Twin-Width of Graphs with Tree-Structured Decompositions

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Abstract

The twin-width of a graph measures its distance to co-graphs and generalizes classical width concepts such as tree-width or rank-width. Since its introduction in 2020 [13, 12], a mass of new results has appeared relating twin width to group theory, model theory, combinatorial optimization, and structural graph theory.

We take a detailed look at the interplay between the twin-width of a graph and the twin-width of its components under tree-structured decompositions: We prove that the twin-width of a graph is at most twice its strong tree-width, contrasting nicely with the result of [7, 6], which states that twin-width can be exponential in tree-width. Further, we employ the fundamental concept from structural graph theory of decomposing a graph into highly connected components, in order to obtain optimal linear bounds on the twin-width of a graph given the widths of its biconnected components. For triconnected components we obtain a linear upper bound if we add red edges to the components indicating the splits which led to the components. Extending this approach to quasi-4-connectivity, we obtain a quadratic upper bound. Finally, we investigate how the adhesion of a tree decomposition influences the twin-width of the decomposed graph.

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1 Introduction

Twin-width is a new graph parameter introduced in [13, 12]. Since its introduction it has gained considerable attention. The *twin-width*[1] of a graph G, denoted by $\mathrm{tww}(G)$, is the minimum width over all contraction sequences of G and a *contraction sequence* of G is roughly defined as follows: we start with a discrete partition of the vertex set of G into n singletons where n is the order of G. Now we perform a sequence of $n-1$ merges, where in each step of the sequence precisely two parts are merged causing the partition to become coarser, until eventually, we end up with just one part – the vertex set of G. Two parts of a partition of $V(G)$ are *homogeneously connected* if either all or none of the possible cross

[1] We refer to the preliminaries of this paper (subsections *graphs and trigraphs* as well as *twin-width*) for an equivalent definition of twin-width which is based on merging vertices instead of vertex subsets.

edges between the two parts are present in G. The red degree of a part is the number of other parts to which it is not homogeneously connected. Finally, the width of a contraction sequence is the maximum red degree amongst all parts of partitions arising when performing the sequence.

In [13, 12] the authors show that twin-width generalizes other width parameters such as rank-width, and, hence also clique-width and tree-width. Furthermore, given a graph H, the class of H-minor free graphs has bounded twin-width and FO-model checking is FPT on classes of bounded twin-width, see [13, 12]. Many combinatorial problems which are NP-hard in general allow for improved algorithms if the twin-width of the input graph is bounded from above and the graph is given together with a width-minimal contraction sequence [9, 8].

Motivation. To decompose a graph into smaller components and estimate a certain parameter of the original graph from the parameters of its components is an indispensable approach of structural graph theory, which serves in taming branch-and-bound trees as well as for theoretical considerations because it allows for stronger assumptions (i.e., high connectivity) on the considered graphs. There are various ways to decompose a graph, e.g., bi-, tri-, or quasi-4-connected components, tree decompositions of small adhesion (the maximum cardinality of the intersection of to adjacent bags), modular decomposition, or decomposition into the factors of a graph product.

So far there is no detailed analysis of the relation between the twin-width of a graph and the twin-width of its biconnected, triconnected, or quasi-4-connected components. The only result towards (k-)connected components is the basic observation that the twin-width of a graph is obviously the maximum over the twin-width over its (1-connected) components. While there already exists a strong analysis of the interplay of tree-width and twin-width (cf. [19, 20]), it is still open how twin-width behaves with respect to the adhesion of a given tree decomposition, which can be significantly smaller than the tree-width of a graph (as an example, consider a graph whose biconnected components are large cliques – the adhesion is 1 whereas the tree-width is the maximum clique size). Further, there exist many variants of tree-width, for example, strong tree-width [23, 16] for which the interplay with twin-width has not yet been discussed in the literature.

Our results. We prove the following bound on the twin-width of a graph:

▶ **Theorem 1.** *If G is a graph of strong tree-width k, then*

$$\operatorname{tww}(G) \leq \frac{3}{2}k + 1 + \frac{1}{2}(\sqrt{k + \ln k} + \sqrt{k} + 2\ln k).$$

This is a strong contrast to the result of [7, 6] that twin-width can be exponential in tree-width. We further provide a class of graphs which asymptotically satisfies that the twin-width equals the strong tree-width. Further, we investigate how to bound the twin-width of a graph in terms of the twin-width of its highly connected components starting with biconnected components.

▶ **Theorem 2.** *If G is a graph with biconnected components $C_1, C_2 \ldots, C_\ell$, then*

$$\max_{i\in[\ell]} \operatorname{tww}(C_i) \leq \operatorname{tww}(G) \leq \max_{i\in[\ell]} \operatorname{tww}(C_i) + 2.$$

Next, we consider decompositions into triconnected components:

▶ **Theorem 3.** *Let $C_1, C_2, \ldots, C_\ell$ be the triconnected components of a biconnected graph G. For $i \in [\ell]$ we construct a trigraph $\overline{C}_i$ from C_i as follows: all virtual edges[2] of C_i are colored red and all other edges remain black. If C_i contains parallel edges, then we remove all but one of the parallel edges such that the remaining edge is red whenever one of the parallel edges was red. Then*

$$\operatorname{tww}(G) \leq \max\left(8 \max_{i\in[\ell]} \operatorname{tww}(\overline{C}_i) + 6, 18\right).$$

Similarly clean decompositions into k-connected graphs with $k > 3$ cannot exist [14, 15]; but we move on one more step and consider the twin-width of a graph with respect to its quasi-4 connected components, introduced by [14, 15].

▶ **Theorem 4.** *Let G be a triconnected graph with quasi-4-connected components $C_1, C_2, \ldots, C_\ell$.*

1. *For $i \in [\ell]$ we construct a trigraph $\widehat{C}_i$ by adding for every 3-separator S in C_i along which G was split a vertex v_S which we connect via red edges to all vertices in S. Then*

$$\operatorname{tww}(G) \leq \max\left(8 \max_{i\in[\ell]} \operatorname{tww}(\widehat{C}_i) + 14, 70\right).$$

2. *For $i \in [\ell]$, we construct a trigraph $\overline{C}_i$ by coloring all edges in 3-separators in C_i along which G was split red. Then*

$$\operatorname{tww}(G) \leq \max\left(4 \max_{i\in[\ell]} \left(\operatorname{tww}(\overline{C}_i)^2 + \operatorname{tww}(\overline{C}_i)\right) + 14, 70\right).$$

For the general case of tree decompositions of bounded adhesion, we get the following:

▶ **Theorem 5.** *For every $k \in \mathbb{N}$ there exist explicit constants D_k and D'_k such that for every graph G with a tree decomposition of adhesion k and parts $P_1, P_2, \ldots, P_\ell$, the following statements are satisfied:*

1. *For each P_i, we construct a trigraph $\widehat{P}_i$ by adding for each adhesion set S in P_i a new vertex v_S which we connect via red edges to all vertices in S. Then*

$$\operatorname{tww}(G) \leq 2^k \max_{i\in[\ell]} \operatorname{tww}(\widehat{P}_i) + D_k.$$

2. *Assume $k \geq 3$. For each P_i, we construct the torso $\overline{P}_i$ by completing every adhesion set in P_i to a red clique. Then*

$$\operatorname{tww}(G) \leq \frac{2^k}{(k-1)!} \max_{i\in[\ell]} \operatorname{tww}(\overline{P}_i)^{k-1} + D'_k.$$

Finally, we refine the result of [19, 20], where the authors bound the twin-width of a graph given its tree-width.

▶ **Theorem 6.** *Let G be a graph with a tree decomposition of width w and adhesion k. Then*

$$\operatorname{tww}(G) \leq 3 \cdot 2^{k-1} + \max(w - k - 2, 0).$$

[2] That is, the pairs of vertices along which G was split to obtain the triconnected components.

Bounding the red degree of decomposition trees. The underlying structure of all the decompositions that we consider in this paper is a tree. We generalize the optimal contraction sequence (cf. [3, 2]) for trees which works as follows: choose a root for the tree. If possible, choose two sibling leaves and contract them (which implies a red edge from the new vertex to its parent). Whenever a parent is joined to two of its leaf-children via red edges, these two children are merged. This ensures that the red degree of any parent throughout the whole sequence never exceeds 2. If there are no sibling leaves, then choose a leaf of highest distance to the root and contract it with its parent. This yields a red edge between the new merged vertex and the former grandparent. Repeat this until we end up with a singleton. We preserve this idea in our proofs to ensure that at no point in time three distinct bag-siblings contribute to the red degree of the vertices in their parent bag.

Further related work. A standard reference on tree-width is [5]. For the basics on graph connectivity and decomposition we refer text books on graph theory such as [24]. The twin-width of a graph given the twin-width of its modular decomposition factors (and in particular, also the twin-width given the width of the factors of a lexicographical product) already has been investigated in [11, 10]. In contrast to the linear-time solvable tree-width decision problem [4] (for a fixed k: is the tree-width of the input graph at most k?), deciding whether the twin-width of a graph is at most 4 is already NP-complete [3, 2]. The twin-width of a graph in terms of its biconnected components has already been considered in [21], where the author obtains a slightly weaker upper bound than Theorem 2.

Organization of the paper. We provide the preliminaries in Section 2. Our results on strong tree-width can be found in Section 3. In Section 4 we prove new bounds on the twin-width of a graph given the twin-widths of its highly connected components, and, we generalize our approach to graphs which allow for a tree-decomposition of small adhesion. Due to space limitations, some of the proofs are omitted. We refer to [17] for the full version of this paper.

2 Preliminaries

For a natural number n, we denote by $[n]$ the n-element set $\{1, \dots, n\}$. For a set A, we write $\mathcal{P}(A)$ for the power set of A. For a natural number $k \leq |A|$, we write $\binom{A}{k}$ for the set of k-element subsets of A.

Graphs and trigraphs. All graphs in this paper are finite, undirected and contain no loops. For a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. We write $|G| := |V(G)|$ for the *order* of G.

A *trigraph* is an undirected, edge-colored graph G with disjoint sets $E(G)$ of *black edges* and $R(G)$ of *red edges*. We can interpret every graph as a trigraph by setting $R(G) = \emptyset$. For a vertex subset A of a trigraph G, we denote by $G[A]$ the *subgraph induced on* A and by $G - A$ the subgraph induced on $V(G) \setminus A$. For a vertex $v \in V(G)$, we also write $G - v$ instead of $G - \{v\}$. If G is a graph, then the *degree* of a vertex $v \in V(G)$ is denoted by $d_G(v)$ (or $d(v)$ if G is clear from context). For trigraphs, we write red-deg$_G(v)$ for the *red degree* of v, i.e., the degree of v in the graph $(V(G), R(G))$. We write $\Delta(G)$ or $\Delta_{\text{red}}(G)$ for the maximum (red) degree of a (tri-)graph G.

A *multigraph* is a graph where we allow multiple edges between each pair of vertices.

Twin-width. Let G be a trigraph and $x, y \in V(G)$ two distinct, not necessarily adjacent vertices of G. We *contract* x *and* y by merging the two vertices to a common vertex z, leaving all edges not incident to x or y unchanged, connecting z via a black edge to all common black neighbors of x and y, and via a red edge to all red neighbors of x or y and to all vertices which are connected to precisely one of x and y. We denote the resulting trigraph by G/xy. A *partial contraction sequence* of G is a sequence of trigraphs $(G_i)_{i\in[k]}$ where $G_1 = G$ and G_{i+1} can be obtained from G_i by contracting two distinct vertices $x_i, y_i \in V(G_i)$. By abuse of notation, we also call the sequence $(x_i y_i)_{i<|G|}$ of contraction pairs a partial contraction sequence. The width of a partial contraction sequence is the maximal red degree of all graphs $G_1, \ldots, G_k$. If the width of a sequence is at most d, we call it a *d-contraction sequence*. A *(complete) contraction sequence* is a partial contraction sequence whose final trigraph is the singleton graph on one vertex. The minimum width over all complete contraction sequences of G is called the *twin-width* of G and is denoted by $\mathrm{tww}(G)$. We often identify a vertex $v \in V(G)$ with the vertices in the graphs G_i that v gets contracted to and sets of vertices with the sets of vertices they get contracted to.

Twin-width has many nice structural properties. For example, it is monotone with respect to induced subgraphs: for every induced subgraph $H \subseteq G$ it holds that $\mathrm{tww}(H) \leq \mathrm{tww}(G)$. Moreover, the twin-width of a disconnected graph is just the maximum twin-width of its connected components.

Tree decompositions and tree-width. Let G be a graph. A *tree decomposition* of G is a pair $\mathcal{T} = (T, \{B_i : i \in V(T)\})$ consisting of a tree T and a family $(B_i)_{i\in V(T)}$ of subsets of $V(G)$, called *bags* satisfying the following conditions

1. every vertex of G is contained in some bag,
2. for every vertex $v \in V(G)$, the set of tree vertices $i \in V(T)$ such that $v \in B_i$ forms a subtree of T,
3. for every edge $e \in E(G)$, there exists some bag which contains both endpoints of e.

The subgraphs $G[B_i]$ are called the *parts* of the tree decomposition. The width of a tree-decomposition is $\max_{i\in V(T)} |B_i| - 1$ and the minimum width over all tree decompositions of G is the *tree-width* of G and is denoted by $\mathrm{tw}(G)$.

For an edge $ij \in E(T)$, the sets $B_i \cap B_j$ are the *adhesion sets* or *separators* of $\mathcal{T}$ and the maximal size of an adhesion set is the *adhesion* of $\mathcal{T}$. The graphs obtained from a part $G[B_i]$ by completing all adhesion sets $B_i \cap B_j$ to cliques is called the *torso* of $G[B_i]$.

Strong tree-width. Strong tree-width, which is also called tree-partition width, is a graph parameter independently introduced by [23] and [16]. A *strong tree decomposition* of a graph G is a tuple $(T, \{B_i : i \in V(T)\})$ where T is a tree and $\{B_i : i \in V(T)\}$ is a set of pairwise disjoint subsets of $V(G)$, one for each node of T such that

1. $V(G) = \bigcup_{i\in V(T)} B_i$ and
2. for every edge uv of G there either exists a node $i \in V(T)$ such that $\{u, v\} \subseteq B_i$ or there exist two adjacent nodes i and j in T with $u \in B_i$ and $v \in B_j$.

The sets B_i are called *bags* and $\max_{i\in V(T)} |B_i|$ is the *width* of the decomposition. The minimum width over all strong tree decompositions of G is the *strong tree-width* $\mathrm{stw}(G)$ of G.

The strong tree-width of a graph is bounded in its tree-width via $\mathrm{tw}(G) \leq 2\,\mathrm{stw}(G) - 1$, see [25]. In the other direction, there is no bound: the strong-tree width of a graph is unbounded in its tree-width [25]. However, it holds that $\mathrm{stw}(G) \in O(\Delta(G) \cdot \mathrm{tw}(G))$, see [25]. Thus, for graphs of bounded degree, the two width notions are linearly equivalent.

▶ Remark 7. In general, the strong tree-width is unbounded in the twin-width of a graph. For example, consider a complete graph on $2n$ vertices. A width-minimal strong tree decomposition of this graph has two bags, each containing n vertices. However the twin-width of a complete graph is 0.

Highly connected components. A *cut vertex* of a graph G is a vertex $v \in V(G)$ such that $G - v$ contains more connected components that G. A maximal connected subgraph of G that has no cut vertex is a *biconnected components* of G. The *block-cut-tree* of G is a bipartite graph where one part is the set of biconnected components of G and the other part is the set of cut vertices of G and a biconnected component is joined to a cut vertex precisely if the vertex is contained in the¨ component. This graph is a forest, and even a tree if G is connected [24]. If we choose a biconnected component as a root of this tree, we can restrict this tree structure to a tree structure on the biconnected components of G. Thus, the decomposition of a graph into biconnected components can also be phrased as follows:

▶ **Theorem 8** (see [24]). *For every connected graph G, there exists a tree decomposition $\mathcal{T}$ of G such that $\mathcal{T}$ has adhesion at most 1, and every part is either 2-connected or a complete graph of order 2. Moreover, the set of bags of this tree decomposition is isomorphism-invariant.*

Similarly, by splitting a graph at certain separators of size at most 2, we obtain the following:

▶ **Theorem 9** ([18]). *For every 2-connected graph, there exists a tree decomposition $\mathcal{T}$ of G such that $\mathcal{T}$ has adhesion at most 2, and the torso of every bag is either 3-connected, a cycle, or a complete graph of order 2. Moreover, the set of bags of this tree decomposition is isomorphism-invariant.*

The *triconnected components of G* are multigraphs constructed from the torsos of this tree decomposition. In this work, these multigraphs are not important and we also call the torsos themselves *triconnected components.*

A similarly clean decomposition into 4-connected components arranged in a tree-like fashion does not exist [14, 15]. This motivated Grohe to introduce the notion of quasi-4-connectivity [14, 15]: A graph G is called *quasi-4-connected* if it is 3-connected and all 3-separators split off at most a single vertex. That is, for every separator S of size 3, the graph $G - S$ splits into exactly two connected components, at least one of which consists of a single vertex. The prime example of quasi-4-connected graphs which are not 4-connected are hexagonal grids. For quasi-4-connectivity, we once again get a tree-like decomposition into components:

▶ **Theorem 10** ([14, 15]). *For every 3-connected graph G, there exists a tree decomposition $\mathcal{T}$ of G such that $\mathcal{T}$ has adhesion at most 3, and the torso of every bag is either quasi-4-connected or of size at most 4.*

The torsos of this tree decomposition are called *quasi-4-connected components of G.*

3 Twin-width of graphs of bounded strong tree-width

▶ **Theorem 1.** *If G is a graph of strong tree-width k, then*

$$\operatorname{tww}(G) \leq \frac{3}{2}k + 1 + \frac{1}{2}(\sqrt{k + \ln k} + \sqrt{k} + 2\ln k).$$

Proof. For a graph H and a vertex subset $U \subseteq V(H)$ a partial contraction sequence s of H is a *U-contraction sequence* if only vertices of U are involved in the contractions in s and s is of length $|U|$, that is, performing all contractions of s yields a partition of $V(H)$ where U forms one part and the rest of the parts are singletons. We denote the minimum width over all U-contraction sequences of H by $\mathrm{tww}_U(H)$.

Let $\mathcal{T} = (T, \{B_i : i \in V(T)\})$ be a strong tree decomposition of G of width k. Fix $r \in V(T)$ and consider T to be a rooted tree with root r from now on. If a bag B_i contains only one vertex v, then we set $v_i := v$. We label all nodes i of T with $|B_i| = 1$ as *merged*. All other nodes of T are labeled as *unmerged*. A node p of T is a *leaf-parent* if all of its children are leaves. If B_i is a bag of $\mathcal{T}$, then *contracting* B_i means to apply a width-minimal B_i-contraction sequence and then relabel i as *merged*. After a contraction of two vertices u and v to a new vertex x we *update* the strong tree decomposition $\mathcal{T}$, that is, if u and v were contained in the same bag, then we simply replace u and v by x. If, otherwise, u and v are contained in adjacent bags, then we remove u and v from its bags and insert x to the bag which is closer to the root. If this causes an empty bag, we remove the bag as well as the corresponding tree-vertex. Observe that updating preserves the strong tree-width. We claim that the algorithm Contract merges G into a single vertex via a contraction sequence of the required width.

Algorithm 1 Contract(G, $\mathcal{T} = (T, \{B_i, i \in V(T)\})$).

while $|V(T)| \geq 2$ **do**
    Choose a leaf ℓ which maximizes the distance to r
    if *the parent p of ℓ has two merged children ℓ_1, ℓ_2,* **then**
        contract v_{ℓ_1} with v_{ℓ_2},
        update $\mathcal{T}$
    if *ℓ is the only child of its parent p and ℓ is merged,* **then**
        contract B_p and denote the resulting vertex b_p,
        contract b_p with v_ℓ,
        update $\mathcal{T}$
    if *amongst ℓ and its siblings there is an unmerged leaf ℓ' and at most one merged leaf,* **then**
        contract $B_{\ell'}$,
        update $\mathcal{T}$
Apply a width-minimal contraction sequence to the remaining graph.

First, we check that that the algorithm terminates. Observe that the root r is not part of any of the contractions in the while-loop. In particular, as long as the loop is executed, there exists at least one leaf-parent. In every iteration of the loop at least one of the if-conditions is satisfied and hence, $|V(G)|$ shrinks with every iteration, which proves that the algorithm terminates with a singleton graph, that is, it provides a contraction sequence.

It remains to bound the width of the sequence. For $a \in \mathbb{N}$ we set $f(a) := (a + \sqrt{a + \ln a} + \sqrt{a} + 2\ln a)/2$. We will exploit the result of [19, 20] that an a-vertex graph has twin-width at most $f(a)$.

Let $(G_i)_{i \leq |G|}$ be the contraction obtained by the algorithm. Fix $i \in [|G|]$ and $v \in G_i$ and let $\mathcal{T}_i = (T_i, \mathcal{B}_i)$ be the strong tree decomposition corresponding to G_i and B_j the bag containing v in $\mathcal{T}$.

If j is neither a leaf, nor in a leaf-parent, nor the parent of a leaf-parent in T_i, then red-deg$(v) = 0$.

Assume that j is a leaf of T_i, then all red edges incident to v are either internal edges of B_j or joining v with a vertex of B_p where p is the parent of j in T_i. Since $\mathrm{stw}(G) \leq k$ there are at most k red edges of the latter form. Internal red edges of a bag may only arise during a the contraction of this leaf-bag in Line 10. Since the corresponding partial contraction sequence is chosen to be width-minimal and by the bound of [19, 20] we obtain that red-deg$_{G_i}(v) \leq k + f(k)$.

Now assume that j is a leaf-parent in T_i. If the bag B_j of $\mathcal{T}_i$ was already contained in $\mathcal{T}$, then there are no internal red edges in B_j and the only red edges incident to v are incident to the vertices of precisely one leaf-bag, or, to the vertices of precisely two leaf-bags one of which is merged. In each of the two cases, the red degree of v in G_i is bounded by $k+1$. Otherwise B_j is obtained during the contraction in Line 6. In this case, j has precisely one child ℓ in T_i and ℓ is merged. Hence, j has at most $k + f(k) + 1$ red neighbors.

Finally, assume that j is neither a leaf nor a leaf-parent but parent of a leaf-parent in T_i. Let $j_1, \dots, j_h$ be the children of j in T_i. Observe that there are at most two children of j, say, j_1 and j_2 such that v is joined to vertices of the corresponding bags and there are no internal red edges in B_j. The only red edges incident to v are arsing during the contraction of B_{j_1} or B_{j_2} in Line 6. Since first, one of the two children is contracted to one vertex before any contraction in the other bag happens, the red degree of v is bounded by $k+1$. ◀

▶ **Lemma 11.** *There exists a family of graphs $(H_n)_{n \in \mathbb{N}}$ such that $\lim_{n\to\infty} \frac{\mathrm{tww}(H_n)}{\mathrm{stw}(H_n)} \geq 1$.*

Proof. For each $n \in \mathbb{N}$ let H_n be the n-th Paley graph. Fix $n \in \mathbb{N}$. It is known that $\mathrm{tww}(H_n) = \frac{|V(H_n)|-1}{2}$, see [19, 20]. By distributing the vertices of H_n to two bags, one of cardinality $\frac{|V(H_n)|+1}{2}$, the other one of cardinality $\frac{|V(H_n)|-1}{2}$, we obtain a strong tree decomposition of H_n of width $\frac{|V(H_n)|+1}{2}$. ◀

4 Twin-width of graphs with small separators

4.1 Biconnected components

We start our investigation of graphs of small adhesion by proving a bound on the twin-width of graphs in terms of the twin-width of their biconnected components. This proof contains many of the ideas we will generalize later to deal with tri- and quasi-4-connected components as well as general graphs with a tree decomposition of bounded adhesion.

The main obstacle to constructing contraction sequences of a graph from contraction sequences of its biconnected components is that naively contracting one component might increase the red degree of the incident cut vertices in the neighboring components arbitrarily. Thus, we need to find contraction sequences of the biconnected components not involving the incident cut vertices.

Let G be a trigraph and $\mathcal{P}$ be a partition of G. Denote by $G/\mathcal{P}$ the trigraph obtained from G by contracting each part of $\mathcal{P}$ into a single vertex. For a vertex $v \in V(G)$ we denote by $\mathcal{P}(v)$ the part of $\mathcal{P}$ that contains v. If $\mathcal{P}(v) \neq \{v\}$, then we obtain a refined partition $\mathcal{P}_v$ by replacing $\mathcal{P}(v)$ in $\mathcal{P}$ by the two parts $\mathcal{P}(v)$ and $\{v\}$. Otherwise, we set $\mathcal{P}_v = \mathcal{P}$. Since $G/\mathcal{P}$ can be obtained from $G/\mathcal{P}_v$ by at most one contraction, and one contraction of a trigraph reduces the maximum red degree by at most 1 we have

$$\Delta_{\mathrm{red}}(G/\mathcal{P}_v) \leq \Delta_{\mathrm{red}}(G/\mathcal{P}) + 1. \tag{1}$$

▶ **Lemma 12.** *For every trigraph G and every vertex $v \in V(G)$,*

$$\operatorname{tww}_{V(G-v)}(G) \leq \operatorname{tww}(G) + 1.$$

Proof. Let $(\mathcal{P}^{(i)})_{i\in[|G|]}$ be a sequence of partitions corresponding to a width-minimal contraction sequence of G. Further, let j be the maximal index with $\{v\} \in \mathcal{P}^{(j)}$. Then $(\mathcal{P}^{(i)})_{i\in[j]}$ is a partial $\operatorname{tww}(G)$-contraction sequence which does not involve v, and by (1) the sequence $(\mathcal{P}_v^{(i)})_{i\in[|G|]\setminus[j+1]}$ is a partial $(\operatorname{tww}(G)+1)$-contraction sequence which contracts the resulting trigraph until v and one further vertex remain. Combining these two sequences yields the claim. ◀

▶ **Theorem 2.** *If G is a graph with biconnected components $C_1, C_2 \ldots, C_\ell$, then*

$$\max_{i\in[\ell]} \operatorname{tww}(C_i) \leq \operatorname{tww}(G) \leq \max_{i\in[\ell]} \operatorname{tww}(C_i) + 2.$$

Proof. The lower bound follows from the fact that all biconnected components are induced subgraphs of G together with the monotonicity of twin-width.

For the upper bound we may assume that G is connected since the twin-width of a disconnected graph is the maximum twin-width of its connected components [13, 12]. Consider the block-cut-tree of G, i.e., the tree T whose vertex set is the union of the biconnected components of G and the cut vertices of G, where every cut vertex joined to precisely those biconnected components containing it. In particular, the biconnected components and the cut vertices form a bipartition of T.

We choose a cut vertex r as a root of T. For every biconnected components $C \in V(T)$, we let v_C be the parent of C in T.

To make our argument simpler, let $\widehat{G}$ be the graph obtained from G by joining a new vertex r_v to every vertex $v \in V(G)$ via a red edge. Similarly, for a biconnected component C, we let $\widehat{C}$ be the graph obtained from C by attaching a new vertex r_v to every vertex v of C. For each cut vertex c, we let $\widehat{G}_c$ be the graph induced by $\widehat{G}$ on the union of all blocks in the subtree T_c of T rooted at c together with all vertices r_v adjacent to these blocks.

We show that $\operatorname{tww}(\widehat{G}) \leq \max_{i\in[\ell]} \operatorname{tww}(C_i) + 2$. The claim then follows since G is an induced subgraph of $\widehat{G}$.

▷ Claim 13. For every biconnected component C of G,

$$\operatorname{tww}_{V(\widehat{C}-v_C)}(\widehat{C}) \leq \operatorname{tww}(C) + 2.$$

Proof of the Claim. By applying Lemma 12 to C and v_C, we find a $V(C - v_C)$-contraction sequence S of C of width at most $\operatorname{tww}(C) + 1$. We show how this contraction sequence can be adapted to also contract the vertices r_v for all cut vertices v incident to C. Indeed, before every contraction vw of S, we insert the contraction of r_v and r_w. This keeps the invariant that we never contract a vertex from C with a vertex r_v, and further, every vertex of C is incident to at most one vertex r_v (or a contraction of those vertices). Moreover, the red degree among the vertices r_v also stays bounded by 2. The entire partial contraction sequence constructed so far thus has width at most $\operatorname{tww}(C) + 2$.

After applying this sequence, we end up with at most four vertices: v_C, r_{v_C}, the contraction of $C - v_C$ and the contraction of all vertices r_v for vertices $v \neq v_C$. As r_{v_C} is only connected to v_C and the contraction of all other vertices r_v is not connected to v_C, these four vertices form a path of length four. Thus, the contraction sequence can be completed with trigraphs of width at most 2. ◁

Now, consider again the whole graph $\widehat{G}$ and choose a leaf block C of T. We can apply the partial contraction sequence given by the previous claim to $\hat{C}$ in $\hat{G}$. Because we never contract v_C with any other vertex, this does not create red edges anywhere besides inside $\hat{C}$. Thus, it is still a partial $(\mathrm{tww}(C) + 2)$-contraction sequence of $\widehat{G}$. Moreover, the resulting trigraph is isomorphic to $\widehat{G} - V(\widehat{C} - v_C)$, i.e., the graph obtained from $\widehat{G}$ by just removing the biconnected component C (but leaving the cut vertex v_C). By iterating this, we can remove all biconnected components one after the other using width at most $\max_{i\in[\ell]} \mathrm{tww}(C_i) + 2$. Finally, we end up with just two vertices: The root cut vertex r, together with its red neighbor r_r, which we can simply contract. ◀

Note that the bounds in Theorem 2 are sharp even on the class of trees: the biconnected components of a tree are just its edges which have twin-width 0. As there are trees both of twin-width 0 and of twin-width 2, both the upper and the lower bound can be obtained.

▶ **Corollary 14.** *Let $\mathcal{C}$ be a graph class closed under taking biconnected components. Then $\mathcal{C}$ has bounded twin-width if and only if the subclass of 2-connected graphs in $\mathcal{C}$ has.*

Moreover, Theorem 2 also reduces the algorithmic problem of computing or approximating the twin-width of a graph to within some factor to the corresponding problem on biconnected graphs.

4.2 Apices and contractions respecting subsets

To deal with adhesion sets of size at least 2, it no longer suffices to find contraction sequences of the parts that just don't contract vertices in the adhesion sets. Indeed, as those vertices can appear parts corresponding to a subtree of unbounded depth, this could create an unbounded number of red edges incident to vertices in adhesion sets. Instead, we want contraction sequences that create no red edges incident to any vertices of adhesion sets.

For a trigraph G and a set of vertices $A \subseteq V(G)$ of red degree 0, we say that a partial sequence of d-contractions $G = G_0, G_1, \ldots, G_\ell$ *respects* A if $G_i[A] = G[A]$ and $\text{red-deg}_{G_i}(a) = 0$ for all $i \leq \ell$ and $a \in A$. Thus, for every contraction xy in the sequence, we have $x, y \notin A$ and $N(x) \cap A = N(y) \cap A$, which implies that the vertices in A are not incident to any red edges all along the sequence.

A *complete d-contraction sequence respecting A* is a sequence of d-contractions that respects A of maximal length, i.e., one whose resulting trigraph G_ℓ does not allow a further contraction respecting A. This is equivalent to no two vertices in $V(G_\ell) \setminus A$ having the same neighborhood in A. In particular, a complete contraction sequence respecting A leaves at most $2^{|A|}$ vertices besides A.

We write $\mathrm{tww}(G, A)$ for the minimal d such that there exists a complete d-contraction sequence respecting A. For a single vertex $v \in V(G)$, we also write $\mathrm{tww}(G, v)$ for $\mathrm{tww}(G, \{v\})$. Note that $\mathrm{tww}(G) = \mathrm{tww}(G, \emptyset)$.

It was proven in [13, 12, Theorem 2] that adding a single apex to a graph of twin-width d raises the twin-width to at most $2d + 2$. The proof given there readily works in our setting without any modifications.

▶ **Theorem 15.** *Let G be a trigraph, $v \in V(G)$ a vertex not incident to any red edges and $A \subseteq V(G) \setminus \{v\}$ a set of vertices. Then*

$$\mathrm{tww}(G, A \cup \{v\}) \leq 2\,\mathrm{tww}(G - v, A) + 2.$$

▶ **Corollary 16.** *Let G be a trigraph and $A \subseteq V(G)$ a subset of vertices with* $\text{red-deg}(a) = 0$ *for all vertices $a \in A$. Then*

$$\operatorname{tww}(G, A) \leq 2^{|A|} \operatorname{tww}(G) + 2^{|A|+1} - 2.$$

4.3 Tree decompositions of small adhesion

We are now ready to generalize the linear bound on the twin-width of a graph in terms of its biconnected components to allow for larger separators of bounded size. This is most easily expressed in terms of tree decompositions of bounded adhesion.

In all of the following two sections, let G be a graph, $\mathcal{T} = ((T, r), \{B_t : t \in V(T)\})$ a rooted tree decomposition with adhesion $k \geq 1$.

For a vertex $t \in V(T)$, we write $P_t := G[B_t]$ for the part associated to t. For a vertex $t \in V(T)$ with parent $s \in V(T)$ we write $S_t := B_t \cap B_s$ and call S_t the *parent separator* of P_t or a *child separator* of P_s. Moreover, we set $S_r := \emptyset$ to be the *root separator*. For a tree vertex $t \in V(T)$, we write T_t for the subtree of T with root t, $G_t := G[\bigcup_{s \in V(T_t)} B_s]$ for the corresponding subgraph and $\mathcal{T}_t$ for the corresponding tree decomposition of G_t.

We can assume w.l.o.g. that every two vertices $s, t \in V(T)$ with $S_s = S_t$ are siblings. Indeed, if they are not, let s' be a highest vertex in the tree with parent separator S_s and construct another tree decomposition by attaching all vertices t with $S_t = S_s$ directly to the parent of s' instead of their old parent. By repeating this procedure if necessary, we obtain the required property.

For a vertex $t \in T$ with children $c_1, \ldots, c_\ell$, we set $N_{c_i} := \{N(v) \cap S_{c_i} : v \in V(G_{c_i}) \setminus S_{c_i}\}$ to be the set of (possibly empty) neighborhoods that vertices in $G_{c_i} - S_{c_i}$ have in the separator S_{c_i} (and thus in P_t). We now define a trigraph $\widetilde{P}_t$ with vertex set

$$V(\widetilde{P}_t) := V(P_t) \mathbin{\dot\cup} \{s_M^{c_i} : i \in [\ell], M \in N_{c_i}\}.$$

We will often abuse notation and also denote the set $\{s_M^{c_i} : M \in N_{c_i}\}$ by N_{c_i}.

We define the edge set of $\widetilde{P}_t$ such that

1. $\widetilde{P}_t[V(P_t)] = P_t$,
2. $\widetilde{P}_t[N_{c_i}]$ is a red clique for every i,
3. $s_M^{c_i}$ is connected via black edges to all vertices in M,

and there are no further red or black edges. Note that in $\widetilde{P}_t$, there are no red edges incident to any vertices in P_t and thus in particular not to any vertices in S_t. A drawing of the gadget attached to S_{c_i} in $\widetilde{P}_t$ in comparison with the simpler gadgets we will reduce to later can be seen in Figure 1.

▶ **Lemma 17.** *Let G and $\mathcal{T}$ be as above. For every $t \in V(T)$, it holds that*

$$\operatorname{tww}(G_t, S_t) \leq \max_{s \in V(T_t)} \operatorname{tww}(\widetilde{P}_s, S_s).$$

In particular, $\operatorname{tww}(G) \leq \max_{s \in V(T)} \operatorname{tww}(\widetilde{P}_s, S_s) \leq 2^k \max_{s \in V(T)} \operatorname{tww}(\widetilde{P}_s) + 2^{k+1} - 2.$

Proof sketch. We proceed inductively, starting by contracting the leaf bags and then moving up the tree. The graphs $\widetilde{P}_t$ are defined precisely so that contracting all child bags of some bag yields $\widetilde{P}_t$. ◀

If $\mathcal{T}$ has bounded width, we can proceed as in [19, 20, Lemma 3.1] to bound the twin-width of the graphs $\widetilde{P}_t$ and thus the twin-width of G:

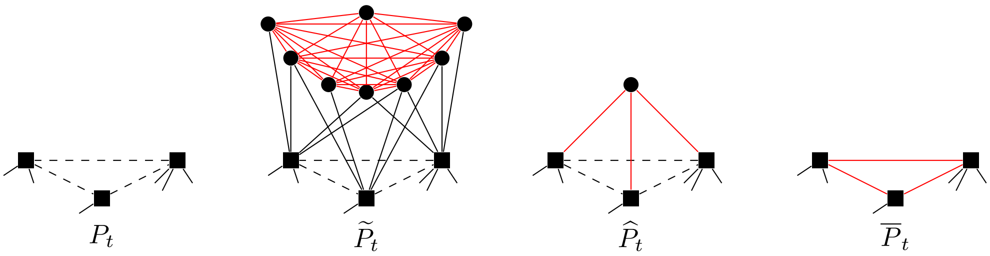

Figure 1 A separator $S_{t'}$ on three (square) vertices together with the three versions of gadgets we attach to it. Dashed edges represent either edges or non-edges. In $\widetilde{P}_t$, we add a red clique consisting of one vertex for every neighborhood of vertices in $G_{t'} - S_{t'}$ in $S_{t'}$. In $\widehat{P}_t$, we only add a single vertex with red edges to all vertices in $S_{t'}$. In $\overline{P}_t$, we add no new vertices but complete all child separators to red cliques.

▶ **Lemma 18.** *Let G and $\mathcal{T}$ be as above and additionally assume that $\mathcal{T}$ has width at most w. For every $t \in V(T)$, it holds that*

$$\operatorname{tww}(\widetilde{P}_t, S_t) \leq 3 \cdot 2^{k-1} + \max(w - k - 2, 0).$$

Proof. We first note that the red degree of $\widetilde{P}$ itself is bounded by $2^k - 1$.

Now, let $c_1, \dots, c_\ell$ be the (possibly empty) list of children of t in T. We first find a contraction sequence of $\bigcup_{i=1}^{\ell} N_{c_i}$ respecting S_t. For this, we argue by induction that $\bigcup_{i=1}^{j-1} N_{c_i}$ can be contracted while preserving the required width. This claim is trivial for $j = 1$. Hence assume we have already contracted $\bigcup_{i=1}^{j-1} N_{c_i}$ to a set B_{j-1} of size at most $2^{|S_P|}$. The vertices of B_{j-1} may be connected via red edges to vertices in B_{j-1} itself and in $P_t \setminus S_t$. Thus, the red degree of vertices in B_{j-1} is bounded by

$$|B_{j-1}| - 1 + |P_t| - |S_t| \leq 2^{|S_t|} + |P_t| - |S_t| - 1 \leq 2^k + w - k - 1,$$

while the red degree of vertices in $P_t \setminus S_t$ is bounded by $|B_j| \leq 2^{|S_t|} \leq 2^k$.

Now, we first apply a maximal contraction sequence of N_{c_j} respecting S_t resulting in a quotient $\bar{N}_{c_j}$. Because of our assumption on the tree decomposition $\mathcal{T}$, we know that $S_t \neq S_{c_j}$ for all $j \in [\ell]$. In particular, this implies that $|S_{c_j} \cap S_t| < k$. and thus $|\bar{N}_{c_j}| \leq 2^{k-1}$. During this contraction sequence, there can appear red edges between the contracted vertices of N_{c_j} and vertices in $P_t \setminus S_t$. The vertices of N_{c_j} thus have red degree bounded by $|N_{c_j}| - 1 + |P_t| - |S_t| \leq 2^k + w - k - 1$. Every red neighbor of vertices in $P_t \setminus S_t$ in a quotient of N_{c_j} must be the contraction of at least two vertices of N_{c_j}. Thus, the red degree of these vertices is bounded by

$$|B_{j-1}| + |N_{c_j}|/2 \leq 2^k + 2^{k-1} = 3 \cdot 2^{k-1}.$$

Next, we contract vertices from B_{j-1} and $\bar{N}_{c_j}$ which have equal neighborhoods in S_t. As our bounds already allow every vertex in $P_t \setminus S_t$ to be connected via red edges to all of $B_{j-1} \cup \bar{N}_{c_j}$, it suffices to argue that this keeps the red degree of vertices in $B_{j-1} \cup \bar{N}_{c_j}$ within our bounds. But this set has size at most $3 \cdot 2^{k-1}$. Hence, after one contraction, the red degree is bounded by

$$|B_{j-1}| + |\bar{N}_{c_j}| - 2 + |P_t| - |S_t| \leq 3 \cdot 2^{k-1} + w - k - 2.$$

We have now successfully contracted N_{c_j} into B_{j-1} while keeping the red degree bounded by

$$\max(2^k + w - k - 1, 2^k, 3 \cdot 2^{k-1}, 3 \cdot 2^{k-1} + w - k - 2) = 3 \cdot 2^{k-1} + \max(w - k - 2, 0).$$

By repeating this procedure for all $j \in [\ell]$, we find a contraction sequence of $\bigcup_{i=1}^{\ell} N_{c_i}$ respecting S_t within this width. The resulting graph thus consists of S_t, the vertices of $P_t \setminus S_t$ and the vertices from B_ℓ. In total, these are at most $2^{|S_t|} + |P_t| - |S_t| \leq 2^k + w - k$ vertices besides those in S_t. These can further be contracted while keeping the red degree bounded by

$$2^k + w - k - 1 \leq 3 \cdot 2^{k-1} + w - k - 2.$$

In total, our contraction sequence thus has width at most $3 \cdot 2^{k-1} + \max(w-k-2, 0)$, proving the claim. ◀

By combining Lemma 18 with Lemma 17, we obtain a general bound on the twin-width of graphs admitting a tree decomposition of bounded width and adhesion:

▶ **Theorem 6.** *Let G be a graph with a tree decomposition of width w and adhesion k. Then*

$$\operatorname{tww}(G) \leq 3 \cdot 2^{k-1} + \max(w - k - 2, 0).$$

This upper bounds sharpens the bound given in [19, 20] by making explicit the dependence on the adhesion of the tree decomposition. Our bound shows that, while the twin-width in general can be exponential in the tree-width [7, 6], the exponential dependence comes from the adhesion of the tree decomposition and not from the width itself.

Moreover, our bound is asymptotically sharp. As already mentioned, it is known that there are graphs whose twin-width is exponential in the adhesion of some tree decomposition [7, 6]. By adding into some bag a Paley graph whose twin-width is linear in its size [1], we also achieve asymptotic sharpness in the linear width term.

4.4 Simplifying the parts

Before we apply this general lemma to the special case of the tree of bi-, tri- or quasi-4-connected components, we show that we can simplify the gadgets attached in the graphs $\widetilde{P}$ to all separators while raising the twin-width by at most a constant factor.

In a first step, we replace the sets N_{c_i} from the definition of the parts $\widetilde{P}_t$ by a single common red neighbor for every separator. For every vertex $t \in V(T)$ with children $c_1, \ldots, c_\ell$, we define the trigraph $\widehat{P}_t$ as follows: we set $\mathcal{S}(t) \coloneqq \{S_{c_i} \colon i \in [\ell], S_{c_i} \nsubseteq S_{c_j} \text{ for all } j \in [\ell]\}$ to be the set of subset-maximal child separators of P_t. Now, we take a collection of fresh vertices $V_S \coloneqq \{v_S \colon S \in \mathcal{S}(t)\}$ and set

$$V(\widehat{P}_t) \coloneqq V(P_t) \mathbin{\dot\cup} V_S.$$

The subgraph induced by $\widehat{P}_t$ on $V(P_t)$ is just P_t itself. The vertex v_S is connected via red edges to all vertices in S and has no further neighbors. A drawing of the gadget attached to S_{c_i} in $\widehat{P}_t$ can be found in Figure 1.

▶ **Lemma 19.** *Let G and $\mathcal{T}$ be as before. Then for every $t \in V(T)$, it holds that*

$$\operatorname{tww}(\widetilde{P}_t, S_t) \leq \max(2^k \operatorname{tww}(\widehat{P}_t) + 2^{k+1} - 2, 4^k + 2^k - 2).$$

In particular, $\operatorname{tww}(G) \leq \max(2^k \max_{t \in V(T)} \operatorname{tww}(\widehat{P}_t) + 2^{k+1} - 2, 4^k + 2^k - 2)$.

Proof sketch. If no two child separators of P_t are contained in each other, the claim can be proven by first applying Corollary 16 and then carefully contracting $\widetilde{P}_t$ to $\widehat{P}_t$. The general case can be reduced to this special case. ◀

Next, we want to define a version $\overline{P}_t$ of the parts which does not need extra vertices in P_t but instead marks the separators via red cliques. Indeed, let $\overline{P}_t$ be the trigraph obtained from P_t by completing each of the sets $S \in \mathcal{S}_t$ to a red clique. Thus, the underlying graphs of the trigraphs $\overline{P}_t$ are just the *torsos* of the tree decomposition. We thus call the graphs $\overline{P}_t$ the *red torsos* of the tree decomposition $\mathcal{T}$.

▶ **Lemma 20.** *For every $t \in V(T)$, it holds that*

$$\operatorname{tww}(\widehat{P}_t) \leq \max\left(k+1, \operatorname{tww}(\overline{P}_t) + \binom{\operatorname{tww}(\overline{P}_t)}{k-1}, \operatorname{tww}(\overline{P}_t) + \binom{2k-3}{k-1}\right).$$

In particular,

$$\operatorname{tww}(G) \leq \max\left(\begin{array}{l} 2^k \max_{t\in V(T)}\left(\operatorname{tww}(\overline{P}_t) + \binom{\operatorname{tww}(\overline{P}_t)}{k-1}\right) + 2^{k+1} - 2, \\ 2^k \max_{t\in V(T)} \operatorname{tww}(\overline{P}_t) + 2^k\binom{2k-3}{k-1} + 2^{k+1} - 2, \\ 4^k + 2^k - 2 \end{array}\right)$$

Proof sketch. We extend a contraction sequence of $\overline{P}_t$ to a contraction sequence of $\widehat{P}_t$ while ensuring that the neighborhoods of vertices v_S are not contained in each other. Then, the bound on the red degree of vertices can be proven via a variant of Sperner's theorem [22]. ◀

Combining Lemma 19 and Lemma 20, we get the following two asymptotic bounds on the twin-width of a graph admitting a tree decomposition of small adhesion.

▶ **Theorem 5.** *For every $k \in \mathbb{N}$ there exist explicit constants D_k and D'_k such that for every graph G with a tree decomposition of adhesion k and parts $P_1, P_2, \ldots, P_\ell$, the following statements are satisfied:*

1. $\operatorname{tww}(G) \leq 2^k \max_{i\in[\ell]} \operatorname{tww}(\widehat{P}_i) + D_k$,
2. *if $k \geq 3$, then* $\operatorname{tww}(G) \leq \dfrac{2^k}{(k-1)!} \max_{i\in[\ell]} \operatorname{tww}(\overline{P}_i)^{k-1} + D'_k$.

4.5 Tri- and quasi-4-connected components

We now want to apply these general results on the interplay between twin-width and tree decompositions of small adhesion to obtain bounds on the twin-width of graphs in terms of the twin-width of their tri- and quasi-4-connected components.

▶ **Theorem 3.** *Let $C_1, C_2, \ldots, C_\ell$ be the triconnected components of a biconnected graph G. If we write $\overline{C}_i$ for the red torsos of the triconnected components C_i, then*

$$\operatorname{tww}(G) \leq \max\left(8 \max_{i\in[\ell]} \operatorname{tww}(\overline{C}_i) + 6, 18\right).$$

Proof. This follows from Lemma 20 applied to the tree of triconnected components of G together with the observation that for $k = 2$, the second term in the maximum in Lemma 20 is always bounded by the maximum of the first and third term. ◀

Note that in Theorem 3 we cannot hope for a lower bound similar to the lower bound in Theorem 2 without dropping the virtual edges. Indeed, consider a 3-connected graph G of large twin-width (e.g. Paley graphs or Rook's graphs). By [3, 2], a $(2\lceil\log(|G|)\rceil - 1)$-subdivision H of G has twin-width at most 4, but its triconnected components are G and multiple long cycles. Thus, there exist graphs of twin-width at most 4 with triconnected components of arbitrarily large twin-width.

Moreover, the red virtual edges in each separator can also not be replaced by black edges.

▶ **Lemma 21.** *There exists a family of graphs $(G_n)_{n\in\mathbb{N}}$ with unbounded twin-width such that the twin-width of the class of triconnected components of G_n with black virtual edges is bounded.*

Proof. Let G_n be the graph obtained from a clique K_n by subdividing every edge once. The triconnected components of this graph are the K_n and a K_3 for every edge of the K_n, which all have twin-width 0.

In order to show that the twin-width of the family $(G_n)_{n\in\mathbb{N}}$ is unbounded, we show that for every $d \geq 2$ and $n \geq n_d \coloneqq (d+1)\binom{d}{2} + 1$, we have $\operatorname{tww}(G_n) > d$. For this, consider any d-contraction sequence of G_n for $n \geq n_d$ and let $\mathcal{P}$ be the partition of $V(G_n)$ right before the first contraction in the sequence that does not contract two subdivision vertices. We show that every partition class $P \in \mathcal{P}$ has size at most $\binom{d}{2}$. As no subdivision vertices were contracted so far, we only need to consider classes of subdivision vertices. Thus, let $P = \{v_{e_1}, \dots, v_{e_\ell}\}$ be such a class, where $e_1, \dots, e_\ell \in \binom{V(K_n)}{2}$ are edges of the original K_n. If the edges e_i all have a common endpoint, then P has red edges to all ℓ other endpoints of these edges, meaning that $\ell \leq d \leq \binom{d}{2}$. Otherwise, P has red edges to all endpoints of all e_i. If $\ell > \binom{d}{2}$, these have to be more that d, which is a contraction. Thus, $|P| = \ell \leq \binom{d}{2}$.

Now, let xy be the next contraction in the sequence. If neither x nor y is a subdivision vertex, then G_n contains precisely $2(n-2)$-many vertices which are connected to either x or y but not both. In the contracted graph, the contraction would thus create at least $\frac{2(n-2)}{\binom{d}{2}} \geq 2d+2$ red edges incident to the contracted vertex. If, on the other hand, either x or y is a subdivision vertex but the other is not, then x and y have no common neighbors. But as non-subdivision vertices have degree $n-1$ in G_n, contracting these two would create at least $\frac{n-1}{\binom{d}{2}} \geq d+1$ red edges incident to the contracted vertex. Thus, no further contraction keeps the red degree of the sequence bounded by d, which implies $\operatorname{tww}(G_n) > d$. ◀

In the case of separators of size 3, we get two bounds on the twin-width of a graph in terms of its quasi-4-connected components: one linear bound in terms of the subgraphs induced on the quasi-4-connected components together with a common red neighbor for every 3-separator along which the graph was split, and one quadratic bound in terms of the (red) torsos of the quasi-4-connected components.

▶ **Theorem 4.** *Let G be a triconnected graph with quasi-4-connected components $C_1, C_2, \dots, C_\ell$.*

1. *For $i \in [\ell]$ we construct a trigraph $\widehat{C}_i$ by adding for every 3-separator S in C_i along which G was split a vertex v_S which we connect via red edges to all vertices in S. Then*

$$\operatorname{tww}(G) \leq \max\left(8 \max_{i\in[\ell]} \operatorname{tww}(\widehat{C}_i) + 14, 70\right).$$

2. *For $i \in [\ell]$, denote by $\overline{C}_i$ the* red torso *of the quasi-4-connected component C_i. Then*

$$\operatorname{tww}(G) \leq \max\left(4 \max_{i\in[\ell]} \left(\operatorname{tww}(\overline{C}_i)^2 + \operatorname{tww}(\overline{C}_i)\right) + 14, 70\right).$$

Proof. The two claims follow from Lemma 19 and Lemma 20 applied to the tree of quasi-4-connected components of G [14, 15] together with the observation that also for $k = 3$, the second term in the maximum in Lemma 20 is always bounded by the maximum of the first and third term. ◀

5 Conclusion and further research

We proved that $\text{tww}(G) \leq \frac{3}{2}k + 1 + \frac{1}{2}(\sqrt{k + \ln k} + \sqrt{k} + 2\ln k)$ if G is a graph of strong tree-width at most k (Theorem 1). Moreover, we demonstrated that asymptotically the twin-width of a Paley graph agrees with its strong tree-width (Lemma 11).

We provided a detailed analysis of the relation between the twin-width of a graph and the twin-width of its highly connected components. Concerning 2-connected graphs, the twin-width of a graph is linear in the twin-width of its biconnected components (Theorem 2). There is a linear upper bound for a slightly modified version of triconnected components (Theorem 3). By further providing a quadratic upper bound on the twin-width of graph given the twin-widths of its modified quasi-4-connected components (Theorem 4) we took one important step further to complete the picture of the interplay of the twin-width of a graph with the twin-width of its highly connected components. As a natural generalization of the above decompositions we considered graphs allowing for a tree decomposition of small adhesion (Theorem 5 and Theorem 6).

It seems worthwhile to integrate our new bounds for practical twin-width computations, for example, with a branch-and-bound approach.

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