Two-State Spin Systems with Negative Interactions

Yumou Fei
School of Mathematical Sciences, Peking University, China

Leslie Ann Goldberg
Department of Computer Science, University of Oxford, UK

Pinyan Lu
Laboratory of Interdisciplinary Research of Computation and Economics (SUFE), Ministry of Education, Shanghai University of Finance and Economics, China

Abstract

We study the approximability of computing the partition functions of two-state spin systems. The problem is parameterized by a 2×2 symmetric matrix. Previous results on this problem were restricted either to the case where the matrix has non-negative entries, or to the case where the diagonal entries are equal, i.e. Ising models. In this paper, we study the generalization to arbitrary 2×2 interaction matrices with real entries. We show that in some regions of the parameter space, it's #P-hard to even determine the sign of the partition function, while in other regions there are fully polynomial approximation schemes for the partition function. Our results reveal several new computational phase transitions.

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1 Introduction

Spin systems are widely studied in statistical physics, probability theory and theoretical computer science. They can express many natural graph invariants such as the number of independent sets or the number of k-colorings, as well as spin models of statistical physics such as the Ising model or the Potts model.

1.1 The Problem

The partition function of a q-state spin system can be parameterized by a symmetric matrix $A \in \mathbb{R}^{q \times q}$. It associates with every graph $G = (V, E)$ the real number

$$Z(G; A) = \sum_{\sigma \in [q]^V} \prod_{\{u,v\} \in E} A_{\sigma(u),\sigma(v)}.$$

▶ Remark 1. Throughout the paper, the word "graph" refers to undirected multigraph permitting self-loops and parallel edges.

Fixing a symmetric matrix A, the complexity of exactly computing $Z(G; A)$ given input G was studied and settled by [10] (for A with 0/1 entries), [6] (for A with nonnegative entries), [13] (for A with real algebraic entries), and [7] (for A with complex algebraic entries). They proved the remarkable "dichotomy theorem", which states that either computing $Z(G; A)$ can be done in polynomial time or it is #P-hard, and the class of tractable matrices A, although lacking a simple explicit characterization, is polynomial-time decidable.

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In this paper, we study the problem of *approximately* computing $Z(G;A)$. For simplicity of handling models of computation, we restrict our attention to rational numbers. We will deal exclusively with two-state spin systems ($q=2$), as they already appear challenging enough:

▶ **Problem 2.** *For which symmetric matrices* $A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix} \in \mathbb{Q}^{2\times 2}$ *is approximately computing* $Z(G;A)$ *tractable?*

If $A_{01} = A_{10} = 0$, it is easy to see that Z_G can be computed exactly in polynomial time (see also [6]). In the following, assume $A_{01} = A_{10} \neq 0$, and we normalize the matrix A so that $A_{01} = A_{10} = 1$. Then A is given by two parameters $A_{00} = \beta$ and $A_{11} = \gamma$. Whenever β and γ is fixed, we abbreviate $Z(G;A)$ to Z_G.

Problem 2 is well studied for nonnegative matrix entries. In the nonnegative quadrant $\beta, \gamma \geq 0$, [17] gave an FPRAS for the "ferromagnetic" case $\beta\gamma \geq 1$. The "antiferromagnetic" case $\beta\gamma < 1$ was later very much settled by a series of work [17, 33, 31, 32, 23, 30, 12]. They proved a computational phase transition that coincides with the boundary of the "uniqueness region" (uniqueness of Gibbs measure on infinite regular trees). Their results in fact extend much beyond Problem 2: the computational phase transition for the anti-ferromagnetic case holds even when external fields are allowed.

However, much less is known about Problem 2 when β or γ is negative. The only existing results in this direction are about the Ising model, which means the special case $\beta = \gamma$. Embedded in a broader study about Tutte polynomials, the following theorems from [16] and [15] classified the approximation complexity of Ising partition functions with negative β:

▶ **Proposition 3** (Corollary 28 of [16])**.** *Fix rational numbers* β, γ *such that* $\beta = \gamma \in (-1, 0)$*. It is #P-hard to determine the sign of the partition function* Z_G*, given an input graph* G*.*

▶ **Proposition 4** (Lemma 7 of [15])**.** *Fix rational numbers* β, γ *such that* $\beta = \gamma < -1$*. Approximating the partition function* Z_G *for an input graph* G *is equivalent to approximately counting perfect matchings in general graphs in the sense that there are approximation-preserving reductions between these problems, implying that either both problems have an FPRAS or neither problem has an FPRAS. Whether approximately counting perfect matchings is tractable or not is a central open question in the area.*

Note that at the point $(\beta, \gamma) = (-1, -1)$, Z_G can be computed exactly in polynomial time (Z_G is $2^{|V(G)|}$ if all vertex degrees are even and 0 otherwise).

1.2 Our Results

In this paper, we explore Problem 2 in the case $\min\{\beta, \gamma\} < 0$. We obtain the following generalization of Proposition 3, whose proof is given in the full version of the paper.

▶ **Theorem 5.** *Fix rational numbers* β, γ *such that* $\min\{\beta, \gamma\} < 0$ *and* $-2 < \beta + \gamma < 1$*, but* $(\beta, \gamma) \notin \{(1, -1), (-1, 1)\}$*. It is #P-hard to determine the sign of the partition function* Z_G*, given an input graph* G*.*

Note that when $(\beta, \gamma) \in \{(1, -1), (-1, 1)\}$, Z_G can be computed exactly in polynomial time [13, Theorem 1.2].

The techniques used in the proof of Theorem 5 are analogous to those in [5] – see Remark 5 of the full version. To prove Theorem 5, we first prove that all real numbers can be "realized" (up to an exponential accuracy) by ratios of the form $[Z_{G,v}]_1 \,/\, [Z_{G,v}]_0$ where $[Z_{G,v}]_i$ is the contribution to the partition function from configurations in which vertex v gets spin i (see Section 2.1 for the precise definition). This fact is used to obtain a reduction from the problem of *exactly* computing the number of minimum cardinality (s, t) cuts in a graph to the problem of determining the sign of the partition function Z_G.

Of course Theorem 5 has ramifications for the complexity of approximating Z_G. In particular, an FPRAS for approximating Z_G gives a polynomial-time randomised algorithm for computing the sign of Z_G, which is not possible assuming that #P-hard problems cannot be solved in randomised polynomial time.

It is then of great interest to find whether the two lines $\beta + \gamma = -2$ and $\beta + \gamma = 1$ are actual thresholds of approximation complexity. The following two theorems, both of which will be proved in Section 3, show that the former line is indeed an actual threshold:

▶ **Theorem 6.** *Fix rational numbers β, γ such that $\beta \neq \gamma$ and $|\beta + \gamma| > 2$. For any positive integer Δ, there is an FPTAS for Z_G, where G is an input graph of maximum degree no more than Δ (without the bounded degree requirement, there is a quasi-polynomial time approximation scheme).*

▶ **Theorem 7.** *Fix rational numbers β, γ such that $\beta \neq \gamma$ and $|\beta + \gamma| \geq 2$. There is an FPRAS for Z_G, where G is an input graph.*

Note that Theorem 7 contains the boundary case $|\beta + \gamma| = 2$, which Theorem 6 doesn't. What's more, since Theorem 7 doesn't require the input graph to be bounded degree, it is not subsumed by Theorem 6 even for the range $|\beta + \gamma| > 2$.

The algorithm of Theorem 6 is based on the zero-freeness framework of [2] and Asano's contraction method [1], while the algorithm of Theorem 7 relies on the "windability" framework of [25] and a holographic transformation. The zero-freeness framework, achieving notable successes in problems with nonnegative parameters (e.g. [27]), applies naturally in the presence of mixed signs as well. In contrast, the "windability" framework, or more generally Markov-chain-based methods only make sense for problems with positive parameters. It is thus somewhat surprising that, via a holographic transformation, we are able to transform the problem into one with positive parameters and furthermore prove the rapid mixing of a Markov chain, for the *maximum possible* parameter range based on a lower bound on $|\beta + \gamma|$.

Now, the obvious challenge is to determine the approximation complexity in the remaining region, that is, for parameters β, γ such that $\min\{\beta, \gamma\} < 0$ and $1 \leq \beta + \gamma < 2$. Unfortunately, we are unable to fully achieve this goal. Instead, we give some results that might provide some insights into this challenge (see Section 4 for more discussion).

▶ **Theorem 8.** *Let β, γ be real numbers such that $\beta + \gamma \geq 1$. Then for any graph G, the partition function Z_G is positive.*

▶ Remark 9. For $\beta + \gamma \leq -2$, it is easy to find a graph G such that $Z_G < 0$ (e.g. a single self-loop or a triangle). When $-2 < \beta + \gamma < 1$ and $\min\{\beta, \gamma\} < 0$ and $\beta, \gamma \notin \{(-1, 1), (1, -1)\}$, Theorem 5 implies that Z_G is negative for some graph G. When $(\beta, \gamma) \in \{(1, -1), (-1, 1)\}$, Z_G is negative for $G = K_4$ (the 4-clique). Combined with these observations, Theorem 8 completely determines the range of parameters β and γ for which the partition function Z_G is always nonnegative: the union of the half plane $\beta + \gamma \geq 1$ and the first quadrant $\beta, \gamma \geq 0$.

Theorem 8 suggests that approximating the partition function is unlikely #P-hard when $\beta + \gamma \geq 1$, and hence the line $\beta + \gamma = 1$ is likely some threshold of approximation complexity.

The proof of Theorem 8 is by induction on the size of the graph and will be given in the full version of the paper. In fact, such recursion methods have also been widely used to show zero-freeness of some partition functions on the complex plane (e.g. [24]), which in turn leads to deterministic approximation algorithms by the framework of [2]. For our partition function, we show that such recursions can be used to determine the largest zero-free disk around 0 for the range $\{(\beta,\gamma) : \gamma < 0 \text{ and } 1 \le \beta+\gamma \le 2\}$.

▶ **Theorem 10.** *Let β, γ be real numbers such that $\gamma < 0$ and $1 \le \beta+\gamma \le 2$. Then for any graph G, the polynomial $Z_G(x)$ as defined in Section 2.1 is zero-free on the disk $\left\{z \in \mathbb{C} : |z| < \frac{\beta-1}{1-\gamma}\right\}$. Furthermore, $\frac{\beta-1}{1-\gamma}$ is the maximum possible radius such that the zero-freeness holds for all graphs G.*

Using the same type of recursion in a more sophisticated way, we are able to show that the partition function Z_G is efficiently computable if $\beta+\gamma$ is sufficiently close to 2, by slightly extending the zero-free region of Theorem 10. This suggests the line $\beta+\gamma = 2$ is *not* really a computational threshold:

▶ **Theorem 11.** *Let $g : (1, +\infty) \to (0, 1)$ be the following function:*

$$g(\beta) = \max\left\{\frac{\beta-2}{\beta^2-1}, \frac{(\beta-1)^2}{\beta^3+\beta^2-\beta}\right\}. \tag{1}$$

Fix rational numbers β, γ such that $\min\{\beta,\gamma\} < 0$ and $\beta+\gamma > 2 - g(\max\{\beta,\gamma\})$. For any positive integer Δ, there is an FPTAS for Z_G, where G is an input graph of maximum degree no more than Δ (without the bounded degree requirement, there is a quasi-polynomial time approximation scheme).

Theorem 11 breaks the algorithmic barrier $\beta+\gamma = 2$ presented by Theorem 6 and shows that the line $\beta+\gamma = 2$ behaves in a completely different way from the line $\beta+\gamma = -2$. The proof of Theorem 11 will be given in the full version of the paper.

1.3 More Related Work

Most of the literature studying 2-state spin systems is restricted to the case where the edge interactions β and γ and the vertex weights λ (i.e. external fields, see Section 2.1) are all nonnegative. But there are also some related lines of work where negative or even complex parameters have received more attention.

For instance, in the case of the Ising model, besides the results mentioned in Proposition 3 and Proposition 4, [14] studies the approximation complexity of $Z(G; A)$, where $A = \begin{bmatrix}\beta & 1\\ 1 & \beta\end{bmatrix}$ and β is any algebraic *complex* number, partly motivated by the connection with quantum complexity classes.

Another line of research concerns the hard-core model (this corresponds to interactions $\beta = 1$ and $\gamma = 0$ with external fields). Regarding this model there has been much work on the complexity of approximating $Z_G(\lambda)$ for bounded-degree graphs G varying parameter $\lambda \in \mathbb{C}$ [20, 11, 5]. Here the study of the complexity of approximation is intimately related to the study of optimal zero-free regions of the polynomial $Z_G(x)$ [3, 9, 4].

2 Preliminaries

As in Section 1.1, we consider a fixed symmetric matrix $A = \begin{bmatrix}A_{00} & A_{01}\\ A_{10} & A_{11}\end{bmatrix} \in \mathbb{Q}^{2\times 2}$.

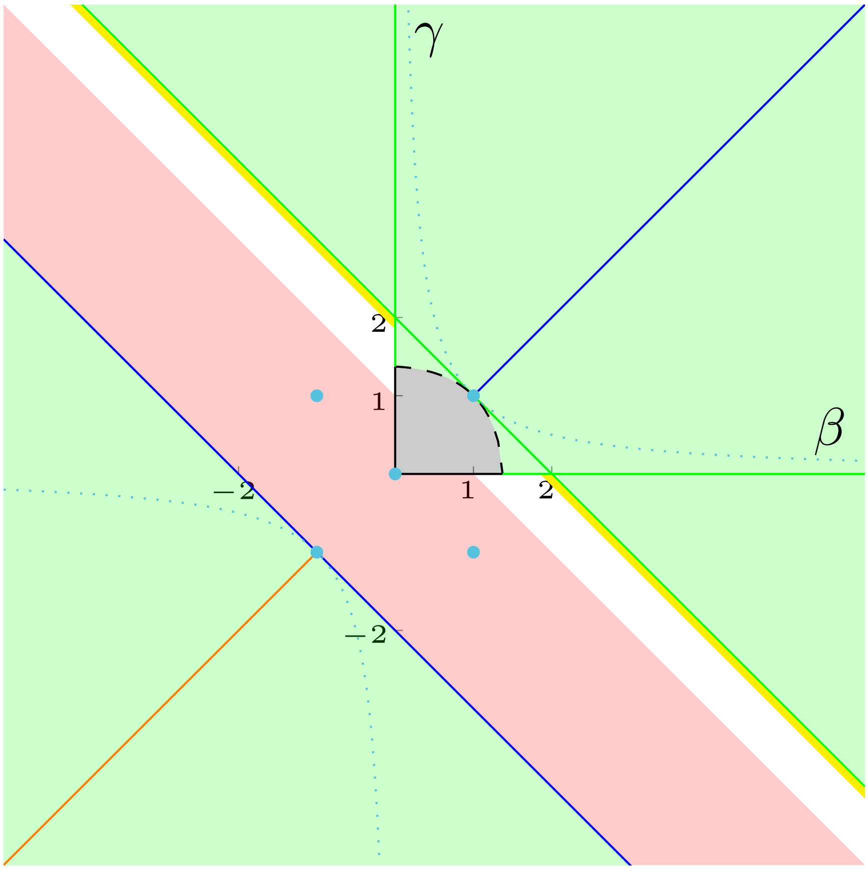

Figure 1 An illustration of the complexity classfication. The sky-blue dots $\{(\beta, \gamma) : \beta\gamma = 1\} \cup \{(-1, 1), (0, 0), (1, -1)\}$ are where Z_G can be computed exactly in polynomial time [6, 13]. Sitting in the bottom-left corner of the first quadrant, the black region is where approximating the partition function is known to be NP-hard [32]. The dashed line stands for the uniqueness boundary for anti-ferromagnetic 2-spin systems. When (β, γ) falls in the green regions, there is an FPTAS for Z_G on bounded degree graphs (due to Theorem 6 and [23]), and an FPRAS for Z_G on all graphs (due to Theorem 7 and [8]). The thin yellow strips to the left of the $\beta + \gamma = 2$ line are where an FPTAS for bounded degree graphs is given by Theorem 11, suggesting that $\beta + \gamma = 2$ is not a threshold. When (β, γ) falls on the blue lines, there is an FPRAS for Z_G (the line $\beta+\gamma = -2$ follows from Theorem 7, while the ray $\beta = \gamma > 1$ is due to [21]). In the red region, apart from the points $(-1, 1)$ and $(1, -1)$, approximating Z_G is #P-hard (Theorem 5). On the orange line, approximating the partition function is equivalent to approximately counting perfect matchings [15].

2.1 Notations

For $G = (V, E)$ and $\boldsymbol{\lambda} \in \mathbb{R}^V$, let

$$Z_G(\boldsymbol{\lambda}) = \sum_{\sigma \in \{0,1\}^V} \left(\prod_{\{u,v\} \in E} A_{\sigma(u),\sigma(v)} \prod_{v \in V} \lambda_v^{\sigma(v)} \right).$$

Here $\boldsymbol{\lambda}$ is the vector of *external fields*. As a special case, we have $Z_G = Z_G(\mathbf{1})$. By setting $\lambda_v = x$ for all $v \in V$, we get a univariate polynomial $Z_G(x)$.

For $v \in V$, let $[Z_{G,v}]$ be a 2×1 vector whose i-th coordinate is

$$[Z_{G,v}]_i = \sum_{\sigma \in \{0,1\}^V} \mathbb{1}\{\sigma(v) = i\} \left(\prod_{\{u,v\} \in E} A_{\sigma(u),\sigma(v)} \prod_{v \in V} \lambda_v^{\sigma(v)} \right).$$

When $[Z_{G,v}]_0 \neq 0$, we define the ratio $R_{G,v} = [Z_{G,v}]_1 / [Z_{G,v}]_0$.

2.2 #CSP and Holant Problems

The problem of computing the partition function of a spin system can be seen as an instance of #CSP problem with a single symmetric binary constraint function. In fact, we may identify the symmetric matrix A with the binary function ψ defined by $\psi(i,j) = A_{ij}$. Then we can denote by #CSP($\{\psi\}$) the problem of computing $Z(G; A)$ given G.

In Sections 3.3 and 3.4, we will utilize the connection between #CSP problems and Holant problems. A Holant instance is a graph $G = (V, E)$ with a variable on each edge and a constraint on each vertex. The constraint on a vertex v is a function $F_v : \{0,1\}^{J_v} \to \mathbb{C}$, where J_v is the set of edges incident to v.

▶ Remark 12. Self-loops might bring in some ambiguity here. But in this paper, we don't consider self-loops in the context of Holant problems, as we're not going to need them.

Let $\mathcal{F}$ be a class of constraint functions. A Holant problem Holant($\mathcal{F}$) asks for computing the partition function

$$\sum_{\sigma \in \{0,1\}^E} \prod_{v \in V} F_v(\sigma|_{J_v})$$

on input $(G, (F_v)_{v \in V})$, where each $F_v \in \mathcal{F}$.

A particular kind of constraint functions we will use in Sections 3.3 and 3.4 is the parity functions. For all positive integer d define $\mathbf{Even}_d, \mathbf{Odd}_d : \{0,1\}^d \to \{0,1\}$ by setting $\mathbf{Even}_d(x_1, \cdots, x_d) = 1$ if and only if $x_1 + \cdots + x_d$ is even and setting $\mathbf{Odd}_d(x_1, \cdots, x_d) = 1$ if and only if $x_1 + \cdots + x_d$ is odd.

3 Approximation Schemes

In this section, we give the two approximation schemes promised in Theorem 6 and Theorem 7.

3.1 Preliminaries for the FPTAS

The deterministic approximation scheme of Theorem 6 will mainly rely on the powerful zero-freeness framework. In particular, our main tool is the following lemma developed and proved in [2] and [26].

▶ **Lemma 13.** *Fix rational numbers β and γ. Let U be an open set in the complex plane that contains the real interval $[0, \lambda]$ for some $\lambda \in \mathbb{Q}^+$. Suppose that for all graphs G the polynomial $Z_G(x)$ has no complex root in U. Then for any positive integer Δ, there exists an FPTAS for $Z_G(\lambda)$, where G is an input graph of maximum degree no more than Δ (without the bounded degree requirement, there is a quasi-polynomial time approximation scheme).*

Our method for showing zero-freeness is the classical contraction method. It was first introduced in [1] to give a simple proof for the Lee-Yang circle theorem [22], and was further extended in [28]. These results have been used previously in the area of approximate counting, e.g., by Sinclair and Srivastava [29] and by Guo, Liao, Lu, and Zhang [18]. Note especially that [18] uses these results for approximation algorithms. We restate the theorem in [28] in the following form:

▶ **Lemma 14.** *For each $i \in [m]$, let K_i be a subset of the complex plane $\mathbb{C}$ that doesn't contain 0. Suppose the complex multi-affine polynomial*

$$P(z_1, \cdots, z_m) = \sum_{I \subseteq [m]} F(I) \prod_{i \in I} z_i,$$

where each $F(I)$ is a complex coefficient, vanishes only when $z_i \in K_i$ for some $i \in [m]$. Write $[m]$ as a disjoint union of subsets $I_1, \cdots, I_n$. Then the complex multi-affine polynomial

$$Q(w_1, \cdots, w_n) := \sum_{J \subseteq [n]} F\left(\bigcup_{j \in J} I_j\right) \prod_{j \in J} w_j$$

can vanish only when $w_j \in (-1)^{|I_j|+1} \prod_{i \in I_j} K_i$ for some $j \in [n]$, where the product is the Minkowski product of sets, meaning that $\prod_{i \in I_j} K_i := \left\{\prod_{i \in I_j} x_i \mid \forall i \in I_j,\ x_i \in K_i\right\}$.

The following corollary is all we need Lemma 14 for:

▶ **Corollary 15.** *Fix real parameters β and γ. Assume that the polynomial $\gamma z_1 z_2 + z_1 + z_2 + \beta$ doesn't vanish when $|z_1|, |z_2| < r$, for some $r > 0$. Then for any graph G, the partition function $Z_G(\boldsymbol{\lambda})$ doesn't vanish if $|\lambda_v| < r^{\deg_G(v)}$ for all $v \in V(G)$.*

Proof. Let $G = (V, E)$ with $|V| = n$. Without loss of generality, assume $V = [n]$. To use Lemma 14, we first need create a ground set $[m]$. For each edge $e = \{u, v\} \in E$, let u_e and v_e be a copy of the vertex u and v, respectively. Then consider the ground set $\bigcup_{e=\{u,v\} \in E} \{u_e, v_e\}$, which has size $m := 2|E(G)|$. Let

$$P(\boldsymbol{z}) = \prod_{e=\{u,v\} \in E} \left(\gamma z_{u_e} z_{v_e} + z_{u_e} + z_{v_e} + \beta\right).$$

Let $K = \{z \in \mathbb{C} : |z| \geq r\}$. The assumption in the statement of the corollary guarantees that $P(\boldsymbol{z})$ vanishes only if some $z_i \in K$.

We can write $P(\boldsymbol{z})$ in the form from Lemma 14 by defining a coefficient $F(I)$ for every subset I of the ground set. To do this, partition E into sets E_0, E_1, and E_2 where E_0 is the set of $e = \{u, v\}$ such at u_e and v_e are both out of I, E_1 is the set of $e = \{u, v\}$ with exactly one of u_e, v_e in I and E_2 is the set of $e = \{u, v\}$ with both of u_e and v_e in I. Then $F(I) = \gamma^{|E_2|} \beta^{|E_0|}$.

Now for each $v \in V$, let I_v be the set of all ground set elements corresponding to vertex v. That is, $I_v = \{v_e \mid e \in E, v \in e\}$. Consider the polynomial

$$Q(w_1, \cdots, w_n) := \sum_{J \subseteq V} F\left(\bigcup_{v \in J} I_j\right) \prod_{j \in J} w_j.$$

We can think of the set J as the set of vertices with spin 1. Then $Q(\boldsymbol{w}) = Z_G(\boldsymbol{w})$. So Lemma 14 guarantees that $Q(w_1, \ldots, w_n)$ vanishes only when, for some $v \in V$, $w_v \in (-1)^{|I_v|+1} \prod_{i \in I_v} K$, proving the corollary. ◀

3.2 Proof of Theorem 6

In light of Corollary 15 and Lemma 13, it only remains to show zero-freeness for the single polynomial $\gamma z_1 z_2 + z_1 + z_2 + \beta$.

▶ **Lemma 16.** *For real numbers β, γ such that $\beta > \gamma$ and $\beta + \gamma > 2$, there exists $r > 1$ such that the polynomial $\gamma z_1 z_2 + z_1 + z_2 + \beta$ doesn't vanish when $|z_1|, |z_2| < r$.*

Proof. Let $D(0, r)$ denote the open disk $\{z \in \mathbb{C} : |z| < r\}$. Let $g : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be the Möbius transformation $z \mapsto -\frac{z+\beta}{\gamma z+1}$. Since $g(z)$ is the unique solution to the equation $\gamma z \cdot g(z) + z + g(z) + \beta = 0$, it suffices to show for some $r > 1$ that g maps $D(0, r)$ into $D(0, r)^c$.

Note that since $\beta, \gamma \in \mathbb{R}$, the transformation g maps the $\mathbb{R} \cup \{\infty\}$ into $\mathbb{R} \cup \{\infty\}$. By conformality, g maps any circle centered on $\mathbb{R} \cup \{\infty\}$ to a circle centered on $\mathbb{R} \cup \{\infty\}$. In particular, $g(D(0, r))$ is a disk centered on $\mathbb{R} \cup \{\infty\}$. So $g(D(0, r))$ and $D(0, r)$ are disjoint as long as their intersections with $\mathbb{R} \cup \{\infty\}$ are disjoint. It suffices to show that g maps the real interval $(-r, r)$ into $(-r, r)^c$, for some $r > 1$. By continuity of g, it also suffices to show that g maps the interval $[-1, 1]$ into $[-1, 1]^c$.

Now take any real number $z \in [-1, 1]$. From $\beta > \gamma$ and $\beta + \gamma > 2$ we know $\beta > 1$. we have

$$\begin{aligned} |g(z)| > 1 &\Leftrightarrow |z + \beta|/|\gamma z + 1| > 1 \\ &\Leftrightarrow (z + \beta)^2 > (\gamma z + 1)^2 && \text{(since } \beta, \gamma, z \in \mathbb{R}) \\ &\Leftrightarrow \left(1 - z\frac{\gamma - 1}{\beta - 1}\right)\left(1 + z\frac{\gamma + 1}{\beta + 1}\right) > 0 && \text{(since } \beta > 1). \end{aligned}$$

It follows from $\beta > \gamma$ that $\frac{\gamma-1}{\beta-1} < 1$ and $\frac{\gamma+1}{\beta+1} < 1$, while it follows from $\beta + \gamma > 2$ that $\frac{\gamma-1}{\beta-1} > -1$ and $\frac{\gamma+1}{\beta+1} > -1$. So both $|\frac{\gamma-1}{\beta-1}|$ and $|\frac{\gamma+1}{\beta+1}|$ are less than 1. Since $|z| \le 1$, we have

$$1 - z\frac{\gamma - 1}{\beta - 1} > 0 \text{ and } 1 + z\frac{\gamma + 1}{\beta + 1} > 0.$$

This proves $|g(z)| > 1$ and hence g maps the interval $[-1, 1]$ into $[-1, 1]^c$. ◀

▶ **Corollary 17.** *For real numbers β, γ such that $\beta < \gamma$ and $\beta + \gamma < -2$, there exists $r > 1$ such that the polynomial $\gamma z_1 z_2 + z_1 + z_2 + \beta$ doesn't vanish when $|z_1|, |z_2| < r$.*

Proof. By Lemma 16, the polynomial $(-\gamma)(-z_1)(-z_2) + (-z_1) + (-z_2) + (-\beta)$ doesn't vanish when $|z_1|, |z_2| < r$. So its negation, $\gamma z_1 z_2 + z_1 + z_2 + \beta$, doesn't vanish for $|z_1|, |z_2| < r$ either. ◀

Now we are ready to prove Theorem 6.

Proof of Theorem 6. The range of parameters can be divided into 4 regions:

Case 1: $\beta > \gamma$ and $\beta + \gamma > 2$. Combining Lemma 16 and Corollary 15, there is a disk $D(0, r)$ containing 1 such that, for all graphs G the polynomial $Z_G(x)$ doesn't vanish on $D(0, r)$. An FPTAS is thus given by Lemma 13.

Case 2: $\beta < \gamma$ and $\beta + \gamma > 2$. This case follows by symmetry from Case 1, as switching β and γ preserves Z_G.

Case 3: $\beta < \gamma$ and $\beta + \gamma < -2$. In a similar way to Case 1, this case follows by combining Corollary 17, Corollary 15 and Lemma 13.

Case 4: $\beta > \gamma$ and $\beta + \gamma < -2$. This case follows by symmetry from Case 3. ◀

3.3 Preliminaries for the FPRAS

Our randomized approximation scheme for Theorem 7 closely resembles the one in [21]. The first main ingredient in [21] is the "subgraphs-world" transformation that reduce a spin system problem to a Holant problem. Here, we need to use a slightly generalized version of the subgraphs-world transformation. Though it has appeared in various forms in the literature (e.g. [19]), we introduce it here for the sake of completeness.

▶ **Definition 18.** *For any function $\psi : \{0,1\}^2 \to \mathbb{R}$, define its Fourier transform to be the function $\widehat{\psi} : \{0,1\}^2 \to \mathbb{R}$ given by*

$$\widehat{\psi}(a,b) = \frac{1}{4}\left(\psi(0,0) + (-1)^b\psi(0,1) + (-1)^a\psi(1,0) + (-1)^{a+b}\psi(1,1)\right), \quad \forall a,b \in \{0,1\}.$$

Let $\chi_{a,b}(x_1,x_2) = (-1)^{ax_1+bx_2}$, for $a,b,x_1,x_2 \in \{0,1\}$. Then we have the identity

$$\psi = \sum_{a,b\in\{0,1\}} \widehat{\psi}(a,b)\cdot\chi_{a,b}.$$

▶ **Proposition 19.** *Let $\psi : \{0,1\}^2 \to \mathbb{Q}^{\geq 0}$. An FPRAS for* Holant $\left(\{\widehat{\psi}\} \cup \{\mathbf{Even}_k : k \geq 1\}\right)$ *implies an FPRAS for* #CSP$(\{\psi\})$.

Proof. Let $G = (V,E)$ be an instance of #CSP$(\{\psi\})$. Let $G' = (V',E')$ be defined by

$$V' = V \cup E \text{ and } E' = \bigcup_{e=\{i,j\}\in E} \{\{i,e\},\{j,e\}\}.$$

For every vertex $v \in V \subset V'$, let $F_v = \mathbf{Even}_d$, where $d := \deg_G v$. For every vertex $e \in E \subset V'$, let $F_e = \widehat{\psi}$. In this way, we form a Holant instance ϕ with base graph G'. We have

$$\begin{aligned}
Z_G &= \sum_{x\in\{0,1\}^V} \prod_{\{i,j\}\in E} \psi(x_i,x_j) \\
&= \sum_{x\in\{0,1\}^V} \prod_{\{i,j\}\in E} \sum_{a,b\in\{0,1\}} \widehat{\psi}(a,b)(-1)^{ax_i+bx_j} \\
&= \sum_{x\in\{0,1\}^V} \sum_{y\in\{0,1\}^{E'}} \prod_{e=\{i,j\}\in E} \widehat{\psi}(y_{i,e},y_{j,e})(-1)^{y_{i,e}x_i+y_{j,e}x_j} \\
&= \sum_{y\in\{0,1\}^{E'}} \left(\prod_{e=\{i,j\}\in E} \widehat{\psi}(y_{i,e},y_{j,e})\right)\left(\sum_{x\in\{0,1\}^V} \prod_{e=\{i,j\}\in E} (-1)^{y_{i,e}x_i+y_{j,e}x_j}\right) \\
&= \sum_{y\in\{0,1\}^{E'}} \left(\prod_{e=\{i,j\}\in E} \widehat{\psi}(y_{i,e},y_{j,e})\right)\left(\sum_{x\in\{0,1\}^V} \prod_{i\in V} (-1)^{x_i(\sum_{\{i,e\}\in E'} y_{i,e})}\right) \\
&= \sum_{y\in\{0,1\}^{E'}} \left(\prod_{e=\{i,j\}\in E} \widehat{\psi}(y_{i,e},y_{j,e})\right)\left(\prod_{i\in V}\left(1 + (-1)^{(\sum_{\{i,e\}\in E'} y_{i,e})}\right)\right) \\
&= 2^{|V|} \sum_{y\in\{0,1\}^{E'}} \prod_{e=\{i,j\}\in E} \widehat{\psi}(y_{i,e},y_{j,e}) \prod_{i\in V} \mathbf{Even}\left((y_{i,e})_{\{i,e\}\in E'}\right). \\
&= 2^{|V|}[\![\phi]\!],
\end{aligned}$$

where $[\![\phi]\!]$ denotes the partition function of the Holant instance ϕ. ◀

In [21], the next step is to prove the rapid mixing of a Markov chain associated to the Holant problem and compute the partition function using an MCMC algorithm. But fortunately for us, we don't even need to define the Markov chain, as the powerful framework of [25] has reduced all these efforts to verifying some simple criteria:

▶ **Definition 20.** *For any finite set J and any configuration $x \in \{0,1\}^J$, define $\mathcal{M}_x$ to be the set of partitions of $\{i|x_i = 1\}$ into pairs and at most one singleton. A function $F : \{0,1\}^J \to \mathbb{Q}^{\geq 0}$ is **windable** if there exist values $B(x,y,M) \geq 0$ for all $x, y \in \{0,1\}^J$ and all $M \in \mathcal{M}_{x\oplus y}$ satisfying:*
1. *$F(x)F(y) = \sum_{M\in\mathcal{M}_{x\oplus y}} B(x,y,M)$ for all $x,y \in \{0,1\}^J$, and*
2. *$B(x,y,M) = B(x\oplus S, y\oplus S, M)$ for all $x, y \in \{0,1\}^J$ and all $S \in M \in \mathcal{M}_{x\oplus y}$.*

Here $x \oplus S$ denotes the vector obtained by changing x_i to $1 - x_i$ for the one or two elements i in S.

▶ **Lemma 21.** *Any function $\{0,1\}^2 \to \mathbb{Q}^{\geq 0}$ is windable.*

Proof. The statement follows directly by combining Lemma 7 and Lemma 15 in [25]. ◀

▶ **Definition 22.** *A function $F : \{0,1\}^J \mapsto \mathbb{Q}^{\geq 0}$ is **strictly terraced** if*

$$F(x) = 0 \implies F(x\oplus e_i) = F(x \oplus e_j) \qquad \text{for all } x \in \{0,1\}^J \text{ and all } i, j \in J.$$

Here $x \oplus e_i$ denotes the vector obtained by changing x_i to $1 - x_i$.

▶ **Lemma 23** (Theorem 4 in [25])**.** *If $\mathcal{F}$ is a finite class of strictly terraced windable functions, then there is an FPRAS for* Holant$(\mathcal{F})$.

▶ **Corollary 24.** *If $\mathcal{F}$ is a finite class of strictly terraced windable functions, then there is an FPRAS for* Holant$(\mathcal{F} \cup \{\mathbf{Even}_k : k \geq 1\})$.

Proof. Since an $\mathbf{Even}_k$ constraint can easily be realized using $(k-2)$ copies of $\mathbf{Even}_3$ or $\mathbf{Odd}_3$ constraints and $(k-3)$ additional variables, it suffices to show that there is an FPRAS for Holant$(\mathcal{F} \cup \{\mathbf{Even}_3, \mathbf{Odd}_3\})$. Since $\mathbf{Even}_3$ and $\mathbf{Odd}_3$ are both windable (see [25, Lemma 17]) and strictly terraced, the claim follows from Lemma 23. ◀

3.4 Proof of Theorem 7

Now, it suffices to verify that certain constraint functions are windable and strictly terraced.

▶ **Lemma 25.** *For rational numbers β, γ such that $\beta > \gamma$ and $\beta + \gamma \geq 2$, the function $\psi : \{0,1\}^2 \to \mathbb{Q}$ defined by $\begin{bmatrix} \psi(0,0) & \psi(0,1) \\ \psi(1,0) & \psi(1,1) \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$ satisfies the property that $\widehat{\psi}$ is windable and strictly terraced.*

Proof. Since $\begin{bmatrix} \widehat{\psi}(0,0) & \widehat{\psi}(0,1) \\ \widehat{\psi}(1,0) & \widehat{\psi}(1,1) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} \beta+\gamma+2 & \beta-\gamma \\ \beta-\gamma & \beta+\gamma-2 \end{bmatrix}$, we have $\widehat{\psi}(x) \geq 0$ for all $x \in \{0,1\}^2$, and the only possibility of $\widehat{\psi}(x) = 0$ is when $\beta + \gamma = 2$ and $x = (1,1)$. In that case, we have $\widehat{\psi}(1,0) = \widehat{\psi}(0,1) = \frac{\beta-\gamma}{4}$. It follows that $\widehat{\psi}$ is strictly terraced.

The windablity of $\widehat{\psi}$ follows from Lemma 21. ◀

▶ **Lemma 26.** *For rational numbers β, γ such that $\beta < \gamma$ and $\beta + \gamma \leq -2$, the function $\psi : \{0,1\}^2 \to \mathbb{Q}$ defined by $\begin{bmatrix} \psi(0,0) & \psi(0,1) \\ \psi(1,0) & \psi(1,1) \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$ satisfies the property that $-\widehat{\psi}$ is windable and strictly terraced.*

Proof. Since $\begin{bmatrix} -\widehat{\psi}(0,0) & -\widehat{\psi}(0,1) \\ -\widehat{\psi}(1,0) & -\widehat{\psi}(1,1) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -2-\beta-\gamma & \gamma-\beta \\ \gamma-\beta & 2-\beta-\gamma \end{bmatrix}$, we have $-\widehat{\psi}(x) \geq 0$ for all $x \in \{0,1\}^2$, and the only possibility of $-\widehat{\psi}(x) = 0$ is when $\beta+\gamma = -2$ and $x = (0,0)$. In that case, we have $-\widehat{\psi}(1,0) = \widehat{\psi}(0,1) = \frac{\gamma-\beta}{4}$. It follows that $-\widehat{\psi}$ is strictly terraced.

The windablity of $-\widehat{\psi}$ follows from Lemma 21. ◀

Now we are ready to prove Theorem 7.

Proof of Theorem 7. The range of parameters can be divided into 4 regions:

Case 1: $\beta > \gamma$ and $\beta+\gamma \geq 2$. Combining Lemma 25 and Corollary 24, there is an FPRAS for $\mathsf{Holant}\left(\widehat{\psi} \cup \{\mathbf{Even}_k : k \geq 1\}\right)$, where $\psi : \{0,1\}^2 \to \mathbb{Q}$ defined by $\begin{bmatrix} \psi(0,0) & \psi(0,1) \\ \psi(1,0) & \psi(1,1) \end{bmatrix} = \begin{bmatrix} \beta & 1 \\ 1 & \gamma \end{bmatrix}$. An FPRAS for $\#\mathsf{CSP}(\{\psi\})$is thus given by Proposition 19.

Case 2: $\beta < \gamma$ and $\beta+\gamma \geq 2$. This case follows by symmetry from Case 1, as switching β and γ preserves Z_G.

Case 3: $\beta < \gamma$ and $\beta+\gamma \leq -2$. In a similar way to Case 1, this case follows by combining Lemma 26, Corollary 24 and Proposition 19.

Case 4: $\beta > \gamma$ and $\beta+\gamma \leq -2$. This case follows by symmetry from Case 3. ◀

4 Concluding Remarks

The obvious problem left open by this work is to fully classify the complexity of approximating Z_G in the parameter range $1 \leq \beta+\gamma < 2$ and (without loss of generality) $\gamma < 0$. Observe that there is an NP-hard region in this range: when (β,γ) is sufficiently close to $(1,0)$, by a 2-thickening (i.e. replacing every edge by 2 parallel edges) we get a reduction from the same problem at $A = \begin{bmatrix} \beta^2 & 1 \\ 1 & \gamma^2 \end{bmatrix}$, which lies in the region of "non-uniqueness" and is known to be NP-hard by [32]. However, this only gives us a small bounded region of NP-hardness, since the region of non-uniqueness is bounded (for a rough image, see Figure 1).

Theorem 11 shows that in the other direction, there also exists some tractable region in the range $\{(\beta,\gamma) : \gamma < 0 \text{ and } 1 \leq \beta+\gamma \leq 2\}$. Although the region where tractability is proved extends to infinity, it is rather thin (having width $g(\beta) \approx 0.1$ for small β) and its width tends to zero as $\beta \to +\infty$ (we have $g(\beta) = O(1/\beta)$). Is it possible to prove larger tractable regions?

▶ **Problem 27.** *Does there exist some $\varepsilon > 0$ such that approximating Z_G is tractable whenever $\min\{\beta,\gamma\} < 0$ and $\beta+\gamma > 2-\varepsilon$?*

Possibly the best hope for a complete classification of approximation complexity in the range $\{(\beta,\gamma) : \gamma < 0 \text{ and } 1 \leq \beta+\gamma \leq 2\}$ is to extend the uniqueness line in the positive quadrant to the negative regime.

▶ **Problem 28.** *Is there a natural extension of the uniqueness/non-uniqueness phase transition to the case where $\min\{\beta,\gamma\} < 0$?*

Note that our method for proving Theorem 7 is to transform the problem to another problem with exclusively nonnegative parameters and use the techniques developed specifically for nonnegative problems. Interestingly, Theorem 8 shows that the partition function is always positive in the range $\beta+\gamma \geq 1$. This points to another direction: can we reduce the problem to an "intrinsically positive" one?

▶ **Problem 29.** *Is it possible to transform the problem of computing Z_G in the range $\beta+\gamma \geq 1$ to a problem with only nonnegative parameters, like the way we did in Section 3.3?*

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