On Generalized Corners and Matrix Multiplication

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──── **Abstract** ────

Suppose that $S \subseteq [n]^2$ contains no three points of the form $(x, y), (x, y + \delta), (x + \delta, y')$, where $\delta \neq 0$. How big can S be? Trivially, $n \leq |S| \leq n^2$. Slight improvements on these bounds are obtained from Shkredov's upper bound for the corners problem [23], which shows that $|S| \leq O(n^2/(\log \log n)^c)$ for some small $c > 0$, and a construction due to Petrov [19], which shows that $|S| \geq \Omega(n \log n/\sqrt{\log \log n})$.

Could it be that for all $\varepsilon > 0$, $|S| \leq O(n^{1+\varepsilon})$? We show that if so, this would rule out obtaining $\omega = 2$ using a large family of abelian groups in the group–theoretic framework of [12, 11] (which is known to capture the best bounds on ω to date), for which no barriers are currently known. Furthermore, an upper bound of $O(n^{4/3-\varepsilon})$ for any fixed $\varepsilon > 0$ would rule out a conjectured approach to obtain $\omega = 2$ of [11]. Along the way, we encounter several problems that have much stronger constraints and that would already have these implications.

1 Introduction

The exponent of matrix multiplication ω is the smallest number such that for any $\varepsilon > 0$, there exists an algorithm for multiplying $n \times n$ matrices using $O(n^{\omega+\varepsilon})$ arithmetic operations. Since Strassen's initial discovery that $\omega < 3$ [25], there has been much work on understanding this fundamental constant, with the end goal being the determination of whether or not $\omega = 2$. It is currently known that $2 \leq \omega < 2.3716$ [26].

The best upper bounds on ω obtained since 1987 [24] can be understood as solutions to the following hypergraph packing problem. Let M_n be the *matrix multiplication hypergraph*, the tripartite 3-uniform hypergraph with parts $X_1 = X_2 = X_3 = [n]^2$, and where $((i, j), (k, l), (m, n)) \in X_1 \times X_2 \times X_3$ is a hyperedge if and only if $j = k, l = m, n = i$. Given an abelian group G, let X_G be its "addition hypergraph" with vertex sets $G \sqcup G \sqcup G$, and where $(a_1, a_2, a_3) \in G \times G \times G$ is a hyperedge exactly when $a_1 + a_2 + a_3 = 0$. Suppose that X_G contains k disjoint induced copies of M_n. Then

$$\omega < \log_n(|G|/k). \tag{1}$$

Phrased in terms of the group–theoretic approach proposed by Cohn and Umans [12] and further developed by Cohn, Kleinberg, Szegedy, and Umans [11], this is equivalent to proving upper bounds on ω via *simultaneous triple product property* (STPP) constructions in abelian groups. The above inequality was established in [11, Theorem 5.5]. It can be also be deduced via the *asymptotic sum inequality* of [21].

From this perspective, the best bounds on ω to date are obtained by taking G to be a large power of a cyclic group – specifically, $\mathbb{Z}_7^\ell$ with $\ell \to \infty$. However, in [8] ideas related to the resolution of the cap-set problem in additive combinatorics [13] were used to show that

one cannot obtain $\omega = 2$ using groups of *bounded exponent* – such as $\mathbb{Z}_7^\ell$ – via this approach. This obstruction is due to the fact that when G has bounded exponent, there is power-savings over the trivial upper bound on the size of the largest induced matching in X_G (also called a *3-matching* [20], or a *tricolored sum-free set* [8]). For example, when $G = \mathbb{Z}_7^\ell$ the largest induced matching has size at most $O(6.16^\ell)$. On the other hand, M_n contains an induced matching of size $n^{2-o(1)}$: if we identify vertices in M_n with edges in the complete tripartite graph $K_{n,n,n}$, an induced matching in M_n corresponds to a tripartite graph on at most $3n$ vertices where every edge is contained in a unique triangle, and the number of vertices in the induced matching equals the number of edges in this graph. A well-known construction in extremal combinatorics yields such a graph with $n^{2-o(1)}$ edges (see [27, Corollary 2.5.2])[1] and hence M_n contains an induced matching of size $n^{2-o(1)}$. Modulo minor details, the claimed barrier then follows, as an efficient packing of copies of M_n into X_G would imply the existence of a large induced matching in X_G, a contradiction.[2] This barrier was identified in [5] and executed in [8].

This is the only obstruction to obtaining $\omega = 2$ via the use of Equation (1) that we are aware of. Unfortunately,[3] this barrier says nothing about the viability of general abelian groups, as their addition hypergraphs may contain large induced matchings. For example, if A is a 3-term arithmetic progression free (hereon abbreviated to 3AP-free) subset of G, then the subsets $A, A, -2A$ of the vertex sets of X_G induce a matching of size $|A|$. Hence this barrier cannot apply to any group containing a 3AP-free subset of size $|G|^{1-o(1)}$, such as $\mathbb{Z}_n$ [7]. Could one achieve $\omega = 2$ using cyclic groups, or perhaps products of cyclic groups of growing orders?

In this paper we identify problems in additive combinatorics whose answer we conjecture would rule out obtaining $\omega = 2$ using a large family of abelian groups for which the induced matching barrier is irrelevant. This family includes abelian groups with a bounded number of direct factors. If abelian groups of bounded exponent are at one end of a spectrum, these groups are at the opposite end. These problems have not been studied before as far as we are aware. Aside from their connections to fast matrix multiplication, we find them intrinsically interesting. We now discuss the simplest-to-state such problem.

1.1   A skew corners problem

The *corners problem* in additive combinatorics asks for the size of the largest subset of $[n]^2$ containing no three points of the form
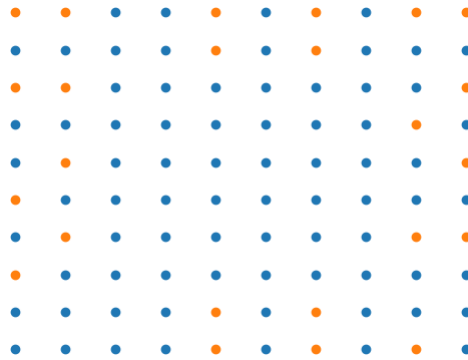
$$(x,y), (x, y+\delta), (x+\delta, y)$$

where $\delta \neq 0$. Ajtai and Szemerédi [1] settled this problem up to factors of $n^{o(1)}$ by proving an upper bound of $o(n^2)$ and a lower bound of $n^{2-o(1)}$. This problem is significant as it was the first multidimensional case of Szemerédi's theorem to be established, and for its application to the number-on-forehead model in communication complexity [9].

Here is a subtle strengthening of the condition of the corners problem for which we know essentially nothing:

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[1] This is the Rusza-Szemerédi problem. The equivalence between induced matchings in M_n and this problem was independently noted in [2].

[2] The techniques involved in the resolution of the cap–set problem (in particular, slice rank) actually give stronger "tensor analogues" of this barrier; see [10, 3].

[3] Or fortunately, for the optimist.

Figure 1 The orange points form a skew corner-free subset of $[10] \times [10]$ of size 24.

▶ **Question 1.** *What is the size of the largest $S \subseteq [n]^2$ which does not contain three points of the form*

$$(x, y), (x, y + \delta), (x + \delta, y')$$

where $x, y, y' \in S$ and $\delta \neq 0$?

That is, not only must S avoid all corners, but given any two points in S lying on the same vertical line, the *entire vertical line* passing through the third point that would form a corner with these two points must be absent from S! Naturally, we call such a set of points *skew corner-free*. See Figure 1 for an example of such a set. Note that any skew corner-free set is clearly corner-free.

Note that there is a trivial lower bound of n, obtained by taking S to be all points lying on a vertical or horizontal line. We conjecture that this is almost optimal:

▶ **Conjecture 2.** *Fix any $\varepsilon > 0$. If S is skew corner-free, then $|S| \leq O(n^{1+\varepsilon})$.*

A construction due to Petrov [19] (Proposition 39) shows that one can have $|S| \geq \Omega(n \log n / \sqrt{\log \log n})$. On the other hand, the best upper bound we know is $O(n^2 / (\log \log n)^{0.0137\cdots})$, which follows immediately from Shkredov's upper bound on the corners problem [23].

Two of the main results of this paper are the following.

▶ **Theorem 3.** *If Conjecture 2 is true, then one cannot obtain $\omega = 2$ via STPP constructions in the family of groups $\mathbb{Z}_q^\ell$, where q is a prime power.*

Furthermore, a weakening of Conjecture 2 would rule out obtaining $\omega = 2$ using a specific type of STPP construction in arbitrary abelian groups. In [11], it was conjectured that this type of construction can be used to obtain $\omega = 2$.

▶ **Theorem 4.** *If the largest skew corner-free subset of $[n]^2$ has size $O(n^{4/3-\varepsilon})$ for some $\varepsilon > 0$, then [11, Conjecture 4.7] is false.*

In fact, seemingly much weaker conjectures than Conjecture 2 would already have these implications. The weakest conjecture we make is the following. Let Δ_n be a triangular array of $n(n+1)/2$ points. Suppose that we delete from Δ_n sets of points lying on lines parallel to the sides of this array, such that the remaining set of points does not contain any equilateral trapezoid with sides parallel to the sides of the array (see Figure 4). For example, we might
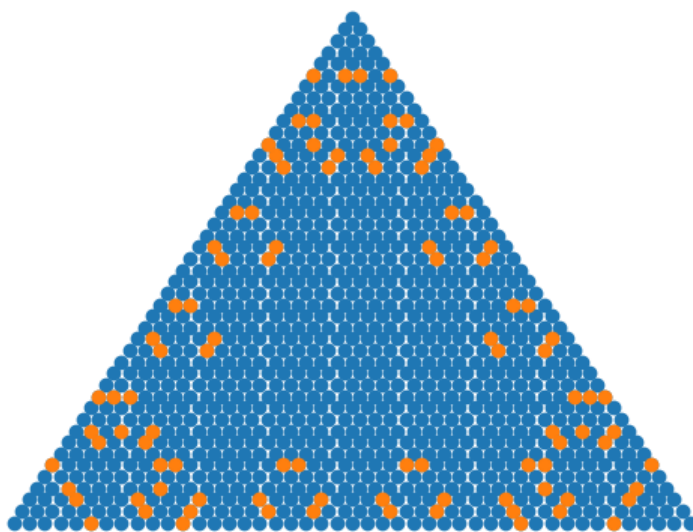
■ **Figure 2** The 90 orange points form a skew corner-free subset of the triangular grid Δ_{45} (Definition 35): for any two orange points on the same line parallel to one of the sides of the grid, the line parallel to this side and passing through a third point that would form an equilateral triangle with these two points contains no orange points. This is largest-possible among subsets of Δ_{45} that are symmetric under the S_3 action on Δ_n.

delete all lines in one direction but one. Then, what is the maximum number of points that can remain? By our example, one can achieve at least n. We conjecture that this is essentially optimal (Conjecture 24). Another condition we introduce, which is intermediate between this and being skew-corner free, is that of a skew corner-free subset of a triangular grid (see Figure 2).

1.2 Paper overview

In Section 2 we review the group–theoretic approach of [12, 11]. In Section 2.1 we record a very weak lower bound for this approach, which follows easily from the removal lemma in groups of [22]. This lower bound becomes much stronger in $\mathbb{Z}_q^\ell$ (Corollary 14), thanks to the improved bounds on the removal lemma of [16], and we make later use of this fact.

In Section 3 we note that the matrix multiplication hypergraph M_n is an extremal solution to a certain forbidden hypergraph problem. This was our motivating observation. We define the "value" of a group, val(G), which captures this forbidden hypergraph problem in a group–theoretic context. This quantity equals the maximum number of triangles in an induced subhypergraph of X_G that does not contain the triforce hypergraph or a cycle of 4 triangles (see Figure 3). This can also be expressed in terms of the group operation slightly awkwardly (Definition 17). The trivial bounds are that $|G| \leq \text{val}(G) \leq |G|^{3/2}$; using the removal lemma of [22], the upper bound can be improved to $o(|G|^{3/2})$ (Proposition 22). STPP constructions yield lower bounds on the quantity val(G) (Proposition 18), so ultimately it is upper bounds on val(G) that we are interested in as a means towards barriers. The quantity val(G) is super-multiplicative under direct product (Proposition 20), which is one reason why power-improvements over the trivial bound seem to be easier to obtain in direct products of groups.

We then focus on the case of abelian groups in Section 4. We show that a bound of $\omega = 2$ using the family of groups $\mathbb{Z}_q^\ell$ would imply that $\mathrm{val}(\mathbb{Z}_n) \geq \Omega(n^{1+c})$ for some $c > 0$ Theorem 27. We also show that a proof of $\omega = 2$ via *simultaneous double product property* constructions [11] in any family of abelian groups would imply that $\mathrm{val}(\mathbb{Z}_n) \geq \Omega(n^{4/3-\varepsilon})$ for any given $\varepsilon > 0$ (Theorem 30). We thank Chris Umans for mentioning a related fact to us, which motivated this result. We then relate $\mathrm{val}(\mathbb{Z}_n)$ to various questions about sets of points in the plane, including Question 1 (Definitions 33, 35, and 37). This gives Theorems 3 and 4. We also give an example which shows that one cannot hope to prove strong upper bounds on $\mathrm{val}(\mathbb{Z}_n)$ via a certain "asymmetric" averaging argument (Proposition 31).

The take-away of this paper is that STPP constructions yield subsets of $G \times G$ which satisfy dramatically stronger properties than that of being corner-free. While subsets satisfying these stronger properties do not imply STPP constructions in any obvious way, we believe that understanding them will be a stepping stone to understanding the power of the group–theoretic approach, and possibly towards improved upper bounds on ω.

2 Background

Bounds on ω from the group–theoretic approach are obtained by designing subsets of groups satisfying the following condition. Throughout this paper, I denotes the identity element of a group G.

▶ **Definition 5.** *A collection of triples of subsets S_i, T_i, U_i of a group G satisfy the simultaneous triple product property (or STPP for short) if*
1. *For each i, the sets S_i, T_i, U_i satisfy the* triple product property*: if $ss'^{-1}tt'^{-1}uu'^{-1} = I$ with $s, s' \in S_i, t, t' \in T_i, u, u' \in U_i$, then $s = s', t = t', u = u'$.*
2. *Setting $A_i = S_i T_i^{-1}, B_j = T_j U_j^{-1}, C_k = U_k S_k^{-1}$,*

$$a_i b_j c_k = I \iff i = j = k$$

for all $a_i \in A_i, b_j \in B_j, c_k \in C_k$.

The crucial fact is the following:

▶ **Theorem 6** ([11, Theorem 5.5]). *If G is abelian and $S_i, T_i, U_i \subseteq G$ satisfy the STPP, then*

$$\sum_i (|S_i||T_i||U_i|)^{\omega/3} \leq |G|.$$

The conditions of the STPP imply that the sets involved satisfy a simple "packing bound" (see the discussion preceding [8, Definition 2.3]).

▶ **Proposition 7** (Packing bound). *If S_i, T_i, U_i satisfy the STPP in a group G, then $\sum_i |S_i||T_i| \leq |G|$, $\sum_i |T_i||U_i| \leq |G|$, and $\sum_i |U_i||S_i| \leq |G|$.*

A particular type of STPP construction can be obtained from pairs of sets satisfying a condition termed the *simultaneous double product property* in [11].

▶ **Definition 8.** *We say that sets $(A_i, B_i)_{i=1}^n$ satisfy the simultaneous double product property (or SDPP for short) if*
1. *For all i, $aa'^{-1} = bb'^{-1}$ only has the solution $a = a', b = b'$ for $a, a' \in A_i, b, b' \in B_i$,*
2. *$a_i(a_j')^{-1} b_j(b_k')^{-1} = 1$ implies $i = k$, where $a_i \in A_i, a_j' \in A_j, b_j \in B_j, b_k' \in B_k$.*

In [11] it was conjectured that one can achieve $\omega = 2$ using SDPP constructions in abelian groups. This amounts to the following.

▶ **Conjecture 9** ([11, Conjecture 4.7]). *For arbitrarily large n, there exists an abelian group G of order $n^{2-o(1)}$ and n pairs of sets A_i, B_i where $|A_i||B_i| > n^{2-o(1)}$ satisfy the SDPP.*

If G is a finite group, we let X_G denote the tripartite 3-uniform hypergraph with vertex parts $X_1 = X_2 = X_3 = G$, and where (g_1, g_2, g_3) is a hyperedge (a *triangle*) whenever $g_1 g_2 g_3 = I$. In the event that G is nonabelian, it is important that we fix some ordering on the parts of X_G here. Recall that a 3-uniform hypergraph is said to be *linear* if any two vertices are contained in at most one hyperedge. For example, X_G is linear. The matrix multiplication hypergraph $M_{p,q,r}$ is defined to be the hypergraph with parts $[p] \times [q], [q] \times [r], [r] \times [p]$, and where $((i,j),(k,l),(m,n))$ is a hyperedge if and only if $j = k, l = m, n = i$. If X is a hypergraph, we sometimes write $E(X)$ for the set of hyperedges of X.

It is convenient to view STPP constructions from a hypergraph perspective.

▶ **Proposition 10.** *There exist sets $S_i, T_i, U_i \subseteq G$, satisfying the STPP if and only if X_G contains as an induced subhypergraph the disjoint union of $M_{|S_i|,|T_i|,|U_i|}$.*

Proof. It follows from the first condition of the STPP that for all i, the subhypergraph induced by $A_i := S_i T_i^{-1}, B_i := T_i U_i^{-1}, C_i := U_i S_i^{-1}$ equals $M_{|S_i|,|T_i|,|U_i|}$. The second condition implies that A_i and A_j are disjoint when $i \neq j$, and similarly for the subsets of the other parts. The second condition also implies that the only hyperedges in the subhypergraph induced by $\sqcup_i A_i, \sqcup_i B_i, \sqcup_i C_i$ are between sets of the form A_i, B_i, C_i, so the claim follows.

Conversely, suppose that $\sqcup_i A_i, \sqcup_i B_i, \sqcup_i C_i$ induce disjoint hypergraphs M_{p_i,q_i,r_i}. Fix some i, and for shorthand write $A := A_i, B := B_i, C := C_i$ and let $p := p_i, q := q_i, r := r_i$. Since A, B, C induce $M_{p,q,r}$, we can by definition write $A = \{a_{ij}\}_{i \in [p], j \in [q]}, B = \{b_{ij}\}_{i \in [q], j \in [r]}, C = \{c_{ij}\}_{i \in [r], j \in [p]} \in G$ where

$$a_{ij} b_{kl} c_{mn} = I \iff j = k, l = m, n = i. \tag{2}$$

We claim that there exist $X = \{x_i\}_{i \in [p]}, Y = \{y_j\}_{j \in [q]}, Z = \{z_k\}_{k \in [r]}$ such that $a_{ij} = x_i y_j^{-1}$, $b_{jk} = y_j z_k^{-1}$, $c_{ki} = z_k x_i^{-1}$ for all $i \in [p], j \in [q], k \in [r]$. This can be accomplished by taking $x_0 = 1, x_i = a_{i0} a_{00}^{-1}$ for $i > 0$, $y_i = a_{0i}^{-1}, z_i = c_{i0}$. Furthermore, Equation (2) implies that X, Y, Z will satisfy the TPP. This shows that for each i there are X_i, Y_i, Z_i such that $A_{1,i} = X_i Y_i^{-1}, A_{2,i} = Y_i Z_i^{-1}, A_{3,i} = Z_i X_i^{-1}$, and X_i, Y_i, Z_i satisfy the TPP. The fact that they induce a disjoint union of hypergraphs implies that if $a \in A_i, b \in B_j, c \in C_k$, then $abc = I$ implies $i = j = k$, which implies the second condition in the definition of the STPP. ◀

▶ Remark 11. The second direction of this proposition is essentially the fact that a complete 2-dimensional simplicial complex has trivial 1-cohomology with coefficients in any group.

2.1 Triangle Removal and the Group-Theoretic approach

In [22], a nonabelian generalization of Green's arithmetic removal lemma [17] was shown to follow from the directed graph removal lemma of Alon and Shapira [4]. Specifically, they showed the following:

▶ **Theorem 12.** *Let G be a finite group of order N. Let $A_1, \ldots, A_m, m \geq 2$, be sets of elements of G and let g be an arbitrary element of G. If the equation $x_1 x_2 \cdots x_m = g$ has $o(N^{m-1})$ solutions with $x_i \in A_i$, then there are subsets $A_i' \subseteq A_i$ with $|A_i'| = o(N)$ such that there is no solution of the equation $x_1 x_2 \cdots x_m = g$ with $x_i \in A_i \setminus A_i'$.*

The best quantitative bounds for this theorem are due to Fox [15], and show that if there are at most δN^{m-1} solutions to $x_1 \cdots x_m = g$, then there exist $A_i' \subseteq A_i$ of size at most εn satisfying the conclusion of the theorem, whenever δ^{-1} is a tower of twos of height $O(\log \varepsilon^{-1})$.

Theorem 12 implies the following:

▶ **Corollary 13.** *If X_i, Y_i, Z_i satisfy the STPP in a group G of order n, then at least one of $\sum |X_i||Y_i|, \sum |X_i||Z_i|, \sum |Y_i||Z_i|$ is at most $o(n)$.*

Proof. Let $A_1 = \sqcup_i X_i Y_i^{-1}, A_2 = \sqcup_i Y_i Z_i^{-1}, A_3 = \sqcup_i Z_i X_i^{-1}$. By definition of the STPP, the equation $x_1 x_2 x_3 = I$ with $x_i \in A_i$ has $\sum_i |X_i||Y_i||Z_i|$ solutions. By the packing bound Proposition 7, $\sum_i |X_i||Y_i| \le n, \sum_i |Y_i||Z_i| \le n, \sum_i |Z_i||X_i| \le n$, so by Cauchy–Schwarz there are at most $n^{3/2} = o(n^2)$ solutions to $a_1 a_2 a_3 = I$.

Now suppose that $B_j \subseteq A_j$ satisfy $|B_j|/|A_j| > 0.9999$; we will show that there is a solution to $b_1 b_2 b_3 = I$. For more than a 0.99 fraction of the values of i we must have $|B_1 \cap X_i Y_i^{-1}|/|X_i Y_i^{-1}| > 0.99$ (because $0.99 \cdot 1 + 0.01 \cdot 0.99 = 0.9999$) and similarly for the other sets. Hence by the pigeonhole principle there is some i for which $|B_1 \cap X_i Y_i^{-1}|/|X_i Y_i^{-1}| > 0.99, |B_2 \cap Y_i Z_i^{-1}|/|Y_i Z_i^{-1}| > 0.99, |B_3 \cap Z_i X_i^{-1}|/|Y_i Z_i^{-1}| > 0.99$. Now consider the tripartite graph with parts X_i, Y_i, Z_i, where (x, y) is an edge between X_i and Y_i if $xy^{-1} \in B_1 \cap X_i Y_i^{-1}$, (y, z) is an edge between Y_I, Z_i when $yz^{-1} \in B_2 \cap Y_i Z_i^{-1}$, and (z, x) is an edge when $zx^{-1} \in B_3 \cap Z_i X_i^{-1}$. Note that the existence of a triangle in this graph implies that there is a solution to $b_1 b_2 b_3 = I$. First, note that at least $0.9|X_i|$ vertices in X_i have at least $0.9|Y_i|$ neighbors in Y_i. (If this were not the case, there would be at most $0.9|X_i||Y_i| + 0.1 \cdot 0.9 \cdot |X_i||Y_i| \le 0.99|X_i||Y_i|$ edges between X_i and Y_i, and hence $|B_1 \cap X_i Y_i^{-1}|/|X_i Y_i^{-1}| \le 0.99$, a contradiction.) Similarly, at least $0.9|X_i|$ vertices in X_i have at least $0.9|Z_i|$ neighbors in Z_i. Hence at least $0.8|X_i|$ vertices in X_i have $0.9|Y_i|$ neighbors in Y_i and $0.9|Z_i|$ neighbors in Z_i. Pick any such vertex $x_0 \in X_i$. There must be an edge between a neighbor of x_0 in Y_i and a neighbor of x_0 in Z_i, since if not, there would be at most $|Y_i||Z_i| - 0.9^2|Y_i||Z_i| = 0.19|Y_i||Z_i|$ edges between Y_i and Z_i. Thus we have found our triangle.

By Theorem 12, we can delete subsets of A_i of size $o(n)$ to eliminate all solutions to $x_1 x_2 x_3 = I$. On the other hand, any three subsets of the A_i's of density 0.9999 contain some such solution. Hence we must have $|A_i| = o(n)$ for some i. ◀

As a corollary of this proof, we have the following.

▶ **Corollary 14.** *There exists an absolute constant $C > 1$ such that if X_i, Y_i, Z_i satisfy the STPP in $\mathbb{Z}_q^\ell$, then at least one of $\sum |X_i||Y_i|, \sum |X_i||Z_i|, \sum |Y_i||Z_i|$ is at most $(q/C)^\ell$.*

Proof. The proof of Corollary 13 shows that $A_1 = \sqcup_i X_i Y_i^{-1}, A_2 = \sqcup_i Y_i Z_i^{-1}, A_3 = \sqcup_i Z_i X_i^{-1}$ have the following properties: there are at most $q^{3n/2}$ solutions to $a_1 + a_2 + a_3 = 0$, and any subsets of A_1, A_2, A_3 of density 0.9999 each contain some such solution. At the same time, by [16, Theorem 1], if $A_1, A_2, A_3 \subseteq \mathbb{Z}_q^\ell$ and there are less than δq^{2n} solutions to $a_1 + a_2 + a_3 = 0$, then we may remove εq^ℓ elements from $A_1 \cup A_2 \cup A_3$ and eliminate all solutions, when $\delta = (\varepsilon/3)^{\Theta(\log q)}$.[4] In our setting, $\delta = q^{-n/2}$ and so $\varepsilon = 3q^{\Theta(-n/\log q)} \le 3C'^{-n}$ for some universal C'. Hence it must have been the case that one of A_1, A_2, A_3 had size at most $(q/C)^\ell$ to begin with, for some universal C. ◀

One can interpret Corollary 13 as saying that the best upper bound on the rank of a direct sum of matrix multiplication tensors provable via the group–theoretic approach is superlinear. We remark the only important property of the matrix multiplication hypergraph

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[4] While [16, Theorem] is only stated for $\mathbb{Z}_p^\ell$, it extends to $\mathbb{Z}_q^\ell$ by the same argument via the use of [8, Theorem A'].
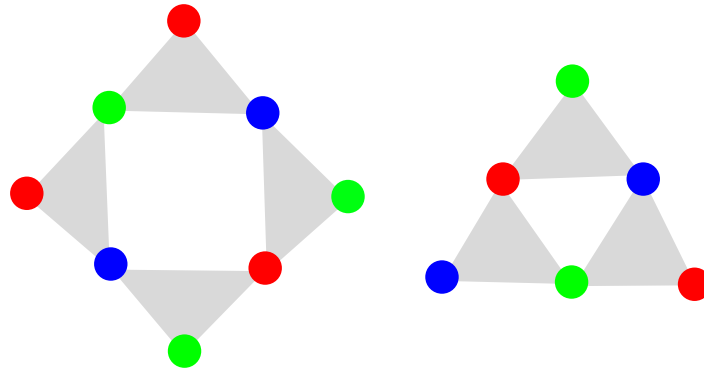
■ **Figure 3** The forbidden hypergraphs in Proposition 16, up to permutations of the three parts (represented by different colors).

for this result was that it satisfies the very weak "regularity" condition that very large subsets of the three vertex parts span at least one triangle. Specifically, considerations similar to those of the proof of Corollary 13 show the following:

▶ **Theorem 15.** *Let $\varepsilon > 0$. Let G be a group of order n. Let $X = \sqcup_{i=1}^{3} A_i$ be a tripartite hypergraph with $o(n^2)$ triangles such that for any $Y_i \subseteq A_i$ with $|Y_i|/n \geq 1 - \varepsilon$, there exists $y_i \in Y_i$ such that $(y_1, y_2, y_3) \in E(X)$. Then if X is an induced subhypergraph of X_G, $|A_i| \leq o(n)$ for $i = 1, 2, 3$.*

3   Equilateral trapezoid-freeness in hypergraphs and groups

We begin with the observation that the matrix multiplication hypergraph is an extremal solution to a certain forbidden hypergraph problem.

▶ **Proposition 16.** *Let X be a linear tripartite hypergraph with parts of size N such that any two vertices from different parts are incident to at most one common vertex in the third part. Then the number of triangles in X is at most $N^{3/2}$. Furthermore, when N is a square, an extremal example is the matrix multiplication hypergraph $M_{N^{1/2}}$.*

The hypergraphs satisfying the condition of Proposition 16 can be equivalently characterized as the linear hypergraphs that do not contain copies of the hypergraphs in Figure 3. We remark that the proof of the upper bound in Proposition 16 is closely related to the upper bound on the Turán density of the 4-cycle.

Proof. Restricting our attention to one of the parts X_1 of X, let d_v be the number of triangles that vertex $v \in X_1$ is contained in. Each $v \in X_1$ is contained in d_v triangles, where the vertices of these triangles belonging to X_2 and X_3 are distinct (as X is linear). Additionally, no pair of such vertices in X_2 and X_3 can be contained in a triangle incident to another vertex $u \in X_1$, so there are $2\binom{d_v}{2}$ pairs of vertices in X_2 and X_3 that are contained in no common triangle. Let (x_2, x_3) be some such pair of vertices. Observe that furthermore, for all $u \neq v \in X_1$, the set of vertices in X_2 and X_3 incident to the set of triangles containing u cannot also contain both x_2 and x_3. For if this happened, there would be triangles $(v, x_2, x_3'), (v, x_2', x_3), (u, x_2, x_3''), (u, x_2'', x_3)$, and then x_2 and x_3 violate the constraint. The total number of triangles equals $m := \sum_{v \in X_1} d_v$, and by the prior observations it follows that $\sum 2\binom{d_v}{2} + m \leq N^2$. So $\sum d_v(d_v - 1) + m = \sum d_v^2 \leq N^2$. The conclusion follows from Cauchy–Schwarz.

To see that $M_{N^{1/2}}$ is extremal, note that it contains $N^{3/2}$ triangles, has parts of size N, and is linear. To see that it satisfies the second condition, let (i, j) be a vertex in the first part, and let (k, l) be a vertex in the second part. Then (i, j) is contained in a common triangle with exactly the vertices in the third part of the form $(*, i)$, and (k, l) is incident to exactly the vertices in the third part of the form $(l, *)$. Hence (l, i) is the unique neighbor of both. The same argument shows the claim for vertices in any two parts. ◄

The key definition in this paper is that of an "equilateral trapezoid-free" triple of subsets of a group. The reason for this name will eventually be explained in Section 4.

▶ **Definition 17.** *Let $A, B, C \subseteq G$. We call (A, B, C) equilateral trapezoid-free if the subhypergraph of X_G induced by $A \subseteq X_1, B \subseteq X_2, C \subseteq X_3$ satisfies the conditions of Proposition 16. Equivalently, (A, B, C) is equilateral trapezoid-free if for any fixed $a' \in A, b' \in B, c' \in C$, the following systems of equations in the variables $a \in A, b \in B, c \in C$ each have at most one solution:*

$$I = a'bc = ab'c,$$
$$I = a'bc = abc',$$
$$I = ab'c = abc'.$$

Let $\mathrm{val}(G)$ be the maximum number of solutions to $abc = I$ over all equilateral trapezoid-free triples (A, B, C).

The relevance of $\mathrm{val}(G)$ to ω is due to the following.

▶ **Proposition 18.** *Suppose that X_G contains disjoint induced subhypergraphs M_{n_i, m_i, p_i}. Then, $\mathrm{val}(G) \geq \sum_i n_i m_i p_i$.*

Proof. By the same reasoning as in the second part of the proof of Proposition 16, M_{n_i, m_i, p_i} satisfies the constraints of Definition 17 and contains $n_i m_i p_i$ hyperedges. As the disjoint union of these hypergraphs satisfies these constraints as well, the claim follows. ◄

In fact, STPP constructions are essentially the only approach we know of for proving lower bounds on $\mathrm{val}(G)$.

To start, we have the following trivial bounds.

▶ **Proposition 19.** *For any group G, $|G| \leq \mathrm{val}(G) \leq |G|^{3/2}$.*

Proof. The lower bound is obtained by the triple $(\{I\}, G, G)$. The upper bound follows from Proposition 16. ◄

The following super-multiplicative behavior of val is easily checked.

▶ **Proposition 20.** *If (A, B, C) is equilateral trapezoid-free in G, and (A', B', C') is equilateral trapezoid-free in H, then $(A \times A', B \times B', C \times C')$ is a equilateral trapezoid-free in $G \times H$.*

It is also easily seen that being equilateral trapezoid-free is preserved by cyclic permutations of the three sets.[5]

▶ **Proposition 21.** *If (A, B, C) is equilateral trapezoid-free, then so is (B, C, A).*

---

[5] This may not hold for all permutations in the case when G is nonabelian.

By an application of Theorem 15 combined with the observation that near-extremal solutions to Proposition 16 are highly "regular", we have the following weak improvement to the trivial upper bound of $|G|^{3/2}$.

▶ **Proposition 22.** *For any group G, $\mathrm{val}(G) \leq o(|G|^{3/2})$.*

Proof. Suppose for contradiction that there exists $\varepsilon_0 > 0$ such that $\mathrm{val}(G) > \varepsilon_0 |G|^{3/2}$, and let $A_0, B_0, C_0 \subseteq G$ witness $\mathrm{val}(G) = \varepsilon_0 |G|^{3/2}$. Next consider the triple $(A, B, C) :=$ $(A_0 \times B_0 \times C_0, B_0 \times C_0 \times A_0, C_0 \times A_0 \times B_0)$, which is equilateral-trapezoid free inside of $H := G^3$ by Proposition 20 and Proposition 21, and witnesses $\mathrm{val}(H) \geq \varepsilon |H|^{3/2}$ where $\varepsilon := \varepsilon_0^3$. Let $|H| = N$. Let X be the tripartite hypergraph with parts A, B, C and where there is a triangle between all triples (a, b, c) where $abc = I$. Let $n := |A| = |B| = |C|$. By Proposition 16 we must have $n \geq \varepsilon^{2/3} N$. Note that the number of triangles in X equals $\varepsilon N^{3/2} \geq \varepsilon n^{3/2}$. In what follows, the degree of a vertex in X refers to the number of triangles containing it.

Let Y be the random variable that is uniformly distributed over the multiset of vertex degrees from one part of X, say A. Then $\mathbb{E}[Y] \geq \varepsilon n^{1/2}$ and $\mathbb{E}[Y^2] \leq n$ (this second inequality follows from the use of Cauchy–Schwarz in the proof of Proposition 16). By the Payley-Zygmund inequality, for any $\theta > 0$, $\Pr(Y > \theta \cdot \varepsilon n^{1/2}) \geq (1 - \theta^2)\varepsilon^2$. Taking $\theta = 1/2$, we conclude that at least $p \cdot n := 3n\varepsilon^2/4$ vertices in A have degree at least $\varepsilon n^{1/2}/2$. This holds for B and C as well.

Now let S, T, and U be any subsets of A, B, C of size at least $n(1 - p/\lambda)$; we'll pick $\lambda \in \mathbb{N}$ later. Then the number of triangles incident to any one of these sets, say S, is at least

$$np(1 - \lambda^{-1}) \cdot \varepsilon n^{1/2}/2 = (3/8)n^{3/2}\varepsilon^3(1 - \lambda^{-1}),$$

and the number of triangles incident to $[n] \setminus T$ or $[n] \setminus U$, sets of size at most np/λ, is at most

$$(n^2 \cdot np/\lambda)^{1/2} = (3^{1/2}/2)n^{3/2}\varepsilon\lambda^{-1/2}$$

by Cauchy–Schwarz. It follows that the number of triangles with one vertex in each of S, T, U is at least

$$(3/8)n^{3/2}\varepsilon^3(1 - \lambda^{-1}) - 2 \cdot (3^{1/2}/2)n^{3/2}\varepsilon\lambda^{-1/2}$$

which is greater than 1 for $\lambda \gg \varepsilon^{-4}$. In summary, between any three subsets of A, B, C size roughly $n(1 - \varepsilon^6)$, there is a triangle.

Recall that $n \geq \varepsilon^{2/3} N$. Since X has at most $N^{3/2} \leq o(N^2)$ triangles, by Theorem 15 we can remove $o(N) = o(n)$ vertices to remove all triangles. But by what we have just shown, after deleting this few vertices some triangle will remain, a contradiction. ◀

▶ Remark 23. By combining this proof with [16], it follows that for fixed n and some $\varepsilon > 0$, $\mathrm{val}(\mathbb{Z}_n^\ell) \leq O(n^{3/2(1-\varepsilon)\ell})$.

4 $\mathrm{val}(\mathbb{Z}_n)$ and its applications

Our weakest conjecture is the following.

▶ **Conjecture 24.** *For all $\varepsilon > 0$, $\mathrm{val}(\mathbb{Z}_n) \leq O(n^{1+\varepsilon})$.*

In this section we give our potential applications of this conjecture. We then introduce several related quantities and make preliminary progress on understanding them.
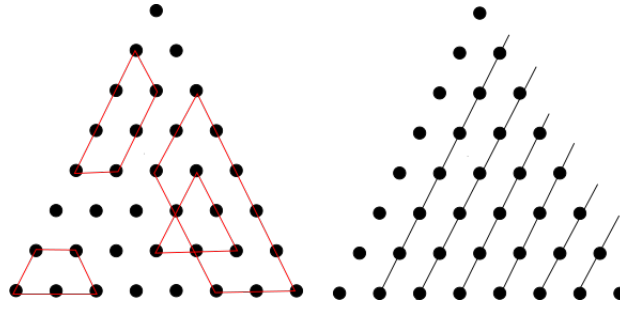
Figure 4 Left: some forbidden trapezoids and triangles in Δ_8. Right: a trapezoid-free subset of Δ_8 of size 8 obtained by deleting all lines but one along one direction.

While the quantity $\text{val}(\mathbb{Z}_n)$ may seem opaque from Definition 17, it can easily be visualized. This is done by first considering the natural notion of an equilateral trapezoid-free subset of the plane, which is convenient to introduce sooner rather than later. Throughout this section, we let $\Delta_{n+1} = \{(a, b, c) \in \mathbb{Z}_{\geq 0}^3 : a + b + c = n\}$. A subset of Δ_{n+1} is said to be corner-free if it contains no configuration $(x + \delta, y, z), (x, y + \delta, z), (x, y, z + \delta)$.

▶ **Definition 25.** *Let $A, B, C \subseteq \{0, \ldots, n\}$. We call (A, B, C) an equilateral trapezoid-free triple if for any fixed a', b', c', the following systems of equations in the variables $a \in A, b \in B, c \in C$ each have at most one solution:*

$$n = a' + b + c = a + b' + c$$
$$n = a' + b + c = a + b + c'$$
$$n = a + b' + c = a + b + c'.$$

Let $\text{val}(n)$ be the maximum number of solutions to $a + b + c = n$ over all equilateral trapezoid-free triples (A, B, C).

We may visualize equilateral trapezoid-free sets as follows. Draw Δ_{n+1} in the plane as a triangular grid of points. Sets A, B, C correspond to collections of lines parallel to the sides of Δ_{n+1}, and a solution $a + b + c = n$ corresponds a point in Δ_{n+1} contained in one line in each of these three directions. Let $S \subseteq \Delta_{n+1}$ be the collection of all such points. A violation of a constraint of Definition 25 corresponds to either a subset of 3 points in S forming an equilateral triangle with sides parallel to the sides of Δ_{n+1}, or a subset of 4 points with sides parallel to the sides of Δ_{n+1} forming an equilateral trapezoid. Equivalently, we are deleting lines parallel to the sides of Δ_{n+1} to eliminate all of such configurations, while leaving as many points as possible. The maximum possible number of points left equals $\text{val}(n)$. See Figure 4.

The following shows that $\text{val}(n)$ and $\text{val}(\mathbb{Z}_n)$ are essentially the same.

▶ **Proposition 26.**
1. $\text{val}(n) \geq \text{val}(n-1)$.
2. $1 + 2 \cdot \text{val}(2n) \geq \text{val}(\mathbb{Z}_n) \geq \text{val}(\lfloor n/3 \rfloor)$.
3. *For $n \geq 6n'$, $\text{val}(\mathbb{Z}_n) \geq \text{val}(\mathbb{Z}'_n)/2 - 1$*

Proof. Suppose that $\text{val}(n)$ is witnessed by sets A, B, C. For $N > n$, $A + (N - n), B, C$ then witness $\text{val}(N) \geq \text{val}(n)$, which shows (1). If we take $N = 3n$, we have that $A + 2n \subseteq \{0, \ldots, N\}$ and $B, C \subseteq \{0, \ldots, N/3\}$, so $a + b + c \leq 5N/3 < 2N$. Since $a + 2n + b + c = 0 \bmod N \iff a + 2n + b + c = N$, this implies that the sets $A + 2n, B, C$ are equilateral

trapezoid-free when viewed as subsets of $\mathbb{Z}_N$. This shows one direction of (2). In the other direction, suppose $\mathrm{val}(\mathbb{Z}_n)$ is witnessed by A, B, C mod n. There are at least $(\mathrm{val}(\mathbb{Z}_n) - 1)/2$ solutions to one of $a + b + c = n, a + b + c = 2n$; let N be the right-hand side of the most frequently satisfied equation. Since every solution to $a + b + c = N$ is a solution to $a + b + c = 0$ mod n, A, B, C must be equilateral trapezoid-free when viewed as subsets of $\{0, \dots, N\}$. This shows the other direction of (2).

Finally, (3) follows from (1) and (2). ◀

▶ **Theorem 27.** *Suppose that one can achieve $\omega = 2$ via STPP constructions in the family of groups $\mathbb{Z}_q^\ell$, q a prime power. Then there exists a constant $c > 0$ such that $\mathrm{val}(\mathbb{Z}_n) \geq \Omega(n^{1+c})$.*

Proof. By Corollary 13 and Corollary 14, any STPP construction with sets X_i, Y_i, Z_i satisfies $\sum |X_i||Y_i| \leq (q/C)^\ell$ (we choose the X and Y sets without loss of generality) where C is an absolute constant. By Hölder's inequality, $\sum(|X_i||Y_i||Z_i|)^{2/3} \leq q^{2\ell/3}(q/C)^{\ell/3} = (q/C^{1/3})^\ell$. If we can obtain $\omega < 3 - \alpha$ via Theorem 6, then

$$
\begin{aligned}
q^\ell < \sum(|X_i||Y_i||Z_i|)^{2/3 \cdot \alpha + (1-\alpha)} &= \sum(|X_i||Y_i||Z_i|)^{2/3 \cdot \alpha}(|X_i||Y_i||Z_i|)^{1-\alpha} \\
&\leq \left(\sum(|X_i||Y_i||Z_i|)^{2/3}\right)^\alpha\left(\sum|X_i||Y_i||Z_i|\right)^{1-\alpha} \\
&\leq (q/C^{1/3})^{\alpha\ell}\mathrm{val}(G)^{1-\alpha}
\end{aligned}
$$

so $\mathrm{val}(G) > q^\ell(C^{\alpha/3(1-\alpha)})^\ell$. By choosing α sufficiently close to 1, $\mathrm{val}(G) > q^\ell 4^\ell$. By taking k-fold products of the sets defining the STPP constructions (using that products of STPPs are STPPs [11, Lemma 5.4]), we find that $\mathrm{val}(\mathbb{Z}_q^{k\ell}) > (4q)^{k\ell}$ for all k.

Let $N = k\ell$. Consider the embedding $\varphi : \mathbb{Z}_q^N \to \mathbb{Z}_{(3q)^N}$ defined by $\varphi(x_1, \dots, x_N) = x_1 + x_2 3q + \dots + x_n(3q)^{N-1}$. Since $\sum y_i(3q)^{i-1}$ has a unique such expression in $\mathbb{Z}_{(3q)^N}$ when $y_i < 3q$, it follows that that

$$
a_1 + a_2 + a_3 \neq a_4 + a_5 + a_6 \implies \varphi(a_1) + \varphi(a_2) + \varphi(a_3) \neq \varphi(a_4) + \varphi(a_5) + \varphi(a_6).
$$

Hence the image of an STPP under φ is an STPP inside of $\mathbb{Z}_{(3q)^N}$, so $\mathrm{val}(\mathbb{Z}_{(3q)^N}) > (4q)^N$. Because this holds for some particular q and all $N = k\ell$, by part (3) of Proposition 26 the theorem follows. ◀

▶ **Corollary 28.** *Suppose that there is a family of STPP constructions obtaining $\omega = 2$ in a family of abelian groups with a bounded number of direct factors. Then there exists a constant $c > 0$ such that $\mathrm{val}(\mathbb{Z}_n) \geq \Omega(n^{1+c})$.*

Proof. Suppose we have a family of STPP construction in groups of the form $G = \mathbb{Z}_{m_1} \times \dots \times \mathbb{Z}_{m_\ell}$, with ℓ fixed. We can then obtain an STPP construction in $\mathbb{Z}_p^\ell$, where p is the smallest prime greater than $\max_{i \in k} 3m_i$, by taking the image of this STPP under the map sending $(x_1, \dots, x_k) \to x_1 + x_2 p + \dots + x_k p^{k-1}$. As k is fixed, it follows from Bertrand's postulate that $p^k \leq O(|G|)$. The inequality Theorem 6 then implies that one can also obtain $\omega = 2$ in the family of groups $\mathbb{Z}_p^\ell$, so we conclude by Theorem 27. ◀

▶ **Remark 29.** Although we expect that Theorem 27 is true when the hypothesis is extended to arbitrary abelian groups, we do not know how to generalize to e.g. $\mathbb{Z}_n^\ell$ for arbitrary n. This is due to the fact that better bounds on the size of 3-matchings in cyclic groups with prime power modului are known than for general moduli (compare Theorems A and A' in [8]). To the best of our knowledge, it is an open problem whether the known bounds for non-prime power moduli are tight. For prime power moduli, the known bounds are tight by [18].

Next we show that sufficiently strong simultaneous double product property constructions, which are known to prove $\omega < 2.48$ [11, Proposition 4.5], imply strong lower bounds on $\mathrm{val}(\mathbb{Z}_n)$. We thank Chris Umans for informing us of the fact that if Conjecture 9 is true, then it is true in cyclic groups, which motivated the following theorem.

▶ **Theorem 30.** *If Conjecture 9 is true, then for any $\varepsilon > 0$, $\mathrm{val}(\mathbb{Z}_n) \geq O(n^{4/3-\varepsilon})$.*

Proof. We begin by recalling how to turn an SDPP construction into an STPP construction [11, Section 6.2]. Let $S \subset \Delta_n$ be corner-free and of size $n^{2-o(1)}$. For all $v = (v_1, v_2, v_3) \in S$, define the following subsets of G^3:

$$A_v = A_{v_1} \times \{1\} \times B_{v_3},$$
$$B_v = B_{b_1} \times A_{v_2} \times \{1\},$$
$$C_v = \{1\} \times B_{v_2} \times A_{v_3}.$$

It can be verified that the sets $(A_v, B_v, C_v)_{v \in S}$ satisfy the STPP. Hence Conjecture 9 yields an STPP with $n^{2-o(1)}$ triples of sets of size $n^{2-o(1)}$, inside a group of size $n^{6-o(1)}$.

Now consider the map from $G^3 = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k}$, where $m_1 \leq m_2 \leq \cdots \leq m_k$, to $G' := \mathbb{Z}_{\prod_i 3m_i}$ sending $(x_1, \ldots, x_k)$ to $x_1 + (3m_1)x_2 + (3m_1)(3m_2)x_3 + \cdots$. First, the image of sets satisfying the STPP under this map still satisfy the STPP. This shows that $\mathrm{val}(G') > n^{2-o(1)} \cdot n^{3(2-o(1))} = n^{8-o(1)}$. Second, for all fixed $c > 0$ and $\ell \in \mathbb{N}$, G^3 cannot contain a subgroup of size $|G^3|^c$ generated by elements of order at most ℓ by [8, Proposition 4.2]. Hence the number of m_i's which are at most ℓ is at most $\log_2(|G^3|^c)$. The number of m_i's which are greater than ℓ is trivially less than $\log_\ell |G^3|$. So,

$$|G'| = \prod_{m_i \leq \ell} 3m_i \prod_{m_i > \ell} 3m_i \leq 3^{\log_2(|G^3|^c) + \log_\ell |G^3|} \cdot |G^3|.$$

By taking c sufficiently small and ℓ sufficiently large, this is at most $n^{6+\delta}$ for any desired $\delta > 0$. The claimed bound follows. ◀

Note that here there is no restriction on the family of abelian groups in consideration, unlike there was in the previous theorem.

4.1 Relaxations of $\mathrm{val}(\mathbb{Z}_n)$

In this section we explore some strengthenings of Conjecture 24 which may be easier to understand. We start by discussing an over-strengthening of Conjecture 24 which *cannot* give any barriers. We then discuss a few strengthenings for which our knowledge is embarrassingly bad, including the notions of skew-corner free sets from the introduction.

Considerations of the proof of the $n^{3/2}$ upper bound of Proposition 19 reveal that it actually held for a (possibly) much weaker problem, where one only requires that the expected number of solutions of one of the three systems of two equations in Definition 17 is at most 1. We begin by noting that this upper bound is essentially best-possible for this weakened problem. In other words, one cannot hope to prove Conjecture 24 via an "asymmetric" averaging argument.

▶ **Proposition 31.** *There exist $A, B, C \subseteq \mathbb{Z}_n$ such that*

$$\mathbb{E}_{a' \in A, b' \in B} \left[\#\{(a, b, c) : 0 = a' + b + c = a + b' + c\} \right] \leq 1$$

and there are $n^{3/2-o(1)}$ solutions to the equation $a + b + c = 0$ with $a \in A, b \in B, c \in C$.

Proof. Let $r(A, B, c)$ denote the number of representations of c as $a + b$. First note that the proposition is equivalent to the statement that $\sum_{c \in C} r(A, B, -c)^2 \leq |A||B|$ and $\sum_{c \in C} r(A, B, -c) = n^{3/2-o(1)}$.

Let $S \subset [n]$ be 3AP-free and of size $n^{1-o(1)}$. Consider the sets

$$A = B = [3n^2, 4n^2] \cup \bigcup_{x \in S} [xn, xn + n/2], C = -\{2xn + y : x \in S, y \in [n]\}$$

regarded as subsets of $\mathbb{Z}_{100n^2}$. By definition, for any $x \in S$ and $y \in [n]$, $-(2xn+y) = c \in C$. If we have any representation $-c = a+b$, then $a, b < 3n^2$. So we have $a = x_1 n + y_1, b = x_2 n + y_1$ with $x_1, x_2 \in S$ and $1 \leq y_1, y_2 \leq n$. So $(x_1+x_2)n+(y_1+y_2) = 2xn+y$, and then we are forced to have $x_1 + x_2 = 2x$ and $y_1 + y_2 = y$. But because S is 3AP-free, we must have $x_1 = x_2 = x$. Hence $r(A, B, -c)$ is exactly the number of solutions to $y = y_1 + y_2$ with $y_1, y_2 \in [n]$, which is $\Omega(n)$ for $\Omega(n)$ choices of $y \in [n]$. Hence $\sum_{c \in C} r(A, B, -c) = \Theta(|S|n^2) = n^{3-o(1)}$. Also, we have that $\sum_{c \in C} r(A, B, -c)^2 = n^{4-o(1)} < |A||B| = \Theta(n^4)$, and we are done. ◄

Can one find a construction achieving $n^{3/2-o(1)}$ for the averaging version of Definition 17 that involves all three systems of equations? That is:

▶ **Question 32.** *What is the maximum over all $A, B, C \subseteq \mathbb{Z}_n$ satisfying*

$$\mathbb{E}_{a' \in A, b' \in B} \left[\#\{(a, b, c) : 0 = a' + b + c = a + b' + c\} \right] \leq 1,$$
$$\mathbb{E}_{a' \in A, c' \in C} \left[\#\{(a, b, c) : 0 = a' + b + c = a + b + c'\} \right] \leq 1,$$
$$\mathbb{E}_{b' \in B, c' \in C} \left[\#\{(a, b, c) : 0 = a + b' + c = a + b + c'\} \right] \leq 1,$$

of the number of solutions to $a + b + c = 0$?

There are a number of relaxations of the quantity val(n) for which we know basically nothing. A first relaxation that still seems very stringent is that of a *triforce-free* triple, defined as follows.

▶ **Definition 33.** *Let $A, B, C \subseteq \{0, \ldots, n\}$. We say that (A, B, C) is triforce-free if there is no solution to*

$$a + b + c' = a + b' + c = a' + b + c = n$$

with $a \neq a', b \neq b', c \neq c'$. We write val$(\blacktriangle, n)$ *for the maximum over all such A, B, C of the number of solutions to $a + b + c = n$.*

This condition just says that $\{(a, b, c) \in A \times B \times C : a+b+c = n\} \subseteq \Delta_{n+1}$ is corner-free. Equivalently, (A, B, C) is triforce-free if the hypergraph with parts A, B, C and triangles between any triples summing to n does not contain the triforce hypergraph (the second hypergraph in Figure 3). As every equilateral trapezoid-free triple of sets also has this property, we have the following.

▶ **Proposition 34.** val$(\blacktriangle, n) \geq$ val(n).

Here is an even weaker notion than that of being triforce-free. We thank Ryan O'Donnell for suggesting this definition.

▶ **Definition 35.** *We call $S \subseteq \Delta_n$ skew-corner free if for $(a, b, c), (a, b', c') \in S$, it holds that $(a + b - b', b'', c'') \notin S$ for all b'', c'', and this remains true after any permutation of the coordinates of S.*

Pictorially, this says that for any two points lying on an axis-aligned line in Δ_n, the parallel line passing through a third point that would form a corner with these two points must contain no points. As Definition 33 yields corner-free subsets of Δ_n obtained by deleting axis-aligned lines, it follows that this is a relaxation of being triforce-free. More formally, we have the following.

▶ **Proposition 36.** *The largest skew-corner free subset of Δ_{n+1} is at least* $\mathrm{val}(\blacktriangle, n)$.

Proof. Suppose that $A, B, C \subset \{0, \ldots, n\}$ satisfy the conditions of Definition 33, and let $S = \{(a, b, c) \subset A \times B \times C : a + b + c = n\} \subseteq \Delta_{n+1}$. Suppose for contradiction that $(a, b, c), (a, b', c') \in S$ and $(a - b + b', b'', c'') \in S$. Since $a - b + b' \in A, b \in B, c' \in C$ and $(a - b + b') + b + c' = a + b' + c' = n$, it follows that $(a - b + b', b, c') \in S$. But this is impossible: the three solutions $a + b' + c' = n, a + b + c = n, (a - b + b') + b + c' = n$ violate Definition 33. One reasons similarly about other permutations of coordinates. ◀

The best lower bound that we know on the size of the largest skew-corner free subset of Δ_n is $\Omega(n)$; n is obtained trivially by taking one line on the side of Δ_n, and it is not hard to improve this to $3n/2$. We have found examples exceeding these bounds with computer search (see Figure 2).

If we weaken Definition 35 by dropping the requirement that the condition holds for all permutations of coordinates, we are led to the following notion.

▶ **Definition 37.** *We say $S \subset [n]^2$ is skew corner-free if it contains no configuration* $(x, y), (x, y + d), (x + d, y')$ *with $d \neq 0$.*

▶ **Proposition 38.** *The largest skew corner-free subset of $[n]^2$ is at least as big as the largest skew corner-free subset of Δ_n.*

Proof. Given a skew corner-free set $S \subseteq \Delta_n$, let S' be its projection onto the first two coordinates. This is a subset of $\{0, \ldots, n-1\}^2$ of size $|S|$. By definition, it contains no points $(a, b), (a, b'), (a + b - b', b'')$. By shifting each point by $(1, 1)$ we obtain a subset of $[n]^2$ with this property. ◀

As a consequence, we have Theorems 3 and 4.

Proof of Theorem 3 and Theorem 4. By Theorem 27, if $\omega = 2$ via STPP constructions in $\mathbb{Z}_q^\ell$, then $\mathrm{val}(\mathbb{Z}_n) \geq \Omega(n^{1+c})$. By Proposition 26, $\mathrm{val}(\mathbb{Z}_n) = \Theta(\mathrm{val}(n))$, and by Propositions 36 and 38, $\mathrm{val}(n)$ is at most the size of the largest skew corner-free subset of $[n]^2$. This proves Theorem 3. One similarly concludes Theorem 4 by using Theorem 30. ◀

We have the following nontrivial lower bound for this relaxed problem, due to a Math-Overflow answer of Fedor Petrov [19].

▶ **Proposition 39.** *There is a skew corner-free subset of $[n]^2$ of size $\Omega(n \log n / \sqrt{\log \log n})$.*

Proof. $A \subseteq [n]$ is called *primitive* if for all $a \neq a' \in A, a \nmid a'$. It is easily seen that if A is primitive then the set of points $(a, ka) \subseteq [n]^2$ for all $k \leq n/a$ avoids the forbidden configurations. This gives a subset of size $n \sum_{a \in A} 1/a$. At the same time, there exists a $c > 0$ and a primitive set A where $\sum_{a \in A} 1/a > c \log n / (\log \log n)^{1/2}$ [14]. We note that this is best-possible, matching (up to the constant) an upper bound on $\sum_{a \in A} 1/a$ for primitive A due to Behrend [6]. ◀

This construction breaks when we strengthen the definition of skew corner-freeness in $[n]^2$ to forbid skew corners with two points parallel to the x axis. This corresponds to the following notion.

▶ **Definition 40.** *We say $S \subset [n]^2$ is bi-skew corner-free if it contains no configurations $(x, y), (x, y + d), (x + d, y')$ or $(x, y), (x + d, y), (x', y + d)$, with $d \neq 0$.*

As far as we know, it is possible that the largest bi-skew corner-free subset of $[n]^2$ has size $O(n)$.

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