Equivocal Blends: Prior Independent Lower Bounds

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Abstract

The prior independent framework for algorithm design considers how well an algorithm that does not know the distribution of its inputs approximates the expected performance of the optimal algorithm for this distribution. This paper gives a method that is agnostic to problem setting for proving lower bounds on the prior independent approximation factor of any algorithm. The method constructs a correlated distribution over inputs that can be described both as a distribution over i.i.d. good-for-algorithms distributions and as a distribution over i.i.d. bad-for-algorithms distributions. We call these two descriptions *equivocal blends*. Prior independent algorithms are upper-bounded by the optimal algorithm for the latter distribution even when the true distribution is the former. Thus, the ratio of the expected performances of the Bayesian optimal algorithms for these two decompositions is a lower bound on the prior independent approximation ratio.

We apply this framework to give new lower bounds on canonical prior independent mechanism design problems. For one of these problems, we also exhibit a near-tight upper bound. Towards solutions for general problems, we give distinct descriptions of two large classes of correlated-distribution "solutions" for the technique, depending respectively on an order-statistic separability property and a paired inverse-distribution property. We exhibit that equivocal blends do not generally have a Blackwell ordering, which puts this paper outside of standard information design.

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1 Introduction

This paper develops a novel method for establishing lower bounds on prior independent approximation algorithms.

Frameworks for stochastic analysis are enabling theoretical understanding of algorithms beyond those provided by classical worst-case treatments (see [19]). These models are especially interesting for algorithm design problems with information theoretic constraints such as online algorithms, mechanism design, streaming algorithms, etc. The Bayesian algorithm design problem can be viewed as a two stage process. In the first stage the input is the prior distribution and an algorithm is constructed for the distribution. In the second stage the constructed algorithm is run on the realized input. The Bayesian optimal algorithm maximizes expected performance.

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The prior independent framework evaluates algorithms, which are ignorant of the (first stage) prior distribution of inputs, against a benchmark defined as the performance of the Bayesian optimal algorithm that is constructed for this prior. With no constraints on the prior distribution, this problem is equivalent to classical worst-case algorithm design. Alternatively, prior independent analyses in mechanism design (e.g., [7]) and online learning (e.g., [3]) restrict the distributions to be independent and identically distributed (i.i.d.), respectively over values of agents in a mechanism and rounds of inputs in an online algorithm.

This paper develops a method for establishing lower bounds on the performance of prior independent algorithms (for classes of i.i.d. distributions). The method is based on Yao's Minimax Principle [20]. The prior independent framework asks for the designer to pick one algorithm that is good on an adversary's chosen worst-case distribution. Yao's minimax principle allows the order of moves of the designer and adversary to be swapped. Thus, the prior independent optimal approximation ratio can be equivalently identified by an adversary choosing a distribution over prior distributions and then the designer choosing a best algorithm. Note that the class of i.i.d. distributions is not closed under convex combination, thus, the adversary's distribution over distributions gives generally a symmetric, correlated distribution over inputs.

Fixing a correlated distribution, a remaining challenge to apply Minimax is to upper bound the performance of all algorithms. This paper gives a concrete method to identify such bounds, specifically by reducing to calculation of expected performances of Bayesian optimal algorithms.

The focus of this paper is *equivocal blends*, which are pairs of distinct distributions over i.i.d. distributions of inputs that induce the same correlated distribution. To establish a prior independent lower bound, we will be considering equivocal blends where one side of the equivocal blend mixes over good-for-algorithms distributions and the other side mixes over bad-for-algorithms distributions. The adversary can choose the mix over good-for-algorithms distributions in which case the expectation over Bayesian optimal performances for this mix defines the benchmark of the prior independent framework. On the other hand, the algorithm cannot tell the two blends apart and thus its expected performance is upper bounded by the expectation over Bayesian optimal performances for the bad-for-algorithms mix. The gap that results from this *Equivocal Blends Technique* between these expected performances is a lower bound on prior independent approximation.

As a simple example, consider the mechanism design problem of posting a price to a single agent with value on $[1, h]$. (Here the restriction to i.i.d. distributions is trivial as there is only one agent.) A class of good-for-algorithms distributions is given by point masses. Note that the Bayesian optimal pricing mechanism for a point mass is to post identically the same price as the value (and the agent always buys – an agent buys if value is at least the price). A class of bad-for-algorithms distributions is given by the equal revenue distribution with cumulative distribution $F(x) = 1 - 1/x$ and a point mass of $1/h$ at h. The equal revenue distribution has the property that the expected revenue from any posted price is 1. Now consider the equivocal blend where on the good-for-algorithms side we have the equal revenue distribution over point masses and on the bad-for-algorithms side we have a point mass on the equal revenue distribution. The expected revenue over Bayesian optimal algorithms from the good-for-algorithms side (in response to point mass distributions) is the expected value of the equal revenue distribution on $[1, h]$, i.e., $1 + \ln h$. The expected revenue from the bad-for-algorithms side is 1. Thus, we have established a lower bound of $1 + \ln h$ on the approximation factor of single-agent posted pricing. (This example analysis is tight due to a matching upper bound from [16].)[2]

[2] This result is summarized as Theorem 18 and proved in detail in Section 6.1.

There are two challenges in establishing prior independent lower bounds via equivocal blends. The first challenge is in sufficiently understanding the Bayesian optimal algorithm for the class of distributions under consideration. In several of the central studied areas of Bayesian algorithms, this first challenge is solved in closed form. Bayesian optimal mechanisms are identified broadly by [18]. For online learning with payoffs that are i.i.d. across rounds, the Bayesian optimal algorithm is trivial: it selects the action with the highest expected payoff (which is the same in each round). Of course, when closed forms are not available, bounds on the Bayesian optimal performance can be employed instead. An important observation of our approach is that not only are Bayesian optimal algorithms used to define the prior independent benchmark, but they can also be used to get non-trivial bounds on any algorithm's approximation ratio.

The second challenge of the blends method is in identifying equivocal blends where the expected Bayesian-optimal performances for good-for-algorithms and bad-for-algorithms distributions are significantly separated. In pursuit of this challenge we give two general approaches for constructing equivocal blends for inputs of size two. (Many of the challenge problems in prior independent mechanism design are for inputs of size two, e.g., [14].) The first approach shows that if the density function of a correlated distribution can be written as a separable product of independent functions *per order statistic* of the inputs, then it can be decomposed into two distinct distributions over i.i.d. distributions. The second approach considers one side of the equivocal blend constructed from any scaled class of distributions with the other side given by the *inverse-distributions* of these (for which, as a class, the roles of values and scales are reversed in comparison to the original class). Both approaches are easily applied to construct novel equivocal blends.

We apply the blends method to two canonical problems in mechanism design. Both are two-agent single-item environments. One considers maximizing revenue under a standard regularity assumption on the distribution. The other considers maximizing residual surplus (i.e., the sum of agents' utilities). Under the restriction to scale invariant mechanisms, [14] identified the prior independent optimal mechanism for revenue (with approximation ≈ 1.907). It is unknown if assuming scale-invariance is with loss. We use the blends method to establish an unconditional lower bound of $^{23}/_{18} \approx 1.2777$; this bound persists when distributions are restricted to a smaller class (of Truncated Uniforms) for which we also prove a near-tight upper bound of 1.292. For residual surplus, an upper bound of $^{4}/_{3}$ exists as a corollary of [16]. We establish a lower bound of 1.00623 (no previous lower bound was known).

Looking forward, the method of equivocal blends is a stochastic framework beyond worst-case analysis of robust algorithms and it exposes important open questions. First, determine whether there are non-trivial settings where the method from equivocal blends is tight. (As just summarized for a standard two-agent auction in the prior independent setting restricting to Truncated Uniform distributions, equivocal blends give a near-tight lower bound which puts the optimal approximation factor in $(1.2777, 1.292)$). Second, develop methods for optimizing the lower bound over classes of equivocal blends. Third, while there are important problems in mechanism design with inputs of size two, other settings (like online algorithms) would benefit from generalizing the method beyond two-input models. Whereas this paper indicates structural obstacles to this generalization, a preliminary study yields a partial extension which we defer to future work.

Related Work

Within the special structure of the prior independent setting, our method applies Yao's Minimax Principle [20].

Alternatively in nonparametric estimation problems, constructions of lower bounds from Minimax exist from Lecam's Method [17] and Assouad's Lemma [2]. While our method bears some similarity to these, there is significant technical deviation. E.g., two optimally-chosen correlated distributions may appear within the lower bound of Lecam's Method, and their *measure of total variation* is assumed to be strictly positive; versus, to the extent we use two correlated distributions, we expect them to be identical. E.g, Lecam's lower bound term uses total variation applied directly to two distributions, whereas ours uses functions mapping product-distributions to optimal-algorithm performances and compares them. We summarize these points of differentiation by noting: we exhibit no obvious dependence on metric spaces. An application of Lecam's Method which approaches the prior independent setting is sample complexity of testing [6] – it effectively uses blends, but other technical distinctions persist.

The prior independent model was introduced in mechanism design by [15] and further refined by [7]. It was conjectured that the second-price auction was the prior independent revenue-optimal mechanism for selling a single item to one of two agents with i.i.d. values from a regular distribution, after [7] showed that it guaranteed an upper bound of 2-approximation. [10] disproved this conjecture by identifying a mechanism with an improved upper bound. [1] – with an additional restriction to scale-invariant mechanisms – proved a weaker version of the conjecture (restriction to monotone hazard rate distributions); and for regular distributions: improved the upper bound and gave the first non-trivial lower bound for prior independent approximation (by establishing a gap for specific distributions). [14] proved the tight result for regular distributions under the scale-invariance restriction. Our Theorem 8 – which lower bounds prior independent revenue approximation – will fall outside of this line of work because (a) we do not assume scale-invariance, and (b) our setting adopts finite value support $[1, h]$ for which the *structure* of algorithms achieving previous bounds is provably sub-optimal.

[14] connected the prior independent model from mechanism design with the standard model for online learning. Most relevantly in relation to our work on prior independent lower bounds, they showed that the simple follow-the-leader algorithm is optimal for expert learning in prior independent settings (by direct analysis rather than by showing a matching lower bound).

Main Paper Outline

Section 2 gives setup of the prior independent setting and proves lower bounds by the Equivocal Blends Technique. Section 3 gives an example of equivocal blends and applies it to two distinct settings within mechanism design to show novel prior independent lower bounds. Section 4 identifies two large classes of blends solutions, each distinctively motivated as a generalization of the example of Section 3. Section 5 connects theoretical optimization of blends to information design and Blackwell ordering. Section 6 gives further blends results and discussion.

2 Prior Independent Setup and Lower Bound Technique

Let $\mathcal{F}$ be a class of probability distributions with known fixed support $\mathcal{V}$ (e.g., $[0, \infty)$). In the *prior independent algorithm design setting*, there is a distribution F which is known to come from the class $\mathcal{F}$ and n inputs are drawn i.i.d. from F (thus input space is $\mathcal{V}^n$). Critically, the algorithm designer does not know the specific $F \in \mathcal{F}$. The notation F is overloaded to be the cumulative distribution function (CDF), and f is its probability density function (PDF).

Fix an algorithm design problem that takes n i.i.d. inputs. Denote a class of feasible algorithms by $\mathcal{A}$ and an algorithm $A \in \mathcal{A}$ with expected performance $A(\boldsymbol{v})$ for inputs $\boldsymbol{v} = (v_1, \ldots, v_n)$. When evaluating the performance in expectation over inputs drawn from a distribution F, we adopt the notation $A(F) = \mathbf{E}_{\boldsymbol{v}\sim F}[A(\boldsymbol{v})]$. An algorithm's performance for an unknown distribution F is measured against the performance of the optimal algorithm which knows F. With these abstractions, we formally define the Bayesian and prior independent optimization problems.

▶ **Definition 1.** *The* Bayesian optimal algorithm design problem *is given by a distribution F and class of algorithms $\mathcal{A}$; and solves for the algorithm* OPT_F *with the maximum expected performance:*

$$\mathrm{OPT}_F = \operatorname*{argmax}_{A\in\mathcal{A}} A(F). \tag{OPT_F}$$

Note that OPT_F is an algorithm. Given a distribution F, the expected performance of the optimal algorithm is $\mathrm{OPT}_F(F)$ and this is the *benchmark* that we use for prior independent algorithms:

▶ **Definition 2.** *The* prior independent algorithm design problem *is given by a class of algorithms $\mathcal{A}$ and a class of distributions $\mathcal{F}$; and searches for the algorithm that minimizes its worst-case approximation:*

$$\alpha^{\mathcal{F}} = \min_{A\in\mathcal{A}} \left[\max_{F\in\mathcal{F}} \frac{\mathrm{OPT}_F(F)}{A(F)}\right] \tag{$\alpha^{\mathcal{F}}$}$$

where the value of the program α^F is the optimal prior independent approximation factor *for class $\mathcal{F}$ and class $\mathcal{A}$ (which we leave implicit).*

2.1 Theoretical Lower Bounds from Minimax

Yao's Minimax Principle (Theorem 3) illustrates the role of the adversary through a direct connection to a 2-player zero-sum game. First we define additional terms for use in Theorem 3 and throughout the paper. Given a space Ω, denote the set of all possible distributions by $\Delta(\Omega)$ – i.e., the *probability simplex*. Denote a distribution over elements $\omega \in \Omega$ by $\gamma \in \Delta(\Omega)$. Given a function $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ where Ω_1 and Ω_2 have arbitrary dimensions, we denote the expectation of f over arguments $\omega_i \in \Omega_i$ according to $\gamma_i \in \Delta(\Omega_i)$ as $f(\gamma_i, \omega_{j\neq i}) = \mathbf{E}_{\omega_i\sim\gamma_i}[f(\omega_i, \omega_j)]$, e.g., in Theorem 3.

▶ **Theorem 3** (Minimax Principle [20]). *Given a 2-player zero-sum game $\mathcal{G}$ in which sequentially player 1 chooses mixed action $\gamma_1 \in \Delta(\Omega_1)$, then player 2 chooses action $\omega_2 \in \Omega_2$. Given the players are cost minimizers and the cost functions on pure actions are (any real-valued function) $C_1(\omega_1, \omega_2) \geq 0$ and $C_2 = -C_1$. Then the* value *of game $\mathcal{G}$ (the left-hand side) satisfies:*

$$\inf_{\gamma_1\in\Delta(\Omega_1)} \sup_{\omega_2\in\Omega_2} C_1(\gamma_1, \omega_2) \geq \sup_{\gamma_2\in\Delta(\Omega_2)} \inf_{\omega_1\in\Omega_1} C_1(\omega_1, \gamma_2) \tag{1}$$

2.2 A Technique for Prior Independent Lower Bounds: Equivocal Blends

There is a detailed explanation of the technique of lower bounds from Yao's Minimax Principle in [5]. A challenge left open is: how to upper bound all algorithms. This section gives an approach that is specific to prior independent design. To outline, we: (a) fix a randomization over adversary strategies in advance; (b) prove an upper bound on the expected performance of the best-response algorithm *from an alternative description of the adversary's induced correlated distribution over inputs*; and (c) measure the gap between the adversary's expected optimal performance and the upper bound of (b). The key idea is the correlation in (b):

▶ **Definition 4.** *A* blend *is a distribution-over-distributions $\delta \in \Delta(\mathcal{F})$. (Thus, $\delta(F)$ is the density at F.) A* blended distribution $\delta^n \in \Delta(\mathcal{V}^n)$ *is the induced density function of the correlated distribution resulting from n i.i.d. draws from a common distribution $\hat{F}$, with $\hat{F} \sim \delta$.*

Two blends $\delta_1,\ \delta_2$ are called equivocal blends *if there exists correlated density g such that:*

$$\delta_1^n(\boldsymbol{v}) = g(\boldsymbol{v}) = \delta_2^n(\boldsymbol{v}) \quad \forall\ \boldsymbol{v}$$

Each of $\delta_1,\ \delta_2$ are a side *of the equivocal blend. Finally, define* $\mathrm{opt}_{n,i} = \mathbf{E}_{F\sim\delta_i}\left[\mathrm{OPT}_F(F)\right]$ *to be the expected performance of an optimal algorithm which knows F over a blend δ_i.*

The point is: an arbitrary blend δ can be "flattened" to describe a specific (symmetric) correlated distribution $g = \delta^n$ over input space $\mathcal{V}^n$. Now suppose in fact two distinct blends δ_1 and δ_2 induce the same correlated distribution, i.e., they satisfy Definition 4. Because both induce the same description of correlated input profiles, every algorithm is limited by the structure of either description. The lower bound of the technique has the following intuition: the adversary chooses $\delta_2 \in \mathcal{F}$ which fixes the benchmark of the current scenario to $\mathrm{opt}_{n,2} = \mathbf{E}_{F\sim\delta_2}\left[\mathrm{OPT}_F(F)\right]$;[3] the blend δ_2 induces the correlated distribution g and the algorithm best responds to g; however the fact that δ_1 also induces g means that every algorithm is upper bounded by $\mathrm{opt}_{n,1}$; if this upper bound is strictly smaller than the benchmark, then a strict gap necessarily ensues. (Note, δ_1 is unrestricted; if it exists, it is a consequence-of-nature of δ_2.) The proof of Theorem 5 appears in Appendix A.

▶ **Theorem 5.** *Consider a prior independent setting with input space $\mathcal{V}^n$, class of algorithms $\mathcal{A}$, and class of distributions $\mathcal{F}$. Let $\mathcal{F}^{\mathrm{all}}$ be all distributions. Assume there exist two distinct equivocal blends $\delta_1 \in \Delta(\mathcal{F}^{\mathrm{all}})$ and $\delta_2 \in \Delta(\mathcal{F})$ and correlated density function g (of Definition 4) such that:*

$$\delta_1^n(\boldsymbol{v}) = g(\boldsymbol{v}) = \delta_2^n(\boldsymbol{v}) \quad \forall\ \boldsymbol{v}$$

Then the optimal prior independent approximation factor $\alpha^{\mathcal{F}}$ is at least the ratio ${}^{\mathrm{opt}_{n,2}}/{}_{\mathrm{opt}_{n,1}}$:

$$\alpha^{\mathcal{F}} = \min_{A\in\mathcal{A}} \max_{F\in\mathcal{F}} \frac{\mathrm{OPT}_F(F)}{A(F)} \geq \frac{\mathrm{opt}_{n,2}}{\mathrm{opt}_{n,1}} \tag{2}$$

▶ **Definition 6.** The Equivocal Blends Technique *is the proof technique for approximation lower bounds which applies Theorem 5 to a specified prior independent design problem.*

A detailed outline of the necessary computations to confirm that descriptions of δ_1 and δ_2 are equivocal blends is given in the full version, which also includes a first *non-trivial* $n = 2$ example. Construction of equivocal blends does not depend on problem domain – e.g., mechanism design or online algorithms – but which side the adversary chooses does depend on the problem. Subsequently in this paper we will (a) give examples of equivocal blends and use them to prove lower bounds per Definition 6, and (b) give general methods for identifying equivocal blends.

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[3] We prove expectation-over-ratios (Theorem 3) transitions to ratio-of-expectations (Theorem 5) in Appendix A.

3 Results in Blends Analysis

The first goal of this section is to exhibit a concrete example of equivocal blends. The example proceeds in two steps: (1) we describe a relaxed solution that allows *infinite weight* which is not directly usable for lower bounds but has simpler algebraic form; and (2), we show that this relaxed solution can be modified to become proper equivocal blends.

The second part of the section uses the equivocal blends example to state novel lower bounds for two distinct problems from mechanism design. We give a lengthy introduction to mechanism design and proofs of these results in the full version of the paper. Interestingly, the distinct objectives of these two problems results in *the two sides of the equivocal blends playing opposite roles* (as either choice of the adversary, or inducing the upper bound). Later in Section 5, we discuss the implications of this observation in terms of precluding *Blackwell ordering* between the two sides.

3.1 A Concrete Equivocal Blends Example

This section provides an explicit example of equivocal blends – with motivation for the chosen distributions from themes in mechanism design. First, we will describe a blends-type solution that has unbounded input support and infinite total weight (so it is not a probability distribution and it is not possible to re-normalize its weights to become one).[4] Second, we modify the infinite-weight solution to have finite weight in a bounded input space (thus, the total weight can be normalized to 1, in particular without affecting our ratio-calculations).

For this running equivocal blends example, the δ_1 side will be parameterized by a base class of upward-closed Quadratics (called "equal revenue" in the mechanism design literature), with PDF given by $\mathrm{qud}_z(x) = {}^{z}/_{x^2}$ and CDF given by $\mathrm{Qud}_z(x) = 1 - {}^{z}/_{x}$ on $[z, \infty)$. The δ_2 side will be a base class of downward-closed Uniforms, with PDF given by $\mathrm{ud}_{0,z}(x) = {}^{1}/_{z}$ and CDF given by $\mathrm{Ud}_{0,z}(x) = {}^{x}/_{z}$ on $[0, z]$. (Generally, let $\mathrm{Ud}_{a,b}$ be the Uniform distribution on $[a, b]$.)

Infinite-weight Blends

We start by describing the weights o_F corresponding to δ_1 and weights ω_F corresponding to δ_2. Because we first allow the total weight to be infinite, we only require the function g (relaxed to be a "correlated function" rather than a correlated distribution) to match up its output *mass* at every input (cf., density of a correlated distribution).

The weights of the upward-closed Quadratics blend (δ_1) are:

- weights $o_{Qz} = \frac{2}{z}dz$ on all upward-closed distributions Qud_z for $z \in (0, \infty)$.

The weights of the downward-closed Uniforms blend (δ_2) are:

- weights $\omega_{Uz} = \frac{2}{z}dz$ on all downward-closed distributions $\mathrm{Ud}_{0,z}$ for $z \in (0, \infty)$.

Using symmetry, we analyze mass in the cone $v_1 \geq v_2 \geq 0$. The calculations of total mass at any $\boldsymbol{v} \in (0, \infty)^2$ are confirmed to be equal from either blend description of common function g:

[4] The elements of a blend δ are technically densities but we generally refer to them as *weights*, i.e., the weight corresponding to a distribution F within the mixture over $\mathcal{F}$ according to δ. We do this to accommodate a relaxed definition for *blend* which allows arbitrary total weight (including infinite).

$$\text{mass of } \mathrm{Qud}_z \text{ blend} = \int_0^{v_2} o_{Qz} \cdot \mathrm{qud}_z(v_1) \cdot \mathrm{qud}_z(v_2) = \int_0^{v_2} \frac{2}{z} \cdot \frac{z}{v_1^2} \cdot \frac{z}{v_2^2}\, dz \;= \frac{1}{v_1^2} = g(\boldsymbol{v}) \tag{3}$$

$$\text{mass of } \mathrm{Ud}_{0,z} \text{ blend} = \int_{v_1}^{\infty} \omega_{Uz} \cdot \mathrm{ud}_{0,z}(v_1) \cdot \mathrm{ud}_{0,z}(v_2) = \int_{v_1}^{\infty} \frac{2}{z} \cdot \frac{1}{z} \cdot \frac{1}{z}\, dz \;= \frac{1}{v_1^2} = g(\boldsymbol{v}) \tag{4}$$

The setup of these calculations is expanded in the full version. As desired, each side of the equivocal blends describes exactly the same function g over $\mathcal{V}^2$. The remaining issue to be addressed is that the total weight of all included distributions is divergent: $\int_0^\infty \frac{2}{z} dz = \infty$.

Modification to Finite-weight Blends

Next we show how to modify the infinite-weight solution above to a proper equivocal blends solution with approximately the same elements. Consider input support $\mathcal{V} = [1, h]$ for $1 < h < \infty$. First we define the weights o_F and ω_F, largely informed by the infinite-weight solution. We let the total weight in the system be any constant and can assume that it gets normalized to 1 later. In fact the total weight will be: $1 + \int_1^h \frac{2}{z} dz = 1 + 2\ln h$.

The Quadratics have the same general description as the infinite-weight case but are now top-truncated at h, with truncated density moved to a point mass at h.[5] Formally, Quadratics have PDF $\overline{\mathrm{qud}}^{\bullet h}_z(x) = z/x^2$ on $[1, h)$ and point mass $\overline{\mathrm{qud}}^{\bullet h}_z(h) = 1/h$, correspondingly CDF $\overline{\mathrm{Qud}}^{\bullet h}_z(x) = 1 - z/x$ on $[1, h)$ and $\overline{\mathrm{Qud}}^{\bullet h}_z(h) = 1$.

The Uniforms have the same general description as the infinite-weight case but now have domain lower bound at 1 and *allow* top-truncation at h. Formally, Uniforms without truncation have PDF $\mathrm{ud}_{1,z}(x) = 1/z-1$ and CDF $\mathrm{Ud}_{1,z}(x) = x-1/z-1$ on $[1, z]$. Uniforms with truncation have PDF $\overline{\mathrm{ud}}^{\bullet h}_{1,b}(x) = 1/b-1$ on $[1, h)$ and point mass $\overline{\mathrm{ud}}^{\bullet h}_{1,b}(h) = b-h/b-1$, correspondingly $\overline{\mathrm{Ud}}^{\bullet h}_{1,b}(x) = x-1/b-1$ on $[1, h)$ and $\overline{\mathrm{Ud}}^{\bullet h}(h) = 1$.

The weights of the upward-closed Quadratics blend (δ_1) are:

- point mass of weight $o_{\mathrm{pm}} = 1$ on (truncated) distribution $\overline{\mathrm{Qud}}^{\bullet h}_1$;
- weights $o_{Qz} = \frac{2}{z} dz$ on all upward-closed (truncated) distributions $\overline{\mathrm{Qud}}^{\bullet h}_z$ for $z \in [1, h]$.

The weights of the downward-closed Uniforms blend (δ_2) are:

- point mass of weight $\omega_{\mathrm{pm}} = \frac{(2h-1)^2}{h^2}$ on (truncated) distribution $\overline{\mathrm{Ud}}^{\bullet h}_{1,2h}$;
- weights $\omega_{Uz} = \frac{2(z-1)^2}{z^3} dz$ on all downward-closed distributions $\mathrm{Ud}_{1,z}$ for $z \in [1, h]$.

(In fact, we use only one uniform distribution with truncation: $\overline{\mathrm{Ud}}^{\bullet h}_{1,2h}$.) Calculations to show that these blends give the same g over $[1, h]^2$ are given in the full version.

3.2 Mechanism Design Lower Bounds for Revenue and Residual Surplus

We show two prior independent lower bounds in mechanism design from the same equivocal blends solution (using finite-weight Quadratics-versus-Uniforms of Section 3.1 and the Equivocal Blends Technique of Definition 6). *Revenue* and *residual surplus* are two objectives within mechanism design (see definitions in the full version). Theorem 8 (below, for a revenue

[5] We explain the notation $\overline{F}^{\bullet h}$. The overline modifies F with truncation. A circle-mark truncates to a pointmass, a tick-mark truncates and re-normalizes density (e.g., $\overline{F}^{1}$); a mark at the left end point is bottom-truncation, at the right end point is top-truncation. The superscipts indicate the input(s) of truncation, in order. Thus, $\overline{F}^{\bullet h}$ is: top-truncated to a pointmass at h, i.e., the original CDF jumps to 1 at h.

objective) uses an adversarial distribution over the Uniforms side of the equivocal blend. By contrast, Theorem 10 (for a residual surplus objective) uses an adversarial distribution over the Quadratics side. This dichotomy of the respective adversaries' choices highlights how our single example of equivocal blends can be distinctly applied – with blends' roles reversed – in distinct algorithm-objective settings (which moreover is sufficient to prove that there is no *Blackwell ordering* between the blends, see Corollary 17).

A given prior independent lower bound is stronger if it holds for a smaller class of distributions. Let $L^{\mathcal{F}}$ be a lower bound on the optimal approximation factor $\alpha^{\mathcal{F}}$ for a class $\mathcal{F}$. Fact 7 makes clear that $L^{\mathcal{F}}$ holds additionally for a superclass $\mathcal{E}$:

▶ **Fact 7.** *Given two classes of distributions $\mathcal{E}$ and $\mathcal{F}$ such that $\mathcal{E} \supset \mathcal{F}$. Then $\alpha^{\mathcal{E}} \geq \alpha^{\mathcal{F}} \geq L^{\mathcal{F}}$.*

Thus, we give our results for the smallest classes of distributions in order to state the strongest bounds from our analysis. Define two sub-classes: Uniforms $\mathcal{F}^{\text{unif}}[1,h] = \{\overrightarrow{\text{Ud}}^{h}_{1,b} \ : \ 1 \leq b\} \equiv$ uniforms on $[1,b]$ truncated at h; and Quadratics $\mathcal{F}^{\text{quad}}[1,h] = \{\overrightarrow{\text{Qud}}^{h}_{a} \ : \ 1 \leq a \leq h\} \equiv$ quadratics on $[a,h]$ truncated at h. We explain the approach for both theorems but full proofs are deferred to the full version of the paper.

▶ **Theorem 8.** *Given a single-item, 2-agent, truthful auction setting with a revenue objective and with agent values restricted to support $[1,h]$ for $h > 2$. For the Uniforms class $\mathcal{F}^{\text{unif}}$, the optimal prior independent approximation factor $\alpha_h^{\mathcal{F}^{\text{unif}}}$ has upper and lower bounds:*

$$1.292 > \alpha_h^{\mathcal{F}^{\text{unif}}} \geq \frac{\text{opt}_{2,2}}{\text{opt}_{2,1}} = \frac{\frac{23h}{6} - \frac{7}{2} - \ln({}^{h}\!/_{2})}{3h-2} = L_h^{\mathcal{F}^{\text{unif}}} \text{ where } L_\infty^{\mathcal{F}^{\text{unif}}} \approx 1.2777 \tag{5}$$

The 1.292-upper bound is obtained by the Second Price Auction. The lower bound $L_h^{\mathcal{F}^{\text{unif}}} \to {}^{23}\!/_{18} \approx 1.2777$ as $h \to \infty$ and this is the supremum of $L_h^{\mathcal{F}^{\text{unif}}}$ over $h \geq 1$.

Lower bounds for Uniforms (without truncation) have previously been considered within mechanism design, e.g., for a Bayesian, multi-dimensional setting [11]; and for prior independence, with budgeted agents [8]; and with distribution samples [9]. Canonical revenue maximization measures worst-case approximation with respect to the class of *regular* distributions $\mathcal{F}^{\text{reg}}$ (formally defined in the full version). All of our Uniforms are regular: $\mathcal{F}^{\text{reg}} \supset \mathcal{F}^{\text{unif}}$. As a corollary, we lower bound regular distributions: $\alpha_h^{\mathcal{F}^{\text{reg}}} \geq L_h^{\mathcal{F}^{\text{unif}}}$.

The proof of Theorem 8 follows the script of the Equivocal Blends Technique. We set $\delta_2 \in \Delta(\mathcal{F}^{\text{unif}})$ to be the Uniforms blend with finite weights (page 8) and we set $\delta_1 \in \Delta(\mathcal{F}^{\text{all}})$ to be the corresponding Quadratics equivocal blend. The Second Price Auction (SPA) is optimal for all Quadratics in $\mathcal{F}^{\text{quad}} \subset \mathcal{F}^{\text{all}}$; the lower bound $h > 2$ is necessary so that the Second Price Auction is not also optimal for all Uniform distributions with positive weight in δ_2 (for $h \leq 2$ there is no gap: ${}^{\text{opt}_{2,2}}\!/_{\text{opt}_{2,1}} = 1$). Given these, the right-hand side of equation (5) is simply the result of evaluating ${}^{\text{opt}_{2,2}}\!/_{\text{opt}_{2,1}}$ (and recalling from Definition 4 that $\text{opt}_{n,i} = \mathbf{E}_{F \sim \delta_i}[\text{OPT}_F(F)]$).

The 1.292-upper bound for the Uniforms class in Theorem 8 follows from lemmas of [14] and [7].

Previously for 2-agent auctions for revenue and *unbounded value space*, with the additional restriction to *scale-invariant* mechanisms, [1] proved for monotone hazard rate distributions ($\mathcal{F}^{\text{mhr}}$) that the SPA is optimal and gave the optimal approximation $\alpha^{\mathcal{F}^{\text{mhr}}} \approx 1.398$; and also proved for regular distributions ($\mathcal{F}^{\text{reg}}$) the first-ever prior independent lower bound. [14] gave the optimal mechanism and approximation $\alpha^{\mathcal{F}^{\text{reg}}} \approx 1.907$; comparing this result to Theorem 8, we have established an upper bound on the gap between optimal prior independent approximation factors for infinite and finite value support:

▶ **Corollary 9.** *Fix a single-item, 2-agent, truthful auction setting with a revenue objective. Consider alternatively unbounded agent values in* $(0,\infty)$ *and agent values restricted to support* $[1,h]$ *for* $h>2$*. The gap between* $\mathcal{F}^{reg}_{(0,\infty)}$ *and* $\mathcal{F}^{reg}_{[1,h]}$ *is at most (using conservative rounding):* ${}^{1.908}/_{1.2777} \approx 1.495$*.*

We switch now to the residual surplus objective:

▶ **Theorem 10.** *Given a single-item, 2-agent, truthful auction setting with a residual surplus objective and with agent values restricted to support* $[1,h]$ *for* $h \geq 8.56$*. For Quadratics class* $\mathcal{F}^{\text{quad}}$*, the optimal prior independent approximation factor* $\alpha_h^{\mathcal{F}^{\text{quad}}}$ *has a lower bound:*

$$\alpha_h^{\mathcal{F}^{\text{quad}}} \geq \frac{\text{opt}_{2,2}}{\text{opt}_{2,1}} > \frac{4h^2-2h-h\ln h-e\ln h-e}{4h^2-3h-h\ln h} = L_h^{\mathcal{F}^{\text{quad}}} \text{ where } L_{18}^{\mathcal{F}^{\text{quad}}} \approx 1.00623 \qquad (6)$$

The lower bound $L_h^{\mathcal{F}^{\text{quad}}} \to 1$ *as* $h\to\infty$*. As an example bound: for* $h\in\mathbb{N}$*, the maximum of* $L_h^{\mathcal{F}^{\text{quad}}}$ *is achieved at* $h=18$ *with* $L_{18}^{\mathcal{F}^{\text{quad}}} \approx 1.00623$*.*

Residual surplus maximization measures worst-case approximation with respect to the class of all distributions $\mathcal{F}^{\text{all}}$.[6] As a corollary, we lower bound all distributions: $\alpha_h^{\mathcal{F}^{\text{all}}} \geq L_h^{\mathcal{F}^{\text{quad}}}$.

Once again, the proof of Theorem 10 uses the Equivocal Blends Technique. This time we set $\delta_2 \in \Delta(\mathcal{F}^{\text{quad}})$ to be the Quadratics blend with finite weights and set $\delta_1 \in \Delta(\mathcal{F}^{\text{all}})$ to be the corresponding Uniforms. The Lottery (i.e., uniform-random allocation) is optimal for all Uniforms in $\mathcal{F}^{\text{unif}} \subset \mathcal{F}^{\text{all}}$; the lower bound $h \geq 8.56$ is necessary so that the Lottery is not also optimal for all Quadratics with positive weight in δ_2 (for $h \leq 8.55$ there is no gap). Note, the right-hand side of equation (6) is a simplified lower bound on the ratio ${}^{\text{opt}_{2,2}}/_{\text{opt}_{2,1}}$ as shown in the statement.

For residual surplus, there is no previous lower bound. Our mechanism design results have not been optimized in order to identify best lower bounds from the Equivocal Blends Technique.

4 General Equivocal Blends Solutions: Order-statistic Separability and Inverse-distributions

This section describes two broad approaches for infinite-weight equivocal blends solutions that may be useful for identifying good lower bounds for problems of interest, i.e., within a search over equivocal blends for the one that yields the best lower bound.

The first blends structure exists when the common function g (describing mass) can be written as *multiplicatively-separable functions per order-statistic of the inputs* (for $n=2$). The second blends structure generates one side of the equivocal blend by parameterizing over scales of a fixed, base function F, and the other side is then automatically generated by parameterizing over scales of the *inverse-distribution* of F. The example of Section 3.1 is a special case of both approaches.

For simplicity, we describe these constructions allowing for infinite-weight blends. Similar methods as used in the example of Section 3.1 can modify them to proper probability distributions.

[6] We note the contrast: $\mathcal{F}^{\text{all}}$ is standard for prior independent design with a residual surplus objective, whereas $\mathcal{F}^{\text{reg}}$ is standard with a revenue objective. As partial explanation: for the class $\mathcal{F}^{\text{all}}$, [16] show that constant-approximation is possible for residual surplus, and also show a super-constant lower bound for revenue. Revenue maximization restricts to regular distributions which satisfy a natural concavity property, and for which constant-approximation is possible. The first upper bound was from [7].

4.1 Blends from Order-statistic Separability

This section introduces *order-statistic-separable* functions and subsequently describes a class of equivocal blends based on these functions. Fix $n = 2$ and our inputs in the cone $v_1 \geq v_2 \geq 0$ in which v_1 represents the first (largest) order statistic and v_2 the second (smaller) order statistic.

▶ **Definition 11.** *Given $n = 2$. An* order-statistic-separable function *(with domain $\mathcal{V}^2$) is symmetric across the line $v_1 = v_2$ and for inputs subject to $v_1 \geq v_2 \geq 0$, has the form:*

$$g(\boldsymbol{v}) = g_1(v_1) \cdot g_2(v_2)$$

for which both g_1 and g_2 adopt the domain $\mathcal{V}$.

To be clear, the separate functions g_1 and g_2 are <u>not</u> independent factors of g because of the condition $v_1 \geq v_2$. The function g is correlated and is not a product itself. Let $G_1(z) = \int_z^\infty g_1(y)dy$ and $G_2(z) = \int_0^z g_2(y)dy$ be respectively upward-cumulative and downward-cumulative functions. (Intuitively, if $G_1(z)$ is finite, then a "normalized" function ${g_1(x)}/{G_1(z)}$ gives the PDF of a conditional probability distribution parameterized by z, on domain $[z, \infty)$; and the analogous statement is true for finite $G_2(z)$ on domain $(0, z]$.)

Before stating a formal result in Theorem 12 to construct equivocal blends, we show that the Quadratics-versus-Uniforms example of Section 3.1 exhibits order-statistic separability. The blends' correlated density at every point $\boldsymbol{v} \in \mathbb{R}^2_+$ for $v_1 \geq v_2$ was calculated in equations (3) and (4) to be $g(\boldsymbol{v}) = {1}/{v_1^2}$. It is easy to verify that $g_1(v_1) = {1}/{v_1^2}$ and $g_2(v_2) = 1$ satisfy Definition 11.

▶ **Theorem 12.** *Consider non-negative functions $g_1(\cdot)$ and $g_2(\cdot)$ each with domain $(0, \infty)$. For every $z > 0$, let $g_{1,z}$ be g_1 restricted to the domain $[z, \infty)$ and $g_{2,z}$ be g_2 restricted to the domain $(0, z]$.*

Each δ_i blend is a distribution over the set $\{g_{i,z} \ : \ z > 0\}$. Let $o_{g_1}(z)$ and $\omega_{g_2}(z)$ be functions (as free parameters which we may design) *to describe weights corresponding respectively to each $g_{1,z}$ and to each $g_{2,z}$.*

First, assume $g_1(\cdot)$ and $g_2(\cdot)$ satisfy the following conditions:
1. *The function $\chi(z) = \frac{g_1(z)}{g_2(z)}$ evaluated in the limit at ∞ is 0, i.e., $\lim_{z\to\infty} \chi(z) = 0$;*
2. *the function $\psi(z) = \frac{g_2(z)}{g_1(z)}$ evaluated in the limit at 0 is 0, i.e., $\lim_{z\to 0} \psi(z) = 0$;*
3. *$\chi(z)$ must be weakly decreasing, equivalently, $\psi(z)$ must be weakly increasing;*

Then the weights functions $o_{g_1}(z) = d\psi(z)$ and $\omega_{g_2}(z) = -d\chi(z)$ give an equivocal blends solution with:

$$g(\boldsymbol{v}) = g_1(v_1) \cdot g_2(v_2) \text{ for } \boldsymbol{v} = (v_1, v_2 \leq v_1)$$

If the following condition additionally holds:

4. *the integrals $G_1(z) = \int_z^\infty g_1(y)\ dy$ and $G_2(z) = \int_0^z g_2(y)\ dy$ are positive and finite for all $z \in (0, \infty)$;*

then for the same function g, there exists an equivocal blends solution (by modification from the original solution) for which all of the $g_{1,z}$ and $g_{2,z}$ functions are distributions.

(The proof and discussion of Theorem 12 are given in the full version of the paper due to space constraints.) The modification for the last part of Theorem 12 is defined by: the distributions making up the blends classes are $\tilde{g}_{1,z}(x) = {g_{1,z}(x)}/{G_1(z)}$ and $\tilde{g}_{2,z}(x) = {g_{2,z}(x)}/{G_2(z)}$ and the weights are $\tilde{o}_{g_1}(z) = d\psi(z) \cdot (G_1(z))^2$ and $\tilde{\omega}_{g_2}(z) = -d\chi(z) \cdot (G_2(z))^2$. Two nested classes of equivocal blends from Theorem 12 are given in Section 6.2.

4.2 *Dual* Blends from Inverse-distributions

It is a remarkable feature of the infinite-weight Quadratics-versus-Uniforms equivocal blends *that both sides use the exact same weights parameters per* z, namely $o_{Qz} = \omega_{Uz} = {}^{2}/_{z} \cdot dz$. This structure is not an anomaly – it is indicative of a special-case class of infinite-weight *dual* blends solutions which we formalize in Theorem 13 (and give the key definitions and proof below).

The critical structure is the multiplicative inverse "${}^{1}/_{z}$". Its importance is highlighted from two perspectives: inverse-distributions and arbitrary distribution rescaling. Notably, Quadratics and Uniforms are inverse-distributions to each other, which we see directly from $\mathrm{Qud}_1(x) = 1 - {}^{1}/_{x}$ on $[1,\infty)$ for which the inverse-distribution CDF is $1 - \mathrm{Qud}_1({}^{1}/_{x}) = 1 - (1 - {}^{1}/_{1/x}) = x = \mathrm{Ud}_{0,1}(x)$ on $[0,1]$. Additionally, the Quadratics blend assigns weights to all rescalings of Qud_1 and the Uniform blend assigns weights to all rescalings of $\mathrm{Ud}_{0,1}$. Fundamentally, Theorem 13 shows that there is a duality between distribution values and distribution scales, as observed in equations (7), (8).

▶ **Theorem 13** (Dual Blends Theorem). *Given distribution* F*, define members* F_y *of its parameterized class of all possible rescalings* $y > 0$*, and its inverse-distribution* i-F *by*

$$\mathrm{F}_z(x) = \mathrm{F}({}^{x}/_{z}) = 1 - \text{i-F}({}^{z}/_{x}) = 1 - \text{i-F}_x(z) \tag{7}$$

For $n = 2$*,* F_z *and* i-F_z *give classes that are dual blends using weights* $o_z = \omega_z = {}^{1}/_{z}$*, i.e., they describe a common function* g *at every* $\boldsymbol{v} = (v_1, v_2 \leq v_1)$*:*

$$\int_0^\infty \frac{1}{z} \cdot \mathrm{f}_z(v_1) \cdot \mathrm{f}_z(v_2)\ dz = g(\boldsymbol{v}) = \int_0^\infty \frac{1}{z} \cdot \text{i-f}_z(v_1) \cdot \text{i-f}_z(v_2)\ dz \tag{8}$$

▶ **Definition 14.** *Given a distribution* F *with domain* $[a,b]$ *(or domain* $[a,\infty)$*), i.e.,* $F(a) = 0$ *and* $F(b) = 1$*. The* inverse-distribution *of* F *is defined by the CDF function* $\text{i-}F(x) = 1 - F({}^{1}/_{x})$ *on domain* $[{}^{1}/_{b}, {}^{1}/_{a}]$ *(respectively domain* $(0, {}^{1}/_{a}]$*). The PDF of the inverse-distribution is denoted* i-f*. (Fact: as an operation,* distribution inversion *is its own inverse, i.e., it respects the identity* $\text{i-}(\text{i-}F) = F$*.)*

▶ **Fact 15.** *Given a distribution* $F_{z=1}$ *with default scaling parameter* $z = 1$ *and with domain* $[a,b]$ *(or domain* $[a,\infty)$*). The distribution* F_1 *can be arbitrarily re-scaled for* $z \in (0,\infty)$ *to* $F_z(x) = F_1({}^{x}/_{z})$ *with domain* $[z \cdot a, z \cdot b]$ *(respectively domain* $[z \cdot a, \infty)$*).*

These concepts come together in Theorem 13 which proves that an infinite-weight blends solution always exists effectively from fixing symmetric weights $o_z = \omega_z = {}^{1}/_{z} \cdot dz$ and then choosing the g_1 and g_2 as inverse-distributions of each other. In Theorem 12 by comparison, g_1 and g_2 were (relatively) free parameters to be chosen first, for which weights could then be identified to complete an equivocal blends solution. We give a concise proof of Theorem 13 from the key ideas of this section (inverse-distributions and rescaling):

Proof. Given distribution F and its inverse-distribution i-F, the rescaled CDFs and PDFs are:

$$\mathrm{F}_z(x) = \mathrm{F}({}^{x}/_{z}) \qquad \text{i-F}_z(x) = \text{i-F}({}^{x}/_{z}) = 1 - \mathrm{F}({}^{z}/_{x})$$

$$\mathrm{f}_z(x) = \frac{1}{z} \cdot \mathrm{f}({}^{x}/_{z}) \qquad \text{i-f}_z(x) = \frac{z}{x^2} \cdot \mathrm{f}({}^{z}/_{x})$$

Starting from the right-hand side of equation (8), the following sequence completes the proof:

$$\int_0^\infty \frac{1}{z} \cdot \text{i-f}_z(v_1) \cdot \text{i-f}(v_2)\ dz = \int_0^\infty \left[\frac{1}{z} \cdot dz\right] \cdot \left(\frac{z}{v_1^2} \cdot \mathrm{f}({}^{z}/_{v_1})\right) \cdot \left(\frac{z}{v_2^2} \cdot \mathrm{f}({}^{z}/_{v_2})\right)$$

(here we perform calculus-change-of-variables using $z = \zeta(y) = \frac{v_1 \cdot v_2}{y}$; recall that part of the substitution is $dz = \zeta'(y) \cdot dy$, and integral endpoints get mapped by $\zeta^{-1}(\cdot)$)

$$= \int_{\infty}^{0} \left[\frac{1}{\frac{v_1 \cdot v_2}{y}} \cdot \left(\frac{-v_1 \cdot v_2}{y^2} \cdot dy \right) \right] \cdot \left(\frac{\frac{v_1 \cdot v_2}{y}}{v_1^2} \cdot \mathrm{f}({}^{v_2}/{}_{y}) \right) \cdot \left(\frac{\frac{v_1 \cdot v_2}{y}}{v_2^2} \cdot \mathrm{f}({}^{v_1}/{}_{y}) \right)$$

$$= \int_{0}^{\infty} \left[\frac{1}{y} \cdot dy \right] \cdot \left(\frac{1}{y} \cdot \mathrm{f}({}^{v_2}/{}_{y}) \right) \cdot \left(\frac{1}{y} \cdot \mathrm{f}({}^{v_1}/{}_{y}) \right) = \int_{0}^{\infty} \frac{1}{y} \cdot \mathrm{f}_y(v_2) \cdot \mathrm{f}_y(v_1)\ dy \qquad \blacktriangleleft$$

An interesting property of (infinite-weight) dual blends that emerges from the proof of Theorem 13 is: *we do not have to solve for a closed-form expression* for the function g in order to prove equality of its dual descriptions. As a consequence, obtaining prior independent lower bounds from dual blends may reduce to computation of expectations over optimal performances $\mathrm{OPT}_F(F)$.

A dual blend from Theorem 13 based on optimal mechanism design is outlined in Section 6.3.

5 Equivocal Blends Design is Information-Design-Design

This section connects *theoretical optimization* of the Blends Technique to the economics topic of information design, specifically as a procedure of *information-design-design*. For a given prior independent problem (parameterized by a class of distributions $\mathcal{F}$), the main idea is to separate into two modular problems the search for the optimal equivocal blend (i.e., the one that yields the largest lower bound of any equivocal blend). (1) An "outer" problem identifies an optimal correlated distribution $g^* \in \mathcal{G} = \{\delta^n \mid \delta \in \Delta(\mathcal{F})\}$. The outer problem searches over: (2) for any exogenous $g \in \mathcal{G}$, an "inner" problem identifies two blends that induce g – cf., the Equivocal Blends Technique: respectively the blends are an adversary distribution over $\mathcal{F}$ and an alternative distribution over $\mathcal{F}^{\text{all}}$ – to maximally separate the ratio of optimal performances given each blend.

Effectively, the distributions that compose each blend are randomized *signals* (i.e, signal a distribution F) whose realizations are used to resolve the optimal algorithm OPT_F. If signals can be designed as outputs of a mapping from *fixed states*, then such signal-response games are called *information design*. We exhibit the separation of problems first and defer the formal presentation of information design to the full version of the paper.

Describing the sequence of inequalities below, the first line starts with a prior independent problem and its right-hand side optimizes over lower bounds from the Blends Technique. This step removes the algorithm design problem of the min-player and gives a new problem (which is constrained with respect to the original, possibly with loss). Next, where an adversary optimizes a $\sup - \sup$ program, we rearrange these two successive choices to: (a) optimize the correlated distribution g which represents both (flattened) sides of the equivocal blends simultaneously; and then (b) optimize over blends which induce g to maximize the numerator (using $\mathcal{F}$) and minimize the denominator (using $\mathcal{F}^{\text{all}}$).[7] The final line is a reorganization using independence of numerator and denominator which each comprise a *sub-problem of the Equivocal Blends Technique*.

[7] This optimization may be non-trivial – for a single exogenous g, there are generally multiple candidate blends which induce g. Intuitively, this is true because the set $\{\delta \mid \delta^n = g\}$ is closed under convex combination. To illustrate, first consider two distinct equivocal blends examples $g^a = \delta_1^n = \delta_2^n$ and $g^b = \delta_3^n = \delta_4^n$. Then $g^{ab} = {}^{g^a}/{}_2 + {}^{g^b}/{}_2$ has four blends solutions: ${}^{\delta_i^n}/{}_2 + {}^{\delta_j^n}/{}_2$ for all $i \in \{1, 2\},\ j \in \{3, 4\}$. (We ignore that, e.g., the ${}^{\delta_i^n}/{}_2$ term may mix over ${}^{\delta_1^n}/{}_2$ and ${}^{\delta_2^n}/{}_2$ – an optimization never needs this mix by linearity of expectation.) More generally the convex set $\{\delta \mid \delta^n = g\}$ is a Hilbert space, e.g., if g is a continuous mixture over a continuum of equivocal blends.

$$\alpha^{\mathcal{F}} = \min_{A\in\mathcal{A}} \max_{F\in\mathcal{F}} \frac{\mathrm{OPT}_F(F)}{A(F)} \geq \sup_{\delta_2\in\Delta(\mathcal{F})} \left[\sup_{\delta_1\in\{\delta \ \mid \ \delta^n=g=\delta_2^n\}} \left[\frac{\mathbf{E}_{F\sim\delta_2}\left[\mathrm{OPT}_F(F)\right]}{\mathbf{E}_{F\sim\delta_1}\left[\mathrm{OPT}_F(F)\right]} \right] \right]$$

$$= \sup_{g\in\mathcal{G}} \left[\sup_{\substack{\delta_2\in\{\delta \ \mid \ \delta\in\Delta(\mathcal{F}) \text{ and } \delta^n=g\} \\ \delta_1\in\{\delta \ \mid \ \delta^n=g\}}} \left[\frac{\mathbf{E}_{F\sim\delta_2}\left[\mathrm{OPT}_F(F)\right]}{\mathbf{E}_{F\sim\delta_1}\left[\mathrm{OPT}_F(F)\right]} \right] \right]$$

$$= \sup_{g\in\mathcal{G}} \left[\frac{\sup_{\delta_2\in\{\delta \ \mid \ \delta\in\Delta(\mathcal{F}) \text{ and } \delta^n=g\}} \left(\mathbf{E}_{F\sim\delta_2}\left[\mathrm{OPT}_F(F)\right]\right)}{\inf_{\delta_1\in\{\delta \ \mid \ \delta^n=g\}} \left(\mathbf{E}_{F\sim\delta_1}\left[\mathrm{OPT}_F(F)\right]\right)} \right] \tag{9}$$

▶ **Definition 16.** *The optimization problem of equation* (9) *is* Information-Design-Design*. Within the brackets, we refer to the optimizations respectively as the* Numerator *and* Denominator Games*.*

Thus, when g is fixed exogenously by an outer maximization, there is a reduction to diametrically-opposite questions of constrained information design.

▶ **Proposition 1.** *Consider the prior independent design problem (Definition 2) given a class of distributions $\mathcal{F}$, a class of algorithms $\mathcal{A}$, and n inputs. Optimization of the Blends Technique approach to prior independent lower bounds is described by:*

$$\alpha^{\mathcal{F}} \geq \sup_{g\in\mathcal{G}} \left[\frac{\sup_{\delta_2\in\{\delta \ \mid \ \delta\in\Delta(\mathcal{F}) \ and \ \delta^n=g\}} \left(\mathbf{E}_{F\sim\delta_2}\left[\mathrm{OPT}_F(F)\right]\right)}{\inf_{\delta_1\in\{\delta \ \mid \ \delta\in\Delta(\mathcal{F}^{all}) \ and \ \delta^n=g\}} \left(\mathbf{E}_{F\sim\delta_1}\left[\mathrm{OPT}_F(F)\right]\right)} \right]$$

Further, its Numerator Game and its Denominator Game can be independently instantiated as problems of constrained information design.

To re-summarize: constraining the design is the key step – informally information design is a *signalling game*, to which we add the requirement that signals be distributions $F \in \mathcal{F}$ or $F \in \mathcal{F}^{\text{all}}$ (which each induce an i.i.d. product distribution). At the same time, (a) each distribution over signals is a blend, and (b) an optimal algorithm can be run in response to a given signal $\hat{F}$ (i.e., we recognize the use of distributions-as-signals in $\mathrm{opt}_{n,i} = \mathbf{E}_{F\sim\delta_i}\left[\mathrm{OPT}_F(F)\right]$).

In the full version of the paper, we evaluate equivocal blends from the perspective of *Blackwell (partial) ordering*, which compares two designs of signalling strategies, equivalently in terms of both (a) a strong measure of their information content, and (b) a strong measure of their usefulness for arbitrary optimization objectives. If two distinct optimizations prefer expectation over optimal performances from distinct sides of an equivocal blend, then Blackwell ordering is precluded ([4], the theorem is in the full version). Our example of Quadratics-versus-Uniforms equivocal blends meets this condition and Corollary 17 (next) is a consequence of our prior independent lower bounds in Section 3.2 whereby Theorem 8 (for revenue) used an adversarial distribution over the Uniforms side of the equivocal blend, versus, Theorem 10 (for residual surplus) used an adversarial distribution over the Quadratics side.

▶ **Corollary 17.** *Finite-weight Quadratics-versus-Uniforms equivocal blends are an example for which there is no relationship according to Blackwell ordering.*

Most of this Section 5 has been deferred. In the full version of the paper, we give a formal introduction to information design and describe the respective reductions of the Numerator and Denominator Games to information design (thereby providing the proof for Proposition 1).

6 Further Analysis of Equivocal Blends

This section gives further detail of example equivocal blends, corollaries, and follow-up discussion. Generally, material in this section was deferred to bring forward more-critical material.

6.1 A Single-agent Auction with a Tight Bound

This section gives the extended details to support the 1-agent, tight result from blends analysis as presented in the Introduction (Section 1). Although formal introduction of mechanism design is deferred to the full version, the setting here is simple enough to outline and provide the details. The result was for a 1-agent price-posting auction, in which the auction commits to a fixed, take-it-or-leave-it price π to the agent who has private value v from known support, and the agent buys if and only if $v \geq \pi$. The following theorem states the result as summarized in the Introduction:

▶ **Theorem 18.** *Given a single-item, 1-agent* price-posting *auction with a revenue objective and agent value in support $\mathcal{V} = [1, h]$. The optimal prior independent approximation factor is $1 + \ln h$.*

Theorem 18 exists from matching upper and lower bounds. The upper bound is from [16]. We prove the lower bound from equivocal blends analysis as follows (consisting of Quadratics-versus-Point-Masses).

The (truncated) Quadratics were defined in Section 3.1. For convenience: they have PDF $\overline{\mathrm{qud}}_z^h(x) = {}^z/_{x^2}$ on $[1, h)$ and point mass $\overline{\mathrm{qud}}_z^h(h) = {}^1/_h$, correspondingly CDF $\overline{\mathrm{Qud}}_z^h(x) = 1 - {}^z/_x$ on $[1, h)$ and $\overline{\mathrm{Qud}}_z^h(h) = 1$.

Point mass distributions Pmd_z are paramaterized by a single point z with probability 1. They have CDF given by $\mathrm{Pmd}_z(x) = 0$ on $[0, x)$ and $\mathrm{Pmd}_z(x) = 1$ on $[x, \infty)$. (Regarding the PDF of point masses, we make the obvious simplifications; for full formality, see the full version.)

The blends are simple: on one side we have a point mass on $\overline{\mathrm{Qud}}_1^h$; and on the other side we have the distribution $\overline{\mathrm{Qud}}_1^h$ over weights of parameter z corresponding to point mass distributions Pmd_z. Note, this simple structure is possible exclusively because equivocal blends here for input-size $n = 1$ have a reduction that does not exist for $n \geq 2$: for $n = 1$, the underlying set of distributions – which are one-dimensional "product" distributions – is closed under convex combination. Formally:

The (singular) weight of the Quadratic blend (δ_1) is:

- weight $o_{\mathrm{pm}} = 1$ on $\overline{\mathrm{Qud}}_1^h$.

The weights of the Point-Mass blend (δ_2) are:

- weight $\omega_z = \overline{\mathrm{qud}}_1^h(z) = {}^1/_{z^2} \cdot dz$ on Pmd_z for $z \in [1, h)$;
- weight $\omega_h = {}^1/_h$ on Pmd_h.

To confirm the blend at each $v \in [1, h)$, we calculate:

$$\text{result of } \overline{\mathrm{Qud}}_1^h \text{ blend} = o_{\mathrm{pm}} \cdot \overline{\mathrm{qud}}_1^h(v) \qquad = \frac{1}{v^2} = g(v) \tag{10}$$

$$\text{result of } \mathrm{Pmd}_z \text{ blend} = \omega_v \cdot [\text{density of point mass at } v] \qquad = \frac{1}{v^2} = g(v)$$

and at $v = h$, we calculate:

$$\begin{aligned}
&\text{result of } \overset{\bullet}{\overline{\mathrm{Qud}}}{}_1^h \text{ blend} = o_{\mathrm{pm}} \cdot \overset{\bullet}{\overline{\mathrm{qud}}}{}_1^h(h) &&= \frac{1}{h} = g(h) \qquad (11)\\
&\text{result of } \mathrm{Pmd}_z \text{ blend} = \omega_h \cdot [\text{density of point mass at } h] &&= \frac{1}{h} = g(h)
\end{aligned}$$

We evaluate the lower bound resulting from inserting this blend into Theorem 5. For $\overset{\bullet}{\overline{\mathrm{Qud}}}{}_1^h$, posting any price $\pi \in [1, h]$ gets the same revenue of 1, due to the following. The agent buys if and only if random value $v \sim \overset{\bullet}{\overline{\mathrm{Qud}}}{}_1^h$ is at least the price (we break ties in favor of the auction). The calculated probability of buying for any π is $1 - \overset{\bullet}{\overline{\mathrm{Qud}}}{}_1^h(\pi) = 1 - (1 - {}^1/_\pi) = {}^1/_\pi$, and the revenue conditioned on buying is the posted price π. Thus independent of π, revenue is $\pi \cdot {}^1/_\pi = 1$.

For Pmd_z, the optimal price to post is z – when the agent identically has value z from the point mass distribution, revenue is trivially maximized by always selling at posted price $\pi = z$.

The lower bound ratio to calculate is ${}^{\mathrm{opt}_{2,2}}/_{\mathrm{opt}_{2,1}}$ (and we recall from Definition 4 that $\mathrm{opt}_{n,i} = \mathbf{E}_{F \sim \delta_i}[\mathrm{OPT}_F(F)]$). Thus we complete the proof of Theorem 18 by calculating:

$$\frac{\mathbf{E}_{F\sim\delta_2}[\mathrm{OPT}_F(F)]}{\mathbf{E}_{F\sim\delta_1}[\mathrm{OPT}_F(F)]} = \frac{\omega_h \cdot h + \int_1^h \omega_z \cdot z}{o_{\mathrm{pm}} \cdot 1} = \frac{\frac{1}{h} \cdot h + \int_1^h \frac{1}{z^2} \cdot z \, dz}{1} = 1 + \ln h$$

6.2 Sub-classes of Equivocal Blends from Order-statistic Separability

Theorem 12 stated that equivocal blends exist from order-statistic-separable functions ($g(\boldsymbol{v}) = g_1(v_1) \cdot g_2(v_2)$ for $v_1 \geq v_2$) under modest conditions. To apply Theorem 12, we need the elements of an equivocal blend to be proper distributions. The theorem's Conditions (1, 2, 3) are sufficient for an algebraic solution. Additionally satisfying Condition (4) which states, "the integrals $G_1(z) = \int_z^\infty g_1(x)dx$ and $G_2(z) = \int_0^z g_2(x)dx$ are positive and finite for all z," is sufficient for the special case in which we construct $\tilde{g}_{1,z}$ and $\tilde{g}_{2,z}$ to necessarily be probability distributions. This is sufficient to avoid, e.g., $\tilde{g}_{1,z}(x) = {}^{g_{1,z}(x)}/_{G_1(z)}$ being not well-defined or 0. (We give an example failing Condition (4) in the full version.)

The interpretation of Condition (4) is that g_1 must be everywhere "positive and upward-finite" and g_2 must be everywhere "positive and downward-finite."

▶ **Definition 19.** *Given a non-negative function $g_i(\cdot)$ with domain $(0, \infty)$. The function $g_i(\cdot)$ is* upward-finite *if $\int_z^\infty g_i(x)\, dx$ is finite for every z, and it is* downward-finite *if $\int_0^z g_i(x)\, dx$ is finite for every z.*

The simple structure of Conditions (1-4) in Theorem 12 are remarkably easy to satisfy. We give a first example-class of solutions based on monomials (whose finiteness is easy to verify):

▶ **Corollary 20.** *Consider parameterized functions $g^\eta(x) = {}^1/_{x^\eta}$ for any $\eta \in \mathbb{R}$. Setting $g_1 = g^{\eta_+}$ for $\eta_+ > 1$ and $g_2 = g^{\eta_-}$ for $\eta_- < 1$ will meet conditions (1-4) of Theorem 12. Thus, there is an equivocal blends solution for which elements are distributions (as modified) from any g^{η_+} and g^{η_-}.*

Moreover, assume we drop Condition (4). Resetting $g_1 = g^{\eta_+}$ and $g_2 = g^{\eta_-}$ subject to only the weaker condition $\eta_- < \eta_+$ will meet all conditions (1-3) of Theorem 12.

The first statement of Corollary 20 (setting $g_1 = g^{\eta_+}$ for $\eta_+ > 1$ and $g_2 = g^{\eta_-}$ for $\eta_- < 1$) is itself a subset of a much larger class of (infinite-weight) equivocal blends whose elements will be probability distributions (through the constructions of the $\tilde{g}_{i,z}$):

▶ **Corollary 21.** *Consider arbitrary function g_1 that is monotone strictly decreasing and upward-finite; and arbitrary function g_2 that is monotone strictly increasing and downward-finite and also satisfies $\lim_{x\to 0} g_2(x) = 0$ and $\lim_{x\to\infty} g_2(x) = \infty$. The functions g_1 and g_2 together satisfy all Conditions (1-4) of Theorem 12. Thus, there is an equivocal blend solution for which the elements are distributions (as modified) from g_1 and g_2.*

The class of solutions in Corollary 21 is still smaller than the class of all solutions to Theorem 12 (for example, g_1 need not be monotone). As just one of literally-unlimited examples, Corollary 21 is satisfied by assigning $g_1(x) = {}^1/_{x^3}$ and $g_2(x) = (\ln(x+1))^2$. Additional discussion of Theorem 12 is given in the full version of the paper, *including discussion of structure – evident from this theorem – that suggests that there is no direct generalization* of our equivocal blends framework to $n > 2$.

6.3 A Dual Blend Example from Inverse-distributions

The goal of this section is to *overview* a dual blend example from the inverse-distribution structure of Theorem 13, which states that a dual blend exists from using: all scales of a distribution F on one side, all scales of its inverse-distribution i-F on the other, and weights identically ${}^1/_z \cdot dz$ on both sides. For our example, we use the *exponential distribution*, which is motivated by a distribution chosen by a minimax-optimal adversary within optimal prior independent mechanism design. (This adversary occurs in [1], in a setting that restricts to the class monotone hazard rate distributions ($\mathcal{F}^{\text{mhr}}$) – a class for which exponentials are on the *boundary*.

The full analysis of the following dual blends is sufficiently technical that we defer it to the full version of the paper, where we give a wholly-contained presentation of the example.

Exponentials versus Inverse-Exponentials

The standard exponential distribution is Exd_1 which has CDF $\text{Exd}_1(x) = 1 - e^{-x}$ on $[0, \infty)$ and PDF $\text{exd}_1(x) = e^{-x}$.

Its inverse-distribution – as calculated from Definition 14 – is the *inverse-exponential* distribution i-Exd_1 which has CDF i-$\text{Exd}_1(x) = e^{-{}^1/_x}$ and PDF i-$\text{exd}_1(x) = \frac{1}{x^2} \cdot e^{-{}^1/_x}$.

Given the arbitrary re-scaling inherent in Theorem 13 (recall Fact 15), we will need all possible scales of these distributions to compose our blend. Scalings of exponentials (and inverse-exponentials) are naturally representable by hazard rate parameters β.

The distributions in the exponentials blend will have general description $\text{Exd}_\beta(x) = 1 - e^{-\beta x}$ and $\text{exd}_\beta(x) = \beta \cdot e^{-\beta x}$ on $x \in [0, \infty)$; and the distributions in the inverse-exponentials blend will have general description i-$\text{Exd}_\beta(x) = e^{-{}^\beta/_x}$ and PDF i-$\text{exd}_\beta(x) = \frac{\beta}{x^2} \cdot e^{-{}^\beta/_x}$ on $[0, \infty)$.

Lastly, towards implementing the desired blends of this section which will further support *modification to finite weight*, we will *truncate* all of the distributions composing each side of our dual blends: we use top-truncation to modify Exponentials into our downward-closed class of Exponentials, and *bottom-truncation* to modify inverse-exponentials into our upward-closed class of Inverse-Exponentials. (For full explanation of distribution note, recall footnote 5.)

To resummarize, we have:

- arbitrary re-scaling, which is naturally represented as hazard rate parameters β;
- truncations, parameterized by truncation point z;
- and note that for each side of the dual blend, there will be a fixed, functional relationship between z and β which also depends on the size of the truncated mass: quantile $\bar{q}$ for top-truncated Exponentials and percentile $\bar{p}$ for bottom-truncated Inverse-Exponentials; and even though we could replace either z or β (by writing one in terms of the other), we keep them both to simplify the notation.

We can now write our (infinite weight) dual blends result that follows from Theorem 13:

▶ **Corollary 22.** *Fix $\bar{q} = \bar{p} \in (0,1)$. Given the class of downward-closed (top-truncated) Exponentials with members $\overline{\mathrm{Exd}}{}^{z}_{\beta}$ and the class of upward-closed (bottom-truncated) Inverse-Exponentials with members $\overline{\text{i-Exd}}{}^{z}_{\beta}$, each class including all $z > 0$ (equivalently all $\beta > 0$). For $n = 2$ and $o_z = \omega_z = {}^1\!/_z \cdot dz$, we have the following dual blends matching up at every $\boldsymbol{v} = (v_1, v_2 \leq v_1)$ to describe a common function g:*

$$\int_0^{v_2} \frac{1}{z} \cdot \overline{\mathrm{exd}}{}^{z}_{\beta}(v_1) \cdot \overline{\mathrm{exd}}{}^{z}_{\beta}(v_2)\ dz = g(\boldsymbol{v}) = \int_{v_1}^{\infty} \frac{1}{z} \cdot \overline{\text{i-exd}}{}^{z}_{\beta}(v_1) \cdot \overline{\text{i-exd}}{}^{z}_{\beta}(v_2)\ dz \tag{12}$$

The completion of this example with full details is deferred to the full version of the paper. A partial list of its contents is:

- full set up of the distributions composing our Exponentials and Inverse-Exponentials blends, including mathematical details of the parameters;
- confirmation of dual blends calculations;
- modification to finite weight dual blends, based on the same elements, including all necessary supporting calculations;
- as a point of interest within mechanism design, confirmation that Exponential distributions remain *monotone hazard rate* after their modification for use in the finite weight dual blends (and thus, they are available to an adversary restricted to the class $\mathcal{F}^{\mathrm{mhr}}$);
- and finally, as a second point of interest, we graph post-truncation revenue curves in Figure 1 (below on page 19; revenue curves are defined along with mechanism design in the full version); the remarkable observation is that we have *concave* revenue curves of the top-truncated Exponentials on one side of the dual blend versus *single-troughed* revenue curves of the bottom-truncated Inverse-Exponentials on the other side of the dual blend.

The contrasting structures of the distributions' revenue curves within the respective sides of the dual blend are surprising and intriguing within mechanism design.

1. Intuitively, geometrically, it seems impossible that the adversary can possibly choose a blend composed of distributions inducing concave revenue curves, and in response the mechanism perceives a blend over distributions inducing inverted structure; but there it is.
2. Generally, non-concave revenue curves are significantly more difficult to analyze and optimize, *especially for robust-design settings like prior independence* in which the relaxation to the class of all distributions $\mathcal{F}^{\mathrm{all}}$ allowing arbitrary non-concave revenue curves results in super-constant lower bounds on approximation [12]; indeed, the definition of the regular class $\mathcal{F}^{\mathrm{reg}}$ is the restriction to (weakly) concave revenue curves; and the analytical-technical and mechanism-performance challenges that arise outside the class $\mathcal{F}^{\mathrm{reg}}$ are the reasons that restriction to $\mathcal{F}^{\mathrm{reg}}$ is frequently assumed in the literature.
3. Technically, the possibility that the mechanism is responding to a distribution over *irregular* distributions – with their associated technical challenges in the prior independent setting – is a salient weakness in its information that may ultimately govern the approximation gaps induced by *optimal* dual blends design.

We give a second example dual blend from Theorem 13 (infinite weight only) in the full version, to which we defer all details. It is also motivated by a distribution chosen by an adversary within optimal mechanism design. Specifically, it is designed from a "shifted-quadratic" distribution $\mathrm{Sqd}_{0,-1}$ which has CDF $\mathrm{Sqd}_{0,-1}(x) = 1 - {}^1\!/_{x+1}$ on $[0, \infty)$ and PDF $\mathrm{sqd}_{0,1}(x) = {}^1\!/_{(x+1)^2}$. Its presentation is largely analogous to the presentation of infinite-weight Exponentials-versus-Inverse-Exponentials.

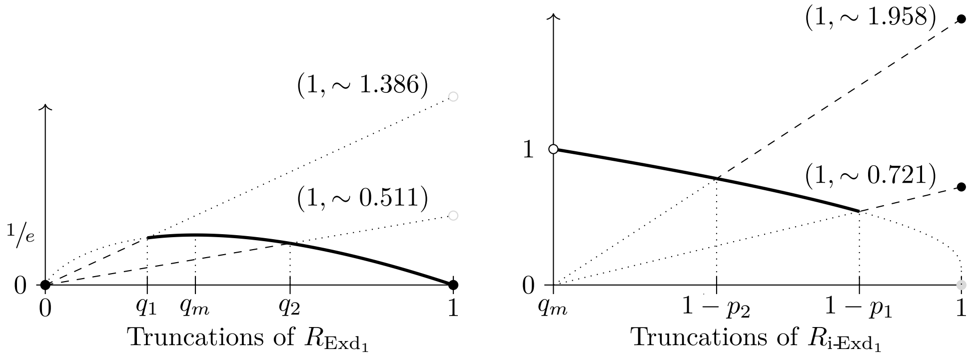

Figure 1 Example Ironings of Exponential and Inverse-Exponential.

The left figure shows the results of two examples of (dashed) top-truncation of the distribution Exd_1, respectively at $q_1 = 0.25 < q_m$ and $q_2 = 0.6 > q_m$. The right figure shows the results of the corresponding two examples of (dashed) bottom-truncation of the distribution i-Exd_1, respectively at $1 - p_1 = 1 - q_1 = 0.75$ and $1 - p_2 = 1 - q_2 = 0.4$; the (approximate) heights of the points on the right describe the *agent-values* of the respective truncations.

An additional point of interest for the base class of (all rescalings of) Shifted-Quadratics is: *its class of inverse-distributions is identically itself*: each element of the class of Shifted-Quadratics which is indexed by a positive, real-valued shape parameter ϕ maps to the Shifted-Quadratic with multiplicatively-inverted shape parameter $1/\phi$.

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A Proof of the Equivocal Blends Lower-Bound Theorem

For use in this section, recall our notation $A(F) = \mathbf{E}_{\boldsymbol{v}\sim F}[A(\boldsymbol{v})]$ for the expected performance of algorithm A on n i.i.d. draws from a distribution F.

First we state Lemma 23 which shows that for any fixed blend $\bar{\delta}$ (as implicit choice of the adversary), we can obtain a lower bound on prior independent approximation [13]. (This lower bound is used as an interim step within the proof of Theorem 5.)

Lemma 23 states that we can replace the adversary's maximization problem within prior independent design (for reference see equation (13)). In its place, the adversary effectively sets a benchmark as the expectation of optimal performance over distributions drawn from $\bar{\delta}$ (thus, the benchmark is $\mathbf{E}_{F\sim\bar{\delta}}[\mathrm{OPT}_F(F)]$). Symmetrically, the algorithm's performance is its expected performance over distributions drawn from $\bar{\delta}$ (thus, its performance is $\mathbf{E}_{F\sim\bar{\delta}}[A(F)]$).

An algorithm's approximation of the benchmark is measured as the ratio of this benchmark to its performance, i.e., as *ratio-of-expectations* (ROE). The lower bound results from the minimum ratio achieved by any algorithm $A \in \mathcal{A}$. Practically, this interim lower bound is only an abstraction because we don't say anything about how to optimize the algorithm A.

▶ **Lemma 23** (The Ratio-of-Expectations Benchmark Lemma [13]). *Consider a prior independent setting with input space $\mathcal{V}^n$, class of algorithms $\mathcal{A}$, and class of distributions $\mathcal{F}$. Let $\bar{\delta} \in \Delta(\mathcal{F})$ be any fixed blend, i.e., a fixed distribution over the distributions of $\mathcal{F}$. Then*

$$\alpha^{\mathcal{F}} = \min_{A\in\mathcal{A}} \max_{F\in\mathcal{F}} \frac{\mathrm{OPT}_F(F)}{A(F)} \geq \min_{A\in\mathcal{A}} \left[\frac{\mathbf{E}_{F\sim\bar{\delta}}[\mathrm{OPT}_F(F)]}{\mathbf{E}_{F\sim\bar{\delta}}[A(F)]}\right], \quad \text{for fixed } \bar{\delta} \tag{13}$$

The key step in the proof of Lemma 23 is carved out as its own lemma and is noteworthy here:

▶ **Lemma 24** (The Randomized-state Relaxation Lemma [13]). *Consider a prior independent setting with input space $\mathcal{V}^n$, class of algorithms $\mathcal{A}$, and class of distributions $\mathcal{F}$. Without loss of generality for the value of prior independent optimization, we can simultaneously: relax* *the distribution space $\mathcal{F}$ to allow any blend $\delta \in \Delta(\mathcal{F})$, and transition EOR to ROE. I.e., the following* min-max *programs necessarily have the same value:*

$$\min_{A\in\mathcal{A}} \max_{F\in\mathcal{F}} \frac{\mathrm{OPT}_F(F)}{A(F)} = \min_{A\in\mathcal{A}} \max_{\delta\in\Delta(\mathcal{F})} \left[\frac{\mathbf{E}_{F\sim\delta}\left[\mathrm{OPT}_F(F)\right]}{\mathbf{E}_{F\sim\delta}\left[A(F)\right]}\right] \tag{14}$$

With the statement Lemma 23 in place (along with partial explanation for it via Lemma 24), we are prepared to restate and prove Theorem 5.

▶ **Theorem 5.** *Consider a prior independent setting with input space $\mathcal{V}^n$, class of algorithms $\mathcal{A}$, and class of distributions $\mathcal{F}$. Let $\mathcal{F}^{\mathrm{all}}$ be all distributions. Assume there exist two distinct equivocal blends $\delta_1 \in \Delta(\mathcal{F}^{\mathrm{all}})$ and $\delta_2 \in \Delta(\mathcal{F})$ and correlated density function g (of Definition 4) such that:*

$$\delta_1^n(\boldsymbol{v}) = g(\boldsymbol{v}) = \delta_2^n(\boldsymbol{v}) \quad \forall\, \boldsymbol{v}$$

Then the optimal prior independent approximation factor $\alpha^{\mathcal{F}}$ is at least the ratio ${}^{\mathrm{opt}_{n,2}}/{}_{\mathrm{opt}_{n,1}}$:

$$\alpha^{\mathcal{F}} = \min_{A\in\mathcal{A}} \max_{F\in\mathcal{F}} \frac{\mathrm{OPT}_F(F)}{A(F)} \geq \frac{\mathrm{opt}_{n,2}}{\mathrm{opt}_{n,1}}$$

Proof. We start with the prior independent design problem and apply Lemma 23 (given above; by assigning $\bar{\delta} = \delta_2$ in its statement). Justifications for the next steps are given afterwards.

$$\begin{aligned}
\min_{A\in\mathcal{A}} \max_{F\in\mathcal{F}} \frac{\mathrm{OPT}_F(F)}{A(F)} &\geq \min_{A\in\mathcal{A}} \left[\frac{\mathbf{E}_{F\sim\delta_2}\left[\mathrm{OPT}_F(F)\right]}{\mathbf{E}_{F\sim\delta_2}\left[A(F)\right]}\right] \\
&= \min_{A\in\mathcal{A}} \left[\frac{\mathrm{opt}_{n,2}}{\mathbf{E}_{\boldsymbol{v}\sim g}\left[A(\boldsymbol{v})\right]}\right] \\
&= \min_{A\in\mathcal{A}} \left[\frac{\mathrm{opt}_{n,2}}{\mathbf{E}_{F\sim\delta_1}\left[A(F)\right]}\right] \\
&\geq \min_{A\in\mathcal{A}} \left[\frac{\mathrm{opt}_{n,2}}{\mathbf{E}_{F\sim\delta_1}\left[\mathrm{OPT}_F(F)\right]}\right] = \frac{\mathrm{opt}_{n,2}}{\mathrm{opt}_{n,1}}
\end{aligned} \tag{15}$$

- The second and third lines substitute using the definition of $\mathrm{opt}_{n,i}$ and the assumption in the theorem statement that $\delta_1^n(\boldsymbol{v}) = g(\boldsymbol{v}) = \delta_2^n(\boldsymbol{v})$.
  Note, the adversary's choice of δ_2 is restricted to the set $\Delta(\mathcal{F})$ up front in the prior independent problem (i.e., the parameter $\mathcal{F}$ is fixed exogenously), and δ_2 induces $g = \delta_2^n$. However given g, there may exist any alternative description δ_1 with $g = \delta_1^n$, including a $\delta_1 \in \Delta(\mathcal{F}^{\mathrm{all}})$ that uses distributions outside the original class $\mathcal{F}$. This freedom to design δ_1 is an inherent *consequence of nature.*
- The fourth line inequality recognizes that expectation over locally optimal performances – each knowing the true F when realized – must weakly dominate the performance of a single algorithm run against all realizations of F (formally: Fact 25 after this proof).
- The final equality substitutes and realizes that the algorithm no longer appears in the function to be minimized, i.e., the objective is constant. ◀

The following holds because each OPT_F algorithm is optimal pointwise per F, whereas running A against each F is itself immediately upper bounded by OPT_F:

▶ **Fact 25.** *Given an arbitrary prior independent algorithm design setting with class of distributions $\mathcal{F}$ and class of algorithms $\mathcal{A}$, and given $\delta \in \Delta(\mathcal{F})$. For any fixed algorithm $A \in \mathcal{A}$:*

$$\mathbf{E}_{F\sim\delta}\left[\mathrm{OPT}_F(F)\right] \geq \mathbf{E}_{F\sim\delta}\left[A(F)\right]$$