

The Non-Uniform Peregbor Conjecture for Time-Bounded Kolmogorov Complexity Is False

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Abstract

The *Peregbor* (Russian for “brute-force search”) conjectures, which date back to the 1950s and 1960s are some of the oldest conjectures in complexity theory. The conjectures are a stronger form of the $\text{NP} \neq \text{P}$ conjecture (which they predate) and state that for “meta-complexity” problems, such as the *Time-bounded Kolmogorov complexity Problem*, and the *Minimum Circuit Size Problem*, there are no better algorithms than brute force search.

In this paper, we disprove the *non-uniform* version of the Peregbor conjecture for the Time-Bounded Kolmogorov complexity problem. We demonstrate that for every polynomial $t(\cdot)$, there exists of a circuit of size $2^{4n/5+o(n)}$ that solves the $t(\cdot)$ -bounded Kolmogorov complexity problem on *every* instance.

Our algorithm is *black-box* in the description of the Universal Turing Machine U employed in the definition of Kolmogorov Complexity and leverages the characterization of one-way functions through the hardness of the time-bounded Kolmogorov complexity problem of Liu and Pass (FOCS’20), and the time-space trade-off for one-way functions of Fiat and Naor (STOC’91). We additionally demonstrate that no such black-box algorithm can have circuit size smaller than $2^{n/2-o(n)}$.

Along the way (and of independent interest), we extend the result of Fiat and Naor and demonstrate that any efficiently computable function can be inverted (with probability 1) by a circuit of size $2^{4n/5+o(n)}$; as far as we know, this yields the first formal proof that a non-trivial circuit can invert any efficient function.

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1 Introduction

In his historical account, Thaktenbrot [22], describes efforts in the 1950s and 1960s in the Russian Cybernetics program to understand problem that requiring *brute-force search* to solve. [22, 23, 24]. The so-called *Peregbor* (Russian for brute-force search) conjectures refer to the conjectures that certain types of, what today are referred to as “meta-complexity”, problems require brute-force search to be solve. These include (a) the *Minimum Circuit*



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Size problem (MCSP) [13, 22] – finding the smallest Boolean circuit that computes a given function x , and (b) the *Time-Bounded Kolmogorov Complexity Problem* [15, 21, 3, 14, 10, 20] – computing the length, denoted $K^t(x)$ of shortest program (evaluated on some particular Universal Turing machine U) that generates a given string x , within time $t(|x|)$, where t is a polynomial. The Perebor conjectures state that solving these problems require an algorithms with running time (in the *uniform* regime), or circuit size (in the *non-uniform* regime) close to 2^n where $n = |x|$ is the size of the given instance x .

The Perebor conjecture can be viewed as an early precursor, and stronger form, of the $\mathbf{NP} \neq \mathbf{P}$ conjecture (as these meta-complexity problems reside in \mathbf{NP}). In this work, we focus on the Time-Bounded Kolmogorov Complexity Problem, and the stronger *non-uniform* version of the Perebor conjecture for the time-bounded Kolmogorov Complexity problem.

1.1 Our Results

Circuit Complexity Upperbounds for Solving K^t . Our main result disproves the non-uniform Perebor conjecture for Time-bounded Kolmogorov Complexity:

► **Theorem 1** (Main theorem). *For every $t \in \text{poly}$, there exists a circuit family $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ of size $\tilde{O}(2^{4n/5})$ such that, for every $n \in \mathbb{N}$ and for every $x \in \{0, 1\}^n$, $C_n(x) = K^t(x)$ (That is, the circuits computes $K^t(x)$ on every instance x .)*

In fact, our theorem is stronger than stated: we not only show how to compute K^t but also to find a so-called K^t -witness (i.e., a shortest-length program generated the string x) – this was referred to as the *constructive* version of the problem in [22].

We highlight that the fact that a non-uniform Perebor conjecture may be false may perhaps not be shocking to experts; indeed, Ren and Santhanam [19] recently considered an (average-case strengthening) of the non-uniform Perebor conjecture for the Time-bounded Kolmogorov complexity and noted that by the recent connection [16] between average-case hardness of the time-bounded Kolmogorov complexity problem and the cryptographic notion of *one-way functions* [6], there is evidence pointing to this conjecture being false:

However, building on Hellman [11], Fiat and Naor [7] showed that no such one-way function exists in the non-uniform RAM model. In particular, for any function $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$, there is an algorithm that runs in $2^{3n/4}$ time, with random access to an advice tape of length $2^{3n/4}$, and inverts f at any point. It is conceivable that a similar attack could also be implemented in circuits, i.e. every function f could be inverted by a circuit of size $2^{99n/100}$ in the worst-case. This gives strong evidence that the non-uniform version of Hypothesis 5.18 is false.

As far as we know, such a circuit implementation of [7] is not known – indeed, as we discuss below, the natural implementation of their algorithm as a circuit leads to a “trivial” size circuit, and does not beat Perebor.

Our main technical result is a non-trivial implementation of the Fiat-Naor algorithm using a circuit of size $\tilde{O}(2^{4n/5})$ (note that this is larger than the running-time/advice size of $\tilde{O}(2^{3n/4})$ achieved by Fiat and Naor):

► **Theorem 2.** *For every efficiently computable function $f : \{0, 1\}^n \rightarrow \{0, 1\}^n$, there exists a circuit family $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ of size $\tilde{O}(2^{4n/5})$ such that, for every $n \in \mathbb{N}$ and for every $x \in \{0, 1\}^n$, $f(C_n(f(x))) = f(x)$ (That is, the circuits inverts f on every point in the range of f .)*

We next combine Theorem 2 with the one-way function construction of [16] to conclude Theorem 1 – leveraging the fact that the one-way function construction is length-preserving and thus there is little loss in the parameters. We additionally note that if we instead rely on the one-way function construction from the MCSP problem of [19] which also is length preserving (but with slightly longer input size), then we directly also get an algorithm for the Minimum Circuit Size (MCSP) problem with circuit size $2^{4n/5+o(n)}$.¹

Circuit Complexity Lowerbounds for *Black-box Solving* K^t . We highlight that our K^t solving circuit is “black-box” in the description of the Universal Turing machine U employed in the definition of Kolmogorov complexity (or more precisely, the circuit can be implemented having simply oracle access to U) – we refer to such circuits as being a *black-box* K^t solver. We finally demonstrate that subexponential-size *black-box* K^t solvers cannot exist, and as such our non-uniform K^t solver is “optimal” up to a constant in the exponent w.r.t. such black-box algorithms.

► **Theorem 3 (Informal).** *Every black-box K^t -solver must have circuit size at least $2^{n/2-o(n)}$.*

1.2 Proof Outline

We here provide a proof outline of Theorems 1 and 2.

1.2.1 Constructing a One-way Function from K^t

As mentioned above, our starting point is the one-way function construction of [16] from the Time-bounded Kolmogorov Complexity problem. We note that this construction trivially works also in the worst-case regime (i.e., any worst-case inverted for the one-way function also solves K^t in the worst-case). But most importantly, this construction is *length preserving*: any algorithm breaking the one-way function of inputs of length $n+O(\log n)$ solves the K^t problem on inputs of length n . This observation will be crucial as it means that the upperbound that we get for inverting the one-way function will directly translate to (essentially) the same bound for K^t . (The same hold also for the one-way function construction from the MCSP problem from [19], except that the input length now is $n+o(n)$, where n is the MCSP input length).

1.2.2 The Hellman/Fiat-Naor Algorithm

The Hellman Algorithm. Before describing the Fiat-Naor algorithm for inverting arbitrary functions, we describe the algorithm of Hellman [11] for inverting a permutation. The main idea of Hellman is that for every permutation $f: [2^n] \rightarrow [2^n]$, and every image $y \in [2^n]$, if we compute the chain $(y, f(y), f(f(y)), \dots)$, at some point we must find a pre-image of y . Namely, there exists some $k \geq 1$ such that $f^k(y) = y$, and thus $f^{k-1}(y) = f^{-1}(y)$. Of course, this k can be large, and it could even be that $k = 2^n$. In this case, computing the chain $(y, f(y), f(f(y)), \dots)$ takes the same time as brute-force search.

To overcome this, Hellman’s algorithm uses an advice string. For a parameter T , this advice string contains S entries of the form $(x, f^T(x))$. Then, to find a pre-image of y , the algorithm computes the chain of length T , $(y, f(y), \dots, f^T(y))$. If one of the points in the chain is a pre-image of y , the algorithm finishes. Otherwise, the algorithm looks for an entry

¹ We thank an anonymous reviewer for noting that we can also apply the attacker to the MCSP problem.

$(x_i, f^T(x_i))$ in the advice string, such that $f^T(x_i)$ appears in the chain $(y, f(y), \dots, f^T(y))$. If it finds such an entry, then it holds that $f^k(y) = f^T(x_i)$ for some $k \leq T$, or equivalently, $y = f^{T-k}(x_i)$. The algorithm then can find the pre-image of y by computing $f^{T-k-1}(x_i)$.

Every such entry $(x_i, f^T(x_i))$ in the advice can be used to invert the T images $(f(x_i), \dots, f^T(x_i))$ using $O(T)$ calls to f . Hellman showed that we can find at most $S = N/T$ points x_1, \dots, x_S , such that the advice $((x_1, f^T(x_1)), \dots, (x_S, f^T(x_S)))$ can be used to invert *any* image with $O(T)$ calls to f . This allows to invert any such permutation in time $\tilde{O}(2^{n/2})$ using an advice of length $\tilde{O}(2^{n/2})$.

When $f: [2^n] \rightarrow [2^n]$ is an arbitrary function, it is no longer true that $y = f^{T-k}(x_i)$ if $f^k(y) = f^T(x_i)$. Moreover, it might be impossible to find a non-trivial number of points x_1, \dots, x_S such that the T -length chains starting from these points cover all the images of the function. However, Hellman showed under some assumptions on the structure of f (that is, that f is “random enough”), that a similar algorithm to the above can work with slightly worse parameters. Fiat and Naor later generalized Hellman’s result to work with any function f .

The Fiat-Naor Algorithm. Let $f: [2^n] \rightarrow [2^n]$ be an arbitrary function. The first step in the Fiat-Naor algorithm is to remove from f any image with too many pre-images. To do so, for a parameter $U \in \mathbb{N}$, let $\mathcal{A} = \{y_1, \dots, y_r\}$ be the set of all images y_i with more than U pre-images. Observe that the size of \mathcal{A} , r , is at most $2^n/U$. The Fiat-Naor algorithm adds to the advice a pre-image x_i of every $y_i \in \mathcal{A}$. By saving $(x_i, y_i)_{y_i \in \mathcal{A}}$ in the advice, we can trivially invert any image y_i . Thus, we are left to invert f over the domain $\mathcal{D} = \{x: f(x) \notin \mathcal{A}\}$ of all inputs with a small number of “siblings”. Note that over the domain \mathcal{D} , every image of f has at most U pre-images. In the following, we restrict attention to functions with a small number of heavy images. That is, we assume that $|\mathcal{D}| \geq 2^n/2$. (Fiat and Naor deal with this issue later using a different parametrization, which would lead to further complications in our setting. Nevertheless, as we shall explain in more detail below, we shall observe that focusing on the above “simplified” case actually is without loss of generality.)

Next, [7] proceed by presenting an algorithm A with advice of length $\tilde{O}(m)$ that makes $\tilde{O}(t)$ calls to f . For some choices of the parameters m and t , [7] presented a distribution over advices, such that for every image $y \in f(\mathcal{D})$ and for a random advice, A successfully invert y with probability roughly mt/N . By running the algorithm $\ell = \tilde{O}(N/mt)$ times with ℓ random advices, [7] get an algorithm that inverts *all* images $y \in f(\mathcal{D})$ simultaneously with high probability. The resulting algorithm has advice of length $S = \ell \cdot m + |\mathcal{A}|$, and it makes $T = \ell \cdot t$ calls to the function f .

We next describe the algorithm A . To make the function f behave more randomly, Fiat-Naor uses a hash function $g: [2^n] \rightarrow [\mathcal{D}]$. Then, the algorithm actually tries to invert the function $h = g \circ f$. Choosing the function g such that it will have a succinct description and yet will be easy to evaluate is an important part of the Fiat-Naor algorithm. But for now, we can think of g as a random function. As in Hellman’s algorithm, the advice contains $(x_i, h^t(x_i))$ for m randomly chosen points x_1, \dots, x_m .

Given the advice $(x_i, h^t(x_i))_{i \in [m]}$, the function g and an input $y = f(x)$ to invert, the algorithm A starts with computing $g(y) = h(x)$. Next, the algorithm proceed by computing the t -length chain $(h(x), h^2(x), \dots, h^t(x))$. Then, for every $i \in [m]$ such that $h^t(x_i) = h^t(x)$ (that is, $h^t(x_i)$ appears in the chain $(h(x), h^2(x), \dots, h^t(x))$), A computes $x' = h^{t-k}(x_i)$ and checks if $f(x') = y$. Importantly, when f is not a permutation, it is possible that $f(x') \neq y$. In this case, we say that i is a false-positive index.

The Correctness Proof. Fiat and Naor showed that for every $y \in f(\mathcal{D})$, when the parameters are chosen such that $mt^2U \approx 2^n$, and over a random choice of the function g and the points $x_1, \dots, x_m \leftarrow [2^n]$, the algorithm succeeds in finding a pre-image of y with probability roughly mt/N . Moreover, to bound the running time of the algorithm Fiat and Naor showed that the expected number of false-positive indexes is constant. For the convenience of the interested reader, we give a high level sketch of the proof in the full version of this paper (but this will not be relevant for the rest of the paper – we will use this part of Fiat-Naor in a black-box way).

The Function g . We next describe how to choose the function g used by A . As mentioned above, we want g to have a succinct description and we need to be able to evaluate it efficiently. Recall we want g to be from $[2^n]$ to \mathcal{D} . However, the range \mathcal{D} is defined by the function f , and it thus does not have an efficient representation. However, given the set \mathcal{A} , and the function f , it is easy to check if a point x is in \mathcal{D} : simply check if $f(x)$ is not in \mathcal{A} . Using this fact, to construct the function g , Fiat and Naor [7] constructed first a function $g': [2^n] \times [2^n] \rightarrow [2^n]$. Then, they defined $g(x)$ to be $g'(x, z)$ for the minimal $z \in [2^n]$ for which $g'(x, z) \in \mathcal{D}$. By our assumption that \mathcal{D} is large (at least $2^n/2$), if g' is a random function (or $\Omega(1)$ -wise independent), the expectation of this z is $O(1)$. Thus, we will not need to evaluate g' too many times to evaluate g . Moreover, in the RAM model, checking if $g'(x, z) \in \mathcal{D}$ can be done efficiently (using hash tables or binary search), and thus evaluating g has roughly the same cost as evaluating g' .

For the function g' , [7] used a random $O(t)$ -degree polynomial. Such a polynomial has a (relatively) succinct description, and the resulting function g' is $O(t)$ -wise independent, which is enough for their analysis to go through. However, evaluating g' on a single point takes $O(t)$ time, which is more than we can afford.

[7] solved this issue by exploiting the fact that we run the algorithm A ℓ times in parallel, and by using the FFT algorithm. Specifically, as we actually need to construct ℓ functions, g'_1, \dots, g'_ℓ . [7] constructed these polynomials in a correlated way, such that the *amortized* cost of evaluating each polynomial g'_i on a point x_i will be $\tilde{O}(1)$. In other words, computing $g'_1(x_1), \dots, g'_\ell(x_\ell)$ simultaneously can be done in time $\tilde{O}(\ell)$.

The Time and Space Complexity. As mentioned above, the Fiat-Naor algorithms has time $T = \tilde{O}(\ell \cdot t)$ and space $S = \tilde{O}(\ell \cdot m + |\mathcal{A}|)$, where $|\mathcal{A}| \leq 2^n/U$ and $\ell = 2^n/mt$. That is, $T = \tilde{O}(2^n/m)$ and space $S = \tilde{O}(2^n/t + 2^n/U)$. Thus, [7] can choose parameters such that $U = t$. Moreover, by the analysis above we need to choose parameters such that $mt^2U < 2^n$. To minimize $T + S$, we want to choose parameters such that $2^n/m = 2^n/t = 2^n/U$, which imply that $m = t = U = 2^{3n/4}$. In this case, we get that $S = T = 2^{3n/4}$.

On Implementing the Attack as a Circuit. When trying to implement the Fiat-Naor algorithm as a circuit, the main difference from the above is in the evaluation of the functions g_1, \dots, g_ℓ . Specifically, to evaluate g_i on a point x , we need to evaluate the function g'_i on x, z for multiple values of z , and for each one, to check if $f(g'_i(x, z))$ appears in the list \mathcal{A} . While in the RAM model checking if $f(g'_i(x, z))$ appears in the list \mathcal{A} can be done in $\log(|\mathcal{A}|)$ time, with a circuit we must use a circuit of linear size. Since we evaluate g_1, \dots, g_ℓ overall $\Omega(t \cdot \ell)$ times, the total cost of the above lookup is $\Omega(|\mathcal{A}| \cdot t \cdot \ell) = \Omega(2^n/U \cdot 2^n/m)$. By the constraint that $mt^2U < 2^n$, we get that the resulting circuit size is at least $2^n \cdot t^2$, which is more than the trivial size.

1.3 Our Circuit Implementation

To solve the above issues, we first construct a new primitive that we call *batched look-up tables*, and show that it is possible to look-up for multiple values in the list \mathcal{A} with a small amortized cost.

Then, we show that by choosing different parameters than Fiat and Naor [7], we can obtain a non-trivial circuit.

Batched Look-Up Tables. Informally, Batched Look-Up circuits allow to search multiple entries in a list, with a small amortized cost. In more detail, let $\mathcal{T} = (a_1, \dots, a_r)$ be a target list, and let $\mathcal{B} = (x_1, \dots, x_\ell)$ be the inputs. We want to check for each x_i if it appears in \mathcal{T} . The batch look-up circuit returns a list of bits (b_1, \dots, b_ℓ) such that b_i indicates if $x_i \in \mathcal{T}$.

We show that there is a look-up circuit of size $\tilde{O}(|\mathcal{T}| + |\mathcal{B}|)$. It follows that, when \mathcal{B} is long enough, the amortized cost of each search is $\tilde{O}(1)$. We implement this circuit in three steps:

First, we modify the list \mathcal{B} such that each element in the list contains the index. That is, we let $\mathcal{B}' = ((x_1, 1), \dots, (x_\ell, \ell))$. We also let $\mathcal{T}' = ((a_1, 0), \dots, (a_t, 0))$. Then, we use a sorting circuit to sort the union of the lists (according to the first entry of each element, and then the second). Let \mathcal{R}_1 be the output of this step.

Next, we check for each element (x, i) in \mathcal{R}' if there exists an element of the form $(x, 0)$ earlier in the list. If there is, we change the entry to $(1, i)$. If there is no such element, we change the entry to $(0, i)$. Since the list is sorted, this can be done with an efficient algorithm that reads the list in one pass, and thus can be implemented with a small circuit. After this step, the list contains one entry of the form (b, i) for every $i \in [\ell]$, where the value of b is 1 if and only if x_i appears in \mathcal{T} .

Lastly, we sort the list again, this time according to the indexes, and output the first entry of each of the last ℓ elements in the sorted list.

Changing the Parametrization. Using the batched look-up circuit, we can now compute the functions g_1, \dots, g_ℓ in a small amortized cost in parallel. Specifically, we can evaluate all the ℓ functions, each on one point, in cost of $\tilde{O}(\ell + |\mathcal{A}|)$. However, to compute a chain of length t using the function g_i , the algorithm needs to evaluate g_i sequentially. Recall that to evaluate g_i on a point x , we need to find z such that $f(g'_i(x, z)) \notin \mathcal{A}$. Thus the look-up step must be performed between every two consecutive calls g_i . So while we can batch calls across chains, in each chain we must perform the look-up separately for each call. This implies that we need to apply the batch look-up circuit t times, each time on ℓ calls. Thus, we need to pay $\tilde{O}((r + \ell)t)$ for this step. For the parameter choice of Fiat and Naor [7], when $r = S = \ell \cdot m$, we get that $t \cdot r = 2^n$, and thus the circuit size is again trivial.

Fortunately, we can choose the parameters differently to get a better circuit size. Specifically, we choose r to be much smaller than $\ell \cdot m$. In more detail, our final circuit size is $\tilde{O}((r + \ell)t + m \cdot \ell)$.² Recall that $\ell = 2^n / mt$, $r = 2^n / U$, and we need to choose parameters such that $mt^2U < 2^n$, or equivalently $mt^2 < r$. Taking $m = t$ and $r = \ell = 2^n / t^2$, we get that for $t < 2^{n/5}$ the above constrain holds. In this case, the circuit size is $\tilde{O}(2^{4n/5})$.

² The term $m \cdot \ell$ is since in the Fiat-Naor algorithm, we need to find all the indexes j such that $(x_j, h^t(x_j))$ is in the advice and $h^t(x_j)$ appears in the chain starting with y . For this step we also use a batch look-up circuit.

Other Issues. We note that the above outline oversimplifies a bit, and we also require to reanalyze certain aspect of the Fiat-Naor algorithm (on top of the above changes).

Inverting General Functions. The above approach only work when most points in the range of the function has a small number of pre-images. Fiat and Naor show how to deal also with the more general case, but this requires a more subtle choice of parameters and this would complicate things for us. To overcome this issue (and of independent interest), we observe that any function $f: [2^n] \rightarrow [2^n]$ can be converted into a function $f': [2 \cdot 2^n] \rightarrow [2 \cdot 2^n]$ satisfying the small-image requirement, in a way that inverting f can be reduced to inverting f' . Indeed, we can simply let $f'(x) = f(x)$ for every $x \in [2^n]$, and $f'(x) = x$ otherwise. As a result, get that every function f can be inverted with a circuit of size $\tilde{O}(2^{4n/5})$.

2 Preliminaries

2.1 Notations

All logarithms are taken in base 2. We use calligraphic letters to denote sets and distributions, uppercase for random variables, and lowercase for values and functions. Let poly stand for the set of all polynomials. Given a vector $v \in \Sigma^n$, let v_i denote its i^{th} entry, let $v_{<i} = (v_1, \dots, v_{i-1})$ and $v_{\leq i} = (v_1, \dots, v_i)$. Similarly, for a set $\mathcal{I} \subseteq [n]$, let $v_{\mathcal{I}}$ be the ordered sequence $(v_i)_{i \in \mathcal{I}}$. For a function $f: \mathcal{D} \rightarrow \mathcal{R}$, and a set $\mathcal{S} \subseteq \mathcal{D}$, we let $f(\mathcal{S}) = \{f(x) : x \in \mathcal{S}\}$.

2.2 Distributions and Random Variables

When unambiguous, we will naturally view a random variable as its marginal distribution. The support of a finite distribution \mathcal{P} is defined by $\text{Supp}(\mathcal{P}) := \{x : \Pr_{\mathcal{P}}[x] > 0\}$. For a (discrete) distribution \mathcal{P} , let $x \leftarrow \mathcal{P}$ denote that x was sampled according to \mathcal{P} . Similarly, for a set \mathcal{S} , let $x \leftarrow \mathcal{S}$ denote that x is drawn uniformly from \mathcal{S} .

We will use k -wise independent functions in the construction.

► **Definition 4** (*k-wise independent*). For a set \mathcal{R} , n random variables Y_1, \dots, Y_n over Ω are k -wise independent if for every indexes i_1, \dots, i_k , the joint distribution Y_{i_1}, \dots, Y_{i_k} is the uniform distribution over \mathcal{R}^k .

A function family $G = \{g: [N] \rightarrow \mathcal{R}\}$ is k -wise independent if the distribution of $g(1), \dots, g(N)$, for $g \leftarrow G$ is k -wise independent.

2.3 Kolmogorov Complexity

Roughly speaking, the t -time-bounded Kolmogorov complexity, $K^t(x)$, of a string $x \in \{0, 1\}^*$ is the length of the shortest program $\Pi = (M, y)$ such that, when simulated by an universal Turing machine, Π outputs x in $t(|x|)$ steps. Here, a program Π is simply a pair of a Turing Machine M and an input y , where the output of P is defined as the output of $M(y)$. When there is no running time bound (i.e., the program can run in an arbitrary number of steps), we obtain the notion of Kolmogorov complexity.

In the following, let $U(\Pi, 1^t)$ denote the output of Π when emulated on U for t steps. We now define the notion of Kolmogorov complexity with respect to the universal TM U .

► **Definition 5.** Let t be a polynomial. For all $x \in \{0, 1\}^*$, define

$$K_U^t(x) = \min_{\Pi \in \{0,1\}^*} \{|\Pi| : U(\Pi, 1^{t(|x|)}) = x\}$$

where $|\Pi|$ is referred to as the description length of Π .

It is well known that for every x , $K^t(x) \leq |x| + c$, for some constant c depending only on the choice of the universal TM U .

► **Fact 6.** *For every universal TM U , there exists a constant c such that for every $x \in \{0,1\}^*$, and for every t such that $t(n) > 0$, $K_U^t(x) \leq |x| + c$.*

We will use the following bound on the Kolmogorov complexity of strings sampled from the uniform distribution.

► **Lemma 7.** *For any universal TM U and every $n \in \mathbb{N}$, it holds that*

$$\Pr_{x \leftarrow \{0,1\}^n} [K_U^t(x) \geq n - i] \geq 1 - 2^{-i}.$$

In this paper, unless otherwise stated, we fix some universal Turing machine U that can emulate any program Π with polynomial overhead, and let $K^t = K_U^t$.

2.4 Circuits

In this paper we consider circuits over the De-Morgan Basis, which contains the following gates: \wedge (“and” gate with fan-in 2), \vee (“or” gate with fan-in 2), and \neg (“not” gate with fan-in one). The size of a circuit C , denoted by $|C|$ is the number of gates in C . We will sometime allow to use also f -gates in the circuits, for a function $f: \{0,1\}^n \rightarrow \{0,1\}^m$.

► **Definition 8** (Circuit with f -gates). *For a function $f: \{0,1\}^n \rightarrow \{0,1\}^m$, a circuit with f -gates is a circuit which have, in addition to \wedge, \vee and \neg gates, a special f gates. Each f -gate has fan-in n , where each in-wire is label (uniquely) by a number in $1, \dots, n$. Each out-wire is labeled by a number in $1, \dots, m$. The value of a out-wire labeled by i is the i -th bit of $f(w_1, \dots, w_n)$, where w_j is the value of the in-wire labeled by j .*

The following lemma is a non-uniform version of the fast evaluation algorithm used by [7] (see for example [2]), stating that it is possible to efficiently evaluate a polynomial on multiple points. This lemma will help us to get better parameters in our construction.

► **Lemma 9** (Evaluating a polynomial on multiplied points.). *There exists a constant c such that the following holds. Let p be an arbitrary degree d polynomial in a field of size 2^{2^n} , for $n \geq 1$. Then there exists a circuit C of size $d \cdot (n + \log d)^c$ such that $C(x_1, \dots, x_d) = (p(x_1), \dots, p(x_d))$ for every d field elements x_1, \dots, x_d .*

The next lemma states that it is possible to sort a list of integers by an nearly linear size circuits.

► **Lemma 10** (Sorting circuits [1]). *There exists a circuit family of size $O(n \cdot \text{poly}(\log n, k))$ that sorts n numbers of k -bits each.*

We will also use the following simple lemma, that states that any computation on a list that can be done using efficient, one-pass algorithm, can be done also with a small circuit.

► **Lemma 11** (Circuit of an one-pass algorithm.). *There exists a universal constant c such that the following holds. Let A be an algorithm that gets as input a list \mathcal{L} of length ℓ of n -bits elements, and output a list \mathcal{L}' of length ℓ of n -bits elements. Furthermore, assume that A reads \mathcal{L} in one pass: for $i \in [\ell]$, in the i -th step of the algorithm, A reads the i -th entry of \mathcal{L} , and apply a s -sized circuit on the entry and its own state to compute the i -th entry of \mathcal{L}' together with its new state. Then the function computed by A can be computed by a $c\ell \cdot s$ -size circuit.*

3 The Fiat-Naor Algorithm

In this part we describe the algorithm of Fiat and Naor[7], and show how to implement it using a small circuit. For this implementation, we need to choose different parameters than the one used in [7]. For a function $f: [2^n] \rightarrow [2^n]$, our goal is to construct a small circuit that inverts *any* image of f . We prove the following theorem.

► **Theorem 12.** *There exists a constant c such that the following holds. Let $n \in \mathbb{N}$ be a number, and let $f: [2^n] \rightarrow [2^n]$ be a function. Then there exists a circuit C of size $c \cdot 2^{4n/5} \cdot n^c$ with f -gates, such that*

$$\Pr_{x \leftarrow [2^n], y := f(x)} [C(y) \in f^{-1}(y)] = 1.$$

Following [7], we start with focusing on functions which are not *trivial to invert*. Informally, for a parameter $U \in \mathbb{N}$, a function f is said to be U -trivial to invert if there are many images with more than U pre-images.

► **Definition 13** (Trivial to invert function, [7]). *Let $U, n \in \mathbb{N}$ be two numbers. A function $f: [2^n] \rightarrow [2^n]$ is U -trivial to invert if*

$$|\{x: |f^{-1}(f(x))| < U\}| < 2^n/2.$$

Notice that if f is U -trivial to invert, it also U' -trivial to invert, for every $U' < U$. Such a function f which is U -trivial to invert, can be inverted with probability $1/2$ (over a random input) using a relatively short advice, by keeping one pre-image for every image with more than U pre-images. As a simplification of independent interest³, we observe that any function $f: [2^n] \rightarrow [2^n]$ can be converted to a function $f': [2 \cdot 2^n] \rightarrow [2 \cdot 2^n]$ which is not U -trivial to invert for any $U > 1$. Indeed, define $f'(x)$ be equal to $f(x)$ if $x \in [2^n]$, or equal to x otherwise. It immediately follows that f is injective on at least $1/2$ of the domain. Moreover, inverting f is equivalent to inverting f' . Thus, the assumption that f is not trivial to invert is without loss of generality.

In the first step, following [7], we construct a *distribution* \mathcal{P} over circuits of size $\tilde{O}(2^{4n/5})$, such that, for every $y \in f([2^n])$, it holds that $\Pr_{C \leftarrow \mathcal{P}} [C(y) \in f^{-1}(y)] \geq 1/10$. Then, by sampling $O(\log n)$ circuits from \mathcal{P} and combining them together, with high probability we get a single circuit that inverts *any* image y . The main lemma and the derivation of Theorem 12 are formally stated next.

► **Lemma 14.** *There exists a constant c such that the following holds. Let $n \in \mathbb{N}$ be a number, and let $f: [2^n] \rightarrow [2^n]$ be a function which is not $2^{n/5}$ -trivial to invert. There exists a distribution \mathcal{P} over circuits of size at most $c \cdot 2^{4n/5} \cdot n^c$ with f -gates, such that for every $y \in f([2^n])$,*

$$\Pr_{C \leftarrow \mathcal{P}} [C(y) \in f^{-1}(y)] \geq 1/10.$$

Theorem 12 follows easily by Lemma 14.

Proof of Theorem 12. Let $n' = n+1$ and $f': [2^{n'}] \rightarrow [2^{n'}]$ be as defined above. Let \mathcal{P} be the distribution promised by Lemma 14 with respect to f' . Consider the following distribution over circuits \mathcal{P}' . To sample a circuit C from \mathcal{P}' , we start by sampling $15n'$ circuits from \mathcal{P} , $C_1, \dots, C_{15n'}$. Then, we let C be the circuit that given y , computes $x_i = C_i(y)$ for every

³ As mentioned in the introduction, Fiat and Naor, required a more subtle analysis to deal with also U -trivial functions.

$i \in [5n']$, and checks if, for some i , $f'(x_i) = y$. If there exists some i , C outputs x_i (for the minimal such i), or arbitrary value otherwise. It is not hard to see that C can be implemented with size $\sum_{i \in [15n']} |C_i| + \text{poly}(n) \leq c2^{4n'/5} \cdot n^{c+2}$ using f' -gates. Since the function f' can be implemented by a circuit of size $5n$ with f -gates, and since $n' = n + 1$, C can be implemented as a circuit with f -gates, in size at most $10c \cdot 2^{4n/5} \cdot n^{c+3}$.

For the correctness, fix $y \in f([2^n])$, and observe that every pre-image of y with respect to f' is also a pre-image of y with respect to f . By the promise that a random sample from \mathcal{P} inverts y successfully with probability $1/10$, we get that a random sample C from \mathcal{P}' fails to invert y with probability at most $(9/10)^{15n'} \leq 1/2^{2n'}$. By the union bound, C inverts simultaneously all $y \in f([2^n])$ with probability at least $1 - 2^{-n}$.

Since with positive probability C is a circuit of size $c2^{4n/5} \cdot n^{c+2}$ that inverts f on any image, such a circuit exists. \blacktriangleleft

3.1 Proving Lemma 14

We now move to prove Lemma 14. In the following, fix $n \in \mathbb{N}$ and a function $f: [2^n] \rightarrow [2^n]$ which is not trivial to invert. Let $N = 2^n$.

To prove Lemma 14, let m, t, ℓ, k and $U \geq 2^{n/5}$ be parameters to be chosen later. Let $\mathcal{A} = \{y_1, \dots, y_r\}$ be the set of all images y_i such that $|f^{-1}(y_i)| \geq U$, and observe that $r := |\mathcal{A}| \leq N/U$. In the Fiat-Naor algorithm, we give as an advice to the inverter a pre-image $x_i \in f^{-1}(y_i)$ of every $y_i \in \mathcal{A}$. As a result, we only left to invert f on $y \notin \mathcal{A}$.⁴ Let $\mathcal{D} = \{x: f(x) \notin \mathcal{A}\}$. Since f is not U -trivial to invert (recall that $U \geq 2^{n/5}$), it holds that $|\mathcal{D}| \geq N/2$.

Next, we need to invert the function f on the domain \mathcal{D} . For the yet to be chosen parameter ℓ , the algorithm of Fiat and Naor [7] uses ℓ hash functions $g_1, \dots, g_\ell: [N] \rightarrow \mathcal{D}$. We will later describe how to choose this functions, but for now this can be thought as ℓ random functions. Finally, for every $i \in [\ell]$, let $h_i: [N] \rightarrow \mathcal{D}$ be defined by $h_i(x) = g_i(f(x))$.

In the following algorithm, for every $i \in [\ell]$ and $j \in [m]$, let $x_{i,j} \in [N]$ be inputs to f , to be chosen (randomly) later. For every $i \in [r]$, let x_i be such that $f(x_i) = y_i$, for $\mathcal{A} = \{y_1, \dots, y_r\}$.

■ **Algorithm 1** Fiat-Naor [7].

-
1. Parameters: $m, t, \ell, U \in \mathbb{N}$
 2. Advice: $\{(x_i, y_i)\}_{i \in [r]}$, $\{g_i\}_{i \in [\ell]}$, $\{(x_{i,j}, h_i^t(x_{i,j}))\}_{i \in [\ell], j \in [m]}$.
 3. Input: $y \in [N]$.
 4. a. Check if there exists $i \in [r]$ with $y_i = y$. If so, output x_i .
 - b. For every $i \in [\ell]$, set $u_i^0 = g_i(y)$.
 - c. For every $v \in [t]$:
 - i. For every $i \in [\ell]$, compute $u_i^v = h_i(u_i^{v-1})$.
 - d. For every $i \in [\ell]$, let $\mathcal{J}_i = \{j: h_i^t(x_{i,j}) \in \{u_i^0, \dots, u_i^t\}\}$. Set $\hat{u}_{i,j}^0 = x_{i,j}$ for every $i \in [\ell]$ and $j \in \mathcal{J}_i$.
 - e. For every $v \in [t]$:
 - i. For every $i \in [\ell]$ and $j \in \mathcal{J}_i$, let $\hat{u}_{i,k}^v = h_i(\hat{u}_{i,k}^{v-1})$.
 - f. For every $i \in [\ell], v \in [t]$ and $j \in \mathcal{J}_i$, compute $f(\hat{u}_{i,k}^v)$, and output $\hat{u}_{i,k}^v$ if $f(\hat{u}_{i,k}^v) = y$.
-

⁴ Actually, in [7], for $r = \tilde{O}(N/U)$, the points x_1, \dots, x_r are chosen uniformly at random from the domain $[N]$, and the set \mathcal{A} is defined to be $\{f(x_1), \dots, f(x_r)\}$. In [7] it is shown that with high probability the set \mathcal{A} contains all the images with more than U pre-images, which is the only property of \mathcal{A} that is used in the proof. We could also choose \mathcal{A} in the same way here.

That is, for every hash function g_i , the advice contains m points, $x_{i,1}, \dots, x_{i,m}$. On each such point $x_{i,j}$ we apply the function $h_i = g_i \circ f$ t times in a chain, to get $h_i^t(x_{i,j})$, and keep the result in the advice.

Giving an input y , the algorithm first check if $y \in \mathcal{A}$. If it does, a pre-image of y appears in the advice, and thus y can be trivially inverted. Otherwise, we want to check if $y = f(h_i^v(x_{i,j}))$ for some i, j and $v < t$. That is, if y appears in the chain starting with $x_{i,j}$. If this is the case, then $g_i(y) = h_i^{v+1}(x_{i,j})$, and thus the chain $(g_i(y), h_i(g_i(y)), \dots, h_i^t(g_i(y)))$ contains the point $h_i^t(x_{i,j})$. The algorithm therefore search for every i, j such that $h_i^t(x_{i,j})$ is in the chain $(g_i(y), h_i(g_i(y)), \dots, h_i^t(g_i(y)))$. For every such i, j , the algorithm then computes the chain starting with $x_{i,j}$, $(x_{i,j}, \dots, h_i^t(x_{i,j}))$, and checks if one of the element in the latter is indeed a pre-image of y .

Choosing g_1, \dots, g_ℓ . We next describe how to chose the functions g_1, \dots, g_ℓ . The idea in Fiat and Naor [7] is to choose these functions, such that, instead of being ℓ random functions, each function will be $O(t)$ -wise independent, and the joint distribution of the functions will be pair-wise independent. This is enough for the analysis of the algorithm, and allows computing the functions efficiently.

One problem we have is that the set $\mathcal{D} \subseteq [N]$ is dependent on the function f , and thus it is not simple to construct a function from $[N]$ to \mathcal{D} . To overcome this, Fiat and Naor [7] constructed, for each $i \in [\ell]$, a function $g'_i: [N] \times [N] \rightarrow [N]$. Then, for every $x \in [N]$, they defined $g_i(x)$ to be equal to $g'_i(x, z)$ for the minimal value of $z \in \mathbb{N}$ for which $g'_i(x, z) \in \mathcal{D}$. Notice that given the set \mathcal{A} we can check efficiently if $g'_i(x, z) \in \mathcal{D}$, by checking whether $f(g'_i(x, z)) \in \mathcal{A}$. Moreover, since $|\mathcal{D}| \geq N/2$, it is not hard to see that the expected value of z is $O(1)$.

To construct the functions g'_i , we think on the domain $[N] \times [N]$ as $[N^2]$. Let $k \in \mathbb{N}$ be a parameter (we will choose $k = \tilde{O}(t)$), and for every $j \in [k]$, let $a_j, b_j \leftarrow [N^2]$ be random numbers. For every $i \in [\ell]$, let $g'_i: [N^2] \rightarrow [N]$ be defined by $g'_i(\alpha) = (\sum_{j \in [k]} (a_j \cdot i + b_j) \alpha^{j-1})_{\leq n}$ (where all the operations are performed over a field of size $N^2 = 2^{2n}$, and we take the first n bits of the result).

By the construction the functions g'_1, \dots, g'_ℓ are pair-wise independent, where each of them is k -wise independent. As described above, for every $x \in [N]$, let $g_i(x)$ to be equal to $g'_i(x, z^*)$ for $z^* = \min\{z: g'_i(x, z) \in \mathcal{D}\}$ (letting $z^* = N$ if no such exists).

Correctness. By Fiat and Naor [7], for a random choice of points $x_{i,j}$ and the polynomials g_i , Algorithm 1 inverts f on any fixed y with a good probability. We need a slightly different version of Fiat and Naor [7] theorem, stated below.

► **Theorem 15.** *Let $m, t, \ell, k, U \in \mathbb{N}$ be parameters such that $m \leq N$, $8 \log(4N) \leq t \leq N$, $k = 8t$, $mk^2U = N$ and $\ell = N/mt$. Let $f: [N] \rightarrow [N]$ be a function which is not U -trivial to invert. Then, the following holds for every $y \in f([N])$.*

Over a random choice of $\{(a_j, b_j)\}_{j \in [k]}$ and $\{(x_{i,j})_{i \in [\ell], j \in [m]}\}$, Algorithm 1 inverts y with probability at least $1/8$. Moreover, under the same distribution, $\mathbb{E}[\sum_{i \in [\ell]} |\mathcal{J}_i|] \leq \ell$.

We remark the [7] proved the above theorem for $U = N/S$. However, their proof directly generalized to the above set of parameters.

Proof. Theorem 15 follows almost directly from the proof in [7]. There, to get the best parameters, U is chosen such that $N/U = m \cdot \ell$, and we do not add this constrain. Theorem 15 follows by setting $S = N/U$ in the proof of [7]. ◀

3.1.1 Circuit implementation

We now show that, for the right choice of parameters, it is possible to implement Algorithm 1 with a relatively small circuit. Recall that we allow f -gates in our circuit.

One main component in the implementation of *Algorithm 1* is the computation of the functions g_i on different inputs. This is done $\Omega(\ell \cdot t)$ times during the algorithm, and thus must be done efficiently. [7] showed that, due to the dependency between them, the evaluation of the functions g_1, \dots, g_ℓ *simultaneously* on one point each, can be done in $\tilde{O}(\ell)$ time (or, in $\tilde{O}(1)$ amortized time). The following claim shows that (for different parameters) this can be done by a small circuit.

▷ **Claim 16.** There exists a constant c such that the following holds for every $n \in \mathbb{N}$ and $f: [2^n] \rightarrow [2^n]$. Let $\mathcal{A}, r, k, g'_1, \dots, g'_\ell$ and g_1, \dots, g_ℓ be as defined above, and let $\tau, p \in \mathbb{N}$ be numbers. There exists a C circuit of size $c \cdot (r + k + \tau \cdot p) \cdot (n + \log(\tau pr))^c$ such that the following holds. For every $i \in [\tau]$, let $\alpha(i) \in [\ell]$ and $u_i \in [N]$ be such that $g'_{\alpha(i)}(u_i, z) \in \mathcal{D}$ for some $z \leq p$. Then given $\alpha(1), \dots, \alpha(\tau)$ and u_1, \dots, u_τ , C computes $g_{\alpha(1)}(u_1), \dots, g_{\alpha(\tau)}(u_\tau)$

It is worth to mention here that in the setting of [7], the main cost in evaluating g_i is the evaluation of the polynomial g'_i . Thus, for [7] it was enough to show how to compute g'_1, \dots, g'_ℓ *simultaneously*. While their technique helps us getting better parameters, the crucial cost in the setting here is to check if the output of $g'_i(x, z)$ is in the domain \mathcal{D} , or equivalently, if $f(g'_i(x, z)) \in \mathcal{A}$. With RAM machine this can be done efficiently, for example using a binary search, but with a circuit we must a circuit with size linear in the length of \mathcal{A} . In the proof we show that this also can be done simultaneously with a smaller amortized cost. An important part of our proof is showing that there is a near linear sized circuit that preform a task for which we call “batch lookup”. This task is now defined.

► **Definition 17** (Batch look-up). Given a list $\mathcal{A} = (a_1, \dots, a_r) \in (\{0, 1\}^n)^r$ of length r , and a list $\mathcal{L} = (y_1, \dots, y_\tau) \in (\{0, 1\}^n)^\tau$ of τ n -bits numbers, the batch look-up functionality, $\text{BLOOKUP}(\mathcal{A}, \mathcal{L})$ returns τ bits b_1, \dots, b_τ , such that $b_i = 1$ if y_i appears in \mathcal{A} .

In Section 3.1.2 we show how to implement the batch look-up functionality with a small circuit. We prove the following lemma.

► **Lemma 18.** There exists a constant c such that the following holds for every $n \in \mathbb{N}, r \in \mathbb{N}$ and $\tau \in \mathbb{N}$. There exists a circuit C of size $c \cdot (r + \tau) \cdot (n + \log(r\tau))^c$ that computes the BLOOKUP functionality.

Using Lemma 18 we prove Claim 16.

Proof. In the following we use the notation $\tilde{O}(\cdot)$ to hide fixed multiplicative polynomial factors in n and $\log(p\tau r)$. Without loss of generality, assume that $\tau \cdot p \geq k$. Recall that to compute $g_{\alpha(i)}(u_i)$, we need to find the minimal $z \in \mathbb{N}$ such that $g'_{\alpha(i)}(u_i, z) \in \mathcal{D}$. Moreover, by assumption, $z \in [p]$. Thus we start with computing $g_{\alpha(i)}(u_i, z)$ for every $z \in [p]$ and for every $i \in [\tau]$.

To do so, observe that for the polynomials $A(x) = \sum a_j x^j$, and $B(x) = \sum b_j x^j$ (for $x \in [N^2]$), we can write $g'_i(x) = i \cdot A(x) + B(x)$ for every $i \in [\ell]$. Thus, to compute $g'_{\alpha(i)}(u_i, z)$ for every $i \in [\tau]$, $z \in [p]$, it is enough to compute $A(u_i, z)$ and $B(u_i, z)$ for every such i, z . This can be done with a circuit of size $\tilde{O}(k + \tau \cdot p)$, using Lemma 9.

Next, given the values $\left\{ g'_{\alpha(i)}(u_i, z) \right\}_{i \in [\tau], z \in [p]}$, we need to find, for every $i \in [\tau]$, the minimal z for which $g'_{\alpha(i)}(u_i, z) \in \mathcal{D}$. This can be done by applying the BLOOKUP circuit on \mathcal{A} and $\mathcal{L} = (f(g'_{\alpha(1)}(u_1, 1)), \dots, f(g'_{\alpha(1)}(u_1, p)), \dots, f(g'_{\alpha(\tau)}(u_\tau, 1)), \dots, f(g'_{\alpha(\tau)}(u_\tau, p)))$ to

compute $(b_{1,1}, \dots, b_{1,p}, \dots, b_{\tau,1}, \dots, b_{\tau,p})$, where $b_{i,z} = 1$ if and only if $f(g'_{\alpha(i)}(u_i, z)) \in \mathcal{A}$ (or, equivalently, if $g'_{\alpha(i)}(u_i, z) \notin \mathcal{D}$). Then, for each block $i \in [\tau]$ of size p , we can output $g'_{\alpha(i)}(u_i, z)$ for the first z in the block for which $b_{i,z} = 0$. Since the last step can be implemented with an efficient one-pass algorithm, and by Lemma 18, the entire process can be done with a circuit of size $\tilde{O}(k + r + p \cdot \tau)$. Moreover, the output of the above process is exactly $g_{\alpha(1)}(u_1), \dots, g_{\alpha(\tau)}(u_\tau)$. \blacktriangleleft

We next use Claim 16 to prove Lemma 14.

Proof of Lemma 14. In the following we assume that $n > 10$, as we can choose the constant c in Lemma 14 such that the theorem will hold trivially for every smaller n . Fix such $n > 10$ and a function f as in Lemma 14, and let $N = 2^n$.

We start with our choice of parameters. Let $U = N^{2/5}/64$, $m = N^{1/5}$, $t = N^{1/5}$ and $\ell = N^{2/5}$. It is not hard to verify that the conditions of Theorem 15 hold with respect to these parameters, and thus the correctness holds. Namely, for every $y \in f([N])$, over a random value of $a_1, \dots, a_\ell, b_1, \dots, b_\ell \leftarrow [N^2]$, and $x_{1,1}, \dots, x_{\ell,m} \leftarrow [N]$, Algorithm 1 finds a pre-image of y with probability at least $1/8$.

To implement Algorithm 1 as a circuit, we need a bound some running-time parameters. That is, we need to bound the worst-case number of times we will need to evaluate g'_i for each time Algorithm 1 evaluates g_i (so we will be able to use Claim 16). Additionally, we will need to bound the size of the sets \mathcal{J}_i defined in Step 4 of the algorithm.

In the following we show that with probability 0.99 over the choice of $a_1, \dots, a_\ell, b_1, \dots, b_\ell \leftarrow [N^2]$ it holds that:

1. For every $x \in [N]$ and $i \in [\ell]$, there exists $z \in [3n]$ such that $g'_i(x, z) \in \mathcal{D}$, and,
2. $\sum_{i \in [\ell]} |\mathcal{J}_i| \leq 200\ell$.

Then, we show that when Items 1 and 2 above hold, we can implement Algorithm 1 with a small circuit. Overall, by the union bound, this shows that for every $y \in f([N])$, the following holds with probability at least $1/8 - 0.01 \geq 1/10$ over a random value of $a_1, \dots, a_\ell, b_1, \dots, b_\ell \leftarrow [N^2]$, and $x_{1,1}, \dots, x_{\ell,m} \leftarrow [N]$: Algorithm 1 can be implemented with a small circuit *and* it finds a pre-image of y . Given this promise, we can easily construct the distribution \mathcal{P} : To sample a circuit from \mathcal{P} , we randomly choose the parameters $a_1, \dots, a_\ell, b_1, \dots, b_\ell \leftarrow [N^2]$, and $x_{1,1}, \dots, x_{\ell,m} \leftarrow [N]$. Then, if Algorithm 1 can be implemented with a circuit of size $\tilde{O}(N^{4/5})$ with the chosen parameters, we output this circuit. Otherwise, we output an arbitrary small circuit.

To see that Items 1 and 2 above holds, first notice that the second item hold with probability $1 - 1/200$ by Markov and the promise that $\mathbb{E}[\sum_i |\mathcal{J}_i|] \leq \ell$. For the first item, randomly choose $a_1, \dots, a_\ell, b_1, \dots, b_\ell \leftarrow [N^2]$. We start with showing that $g'_i(x, z) \in \mathcal{D}$ for every i, x and some $z \in [3n]$. Fix i and x and recall that g'_i is a degree $k > 3n$ polynomial. Thus, for every $z \in [3n]$, $\Pr[g_i(x, z) \in \mathcal{D}] = \mathcal{D}/N \geq 1/2$ (where the probability is taken over the choice of $\{a_i, b_i\}$). By the k -wise independence of g'_i , we get that

$$\Pr[\forall z \in [3n] \text{ s.t. } g'_i(x, z) \notin \mathcal{D}] \leq 1/2^{3n} = 1/N^3.$$

By applying the union bound on all possible values of $i \in [\ell]$ and $x \in [N]$, we get that

$$\Pr[\exists i \in [\ell], x \in [N], \forall z \in [3n] \text{ s.t. } g'_i(x, z) \notin \mathcal{D}] \leq \ell \cdot N \cdot 1/N^3 \leq 1/N.$$

In other words, with all but $1/N$ probability, for every i and x , there exists some $z \leq [2n]$ such that $g'_i(x, z) \in \mathcal{D}$, as we wanted to show.

Next, we show how to implement Algorithm 1 with a small circuit, under the above assumption. We implement each step of Algorithm 1 separately using a small circuit. Recall that $r = N/U = N^{\frac{3}{5}}$. In the following we use the notation $\tilde{O}(\cdot)$ to hide fixed multiplicative polynomial factors in n .

Step 1 can be done trivially by a circuit of size $\tilde{O}(r)$. Similarly, the last step can be done with a circuit of size $\tilde{O}(\ell \cdot t)$ under the assumption in Item 2, and using the f -gates.

By Claim 16, Step 2 can be implemented by a circuit of size $\tilde{O}(r + k + \ell)$. Similarly, as a corollary of Claim 16, Steps 3 and 5 (under the assumption that Item 2 holds) can be done with a circuit of size $\tilde{O}(t \cdot (r + k + \ell))$.

Lastly, Step 4 can be done with circuit of size $\tilde{O}(\ell \cdot t + \ell \cdot m)$, by using batch look-up again. In more detail, let $\mathcal{A} = (u_1^0, \dots, u_1^t, \dots, u_\ell^0, \dots, u_\ell^t)$ and $\mathcal{L} = (h_1^t(x_{1,1}), \dots, h_\ell^t(x_{\ell,m}))$. We first compute $\text{BLOOKUP}(\mathcal{A}, \mathcal{L})$ to get $\mathcal{R} = (b_{1,1}, \dots, b_{\ell,m})$. Then, we can easily change the list \mathcal{R} such that it will contain the block index and the input $x_{i,j}$ in every coordinate, that is $\mathcal{R}' = ((b_{1,1}, x_{1,1}, 1), \dots, (b_{\ell,m}, x_{\ell,m}, \ell))$. We can now sort the list according to the first entry $b_{i,j}$, and keep only the last 100ℓ entries. By the assumption in Item 1, those entries contains all the entries i, j for which $h_i^t(x_{i,j}) \in \{u_i^0, \dots, u_i^t\}$. Each element in the output list contains both $x_{i,j}$ and i , which is enough to apply Claim 16 in Step 5.

Overall, the entire process can be implemented by a circuit of size $\tilde{O}(t(r + k + \ell) + \ell \cdot m) = \tilde{O}(N^{4/5})$, as stated. \blacktriangleleft

3.1.2 Implementing Batch Look-Up

We now prove Lemma 18. The main idea is to sort the lists \mathcal{A}, \mathcal{L} together, and then find all the duplications in the sorted list.

Proof. Let $\tau, r, n, \mathcal{A} = (a_1, \dots, a_r)$ and $\mathcal{L} = (y_1, \dots, y_\tau)$ be as in Definition 17. Let $k = \lceil \log \tau \rceil$. To compute $\text{BLOOKUP}(\mathcal{A}, \mathcal{L})$ we do the following:

1. First, we construct the lists $\mathcal{A}' = ((a_1, 0^k), \dots, (a_r, 0^k)) \in (\{0, 1\}^n \times \{0, 1\}^k)^r$ and $\mathcal{L}' = ((y_1, 1), \dots, (y_\tau, \tau)) \in (\{0, 1\}^n \times \{0, 1\}^k)^\tau$. Namely, we add the index i (represented as a k -bits string) to every element in \mathcal{L} , and 0 for every element in \mathcal{A} . This can be done with a circuit of size $(r + \tau)(n + k)$.
2. Next, we sort $(\mathcal{A}', \mathcal{L}')$ jointly, according to the first and then the second entry. Let \mathcal{R}_1 be the output. This can be done with a circuit of size $c_1 \cdot (r + \tau)(n + k + \log(r + \tau))^{c_1}$, by Lemma 10, for some universal constant c_1 . Let \mathcal{R}_1 be the output of this step.
3. In the next step, we replace with 1 the first entry of every element (y_i, i) in \mathcal{R}_1 , such that an element of the form $(y_i, 0)$ appears earlier in the list \mathcal{R}_1 . We replace by 0 the first entry of every element (y_i, i) in \mathcal{R}_1 , such that $(y_i, 0)$ does not appear earlier in the list. Let \mathcal{R}_2 be the output of this step. The result of this step is that \mathcal{R}_2 contains entries of the form (b_i, i) , where b_i is 1 if and only if y_i appears in \mathcal{A} . As this step can be implemented by an efficient, one-pass algorithm (the algorithm only needs to remember the first entry of the last element with second entry equal to 0 as its state), it can be also be implemented by circuit of size $c_2 \cdot (r + \tau)(n + k + \log(r + \tau))^{c_2}$ by Lemma 11, for some universal constant c_2 .
4. Lastly, we sort the list again according to the second entry, to get the elements in order according to the indexes. We output the first entry of each of the last τ items in the sorted list. It is not hard to see that the values we output are exactly the one we wanted to compute. Moreover, this last step can also be implemented by a circuit of size $c_1 \cdot (r + \tau)(n + k + \log(r + \tau))^{c_1}$.

By concatenating the circuits for each of the above steps, we get a circuit of size $c \cdot (r + \tau)(n + \log r\tau)^c$ that computes the functionality BLOOKUP, for a sufficiently large constant c . ◀

4 Computing t -Bounded Kolmogorov Complexity

In this part we use Theorem 12 to prove our main theorem. The latter is a corollary of the following theorem.

► **Theorem 19.** *For any universal TM U and every function $t = t(n)$ there exists a circuit family $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ of size $O(2^{4n/5} \cdot \text{poly}(n))$ such that the following holds for every $n \in \mathbb{N}$. C_n is a circuit with f_n -gates, for $f_n: \{0, 1\}^{\leq 2n} \rightarrow \{0, 1\}^*$ defined by $f_n(\Pi) = U(\Pi, 1^{t(n)})$, and for every $x \in \{0, 1\}^n$, $C_n(x)$ outputs $K_U^t(x)$.*

Recall that we defined $K^t = K_U^t$ for some fixed U with polynomial running time overhead. We get the following corollary.

► **Corollary 20 (Main theorem).** *For every $t = t(n)$, there exists a circuit family $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ of size $O(2^{4n/5} \cdot \text{poly}(t(n), n))$ (over the DeMorgan basis) such that, for every $n \in \mathbb{N}$ and for every $x \in \{0, 1\}^n$, $C_n(x)$ outputs $K^t(x)$.*

Proof of Corollary 20. Corollary 20 follows by Theorem 19, observing that we can replace every f_n -gate with a circuit of size $\text{poly}(t(n), n)$ that computes f_n . We get a circuit family of size $O(2^{4n/5} \cdot \text{poly}(n, t(n)))$ that computes K^t . ◀

We now prove Theorem 19

Proof of Theorem 19. Let c be a constant such that $K^t(x) \leq |x| + c$ for every x (see Fact 6). Let $f'_n: \{0, 1\}^{n+c} \times [n+c] \rightarrow \{0, 1\}^n \times [n+c]$ be defined as

$$f'_n(\Pi, i) = \begin{cases} (f_n(\Pi_{\leq i}), i) & |f_n(\Pi_{\leq i})| = n \\ 0^n & \text{Otherwise} \end{cases}$$

Let $n' = n + c + \lceil \log(n + c) \rceil$. In the following, we assume that both the domain and the range of f_n is $[2^{n'}]$, by the use of appropriate encoding and padding.

By Theorem 12, the above imply that there is a circuit family $\widehat{\mathcal{C}} = \{\widehat{C}_n\}_{n \in \mathbb{N}}$ with f'_n gates, of size $O(2^{4n'/5} \cdot \text{poly}(n)) = O(2^{4n/5} \cdot \text{poly}(n))$ that inverts f'_n with probability 1.

Given a circuit \widehat{C}_n that inverts f'_n , we can construct a circuit C_n (with f'_n gates) that computes the K^t complexity of any string x of length n . This can be done by computing $f_n^{-1}(x, 1), \dots, f_n^{-1}(x, n+c)$ and taking the output (Π, i) for the minimal value of i such that $U(\Pi_{< i}, 1^{t(n)}) = x$ (the t -bounded Kolmogorov complexity of the string 0^n can be hardcoded in the circuit).

Observe that the size of C_n is $n' \cdot |\widehat{C}_n| + \text{poly}(n)$. Thus, there exists a circuit family of size $O(2^{4n/5} \cdot \text{poly}(n))$, with f'_n gates, that computes K^t .

Lastly, observe that f'_n can be efficiently computed from f_n , thus we can replace the f'_n gates with a small circuit. ◀

4.1 Solving MCSP

As we noted in Section 1.2.1, the only thing that we rely on from the [16] construction is that (1) the construction works also in the worst-case regime and (2) the construction is *length preserving*: any algorithm breaking the one-way function on inputs of length $n + O(\log n)$ solves the K^t problem on inputs of length n , and more generally, if the input length to the worst-case one-way function is $n'(n) > n$, we get a circuit size of $2^{4n'/5} \text{poly}(n')^5$

Thus, if instead of relying on the one-way function construction of [16], we rely on a variant of the construction from [19] – that outputs the evaluation a circuit and its circuit size (instead of outputting the evaluation of a Turing machine and its length) – then the same proof yields a solver for the Minimum Circuit Size Problem (MCSP) with circuit size $2^{4n'/5} \text{poly}(n')$ where n' is an upperbound on the length of the description of a circuit generating x , which as noted in [19] is bounded by $|x| + o(|x|)$ (due to the circuit encoding of [8]).⁶ Relying on [19], we thus also get the existence of a circuit family $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ of size $2^{4/5n+o(n)}$ that given a truth table of a function f , outputs the minimal size of a circuit that computes f .

It is also worthwhile to note why this approach does not work on the (folklore) construction of worst-case one-way functions from the hardness of SAT: $f(\phi, x) = \phi, \phi(x)$. Inverting this function enables deciding SAT; the issue is that the circuit size becomes $2^{4/5 \cdot (|x|+|\phi|)} > 2^{|x|}$.

5 Lower Bound on the Circuit Size

The result presented in Section 4 holds when the complexity measure is defined with respect to any succinct computation model (in particular, where programs can be evaluated in polynomial size in the description length). Moreover, by Theorem 19, the result generalizes to K^t defined also with respect to any universal TM U – even ones that do not have efficient evaluation – if we allow the circuit to have oracle gates to the UTM U .

In this section we prove a lower bound on this type of circuits. Namely, we show that there is a “black-box” universal TM, with respect to which there is no circuit of sub-exponential size that computes K^t using oracle gates to U . In more detail, we consider the black-box universal TM definition from [17].

► **Definition 21** (Black-box universal TM). *A function $U : \{0, 1\}^* \times 1^* \rightarrow \{0, 1\}^* \cup \{\perp\}$, we say that U is a black-box universal Turing machine (black-box UTM) if*

- (Universality) *There exists a universal Turing machine U_0 such that for any $(M, 1^t)$, if M is a valid description of a Turing machine (w.r.t U_0), $U(M, 1^t)$ outputs what M outputs after t steps.*
- (Unique output) *For any $M \in \{0, 1\}^*$, $t_1, t_2 \in \mathbb{N}, t_1 \leq t_2$, if $U(M, 1^{t_1}) \neq \perp$, $U(M, 1^{t_2}) = U(M, 1^{t_1})$.*

We remark that the above definition is black-box in the following two ways: (1) U is defined to be any (not necessarily efficient) function; (2) U is allowed to assign the output of invalid “programs” with an arbitrary string.

For any black-box UTM U , we can define the time-bounded Kolmogorov complexity with respect to U . Formally, for any black-box universal Turing machine U , any string $x \in \{0, 1\}^*$, let

$$K_U^t(x) = \min_{\Pi \in \{0, 1\}^*} \{|\Pi| \mid U(\Pi, 1^{t(n)}) = x\}.$$

⁵ We thank an anonymous reviewer for asking us to clarify this.

⁶ See the definition of, and bound on, C'_{ub} on page 36 in [19]. We thank an anonymous reviewer for asking if our attack applies to the one-way function construction from the MCSP problem.

In the following, for a universal TM U , a function $t: \mathbb{N} \rightarrow \mathbb{N}$, and a number $n \in \mathbb{N}$, we let f_n^U be the function defined by $f_n^U(\Pi) = U(\Pi, 1^{t(n)})$ for any $\Pi \in \{0, 1\}^{\leq 2n}$. Using appropriate encoding, we can see f_n^U as a function from $n + \log n$ bits to $2t(n)$ bits.

A circuit C with oracle gates is a circuit C with special \mathcal{O} -gates. For a function f (with the same input and output length), we let C^f be the circuit received by replacing the \mathcal{O} -gates with f -gates.

We next define black-box K^t -solvers. We give two definitions, of fully black-box and (plain) black-box. The first is now defined.

► **Definition 22** (Fully black-box K^t -solver). *A circuit family $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ is fully black-box K^t -solver if for every black-box universal TM U , for every $n \in \mathbb{N}$ and for every input $x \in \{0, 1\}^n$, $C_n^{f_n^U}(x) = K_U^t$.*

That is, the circuit family \mathcal{C} can be used to compute the K_U^t complexity with respect to any universal TM U . Our construction in Section 4 is not fully black-box. Indeed, the circuit family we construct is dependent in the universal TM with respect to we defined K^t . We next define a weaker notion of black-box solutions, that captures these kinds of constructions.

► **Definition 23** (Black-box K^t -solver). *For any black-box universal TM U , a circuit family $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ is black-box K_U^t -solver if for every $n \in \mathbb{N}$, C_n is a circuit with f_n^U -gates, and for every input $x \in \{0, 1\}^n$, $C_n^{f_n^U}(x) = K_U^t$.*

Here we only require the circuit to compute K_U^t with respect to U , and allow the circuit to use black-box access to the universal TM. Theorem 19 directly implies the following theorem.

► **Theorem 24.** *For any universal TM U and every function $t = t(n)$ there exists a black-box K_U^t solver $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ of size $O(2^{4n/5} \cdot \text{poly}(n))$.*

In this part we prove the following lower bound on the size of black-box K^t solvers. This lower bound immediately generalized to fully black-box solvers.

► **Theorem 25.** *There exists a black-box UTM \widehat{U} and a constant c such that the following holds for every function $t(n) \in \text{poly}$ with $t(n) \geq n^c$. For any black-box $K_{\widehat{U}}^t$ -solver $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ it holds that $|C_n| \geq 2^{n/2 - o(n)}$ for infinitely many $n \in \mathbb{N}$.*

We leave open the question of closing the gap between the lower and upper bounds on the size of black-box K^t -solvers.

In the following we give two proofs for the above theorem. In both proofs, we use an universal TM U_{oracle} with an oracle as our black-box UTM. In the first proof, we give U_{oracle} an oracle to a random function, used as a pseudorandom generator (PRG). We then use the K^t -solver to break the PRG. We use the result of Coretti, Dodis, Guo and Steinberger [5] which states that a random function is a good PRG against circuits of size roughly $2^{n/2}$.

The second proof, which appears in the full version of this paper, gets slightly worse parameters, but is based on more standard results. In this proof we take the oracle to be a random permutation σ , and use a result from [12] that states of that a random permutation cannot be inverted by a circuit of size $2^{n/2}$. Then, we use Goldreich-Levin hardcore functions to construct a PRG from σ .

The first proof. For our first proof, we will use the following theorem from [5].

► **Theorem 26** ([5]). *There exists a constant c such that the following holds for every numbers $m > n$. Let (A_0, A_1) be a pair of oracle-aided algorithms, such that given a oracle $\mathcal{O}: \{0, 1\}^n \rightarrow \{0, 1\}^m$, $A_0^{\mathcal{O}}$ outputs advice a of length $S > n$, and A_1 makes at most T queries to \mathcal{O} . Then*

$$\begin{aligned} & \left| \Pr_{\mathcal{O} \leftarrow \mathcal{F}_{n,m}, x \leftarrow \{0,1\}^n} [A_1^{\mathcal{O}}(A_0^{\mathcal{O}}, \mathcal{O}(x)) = 1] - \Pr_{\mathcal{O} \leftarrow \mathcal{F}_{n,m}, y \leftarrow \{0,1\}^m} [A_1^{\mathcal{O}}(A_0^{\mathcal{O}}, y) = 1] \right| \\ & \leq cm \left(\sqrt{\frac{ST}{2^n}} + \frac{Tn}{2^n} \right) \end{aligned}$$

where $\mathcal{F}_{n,m}$ is the family of all functions from $\{0, 1\}^n$ to $\{0, 1\}^m$.

That is, a random oracle is a good PRG, even against adversaries with non-uniform advice. The following corollary, stating that there exists a function which is a good PRG against small circuits, follows almost directly from Theorem 26.

► **Corollary 27.** *There exists a constant c such that for every number $n \in \mathbb{N}$ and for $m = n + \log n$, there exists a function $G: \{0, 1\}^n \rightarrow \{0, 1\}^m$, such that the following holds for every circuit of size $2^{n/2-2\log^2 n}$ with G -gates.*

$$\left| \Pr_{x \leftarrow \{0,1\}^n} [C(G(x)) = 1] - \Pr_{y \leftarrow \{0,1\}^m} [C(y) = 1] \right| \leq 1/10.$$

Proof of Corollary 27. Fix large enough $n \in \mathbb{N}$, and assume toward a contradiction that for every function $G: \{0, 1\}^n \rightarrow \{0, 1\}^m$, there exists a circuit C with $\mathcal{O}: \{0, 1\}^n \rightarrow \{0, 1\}^m$ oracle gates of size $2^{n/2-2\log^2 n}$, such that

$$\left| \Pr_{x \leftarrow \{0,1\}^n} [C(G(x)) = 1] - \Pr_{y \leftarrow \{0,1\}^m} [C(y) = 1] \right| > 1/10.$$

By flipping the answer of C , we can assume that

$$\Pr_{x \leftarrow \{0,1\}^n} [C(G(x)) = 1] - \Pr_{y \leftarrow \{0,1\}^m} [C(y) = 1] > 1/10.$$

First, observe that the circuit C can be described with less than $S := 3 \cdot |C| \cdot n^2 \leq 2^{n/2+3\log n-2\log^2 n}$ bits (see for example [9]).

Next, consider the oracle-aided pair of algorithms (A_0, A_1) , that given access to an oracle G , A_0^G outputs a description of length $2^{n/2+3\log n-2\log^2 n}$ of the circuit C that distinguish $G(x)$ from uniform. Given the advice, and an input z , A_1^G simulates $C^G(z)$. Clearly A_1 makes at most $T = |C|$ queries to G . By our assumption, for every G ,

$$\Pr_{x \leftarrow \{0,1\}^n} [A_1^G(A_0^G, G(x)) = 1] - \Pr_{y \leftarrow \{0,1\}^m} [A_1^G(A_0^G, y) = 1] > 1/10.$$

By taking expectation on $G \leftarrow \mathcal{F}_{n,m}$, and by Theorem 26, it must holds that

$$1/10 \leq cm \left(\sqrt{\frac{ST}{2^n}} + \frac{Tn}{2^n} \right) \leq 2cm \sqrt{\frac{ST}{2^n}} \leq 4cn \sqrt{2^{3\log n-4\log^2 n}},$$

which is a contradiction for large enough n . ◀

We can now prove Theorem 25.

Proof of Theorem 25. Let $\mathcal{N} = \{n_1, n_2, \dots\} \subseteq \mathbb{N}$ be the infinite set defined by $n_1 = 1$ and $n_{i+1} = 2^{n_i}$. We will show a black-box UTM, with respect to, $|C_{2n+\log n}| \geq 2^{n/4-o(n)}$ for every $n \in \mathcal{N}$, which concludes the theorem.

The black-box UTM we consider is an oracle universal Turing machine U_{oracle} such that for any oracle \mathcal{O} , $U_{\text{oracle}}^{\mathcal{O}}$ simulates any oracle machine Π with the oracle \mathcal{O} . In addition, if Π does not make oracle query, U_{oracle} will also simulate the execution of Π . We next describe

our choice of the oracle \mathcal{O} : For every large enough $n \in \mathcal{N}$, let $G_n: \{0, 1\}^n \rightarrow \{0, 1\}^{n+\log n}$ be the function promised by Corollary 27 (for $m = n + \log n$). For every $x \in \{0, 1\}^n$ with $n \in \mathcal{N}$, let $\mathcal{O}(x) = G_n(x)$. For every $x \in \{0, 1\}^n$ with $n \notin \mathcal{N}$, let $\mathcal{O}(x) = x$. Let $\hat{U} = U_{\text{oracle}\xi}^{\mathcal{O}}$. The choice of the sparse set \mathcal{N} is to make sure that on input of length n , it is not useful to \hat{U} to call the oracle \mathcal{O} on inputs with length different than n (see more detail below). This is a similar technique to the one used in the sparsification lemma in [4].

Intuitively, we use the oracle \mathcal{O} as a PRG. Observe that for every $n \in \mathcal{N}$, G can be computed in polynomial time using the oracle to \mathcal{O} . Let M_G be such an efficient oracle-aided TM that computes G , and let c be a constant such that n^c is an upper bound on the running time of M_G . Fix a function $t(n)$ such that $t(n) \geq n^c$ for every $n \in \mathbb{N}$. Clearly, $K_{\hat{U}}^t(G(x)) \leq n + 2|M_G| \leq n + 0.1 \log n$, for every $x \in \{0, 1\}^n$ and every large enough n .

On the other hand, for random $Y \leftarrow \{0, 1\}^m$, $K_{\hat{U}}^t(Y) \geq n + \log n - 1$ with probability at least $1/2$. Thus, every circuit C that computes $K_{\hat{U}}^t$ on strings of length $n + \log n$ can be used to distinguish between $G(X)$ and Y , for $X \leftarrow \{0, 1\}^n$, with advantage $1/2$ by outputting 1 if the output of C is larger or equal to $n + \log n - 1$. By the choice of G_n , it follows that if C is a circuit with G_n -gates, it must hold that $|C| \geq 2^{n/2-2\log^2 n}/n^c$.

Recall that the circuits in $\mathcal{C} = \{C_n\}_{n \in \mathbb{N}}$ have oracle gates to $f_n := f_n^{\hat{U}}$, and our lower bound above only holds for circuits with G_n gates. To conclude the theorem, we thus need to show that f_n can be implemented with a circuit of size $\text{poly}(n)$ with G_n -gates. Indeed, in this case, taking $C = C_{n+\log n}$, we get that $|C_{n+\log n}| \geq 2^{n/2-2\log^2 n}/\text{poly}(n)$.

To see that we can implement f_n with circuit of polynomial size (with G_n -gates) first observe that if we allow \mathcal{O} -gates, we can simulate f_n with a circuit of size $\text{poly}(t(n))$. Thus, we only left to implement the oracle \mathcal{O} .

To do so, recall that on inputs x of length n , $\mathcal{O}(x) = G_n(x)$. Therefore we only need to simulate \mathcal{O} on other input lengths. For every $n' > n$ with $n' \in \mathcal{N}$, it holds that $n' \geq 2^n$. Since $t(n) \leq 2^n$, we get that for every call to f_n , \hat{U} cannot call to \mathcal{O} on inputs length $n' \in \mathcal{N}$ with $n' > n$. Thus, we do not need to simulate \mathcal{O} on those inputs. Moreover, for every $n' < n$ with $n' \in \mathcal{N}$, it holds that $n' \leq \log n$. Thus, $\mathcal{O}(x)$ for any $x \in \{0, 1\}^{n'}$ can be computed by a circuit of size $O(2^{n'}) = O(n)$. Lastly, for every x with $|x| \notin \mathcal{N}$, it holds that $\mathcal{O}(x) = x$. Thus the oracle can be implemented trivially also in this case. \blacktriangleleft

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