^R = \{f(x) \mid x \in L\}$, and ${}^R L$ is the preimage of L under f, i.e., ${}^R L = \{x \in M \mid f(x) \in L\}$. Often, authors write $L\,R$ for L^R and $R\,L$ for ${}^R L$; I prefer this notation as it stresses the different roles played by the set L and the relation R.

The mapping $2^M \to 2^M\colon L \mapsto L^R$ is the *right-application of* R while the mapping $2^M \to 2^M\colon L \mapsto {}^R L$ is the *left-application of* R. Note that for $R^{-1} = \{(y,x) \mid (x,y) \in R\}$, we have ${}^R L = L^{R^{-1}}$ and, since $(R^{-1})^{-1} = R$, also ${}^{R^{-1}} L = L^R$.

For two sets $K, L \subseteq M$, let $K \leq_{\text{rat}} L$ if there exists a rational relation $R \subseteq M \times M$ such that $K = L^R$. Since the relation $\mathrm{Id}_{\Sigma^*} = \{(u,u) \mid u \in \Sigma^*\}$ is rational, the relation $\leq_{\text{rat}}$ is reflexive.

In this paper, we will regularly consider subsets of and binary relations on Σ^* and $\mathbb{M}$, resp. I hope to simplify understanding by using the following conventions:

- Subsets of Σ^* are denoted by plain capital letters K and L; binary relations on Σ^* are similarly denoted R, R_1, and R_2.
- Subsets of $\mathbb{M}$ are denoted by curly capital letters $\mathcal{K}$ and $\mathcal{L}$; binary relations on $\mathbb{M}$ are similarly denoted $\mathcal{R}$, $\mathcal{R}_1$, and $\mathcal{R}_2$.

3 Rational and left-closed word relations

We first recall four properties of rational word relations. These properties form the basis of many applications in proofs that use rational relations.

▶ **Theorem 3.1.** *Let $R, R_1, R_2 \subseteq \Sigma^* \times \Sigma^*$ be rational. Then*

(R1) *Left- and right-application of rational relations preserve regularity: if $L \subseteq \Sigma^*$ is regular, then L^R and ${}^R L$ are regular [4, Cor. III.4.2].*

[1] In [34], Sakarovitch gives another definition of "recognizability", but he also shows that his definition is equivalent to the one used in this paper [34, Prop. II.2.1].

(R2) *The composition of rational relations*

$$R_1 \circ R_2 := \{(u, w) \in \Sigma^* \times \Sigma^* \mid \exists v \in \Sigma^* \colon u \; R_1 \; v \; R_2 \; w\}$$

is rational [4, Thm. III.4.4]. It follows that the relation $\leq_{\mathrm{rat}}$ is transitive.

(R3) *The inverse of a rational relation $R^{-1} := \{(v, u) \mid (u, v) \in R\}$ is rational.*

(R4) *A relation $R' \subseteq \Sigma^* \times \Sigma^*$ is rational if, and only if, there exists an alphabet Γ, a regular language $K \subseteq \Gamma^*$, and homomorphisms $f, g \colon \Gamma^* \to \Sigma^*$ such that $R' = \{(f(x), g(x)) \mid x \in K\}$ [30], [4, Thm. III.3.2] ("Nivat's theorem").*

By the very definition, property (R3) holds for rational trace relations $\mathcal{R} \subseteq \mathbb{M}^2$. Also property (R4) holds for any monoid M in place of Σ^* and therefore in particular for relations on the trace monoid $\mathbb{M}$ [4, Prop. III.3.4].

The following example shows that (R2) fails for rational trace relations.

▶ **Example 3.2.** Suppose there are $a, b, c, d \in \Sigma$ with $(a, b) \in D$ and $(c, d) \in I$. Consider the rational word relations $R_1 = \{(a^m b^n, c^m d^n) \mid m, n \geq 1\} = \{(a, c)\}^+ \cdot \{(b, d)\}^+$ and $R_2 = \{(d^n c^m, b^n a^m) \mid m, n \geq 1\}$. Since $(c, d) \in I$, we get $d^n c^m \sim c^m d^n$ for any $m, n \geq 0$ and therefore

$$\begin{aligned}
\mathcal{R}_1 &:= \eta(R_1) = \{([a^m b^n], [c^m d^n]) \mid m, n \geq 1\} \text{ and} \\
\mathcal{R}_2 &:= \eta(R_2) = \{([d^n c^m], [b^n a^m]) \mid m, n \geq 1\} \\
& = \{([c^m d^n], [b^n a^m]) \mid m, n \geq 1\}\,.
\end{aligned}$$

Hence $\mathcal{R} := \mathcal{R}_1 \circ \mathcal{R}_2 = \{([a^m b^n], [b^n a^m]) \mid m, n \geq 1\}$. Suppose the trace relation $\mathcal{R}$ is rational. Then there exists a rational word relation R with $\eta(R) = \mathcal{R}$. Since $(a, b) \in D$, $a^m b^n$ is the only word w with $\eta(w) = [a^m b^n]$ (similarly, $[b^n a^m] = \{b^n a^m\}$). Hence $\eta(R) = \mathcal{R}$ implies $R = \{(a^m b^n, b^n a^m) \mid m, n \geq 1\}$. But this relation is not rational which can easily be shown using Theorem 3.1 and pumping in the language K provided there. Hence the composition $\mathcal{R}$ of the two rational trace relations $\mathcal{R}_1$ and $\mathcal{R}_2$ is not rational.

▶ Remark. In a finitely generated free monoid Σ^*, a set is rational if, and only if, it is recognizable. In the trace monoid, any recognizable set is rational, but the converse implication does not hold, i.e., rationality and recognizability do not coincide (e.g., $\mathcal{L} = \{[ab]\}^*$ with $(a, b) \in I$ is rational, but not recognizable since $\eta^{-1}(\mathcal{L}) = \{u \in \{a, b\}^* \colon |u|_a = |u|_b\}$). Hence, there are two possible versions of (R1) for the trace monoid; later, we will see that none of them holds (Lemma 4.9).

We aim at a class $\mathbb{C}_{\mathbb{M}}$ of rational trace relations that is closed under composition. Recall that a trace relation $\mathcal{R}$ is rational if and only if there exists a rational word relation R with $\mathcal{R} = \eta(R)$. We will therefore first define a class $\mathbb{C}_{\Sigma^*}$ of rational word relations R that is closed under composition and satisfies

$$\eta(R_1 \circ R_2) = \eta(R_1) \circ \eta(R_2) \tag{1}$$

for any relations R_1 and R_2 in $\mathbb{C}_{\Sigma^*}$ (setting $\mathbb{C}_{\mathbb{M}} = \{\eta(R) \mid R \in \mathbb{C}_{\Sigma^*}\}$ will then ensure that $\mathbb{C}_{\mathbb{M}}$ is closed under composition, cf. Definition 4.1).

Note that Eq. (1) fails in Example 3.2 since there, $R_1 \circ R_2 = \emptyset$ and $\mathcal{R}_1 \circ \mathcal{R}_2 \neq \emptyset$. The reason is that there are words $t, u, u', v' \in \Sigma^*$ such that $(t, u) \in R_1$, $(u', v') \in R_2$, and $u \sim u'$ distinct such that $([t], [v']) \in \mathcal{R}_1 \circ \mathcal{R}_2$, but $(t, v') \notin R_1 \circ R_2$. The following definition circumvents this problem.

▶ **Definition 3.3.** *A relation $R \subseteq \Sigma^* \times \Sigma^*$ is* left-closed *if $\sim \circ R \subseteq R \circ \sim$, i.e.,*

$$\big(\exists u' \in \Sigma^* : u \sim u' \; R \; v'\big) \implies \big(\exists v \in \Sigma^* : u \; R \; v \sim v'\big)$$

holds for all $u, v \in \Sigma^*$. The relation R is lc-rational *if it is left-closed and rational.*

A very simple example is the identity relation $\mathrm{Id}_{\Sigma^*} = \{(u,u) \mid u \in \Sigma^*\}$: if $u \sim u' \; \mathrm{Id}_{\Sigma^*} \; v'$, then $u' = v'$. Setting $v = u$, we obtain $u \; \mathrm{Id}_{\Sigma^*} \; v = u \sim u' = v'$. Since this relation is clearly rational, it is indeed lc-rational. Other examples are $\Sigma^* \times \{\varepsilon\}$ and $\{\varepsilon\} \times \Sigma^*$.

▶ **Example 3.4.** Another, more demanding example, is the superword-relation: $u \in \Sigma^*$ is a *superword* of $v \in \Sigma^*$ if $u = u_1 v_1 u_2 v_2 \cdots u_n v_n u_{n+1}$ and $v = v_1 v_2 \cdots v_n$ for some $n \in \mathbb{N}$ and $u_1, u_2, \ldots, u_{n+1}, v_1, v_2, \ldots, v_n \in \Sigma^*$. In this case, we write $u \succeq v$. The superword-relation is rational since $\succeq \, = \{(a,a), (a,\varepsilon) \mid a \in \Sigma\}^*$.

Now suppose $u = x\,ab\,y$ with $(a,b) \in I$ and $x\,ba\,y = u' \succeq v'$. If $v' = \varepsilon$, we get $u \succeq \varepsilon \sim v'$. So suppose $v' \neq \varepsilon$. Then there are $n \in \mathbb{N}$, $u_1 \in \Sigma^*$, $u_2, \ldots, u_n, v_1, v_2, \ldots, v_n \in \Sigma^+$, and $u_{n+1} \in \Sigma^*$ such that $v_1 v_2 \cdots v_n = v'$ and $u_1 v_1 u_2 v_2 \cdots u_n v_n u_{n+1} = u' = x\,ab\,y$. This gives two decompositions of the word u' into the blocks u_i and v_i on the one hand, and into the factors x, ba, and y on the other hand. The factor ba from the second factorization can be covered by some factor u_i, by some factor v_i, or it belongs to some consecutive factors u_i and v_i or v_i and u_{i+1} of the former factorisation – in any of these cases, we construct a word v with $u \succeq v \sim v'$ (see appendix). Since $\sim$ is the least equivalence relation identifying $xaby$ with $xbay$ for $(a,b) \in I$, this proves that the superword-relation $\succeq$ is left-closed.

The subword-relation $\preceq$ is the inverse of the superword-relation. Suppose $(a,c) \in I$ and $(a,b), (b,c) \in D$. Then $ca \sim ac \preceq abc$, but there is no superword of ca that is equivalent to abc. Hence the subword-relation is not left-closed.

We next show that the class of lc-rational word relations has the properties desired: it is closed under composition and the homomorphism η commutes with composition.

▶ **Proposition 3.5.** *Let $R_1, R_2 \subseteq \Sigma^* \times \Sigma^*$.*

(i) *If R_2 is left-closed, then Eq.* (1) *holds, i.e., $\eta(R_1 \circ R_2) = \eta(R_1) \circ \eta(R_2)$.*

(ii) *If R_1 and R_2 are lc-rational, then also $R_1 \circ R_2$ is lc-rational.*

Proof. To demonstrate the first claim, let R_2 be lc-rational. For the inclusion $\eta(R_1 \circ R_2) \subseteq \eta(R_1) \circ \eta(R_2)$, let $(u,w) \in R_1 \circ R_2$. Then there exists $v \in \Sigma^*$ with $u \; R_1 \; v \; R_2 \; w$ and therefore $[u] \; \eta(R_1) \; [v] \; \eta(R_2) \; [w]$ implying $([u],[w]) \in \eta(R_1) \circ \eta(R_2)$.

For the converse inclusion, let $(x,z) \in \eta(R_1) \circ \eta(R_2)$. There is some trace y with $x \; \eta(R_1) \; y \; \eta(R_2) \; z$. Hence there are words u, v, v', w with

- $x = [u]$, $y = [v]$, and $(u,v) \in R_1$ and
- $y = [v']$, $z = [w]$, and $(v',w) \in R_2$.

Hence we have $u \; R_1 \; v \sim v' \; R_2 \; w$. Since R_2 is left-closed, there exists a word $w' \in \Sigma^*$ such that $u \; R_1 \; v \; R_2 \; w' \sim w$. Hence we have $(x,z) = ([u],[w]) = ([u],[w']) \in \eta(R_1 \circ R_2)$. This finishes the verification of the first claim.

Now, assume both relations R_1 and R_2 to be left-closed such that $\sim \circ R_i \subseteq R_i \circ \sim$ holds for all $i \in [2]$. Consequently, we get $\sim \circ R_1 \circ R_2 \subseteq R_1 \circ \sim \circ R_2 \subseteq R_1 \circ R_2 \circ \sim$. Hence, indeed, $R_1 \circ R_2$ is left-closed such that the second claim follows using Theorem 3.1(R2). ◀

The following proposition characterises the lc-rational word relations of the form $K \times L$ for languages $K, L \subseteq \Sigma^*$. This characterisation should also explain the name "left-closed".

▶ **Proposition 3.6.** *Let $K, L \subseteq \Sigma^*$ be non-empty.*
(i) *$K \times L$ is rational iff K and L both are regular.*
(ii) *$K \times L$ is left-closed iff K is* closed*, i.e., $u \sim u' \in K$ ensures $u \in K$.*
Consequently, $K \times L$ is lc-rational if, and only if, K and L are regular and K is closed.

Proof. The first claim is well-known, we present a proof in the appendix.

Suppose $K \times L$ to be left-closed and let $u \sim u' \in K$. Since L is non-empty, there exists $v' \in L$ implying $u \sim u' \ (K \times L) \ v'$. Since $K \times L$ is assumed to be left-closed, there exists $v \in \Sigma^*$ with $u \ (K \times L) \ v \sim v'$. This implies in particular $u \in K$. Hence, K is closed.

Conversely, suppose K to be closed and let $u \sim u' \ (K \times L) \ v'$. Then $u \sim u' \in K$ implying $u \in K$ such that (with $v = v'$) we get $u \ (K \times L) \ v \sim v'$, i.e., $K \times L$ is left-closed. ◀

4 The theory of lc-rational trace relations I

4.1 Definition, examples, composition

Recall that a trace relation $\mathcal{R} \subseteq \mathbb{M} \times \mathbb{M}$ is rational iff there exists a rational word relation $R \subseteq \Sigma^* \times \Sigma^*$ with $\eta(R) = \mathcal{R}$. Similarly, we now lift the concept of lc-rational relations from words to traces.

▶ **Definition 4.1.** *A relation $\mathcal{R} \subseteq \mathbb{M} \times \mathbb{M}$ is* lc-rational *if there exists some lc-rational word relation $R \subseteq \Sigma^* \times \Sigma^*$ with $\mathcal{R} = \eta(R)$.*

Simple examples are $\mathbb{M} \times \{[\varepsilon]\}$ and $\{[\varepsilon]\} \times \mathbb{M}$ since $\Sigma^* \times \{\varepsilon\}$ and $\{\varepsilon\} \times \Sigma^*$ are lc-rational word relations.

▶ **Example 4.2.** Another, more involved example, is the supertrace-relation [24]: $x \in \mathbb{M}$ is a *supertrace* of $y \in \mathbb{M}$ if $x = x_1 y_1 \cdots x_n y_n x_{n+1}$ and $y = y_1 y_2 \cdots y_n$ for some $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_{n+1}, y_1, y_2, \ldots, y_n \in \mathbb{M}$. In this case, we write $x \sqsupseteq y$. It is easily checked that $x \sqsupseteq y$ if, and only if, there are words u and v such that $x = [u]$, $y = [v]$, and $u \succeq v$, i.e., $\sqsupseteq = \eta(\succeq)$. Since the superword-relation $\succeq$ is lc-rational by Example 3.4, we obtain that the supertrace-relation $\sqsupseteq$ is lc-rational.

Also the identity relation $\mathrm{Id}_{\mathbb{M}} = \{(x, x) \mid x \in \mathbb{M}\}$ is lc-rational since Id_{Σ^*} is an lc-rational word relation. Note that $\mathrm{Id}_{\mathbb{M}}$ can be seen as a homomorphism from $\mathbb{M}$ to $\mathbb{M}$. We now generalize this example and show that every homomorphism is an lc-rational relation.

▶ **Lemma 4.3.** *Let $\varphi \colon \mathbb{M} \to \mathbb{M}$ be a homomorphism. Then $\mathcal{R} = \{(x, \varphi(x)) \mid x \in \mathbb{M}\}$ is lc-rational.*

Proof. Choose, for each $a \in \Sigma$, a word $w_a \in \Sigma^*$ with $\varphi([a]) = [w_a]$. Let $h \colon \Sigma^* \to \Sigma^*$ be the homomorphism with $h(a) = w_a$ for all $a \in \Sigma^*$. Then $\varphi([v]) = [h(v)]$ for all words $v \in \Sigma^*$.

Now consider h as a relation, i.e., set $R = \{(u, h(u)) \mid u \in \Sigma^*\}$. Then $\eta(R) = \mathcal{R}$ and $R = \{(a, w_a) \mid a \in \Sigma\}^*$ implying that the relation R is rational.

We next verify that R is left-closed. So let $u, u', v' \in \Sigma^*$ with $u \sim u' \ R \ v'$. From $u' \ R \ v'$, we obtain $v' = h(u')$. With $v := h(u)$, we obtain

$$[v'] = \varphi([u']) = \varphi([u]) = [h(u)] = [v]$$

implying $v' \sim v$. Hence we have $u \ R \ v \sim v'$ implying that R is left-closed. ◀

Later, we will characterize the lc-rational relations among the direct products $\mathcal{K} \times \mathcal{L}$ of sets of traces $\mathcal{K}$ and $\mathcal{L}$ (we have done so for word relations in Proposition 3.6). But first, we do this for the special case that one of the sets $\mathcal{K}$ and $\mathcal{L}$ equals $\{[\varepsilon]\}$.

▶ **Lemma 4.4.** *Let $\mathcal{K} \subseteq \mathbb{M}$ be a set of trace.*
(i) *$\mathcal{K} \times \{[\varepsilon]\}$ is lc-rational if, and only if, $\mathcal{K}$ is recognizable.*
(ii) *$\{[\varepsilon]\} \times \mathcal{K}$ is lc-rational if, and only if, $\mathcal{K}$ is rational*

Proof. First, suppose $\mathcal{K} \times \{[\varepsilon]\}$ is lc-rational. Then there exists an lc-rational word relation $R \subseteq \Sigma^* \times \Sigma^*$ with $\eta(R) = \mathcal{K} \times \{[\varepsilon]\}$. Since ε is the only word w with $\eta(w) = [\varepsilon]$, there exists a language $K \subseteq \Sigma^*$ with $R = K \times \{\varepsilon\}$. Now Proposition 3.6 implies that K is closed and regular. From $\eta(K \times \{\varepsilon\}) = \eta(R) = \mathcal{K} \times \{[\varepsilon]\}$, we obtain $\eta(K) = \mathcal{K}$ and therefore $K \subseteq \eta^{-1}(\mathcal{K})$. For the converse inclusion, let $u \in \eta^{-1}(\mathcal{K})$ implying $[u] \in \mathcal{K}$. From $\eta(K) = \mathcal{K}$, we obtain some word $u' \in K$ with $[u'] = [u]$ and therefore $u \sim u' \in K$. Since K is closed, this implies $u \in K$. Hence, we showed $K = \eta^{-1}(\mathcal{K})$. Since K is regular, the set $\mathcal{K}$ is indeed recognizable.

Next, suppose $\mathcal{K}$ is recognizable. Then $K := \eta^{-1}(\mathcal{K})$ is regular. It is easily seen that K is closed such that, again by Proposition 3.6, the relation $R := K \times \{\varepsilon\}$ is lc-rational. But $\eta(R) = \mathcal{K} \times \{[\varepsilon]\}$, hence this direct product is lc-rational as well. This completes the proof of the first claim.

The second claim is shown similarly, one need not show $K = \eta^{-1}(\mathcal{K})$, but only $\eta(K) = \mathcal{K}$ as we have done above. ◀

By Example 3.2, the composition of rational trace relations is, in general, not rational. The following lemma demonstrates that the composition is rational provided the second relation is lc-rational. If both relations are lc-rational, then the composition is so as well.

▶ **Proposition 4.5.** *Let $\mathcal{R}_1, \mathcal{R}_2 \subseteq \mathbb{M}^2$ be rational trace relations.*
(i) *If $\mathcal{R}_2$ is lc-rational, then $\mathcal{R}_1 \circ \mathcal{R}_2$ is rational.*
(ii) *If $\mathcal{R}_1$ and $\mathcal{R}_2$ both are lc-rational, then $\mathcal{R}_1 \circ \mathcal{R}_2$ is even lc-rational.*

Proof. There exists a rational word relation $R_1 \subseteq \Sigma^* \times \Sigma^*$ with $\eta(R_1) = \mathcal{R}_1$, and there exists an lc-rational word relation $R_2 \subseteq \Sigma^* \times \Sigma^*$ with $\eta(R_2) = \mathcal{R}_2$. By Theorem 3.1, the composition $R_1 \circ R_2$ is rational and satisfies $\eta(R_1 \circ R_2) = \eta(R_1) \circ \eta(R_2)$ by Proposition 3.5(i). Hence $\mathcal{R}_1 \circ \mathcal{R}_2$ is the η-image of the rational word relation $R_1 \circ R_2$ and therefore rational.

If, in addition, $\mathcal{R}_1$ is also lc-rational, then the relation R_1 can be assumed to be lc-rational. Hence, by Proposition 3.5(ii), the composition $R_1 \circ R_2$ is even lc-rational. Hence $\mathcal{R}_1 \circ \mathcal{R}_2$ is the η-image of lc-rational word relation $R_1 \circ R_2$ and therefore lc-rational. ◀

Proposition 3.6 characterises the lc-rational word relations of the form $K \times L$. We can now lift this result to the trace setting.

▶ **Proposition 4.6.** *Let $\mathcal{K}, \mathcal{L} \subseteq \mathbb{M}$ be non-empty.*
(i) *$\mathcal{K} \times \mathcal{L}$ is rational iff $\mathcal{K}$ and $\mathcal{L}$ both are rational.*
(ii) *$\mathcal{K} \times \mathcal{L}$ is lc-rational iff $\mathcal{K}$ is recognizable and $\mathcal{L}$ is rational.*

Proof. The first claim can be shown as the first claim of Proposition 3.6 (see appendix).

Next, let $\mathcal{K}$ be recognizable and $\mathcal{L}$ rational. Then the trace relations $\mathcal{K} \times \{[\varepsilon]\}$ and $\{[\varepsilon]\} \times \mathcal{L}$ are lc-rational by Lemma 4.4. Note that

$$\mathcal{K} \times \mathcal{L} = (\mathcal{K} \times \{[\varepsilon]\}) \circ (\{[\varepsilon]\} \times \mathcal{L})\,.$$

Hence the relation $\mathcal{K} \times \mathcal{L}$ is the composition of two lc-rational relations. Now Proposition 4.5(ii) implies that $\mathcal{K} \times \mathcal{L}$ is lc-rational as well.

Conversely, suppose $\mathcal{K}\times\mathcal{L}$ to be lc-rational. Since $\mathbb{M}$ is recognizable, the relation $\mathbb{M}\times\{[\varepsilon]\}$ is lc-rational by Lemma 4.4. Since

$$(\mathcal{K}\times\mathcal{L})\circ(\mathbb{M}\times\{[\varepsilon]\})=\mathcal{K}\times\{[\varepsilon]\})\,,$$

the relation $\mathcal{K}\times\{[\varepsilon]\}$ is lc-rational by Proposition 4.5. Now $\mathcal{K}$ is recognizable by Lemma 4.4(i).

For the rationality of $\mathcal{L}$, we can argue similarly (using that $\mathbb{M}$ is rational this time). ◀

4.2 Preservation of language properties

Recall that rational word relations preserve the regularity of languages under left- and right-application. Since rationality and recognizability are different notions in the trace monoid, this leads to two possible generalisations; later, Lemma 4.9 below will show that none of them holds for rational trace relations.

Now restrict attention to lc-rational trace relations $\mathcal{R}$. Since $\mathcal{R}^{-1}$ need not be lc-rational, we now get four possible preservation results: we could consider rationality or recognizability as well as left- or right-application. The following lemma shows that two of them hold, Lemma 4.9 proves that the other two fail.

▶ **Theorem 4.7.** *Let $\mathcal{R}\subseteq\mathbb{M}^2$ be an lc-rational trace relation.*

(i) *If $\mathcal{L}\subseteq\mathbb{M}$ is recognizable, then also ${}^{\mathcal{R}}\mathcal{L}$ is recognizable.*

(ii) *If $\mathcal{L}\subseteq\mathbb{M}$ is rational, then also $\mathcal{L}^{\mathcal{R}}$ is rational.*

Proof. Let $\mathcal{L}_1$ be recognizable and $\mathcal{L}_2$ rational. Then $\mathcal{L}_1\times\{[\varepsilon]\}$ and $\{[\varepsilon]\}\times\mathcal{L}_2$ are lc-rational by Lemma 4.4 and we have

$${}^{\mathcal{R}}\mathcal{L}_1\times\{[\varepsilon]\}=\mathcal{R}\circ(\mathcal{L}_1\times\{[\varepsilon]\})\text{ and }\{[\varepsilon]\}\times\mathcal{L}_2^{\mathcal{R}}=(\{[\varepsilon]\}\times\mathcal{L}_2)\circ\mathcal{R}\,.$$

Hence ${}^{\mathcal{R}}\mathcal{L}_1\times\{[\varepsilon]\}$ and $\{[\varepsilon]\}\times\mathcal{L}_2^{\mathcal{R}}$ are lc-rational relations by Proposition 4.5. Using Lemma 4.4 again, we obtain that ${}^{\mathcal{R}}\mathcal{L}_1$ is recognizable and $\mathcal{L}_2^{\mathcal{R}}$ is rational. ◀

Define the relation $\leq_{\mathrm{lc-rat}}$ in the same way as the relation $\leq_{\mathrm{rat}}$, but with the restriction to lc-rational relations: $\mathcal{K}\leq_{\mathrm{lc-rat}}\mathcal{L}$ if there exists an lc-rational relation $\mathcal{R}$ such that $\mathcal{K}=\mathcal{L}^{\mathcal{R}}$. Then Proposition 4.5 trivially implies the following.

▶ **Corollary 4.8.** *The relation $\leq_{\mathrm{lc-rat}}$ is transitive.*

We now come to the announced non-preservation results; they hold for lc-rational trace relations and therefore, in particular, for the larger class of rational trace relations.

▶ **Lemma 4.9.** *Suppose there are $a,b,c,d\in\Sigma$ with $(a,b)\in D$ and $(c,d)\in I$.*

There exists an lc-rational relation $\mathcal{R}\subseteq\mathbb{M}^2$, a recognizable set $\mathcal{K}\subseteq\mathbb{M}$, and a rational set $\mathcal{L}\subseteq\mathbb{M}$ such that ${}^{\mathcal{R}}\mathcal{K}$ is not rational and $\mathcal{L}^{\mathcal{R}}$ is not recognizable.

Proof. Let $R=\{(a,c),(b,d)\}^*$. Then R is rational and, since $(a,b)\in D$, even lc-rational. We consider the lc-rational trace relation $\mathcal{R}=\eta(R)$. Since $(c,d)\in I$, we obtain $([u],[v])\in\mathcal{R}$ if, and only if, $u\in\{a,b\}^*$, $v\in\{c,d\}^*$, $|u|_a=|v|_c$, and $|u|_b=|v|_d$.

Consider the regular language $K=\{cd\}^*$ and let $\mathcal{K}$ denote the rational set $\eta(K)$. Since $(c,d)\in I$, we get $[v]\in\mathcal{K}$ iff $v\in\{c,d\}^*$ and $|v|_c=|v|_d$. It follows that $[u]\in{}^{\mathcal{R}}\mathcal{K}$ iff $u\in\{a,b\}^*$ and $|u|_a=|u|_b$. Let $H\subseteq\Sigma^*$ denote the set of words $u\in\{a,b\}^*$ with $|u|_a=|u|_b$. Since $(a,b)\in D$, this language H is the only language with $\eta(H)={}^{\mathcal{R}}\mathcal{K}$. Since H is not regular, it follows that ${}^{\mathcal{R}}\mathcal{K}$ is not rational which proves the first claim.

Next, let $\mathcal{L}=\{[ab]\}^*$. Then $[u]\in\mathcal{L}$ iff $u\in\{ab\}^*$ since $(a,b)\in D$. Hence $\eta^{-1}(\mathcal{L})$ is the regular language $L=\{ab\}^*$, i.e., $\mathcal{L}$ is recognizable. Note that $\mathcal{L}^{\mathcal{R}}$ is the set of traces $[v]$ with $v\in\{c,d\}^*$ and $|v|_c=|v|_d$ (i.e., it equals $\mathcal{K}$). Hence the preimage of $\mathcal{L}^{\mathcal{R}}$ under η is not regular, i.e., $\mathcal{L}^{\mathcal{R}}$ is not recognizable. ◀

4.3 Products of lc-rational relations

By the very definition, the (componentwise) product of two rational relations is rational again: if $\mathcal{R}_1, \mathcal{R}_2 \subseteq \mathbb{M}^2$ are rational, then so is the relation

$$\mathcal{R}_1 \cdot \mathcal{R}_2 = \{(xx', yy') \mid (x, y) \in \mathcal{R}_1, (x', y') \in \mathcal{R}_2\}\,.$$

Lemma A.1 in the appendix demonstrates that this does not hold for lc-rational relations. We now come to two special cases of relations $\mathcal{R}_1$ that ensure the lc-rationality of $\mathcal{R}_1 \cdot \mathcal{R}_2$:

▶ **Lemma 4.10.** *Let $\mathcal{K} \subseteq \mathbb{M}$ be recognizable. Then the relation $\mathcal{R} = (\mathcal{K} \times \{[\varepsilon]\}) \cdot \mathrm{Id}_{\mathbb{M}} = \{(xy, y) \mid x \in \mathcal{K}, y \in \mathbb{M}\}$ is lc-rational.*

Proof. The language $K := \eta^{-1}(\mathcal{K})$ is regular. Let R denote the set of pairs of words

$$(u_1 v_1 u_2 v_2 \cdots u_n v_n, v_1 v_2 \cdots v_n)$$

with $n \in \mathbb{N}$ and $u_1, u_2, \ldots, u_n, v_1, \ldots, v_n \in \Sigma^*$ such that
- **(i)** $u_1 u_2 \cdots u_n \in K$,
- **(ii)** $v_1 \cdots v_n \in \Sigma^*$, and
- **(iii)** $(v_1 v_2 \cdots v_i, u_{i+1}) \in I$ for all $i \in [n-1]$.

We will show that R is rational, that $\eta(R) = \mathcal{R}$, and that R is left-closed.

To verify that R is a rational relation, we present a rational transducer $\mathcal{T}$ which is a non-deterministic finite automaton with edges labeled by elements from $\Sigma^* \times \Sigma^*$ (cf. [4, p. 77]). Since K is regular, there is a non-deterministic finite automaton $\mathcal{A} = (Q, \Sigma, Q_0, T, F)$ (where Q_0 and F are the initial and accepting states and $T \subseteq Q \times \Sigma \times Q$) accepting K. States of the transducer $\mathcal{T}$ are pairs (q, A) of a state $q \in Q$ and a set of letters $A \subseteq \Sigma$. A state (q, A) is initial if $q \in Q_0$ and $A = \emptyset$ and it is accepting if $q \in F$. The transducer has two types of transitions (with $a \in \Sigma$):
- There is a transition from (p, A) to (q, B) labeled (a, a) iff $p = q$ and $B = A \cup \{a\}$.
- There is a transition from (p, A) to (q, B) labeled (a, ε) iff $(p, a, q) \in T$ is a transition of the automaton $\mathcal{A}$, $\{a\} \times A \subseteq I$, and $A = B$.

Let $p \in Q_0$, $q \in Q$, $A \subseteq \Sigma$, and $v, w \in \Sigma^*$. Then the transducer $\mathcal{T}$ has a path labeled (w, v) from $(p, \emptyset)$ to (q, A) iff the following hold:
- A is the set of letters of v.
- w results from v by injecting some letters (using transitions labeled (a, ε)) that are independent from all letters of v read so far.
- The sequence u of injected letters leads from p to q in the automaton $\mathcal{A}$.

Consequently, (w, v) labels a path from some initial to some accepting state iff $(w, v) \in R$. Hence, indeed, R is rational [4, Thm. III.6.1].

Next, we verify $\eta(R) = \mathcal{R}$. First, suppose $(w, v) = (u_1 v_1 \cdots u_n v_n, v_1 \cdots v_n) \in R$ with the properties from above. From (iii), we obtain $u_1 v_1 \cdots u_n v_n \sim u_1 u_2 \cdots u_n\, v_1 v_2 \cdots v_n$ and therefore $([w], [v]) = ([u_1 \cdots u_n][v], [v])$ which belongs to $\mathcal{R}$ since $u_1 \cdots u_n \in K$ by (i). Thus, $\eta(R) \subseteq \mathcal{R}$. Conversely, let $([uv], [v]) \in \mathcal{R}$, i.e., $u \in K$ and $v \in \Sigma^*$. With $n = 1$, $u_1 = u$, and $v_1 = v$, we get $(uv, v) = (u_1 v_1, v_1) \in R$ and therefore $\mathcal{R} \subseteq \eta(R)$. Thus, indeed, $\eta(R) = \mathcal{R}$.

It remains to be shown that R is left-closed. So let $n \in \mathbb{N}$ and $u_1, \ldots, u_n, v_1, \ldots, v_n \in \Sigma^*$ satisfying (i-iii) from above and let $w \in \Sigma^*$ such that

$$w \sim u_1 v_1 u_2 v_2 \cdots u_n v_n \; R \; v_1 \cdots v_n\,.$$

With $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$, (iii) implies

$$w \sim u_1 v_1 u_2 v_2 \cdots u_n v_n \sim u_1 \cdots u_n \, v_1 \cdots v_n = uv \,.$$

Applying Levi's Lemma for traces [11, p. 74] to the equivalence $w \sim uv$ yields $m \in \mathbb{N}$ and words $u'_1, \ldots, u'_m$ and $v'_1, \ldots, v'_m$ such that

(i) $w = u'_1 v'_1 \, u'_2 v'_2 \cdots u'_m v'_m$,
(ii) $u \sim u'_1 u'_2 \cdots u'_m =: u'$,
(iii) $v \sim v'_1 v'_2 \cdots v'_m =: v'$, and
(iv) $(v'_1 v'_2 \cdots v'_i, u'_{i+1}) \in I$ for all $i \in [m-1]$.

Note that $u' \sim u \in K = \eta^{-1}(\mathcal{K})$ implies $u' \in K$. Hence we get $w \; R \; v' \sim v$. Thus, indeed, the relation R is left-closed. ◀

▶ **Lemma 4.11.** *Let $\mathcal{L} \subseteq \mathbb{M}$ be rational. Then $\mathcal{R} = (\{[\varepsilon]\} \times \mathcal{L}) \cdot \mathrm{Id}_{\mathbb{M}} = \{(y, xy) \mid x \in \mathcal{L}, y \in \mathbb{M}\}$ is lc-rational.*

Proof. There exists a regular language $L \subseteq \Sigma^*$ with $\eta(L) = \mathcal{L}$. Furthermore, $R := \{(v, uv) \mid u \in L, v \in \Sigma^*\}$ is the product of $L \times \{\varepsilon\}$ and Id_{Σ^*} and therefore rational.

We first show that R is even lc-rational. So let $u \in L$ and $v \sim v'$ be arbitrary words such that $v \sim v' \; R \; uv'$. Then we have $v \; R \; uv \sim uv'$. Hence, indeed, R is left-closed.

Next we show $\mathcal{R} = \eta(R)$. For the inclusion "$\supseteq$", let $(v, uv) \in R$, i.e., $u \in L$ and $v \in \Sigma^*$. From $\eta(L) = \mathcal{L}$, we obtain $[u] \in \mathcal{L}$ and therefore $(\eta(v), \eta(uv)) = (\eta(v), \eta(u)\,\eta(v)) \in \mathcal{R}$. Thus, $\mathcal{R} \supseteq \eta(R)$. Conversely, let $(y, xy) \in \mathcal{R}$, i.e., $x \in \mathcal{L}$ and $y \in \mathbb{M}$. From $x \in \mathcal{L} = \eta(L)$, we obtain a word $u \in L$ with $\eta(u) = x$. Further, there is a word $v \in \Sigma^*$ with $\eta(v) = y$. It follows that $(v, uv) \in R$ and therefore $(y, xy) = (\eta(v), \eta(uv)) \in \eta(R)$. ◀

Now the following sufficient condition for the lc-rationality of $\mathcal{R}_1 \cdot \mathcal{R}_2$ follows:

▶ **Theorem 4.12.** *Let $\mathcal{K} \subseteq \mathbb{M}$ be recognizable, $\mathcal{L} \subseteq \mathbb{M}$ rational, and $\mathcal{R} \subseteq \mathbb{M}^2$ lc-rational. Then $(\mathcal{K} \times \mathcal{L}) \cdot \mathcal{R}$ is lc-rational.*

Proof. The relations $\mathcal{R}_1 = (\mathcal{K} \times \{[\varepsilon]\}) \cdot \mathrm{Id}_{\mathbb{M}}$ and $\mathcal{R}_2 = (\{[\varepsilon] \times \mathcal{L}\}) \cdot \mathrm{Id}_{\mathbb{M}}$ are lc-rational by Lemmas 4.10 and 4.11, respectively. Hence, by Proposition 4.5(ii), the relation $\mathcal{R}_1 \circ \mathcal{R} \circ \mathcal{R}_2$ is lc-rational as well. But this composition equals $(\mathcal{K} \times \mathcal{L}) \cdot \mathcal{R}$. ◀

5 An application of lc-rational relations

Inspired by asynchronous automata [36] for recognizable trace languages, we introduced "cooperating multi-pushdown systems" or cPDS, i.e., distributed pushdown systems whose pushdown stores a trace as opposed to a word as in classical pushdown systems [23]. Hence, sets of configurations are trace languages. As main result, we proved that backwards reachability in these cPDS preserves the recognizability and forwards reachability preserves the rationality of sets of configurations; these two proofs use very different proof techniques. It is the aim of this section to show that the reachability relation of a cPDS is an lc-rational trace relation which, by Theorem 4.7, gives the two main results from [23] in a uniform manner.

Before we can define cPDS, we need the following notation. Since Σ is finite, there exists a finite set J and a mapping $\lambda \colon \Sigma \to 2^J \setminus \{\emptyset\}$ such that $(a, b) \in D$ iff $\lambda(a) \cap \lambda(b) \neq \emptyset$.

A *cPDS* is a tuple $\mathbb{P} = (\prod_{i \in J} Q_i, \Delta)$ for some finite sets Q_i of local states and a finite set $\Delta \subseteq \prod_{i \in J} Q_i \times \Sigma \times \Sigma^* \times \prod_{i \in J} Q_i$ of transition rules such that the following hold (where $\overline{p}$ denotes a tuple $(p_i)_{i \in J} Q_i$):

(i) If $(\overline{p}, a, w, \overline{q}) \in \Delta$, then $p_i = q_i$ for all $i \in J \setminus \lambda(a)$.
(ii) If $(\overline{p}, a, w, \overline{q}) \in \Delta$ and $\overline{r} \in \prod_{i\in J} Q_i$ with $p_i = r_i$ for all $i \in \lambda(a)$, then

$$\Big(\overline{r}, a, w, \big((q_i)_{i\in\lambda(a)}, (r_i)_{i\notin\lambda(a)}\big)\Big) \in \Delta\,.$$

(iii) If $(\overline{p}, a, ubv, \overline{q}) \in \Delta$, then $\lambda(a) \supseteq \lambda(b)$.

A *configuration* of $\mathbb{P}$ is a pair $(\overline{p}, x)$ with $\overline{p} \in \prod_{i\in J} Q_i$ a global state and $x \in \mathbb{M}$ a trace. For two configurations $c_1 = (\overline{p}, x)$ and $c_2 = (\overline{q}, z)$, there is a transition from c_1 to c_2 (written $c_1 \vdash c_2$) iff there exists a transition rule $(\overline{p}, a, w, \overline{q}) \in \Delta$ and a trace $y \in \mathbb{M}$ such that $x = [a]\cdot y$ and $z = [w]\cdot y$. Combining results from [23] and from the present paper, we can derive the following claim.

▶ **Lemma 5.1.** *Let $\mathbb{P} = (\prod_{i\in J} Q_i, \Delta)$ be a cPDS and $\overline{p}, \overline{q} \in \prod_{i\in J} Q_i$ global states. The trace relation $\mathcal{R}_{\overline{p},\overline{q}} = \{(x, y) \in \mathbb{M}\times\mathbb{M} \mid (\overline{p}, x) \vdash^* (\overline{q}, y)\}$ is lc-rational.*

Proof. A cPDS is *saturated* if $(\overline{p}, a, ubv, \overline{q}), (\overline{q}, b, \varepsilon, \overline{r}) \in \Delta$ with $(u, b) \in I$ implies $(\overline{p}, a, uv, \overline{r}) \in \Delta$. In [23, Section 5.3], we show how to transform a cPDS $\mathbb{P}$ into a saturated $\mathbb{P}'$ that has the same reachability relation $\vdash^*$. Hence, we can assume $\mathbb{P}$ to be saturated.

From [23, Lemma 5.4], one obtains that the relation $\mathcal{R}_{\overline{p},\overline{q}}$ is a finite union of compositions of relations of the form $(\mathcal{K}\times\mathcal{L})\cdot \mathrm{Id}_{\mathbb{M}}$ with $\mathcal{K} \subseteq \mathbb{M}$ recognizable and $\mathcal{L} \subseteq \mathbb{M}$ rational. Now Theorem 4.12 and Proposition 4.5 imply that the reachability relation $\mathcal{R}_{\overline{p},\overline{q}}$ is lc-rational. ◀

Using the above lemma and Theorem 4.7, we obtain the following results from [23] in a uniform manner.

▶ **Theorem 5.2** (Köcher and Kuske [23])**.** *Let $\mathbb{P} = (\prod_{i\in J} Q_i, \Delta)$ be a cPDS, $\overline{p}, \overline{q} \in \prod_{i\in J} Q_i$, and $\mathcal{L} \subseteq \mathbb{M}$.*
(i) *If $\mathcal{L}$ is recognizable, then $\mathcal{L}_{\mathrm{b}} := \{x \in \mathbb{M} \mid \exists y \in \mathcal{L} \colon (\overline{p}, x) \vdash^* (\overline{q}, y)\}$ is recognizable, i.e., backwards reachability in cPDS preserves recognizability [23, Thm. 4.1].*
(ii) *If $\mathcal{L}$ is rational, then $\mathcal{L}_{\mathrm{f}} := \{y \in \mathbb{M} \mid \exists x \in \mathcal{L} \colon (\overline{p}, x) \vdash^* (\overline{q}, y)\}$ is rational, i.e., forwards reachability in cPDS preserves rationality [23, Thm. 5.1].*

Proof. The claims follow from the observations $\mathcal{L}_{\mathrm{b}} = {}^{\mathcal{R}_{\overline{p},\overline{q}}}\mathcal{L}$ and $\mathcal{L}_{\mathrm{f}} = \mathcal{L}^{\mathcal{R}_{\overline{p},\overline{q}}}$. ◀

6 The theory of lc-rational trace relations II

6.1 A Nivat-type characterisation of lc-rational relations

There are three very basic types of rational word relations $R \subseteq \Sigma^*\times\Sigma^*$: (graphs of) homomorphisms $\Sigma^* \to \Sigma^*$, their inverses, and restrictions to regular languages K, i.e., relations of the form $R_K = \{(u, u) \mid u \in K\}$. Nivat's theorem [30] (cf. [4, Thm. III.3.2]) shows that every rational relation R can be composed of these three basic types. More precisely, he showed that any rational relation R can be written as $f^{-1}\circ R_K \circ g$ for a regular language K and homomorphisms f and g. This result does not only hold for rational relations on words, but for arbitrary monoids, in particular for rational relations $\mathcal{R} \subseteq \mathbb{M}\times\mathbb{M}$ [4, Prop. III.3.4]. Our next result builds on this characterisation of all *rational relations* $\mathcal{R}$ and provides a similar characterisation of all *lc-rational relations* $\mathcal{R}$ in the spirit of Nivat's theorem.

▶ **Theorem 6.1.** *Let $\mathcal{R} \subseteq \mathbb{M}\times\mathbb{M}$. Then $\mathcal{R}$ is lc-rational if, and only if, there exists an alphabet Γ, a regular language $K \subseteq \Gamma^*$, and homomorphisms $\boldsymbol{f}, \boldsymbol{g} \colon \Gamma^* \to \mathbb{M}$ such that*
(a) $\mathcal{R} = \{(\boldsymbol{f}(x), \boldsymbol{g}(x)) \mid x \in K\}$ *and*

(b) *for all* $y \in K$*,* $n \in \mathbb{N}$*, and* $t_1, \ldots, t_n \in \mathbb{M}$ *with* $t_1 \cdot t_2 \cdots t_n = \boldsymbol{f}(y)$*, there exist* $y_1, y_2, \ldots, y_n \in \Gamma^*$ *such that*
- **(b1)** $y_1 y_2 \cdots y_n \in K$*,*
- **(b2)** $\boldsymbol{f}(y_i) = t_i$ *for all* $i \in [n]$*, and*
- **(b3)** $\boldsymbol{g}(y) = \boldsymbol{g}(y_1 \cdots y_n)$*.*

As explained above, the relation $\mathcal{R}$ is rational if, and only if, there are Γ, K, and $\boldsymbol{f}$, $\boldsymbol{g}$ satisfying property (a). The left-closedness therefore corresponds to property (b).

Proof. The detailed proof can be found in the appendix.

First, suppose $\mathcal{R}$ is lc-rational, i.e., the η-image of some lc-rational word relation R. Let Γ, K, f, and g be the alphabet, regular language, and homomorphisms that exist by Theorem 3.1(R4). Then $\boldsymbol{f} = \eta \circ f$ and $\boldsymbol{g} = \eta \circ g$ are homomorphisms from Γ^* to $\mathbb{M}$ with the claimed properties.

Conversely, let Γ, K, $\boldsymbol{f}$, and $\boldsymbol{g}$ be the alphabet, regular language, and homomorphisms from the theorem. Then there are homomorphisms $f, g \colon \Gamma^* \to \Sigma^*$ with $\boldsymbol{f} = \eta \circ f$ and $\boldsymbol{g} = \eta \circ g$. Together with K, they define a rational word relation R. One then shows that R is left-closed and satisfies $\mathcal{R} = \eta(R)$. ◀

6.2 Undecidability

So far, we proved that lc-rational trace relations share some of the nice properties of rational word relations. Unfortunately, it is undecidable (not even semi-decidable) whether a given rational relation is left-closed.

▶ **Proposition 6.2.** *Suppose* D *is transitive and* I *is not transitive such that there are* $a, b, c \in \Sigma$ *with* $(a, b), (b, c) \in I$ *and* $(a, c) \in D$*. It is not semi-decidable whether a given rational trace relation* $\mathcal{R} \subseteq \mathbb{M}^2$ *is lc-rational.*

Proof. Let P be an instance of Post's correspondence problem. From P, Muscholl and Petersen [29, proof of Thm. 2] construct (following a similar construction in [33]) a star-free language $L \subseteq \Sigma^*$ such that $\overline{L} = \eta^{-1}\eta(L)$ is star-free iff P has no solution (this construction uses that I is not transitive). Since D is transitive, $\overline{L}$ is either star-free or not regular [29, Thm. 1]. Hence, $\overline{L}$ is regular if, and only if, P has no solution.

Recall that any star-free language L is regular such that $\mathcal{L} := \eta(L)$ is rational. Hence

$$\begin{aligned}
\mathcal{L} \times \{[\varepsilon]\} \text{ is lc-rational} &\iff \mathcal{L} \text{ is recognizable} && \text{by Proposition 4.6}\\
&\iff \eta^{-1}(\mathcal{L}) \text{ is regular}\\
&\iff \overline{L} \text{ is regular} && \text{since } \eta^{-1}(\mathcal{L}) = \overline{L}\\
&\iff P \text{ has no solution}\,.
\end{aligned}$$

Thus, one can construct a rational relation $\mathcal{L} \times \{[\varepsilon]\}$ from P that is lc-rational if, and only if, P has no solution. Since the existence of *no* solution is not semi-decidable, the claim follows. ◀

7 Rational trace relations

We saw that the composition of a rational and an lc-rational trace relation is rational, again. This holds in particular if the first relation is the inverse of an lc-rational relation. We now demonstrate that all rational trace relations arise in this way.

▶ **Theorem 7.1.** *Suppose there are $a, b \in \Sigma$ with $(a, b) \in D$. Let $\mathcal{R} \subseteq \mathbb{M}^2$ be a binary relation. Then the following are equivalent:*
(i) *$\mathcal{R}$ is rational.*
(ii) *There exist lc-rational relations $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $\mathcal{R} = \mathcal{R}_1^{-1} \circ \mathcal{R}_2$.*

Proof. The implication "(ii)⇒(i)" is immediate by Proposition 4.5(i) since $\mathcal{R}_1^{-1}$ is rational for any rational relation $\mathcal{R}_1$.

For the converse implication, suppose $\mathcal{R}$ is rational. Then there exists a rational relation $R \subseteq \Sigma^* \times \Sigma^*$ with $\mathcal{R} = \eta(R)$. By Nivat's theorem, i.e., Theorem 3.1(R4), there exist an alphabet Γ, homomorphisms f and g from Γ^* to Σ^*, and a regular language $K \subseteq \Gamma^*$ such that

$$R = \{(f(u), g(u)) \mid u \in K\}\,.$$

Suppose $\Gamma = \{c_1, c_2, \ldots, c_n\}$. Let $h\colon \Gamma^* \to \Sigma^*$ be the homomorphisms defined by $h(c_i) = a^i b$.

Now consider the relations

$$R_1 = \{(h(u), f(u)) \mid u \in K\} \text{ and } R_2 = \{(h(u), g(u)) \mid u \in K\}\,.$$

We first show that these relations are lc-rational (by symmetry, we only consider the relation R_1). From Nivat's theorem, we obtain that R_1 is rational. To show that it is left-closed, let $v, v', w' \in \Sigma^*$ with $v \sim v' \; R_1 \; w'$.

Since $R_1 \subseteq \{a, b\}^* \times \Sigma^*$, we obtain $v' \in \{a, b\}^*$. Since $(a, b) \in D$, this implies $v = v'$. Hence, setting $w := w'$, we obtain $v \; R_1 \; w \sim w'$. Hence, indeed, the relations R_1 and R_2 are lc-rational.

Next, we show $R = R_1^{-1} \circ R_2$. For the inclusion "$\subseteq$", let $(v, w) \in R$. Then there exists $u \in K$ with $v = f(u)$ and $w = g(u)$. Hence we obtain

$$v = f(u) \; R_1^{-1} \; h(u) \; R_2 \; g(u) = w$$

and therefore $(v, w) \in R_1^{-1} \circ R_2$. For the converse inclusion, suppose $(v, w) \in R_1^{-1} \circ R_2$. Then there exists some word x with $v \; R_1^{-1} \; x \; R_2 \; w$. By the definition of the relations R_1 and R_2, there are words $u_1, u_2 \in K \subseteq \Gamma^*$ such that

$$v = f(u_1)\,, x = h(u_1) \text{ and } x = h(u_2)\,, w = g(u_2)\,.$$

Since the homomorphism h is injective, we get $u_1 = u_2$ and therefore

$$(v, w) = (f(u_1), g(u_2)) = (f(u_1), g(u_1)) \in R\,.$$

Thus, indeed, $R = R_1^{-1} \circ R_2$.

Finally, let $\mathcal{R}_1 = \eta(R_1)$ and $\mathcal{R}_2 = \eta(R_2)$. Note that R_1^{-1} is rational and satisfies $\eta(R_1^{-1}) = \mathcal{R}_1^{-1}$. From Proposition 3.5(i), we obtain that

$$\eta(R_1^{-1} \circ R_2) = \eta(R_1^{-1}) \circ \eta(R_2) = \eta(R_1^{-1}) \circ \mathcal{R}_2$$

since R_2 is lc-rational. Hence, we obtain

$$\mathcal{R} = \eta(R) = \eta(R_1^{-1} \circ R_2) = \eta(R_1^{-1}) \circ \mathcal{R}_2 = \mathcal{R}_1^{-1} \circ \mathcal{R}_2\,. \qquad ◀$$

Recall that the right-application of rational trace relations does neither preserve the rationality nor the recognizability of a set of traces. The above factorisation result allows to prove that the right-application transforms recognizable sets into rational ones.

▶ **Theorem 7.2.** *Let $\mathcal{R} \subseteq \mathbb{M} \times \mathbb{M}$ be rational and $\mathcal{L} \subseteq \mathbb{M}$ recognizable. Then $\mathcal{L}^{\mathcal{R}}$ is rational.*

Proof. By Theorem 7.1, there are lc-rational relations $\mathcal{R}_1$ and $\mathcal{R}_2$ such that $\mathcal{R} = \mathcal{R}_1^{-1} \circ \mathcal{R}_2$. Hence $\mathcal{L}^{\mathcal{R}} = \mathcal{L}^{\mathcal{R}_1^{-1} \circ \mathcal{R}_2} = (\mathcal{L}^{\mathcal{R}_1^{-1}})^{\mathcal{R}_2} = ({}^{\mathcal{R}_1}\mathcal{L})^{\mathcal{R}_2}$. From Theorem 4.7, we know that $\mathcal{K} := {}^{\mathcal{R}_1}\mathcal{L}$ is recognizable and therefore in particular rational. Hence, again using Theorem 4.7, $\mathcal{K}^{\mathcal{R}_2}$ is rational. ◀

Note that the above lemma cannot be improved: by Lemma 4.9, there exist an lc-rational relation $\mathcal{R}$ and a recognizable set $\mathcal{L}$ such that $\mathcal{L}^{\mathcal{R}}$ is not recognizable. Furthermore, there is also a rational set $\mathcal{L}$ such that ${}^{\mathcal{R}}\mathcal{L}$ is not rational. Note that ${}^{\mathcal{R}}\mathcal{L} = \mathcal{L}^{\mathcal{R}^{-1}}$ and that $\mathcal{R}^{-1}$ is rational. Hence, indeed, the above lemma is optimal.

▶ **Proposition 7.3.** *The relation $\leq_{\mathrm{rat}}$ is not transitive.*

Proof. Consider the lc-rational relation $\mathcal{R}$ and the languages $\mathcal{K}$ and $\mathcal{L}$ from the proof of Lemma 4.9. In that proof, we showed in particular that $\mathcal{L}$ is recognizable, $\mathcal{L}^{\mathcal{R}} = \mathcal{K}$, and ${}^{\mathcal{R}}\mathcal{K}$ is not rational. Hence, by Theorem 7.2, there is no rational trace relation $\mathcal{S}$ with $\mathcal{L}^{\mathcal{S}} = {}^{\mathcal{R}}\mathcal{K}$, i.e., ${}^{\mathcal{R}}\mathcal{K} \not\leq_{\mathrm{rat}} \mathcal{L}$. On the other hand, we have ${}^{\mathcal{R}}\mathcal{K} = \mathcal{K}^{\mathcal{R}^{-1}} \leq_{\mathrm{rat}} \mathcal{K}$ since also $\mathcal{R}^{-1}$ is a rational relation as well as $\mathcal{K} = \mathcal{L}^{\mathcal{R}} \leq \mathcal{L}$. ◀

From this proposition, we get immediately that rational trace relations do not compose for otherwise $\leq_{\mathrm{rat}}$ would be transitive (Example 3.2 provides concrete rational trace relations whose composition is not rational).

Thus, rational trace relations enjoy properties (R3) and (R4) from Theorem 3.1, but not the properties (R1) and (R2). On the other hand, lc-rational relations

- satisfy some versions of (R1) (cf. Theorem 4.7), but not all natural versions (cf. Lemma 4.9),
- satisfy (R2) (cf. Proposition 4.5),
- violate (R3), and
- have a characterisation in the spirit of (R4) (cf. Theorem 6.1).

We next restrict the class of lc-rational trace relations further to obtain a class that satisfies all versions of (R1), (R2), (R3), and has a characterisation in the spirit of (R4).

First, a word relation $R \subseteq \Sigma^* \times \Sigma^*$ is *right-closed* if $\sim \circ R \supseteq R \circ \sim$, i.e., for all $u, v' \in \Sigma^*$, we have

$$\big(\exists u' \in \Sigma^* \colon u \sim u' \; R \; v'\big) \iff \big(\exists v \in \Sigma^* \colon u \; R \; v \sim v'\big) .$$

Compared with the definition of left-closedness, we only inverted the inclusion and implication, resp. The relation R is *rc-rational* if it is right-closed and rational. A trace relation $\mathcal{R} \subseteq \mathbb{M} \times \mathbb{M}$ is *rc-rational*, if there exists some rc-rational relation $R \subseteq \Sigma^* \times \Sigma^*$ with $\mathcal{R} = \eta(R)$.

Let $\mathcal{R} \subseteq \mathbb{M} \times \mathbb{M}$. We call $\mathcal{R}$ *closed-rational* if it is both, lc-rational and rc-rational. To prove closed-rationality of a relation $\mathcal{R}$, one has to produce two relations R_1 and R_2 with $\mathcal{R} = \eta(R_1) = \eta(R_2)$ that are lc-rational and rc-rational, resp. In particular, these two relations can be distinct.

▶ **Example 7.4.** With the notions from Lemma 4.10, the relation $\mathcal{R} = \{(xy, y) \mid x \in \mathcal{L}, y \in \mathbb{M}\}$ is lc-rational. Since any recognizable trace language $\mathcal{L}$ is also rational, the relation $\mathcal{R}^{-1} = \{(y, xy) \mid x \in \mathcal{L}, y \in \mathbb{M}\}$ is lc-rational by Lemma 4.11 implying that $\mathcal{R}$ is rc-rational. Hence $\mathcal{R}$ is even closed-rational.

Looking at the proofs of Lemmas 4.11 and 4.10, we see that the relations $R \subseteq \Sigma^* \times \Sigma^*$ used there are indeed very different.

The following lemma shows that we can certify closed-rationality by providing a single word relation R that is both, lc- and rc-rational.

▶ **Lemma 7.5.** *A relation $\mathcal{R} \subseteq \mathbb{M}^2$ is closed-rational iff there exists a single relation $R \subseteq \Sigma^* \times \Sigma^*$ with $\eta(R) = \mathcal{R}$ that is both, lc-rational and rc-rational.*

Proof. The implication "$\Leftarrow$" is immediate.

So let $\mathcal{R}$ be closed-rational. Then there are relations R_1 and R_2 with $\mathcal{R} = \eta(R_1) = \eta(R_2)$ that are lc-rational and rc-rational, resp. One then shows that $R = R_1 \cup R_2$ is left- and right-closed (see appendix) which implies the claim since $\eta(R) = \mathcal{R}$ and R is rational. ◀

Based on this characterisation, the proof of Theorem 6.1 can be extended easily showing the following Nivat-type characterisation of all closed-rational relations:

▶ **Theorem 7.6.** *Let $\mathcal{R} \subseteq \mathbb{M} \times \mathbb{M}$. Then $\mathcal{R}$ is closed-rational if, and only if, there exists an alphabet Γ, a regular language $K \subseteq \Gamma^*$, and homomorphisms $\boldsymbol{f}, \boldsymbol{g} \colon \Gamma^* \to \mathbb{M}$ such that*

(a) $\mathcal{R} = \{(\boldsymbol{f}(x), \boldsymbol{g}(x)) \mid x \in K\}$,

(b) *for all $y \in K$, $n \in \mathbb{N}$, and $t_1, \ldots, t_n \in \mathbb{M}$ with $t_1 \cdot t_2 \cdots t_n = \boldsymbol{f}(y)$, there exist $y_1, y_2, \ldots, y_n \in \Gamma^*$ such that*
  - **(b1)** $y_1 y_2 \cdots y_n \in K$,
  - **(b2)** $\boldsymbol{f}(y_i) = t_i$ *for all $i \in [n]$, and*
  - **(b3)** $\boldsymbol{g}(y) = \boldsymbol{g}(y_1 \cdots y_n)$, *and*

(c) *for all $y \in K$, $n \in \mathbb{N}$, and $t_1, \ldots, t_n \in \mathbb{M}$ with $t_1 \cdot t_2 \cdots t_n = \boldsymbol{g}(y)$, there exist $y_1, y_2, \ldots, y_n \in \Gamma^*$ such that*
  - **(c1)** $y_1 y_2 \cdots y_n \in K$,
  - **(c2)** $\boldsymbol{g}(y_i) = t_i$ *for all $i \in [n]$, and*
  - **(c3)** $\boldsymbol{f}(y) = \boldsymbol{f}(y_1 \cdots y_n)$.

Finally, the results of this paper show that the class of closed-rational relations has all of the properties from Theorem 3.1 that make rational relations on Σ^* interesting:

▶ **Theorem 7.7.** *The class of closed-rational relations on $\mathbb{M}$ is closed under inversion, under composition, and any closed-rational relation preserves both, rationality and recognizability under both, left- and right-application.*

8 Conclusion

We restricted the class of rational trace relation in order to get a notion that has properties similar to those of rational word relations. Our findings were applied in order to improve the central result on systems with a trace pushdown from [23].

For the definition of lc- and closed-rational relations, it was crucial that the trace monoid $\mathbb{M}$ was finitely generated by the alphabet Σ. Left-closedness can similarly be defined for any monoid M generated by some set Σ as follows. Let $\eta \colon \Sigma^* \to M$ be the natural epimorphism. A relation $R \subseteq \Sigma^* \times \Sigma^*$ is left-closed if, for any $u, v' \in \Sigma^*$, we have

$$(\exists u' \in \Sigma^* \colon \eta(u) = \eta(u') \text{ and } (u', v') \in R) \implies (\exists v \in \Sigma^* \colon (u, v) \in R \text{ and } \eta(v) = \eta(v')) \,.$$

Further, a relation $\mathcal{R} \subseteq M \times M$ is lc-rational if there exists a left-closed and rational relation $R \subseteq \Sigma^* \times \Sigma^*$ with $\mathcal{R} = \eta(R)$. It is, at least mathematically, interesting which of our results on lc-rational relations transfer to this more general setting (actually, many proofs should go through without much alteration, but, e.g., the proof of Lemma 4.10 makes explicite use of Levi's lemma).

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A Appendix

Case distinction from Example 3.4.

- Suppose $x = u_1v_1 \cdots u_{i-1}v_{i-1}x'$, $u_i = x'bay'$, and $y = y'v_iu_{i+1}v_{i+1} \cdots u_{n+1}$. Then

$$\begin{aligned} u &= u_1v_1u_2 \cdots u_{i-1}v_{i-1}\, x'aby'\, v_iu_{i+1}v_{i+1} \cdot u_{n+1} \\ &\succeq v_1v_2 \cdots v_n = v' \,. \end{aligned}$$

  Thus, setting $v := v'$ yields $u \succeq v \sim v'$.
- Suppose $x = u_1v_1 \cdots u_{i-1}v_{i-1}u_ix'$, $v_i = x'bay'$, and $y = y'u_{i+1}v_{i+1} \cdots u_{n+1}$. Then

$$\begin{aligned} u &= u_1v_1 \cdots u_{i-1}v_{i-1}u_i\, x'aby'\, u_{i+1}v_{i+1} \cdots v_nu_{n+1} \\ &\succeq v_1 \cdots v_{i-1}\, x'aby'\, v_{i+1} \cdots v_n \\ &\sim v_1 \cdots v_{i-1}\, v_i\, v_{i+1} \cdots v_n = v' \,. \end{aligned}$$

  Thus, setting $v := v_1 \cdots v_{i-1}\, x'aby'\, v_{i+1} \cdots v_n$ yields $u \succeq v \sim v'$.
- Suppose $x = u_1v_1 \cdots u_{i-1}v_{i-1}x'$, $u_i = x'b$, $v_i = ay'$, and $y = y'u_{i+1}v_{i+1} \cdots u_{n+1}$. Then

$$\begin{aligned} u &= u_1v_1 \cdots u_{i-1}v_{i-1}\, x'b\, ay'\, u_{i+1}v_{i+1} \cdots u_{n+1} \\ &\succeq v_1 \cdots v_{i-1}\, ay'\, v_{i+1} \cdots v_n = v' \,. \end{aligned}$$

  Thus, setting $v := v'$ yields $u \succeq v \sim v'$.
- Finally, suppose $x = u_1v_1 \cdots u_ix'$, $v_i = x'b$, $u_{i+1} = ay'$, and $y = y'v_{i+1}u_{i+2}v_{i+2} \cdots u_{n+1}$. Then

$$\begin{aligned} u &= u_1v_1 \cdots u_i\, x'b\, ay'\, v_{i+1}u_{i+2}v_{i+2} \cdots u_{n+1} \\ &\succeq v_1 \cdots v_{i-1}x'bv_{i+1} \cdots v_{n+1} = v \,. \end{aligned}$$

  Thus, setting $v := v'$ yields $u \succeq v \sim v'$. ◀

Proof of first claim of Prop. 3.6. Let $K \times L$ be rational. Then the mapping $\pi_1 \colon \Sigma^* \times \Sigma^* \to \Sigma^* \colon (u, v) \mapsto u$ is a homomorphism and maps the rational set $K \times L$ onto the set K which is therefore rational, i.e., regular since it is a subset of Σ^*. The analogous argument leads to the regularity of L.

Conversely, suppose K and L both to be regular. Then $K \times \{\varepsilon\}$ is the image of the rational set $K \subseteq \Sigma^*$ under the homomorphism $f \colon \Sigma^* \to \Sigma^* \times \Sigma^* \colon u \mapsto (u, \varepsilon)$ and therefore a rational relation. The same applies to $\{\varepsilon\} \times L$. Since

$$K \times L = (K \times \{\varepsilon\}) \cdot (\{\varepsilon\} \times L)\,,$$

the relation $K \times L$ is the product of two rational relations and therefore rational. ◀

So first the counter-example.

▶ **Lemma A.1.** *There exist lc-rational relations $\mathcal{R}$ and $\mathcal{R}'$ such that $\mathcal{R} \cdot \mathcal{R}'$ is not lc-rational.*

Proof. Consider the rational trace relations $\mathcal{R}_1$ and $\mathcal{R}_2$ from Example 3.2. Note that $\mathcal{R}_2$ is the product of the lc-rational trace relations $\mathcal{R} = \{([d^n], [b^n]) \mid n \geq 1\}$ and $\mathcal{R}' = \{([c^m], [a^m]) \mid m \geq 1\}$ and recall that $\mathcal{R}_1 \circ \mathcal{R}_2$ is not rational. Hence, by Proposition 4.5(i), $\mathcal{R}_2$ cannot be lc-rational. ◀

Proof of $\mathcal{R}_1 \circ \mathcal{R} \circ \mathcal{R}_2 = (\mathcal{K} \times \mathcal{L}) \cdot \mathcal{R}$ in proof of Theorem 4.12. Note that $(x, z) \in \mathcal{R}_1 \circ \mathcal{R} \circ \mathcal{R}_2$ iff there are $y_1, y_2 \in \mathbb{M}$ with $(x, y_1) \in \mathcal{R}_1$, $(y_1, y_2) \in \mathcal{R}$, and $(y_2, z) \in \mathcal{R}_2$. But $(x, y_1) \in \mathcal{R}_1$ is equivalent to the existence of $k \in \mathcal{K}$ with $x = k \cdot y_1$. Similarly, $(y_2, z) \in \mathcal{R}_2$ iff there is $\ell \in \mathcal{L}$ with $z = \ell \cdot y_2$. In summary, we have $(x, z) \in \mathcal{R}_1 \circ \mathcal{R} \circ \mathcal{R}_2$ iff there exist $k \in \mathcal{K}$, $(y_1, y_2) \in \mathcal{R}$, and $\ell \in \mathcal{L}$ with $(x, z) = (k\, y_1, \ell\, y_2)$. But this holds iff $(x, z) \in (\mathcal{K} \times \mathcal{L}) \cdot \mathcal{R}$. ◀

Proof of Theorem 6.1. First, suppose $\mathcal{R}$ is lc-rational. Then there exists an lc-rational relation $R \subseteq \Sigma^* \times \Sigma^*$ with $\mathcal{R} = \eta(R)$. Hence, by Nivat's theorem [30] (cf. [4, Thm. III.3.2]), there exists an alphabet Γ, a regular language $K \subseteq \Gamma^*$, and homomorphisms $f, g \colon \Gamma^* \to \Sigma^*$ such that

$$R = \{(f(x), g(x)) \mid x \in K\}\,.$$

Because of the equivalence of (i) and (iii) in [4, Thm. III.3.2], we can even assume that the homomorphisms f and g are *alphabetic*, i.e., $|f(a)| \leq 1$ and $|g(a)| \leq 1$ for all $a \in \Gamma$. Let $\boldsymbol{f} = \eta \circ f \colon \Gamma^* \to \mathbb{M}$ and $\boldsymbol{g} = \eta \circ g$. From $\mathcal{R} = \eta(R)$, we immediately get property (a). To also show (b), let $y \in K$, $n \in \mathbb{N}$, and $t_1, \ldots, t_n \in \mathbb{M}$ with $t_1 t_2 \cdots t_n = \boldsymbol{f}(y)$. Then there are words $u_1, \ldots, u_n \in \Sigma^*$ with $\eta(u_i) = t_i$ for all $i \in [n]$. Note that

$$u_1 u_2 \cdots u_n \sim f(y)\ R\ g(y)$$

since $\eta(u_1 u_2 \cdots u_n) = \boldsymbol{f}(y) = \eta(f(y))$. Since the relation R is left-closed, there exists a word $v \in \Sigma^*$ such that

$$u_1 \cdots u_n\ R\ v \sim g(y)\,.$$

It follows that there is $y' \in K$ with

$$u_1 u_2 \cdots u_n = f(y')\ R\ g(y') = v \sim g(y)\,.$$

Since the homomorphism f is alphabetic, we can split the word y' into

$$y' = y_1 y_2 \cdots y_n \text{ with } u_i = f(y_i) \text{ for all } i \in [n]\,.$$

We can now verify conditions (b1–3):

(b1) $y_1 \cdots y_n = y' \in K$,
(b2) $\boldsymbol{f}(y_i) = \eta(f(y_i)) = \eta(u_i) = t_i$ for all $i \in [n]$, and
(b3) $\boldsymbol{g}(y) = \eta(g(y)) \overset{v \sim g(y)}{=} \eta(v) = \eta(g(y')) = \boldsymbol{g}(y') = \boldsymbol{g}(y_1 \cdots y_n)$

Thus, we have indeed an alphabet Γ, a regular language K, and homomorphisms $\boldsymbol{f}$ and $\boldsymbol{g}$ such that (a) and (b) hold.

Conversely, suppose Γ is an alphabet, $K \subseteq \Gamma^*$ a regular language, and $\boldsymbol{f}, \boldsymbol{g} \colon \Gamma^* \to \mathbb{M}$ homomorphisms such that (a) and (b) hold.

Choose, for every letter $x \in \Gamma$, words $v_x, w_x \in \Sigma^*$ with $\boldsymbol{f}(x) = \eta(v_x)$ and $\boldsymbol{g}(x) = \eta(w_x)$. Let $f, g \colon \Gamma^* \to \Sigma^*$ be the homomorphisms with $f(x) = v_x$ and $g(x) = w_x$ for all $x \in \Gamma$. Then we obtain $\boldsymbol{f} = \eta \circ f$ and $\boldsymbol{g} = \eta \circ g$. Let

$$R = \{(f(x), g(x)) \mid x \in L\} \subseteq \Sigma^* \times \Sigma^* .$$

From Nivat's theorem, we obtain that R is a rational relation. By its very definition and property (a), we also get $\mathcal{R} = \eta(R)$.

It remains to be shown that the word relation R is left-closed. So let $u, u', v' \in \Sigma^*$ with

$$u \sim u' \, R \, v' .$$

With $n = |u|$, there are letters $a_i \in \Sigma$ for all $i \in [n]$ with $u = a_1 a_2 \cdots a_n$. For $i \in [n]$, we set $t_i = \eta(a_i)$. From $(u', v') \in R$, we obtain some word $y \in K$ with $u' = f(y)$ and $v' = g(y)$. Note that

$$\boldsymbol{f}(y) = \eta(u') = \eta(a_1 \cdots a_n) = t_1 \cdot t_2 \cdots t_n .$$

From (b), we obtain words $y_1, \ldots, y_n \in \Gamma^*$ such that
(b1) $y_1 \cdots y_n \in K$,
(b2) $\boldsymbol{f}(y_i) = t_i$ for all $i \in [n]$, and
(b3) $\boldsymbol{g}(y) = \boldsymbol{g}(y_1 \cdots y_n)$.

Since a_i is the only word with $t_i = \eta(a_i)$, property (b2) implies $f(y_i) = a_i$ for all $i \in [n]$. We therefore get

$$\begin{aligned} u = a_1 \cdots a_n = f(y_1 \cdots y_n) \, R \, g(y_1 \cdots y_n) & \qquad \text{by property (b1)} \\ \sim g(y) & \qquad \text{by property (b3)} \\ = v' . & \end{aligned}$$

With $v = g(y_1 \cdots y_n)$, we therefore get

$$u \, R \, v \sim v' .$$

Hence R is left-closed which completes the proof of the lc-rationality of $\mathcal{R}$. ◀

Proof of Lemma 7.5. The implication "$\Leftarrow$" is immediate.

So let $\mathcal{R}$ be closed-rational. Then there are relations R_1 and R_2 with $\mathcal{R} = \eta(R_1) = \eta(R_2)$ that are lc-rational and rc-rational, resp. Set $R = R_1 \cup R_2$. Clearly, this relation is rational. We show that it is left-closed (the proof of its right-closedness is symmetric). So let $u, u', v' \in \Sigma^*$ with

$$u \sim u' \, (R_1 \cup R_2) \, v' .$$

If $(u', v') \in R_1$, the left-closedness of R_1 yields some word v with $u \; R_1 \; v \sim v'$ and therefore $u \; R \; v \sim v'$. So suppose $(u', v') \in R_2$. Since $\eta(R_2) = \mathcal{R} = \eta(R_1)$, there exists $(u'', v'') \in R_1$ with $u' \sim u''$ and $v' \sim v''$. Hence we get

$$u \sim u' \sim u'' \; R_1 \; v'' \sim v' \, .$$

Now the left-closedness of R_1 yields a word v with $u \; R_1 \; v \sim v'' \sim v'$ and therefore $u \; R \; v \sim v'$. ◀