From Local to Global Optimality in Concurrent Parity Games

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Abstract

We study two-player games on finite graphs. Turn-based games have many nice properties, but concurrent games are harder to tame: e.g. turn-based stochastic parity games have positional optimal strategies, whereas even basic concurrent reachability games may fail to have optimal strategies. We study concurrent stochastic parity games, and identify a local structural condition that, when satisfied at each state, guarantees existence of positional optimal strategies for both players.

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1 Introduction

Two-player games played on finite graphs have been a helpful model in areas of computer science. In such games, states/vertices of the graph are colored; the actions of the players induce an infinite path in the graph, thus inducing an infinite sequence of colors. Who wins depends on the color sequence. These games can be turn-based, i.e. at each state a unique player chooses an outgoing edge leading to a probability distribution over successor states, or concurrent, i.e. at each state, the combination of one action per player determines the probability distribution over successor states. In such stochastic settings, Player A wants to maximize her probability to win, and Player B to minimize the very same probability.

We study the above games in the case of parity objective: colors are natural numbers, and a sequence is winning for Player A iff the maximal color seen infinitely often is even. This objective has been well-studied in connection with model-checking. The turn-based version of these games has nice properties involving deterministic positional strategies, where " positional" means that the played action depends only on the current state: with only deterministic probability distributions over successor states, either of the players has a winning such strategy [15]; with arbitrary probability distributions over successor states, both players have optimal such strategies [16, 7]. Note that in concurrent parity games, positional optimal strategies for distinct starting states yield one positional strategy optimal for all states uniformly. So we omit the word " uniform" in this paper.

The above properties and others break in basic concurrent games except in safety games, as recorded in Table 1, where safety and reachability are special cases of (co-)Büchi, and parity subsumes (co-)Büchi. Büchi games may not have optimal strategies but when they

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Table 1 Memory status of (almost-)optimal strategies in concurrent games. The second column says for each objective whether optimal strategies always exist or not; the third column gives the nature (positional or infinite memory) of optimal strategies when such optimal strategies exist; the fourth column gives the nature of ε-optimal strategies; the last column gives the nature of subgame-optimal strategies, a refinement of optimal strategies. This shows the diversity of memory requirements for the various objectives listed on the left.

Objectives	$\exists$ opt strat?	opt	ε-opt	SubG-opt
Safety	always [9]	pos. [9]	pos. [9]	pos. [9]
Reach.	not always [11]	pos. [2]	pos. [10]	pos. [2]
Büchi	not always	pos. [3]	∞ [8]	pos. [3]
Co-B.	not always	∞ [3]	pos. [6]	pos. [5]
Parity	not always	∞ [8]	∞ [8]	∞ [8]

do, they have positional ones; co-Büchi games may not have optimal strategies, and they may have only infinite-memory ones; and the other way around for ε-optimal strategies, hence incomparable difficulty between Büchi and co-Büchi. This shows that concurrent parity is strictly harder than (co-)Büchi, since parity games may have only infinite-memory (ε-)optimal strategies. (See also the column on subgame optimal strategies.)

Nevertheless, concurrent games' poor behavior in general should not deter us from studying them, as many complex systems are inherently concurrent. See [12] for further arguments. Continuing a recent line of research [1, 2, 3] we seek local structural good behaviors that scale up to the whole game.

The following key property still holds in parity concurrent games, by determinacy of Blackwell games [13]: each state has a value $u \in [0, 1]$, i.e. for all $\varepsilon > 0$, Player A has an ε-optimal strategy for plays starting in the state, i.e. guaranteeing winning probability at least $u - \varepsilon$ to win, regardless of Player B's strategy; and likewise Player B can guarantee that Player A wins with probability at most $u + \varepsilon$. Here we are interested in strategies that are optimal, i.e. that realize exactly the value u, and that are positional. They do not exist in general (see [2, Figure 2]), but our main result is a transfer property from local to global positional optimality, a weak version being the next theorem; the terminology that it uses is explained afterwards.

▶ **Theorem 1.** *If at every state the induced game form is positionally optimizable, both players have positional optimal strategies in the game.*

A game form is a map from pairs of actions to probability distributions of successor states, see Fig. 1 to the left, or [1, p. 5] for more examples. So each state induces a game form. Given a game form $\mathcal{F}$, an $\mathcal{F}$-game has a unique non-trivial state which induces $\mathcal{F}$ and the other player states are of two kinds: first kind, they loop back to the non-trivial state; second kind, they are not colored, but they have an explicit value in $[0, 1]$, and the game stops there. Moreover Player A prefers higher explicit values. See Fig. 1 or Fig. 5. Finally, $\mathcal{F}$ is called *positionally optimizable* if both players have positional optimal strategies in all $\mathcal{F}$-games, which we show to be decidable.

The proof of Theorem 1 involves an extraction of an *environment function*, which gives for each state a summary of some local information sufficient to globally play optimally in the game; the summary of some state q is a (small) $\mathcal{F}$-game, where $\mathcal{F}$ is the original game form of state q. The extraction of this environment function is made by analyzing in an appropriate order the immediate neighbors of the states, and propagating gathered information further away.

As a corollary of Thm. 1, we among other things recover the known result that positional strategies are sufficient to play optimally in turn-based stochastic games [16, 7], since turn-based game forms are positionally optimizable. Let us rephrase and strengthen Thm. 1: if the induced game forms of a game behave well in all (small) $\mathcal{F}$-games, they behave well in all larger games; conversely, by definition an arena inducing a poorly-behaved game form yields at least one poorly-behaved small game. Let us highlight two benefits of Thm. 1: First, designing a game using only well-behaved components, i.e. game forms, ensures the existence of positional optimal strategies, which was our main purpose. Second, it provides a local, structural criterion for games to have positional optimal strategies.

An extended version of this paper with all the technical details can be found in [4].

2 Preliminaries and game forms

If Q is a non-empty set, we denote by Q^* (resp. Q^+, Q^ω) the set of finite (resp. non-empty finite, infinite) sequences of Q. A *(discrete probability) distribution* over a set Q is a function $\mu : Q \to [0,1]$ with a finite support $\mathsf{Sp}(\mu) := \{x \in Q \mid \mu(x) > 0\}$, such that $\sum_{x \in \mathsf{Sp}(\mu)} \mu(x) = 1$. The distribution μ is *deterministic* if its support is a singleton. For all $S \subseteq Q$, we let $\mu[S] := \sum_{x \in S} \mu(x)$. The set of distributions over the set Q is denoted $\mathcal{D}(Q)$.

For all $i \leq j \in \mathbb{N}$, we write $[\![i,j]\!]$ for the set $\{k \in \mathbb{N} \mid i \leq k \leq j\}$. This set is typed in the sense that these are seen as integers and not reals numbers, so that we will be able to consider the disjoint union of $[0,1]$ with such a set of integers which may include 0 or 1. For all finite sets $S \subseteq \mathbb{N}$, we let $\mathsf{Even}(S)$ (resp. $\mathsf{Odd}(S)$) be the smallest even (resp. odd) integer that is greater than or equal to all elements in S.

We recall the definition of game forms and of games in normal forms.

▶ **Definition 2** (Game form and game in normal form)**.** *Let* O *be a non-empty set of outcomes. A* game form *(*GF *for short) on* D *is a tuple* $\mathcal{F} = \langle \mathsf{Act}_\mathsf{A}, \mathsf{Act}_\mathsf{B}, \mathsf{O}, \varrho \rangle$ *where* Act_A *(resp.* Act_B*) is the non-empty finite set of actions available to Player* A *(resp.* B*) and* $\varrho : \mathsf{Act}_\mathsf{A} \times \mathsf{Act}_\mathsf{B} \to \mathcal{D}(\mathsf{O})$ *maps each pair of actions to a distribution over the outcomes. We denote by* $\mathsf{Form}(\mathsf{O})$ *the set of game forms on* O*. A Player-*A *(resp. Player-*B*) game form* $\mathcal{F}$ *is such that* $|\mathsf{Act}_\mathsf{B}| = 1$ *(resp.* $|\mathsf{Act}_\mathsf{A}| = 1$*). A game form is trivial if* $|\mathsf{Act}_\mathsf{B}| = 1$ *and* $|\mathsf{Act}_\mathsf{A}| = 1$*.*

When $\mathsf{O} = [0,1]$*, we say that* $\mathcal{F}$ *is a* game in normal form*. For a valuation* $v : \mathsf{D} \to [0,1]$*,* $\langle \mathcal{F}, v \rangle$ *denotes the game in normal form* $\langle \mathsf{Act}_\mathsf{A}, \mathsf{Act}_\mathsf{B}, [0,1], v \circ \varrho \rangle$ *induced from* $\mathcal{F}$ *by* v*.*

An example of a game form is depicted on the left of Fig. 1 (page 7) where the actions available to Player A are the rows and the actions available to Player B are the columns. Strategies available to Player-A (resp. B) are then the probability distributions over their respective sets of actions. In a game in normal form, Player A tries to maximize the outcome, whereas Player B tries to minimize it.

▶ **Definition 3** (Outcome and value of a game in normal form)**.** *Let* $\mathcal{F} = \langle \mathsf{Act}_\mathsf{A}, \mathsf{Act}_\mathsf{B}, [0,1], \varrho \rangle$ *be a game in normal form. For* $\mathsf{C} \in \{\mathsf{A}, \mathsf{B}\}$*, the set of strategies of Player* C *is* $\mathcal{D}(\mathsf{Act}_\mathsf{C})$ *and is thereafter denoted* $\Sigma_\mathsf{C}(\mathcal{F})$*. For a pair of strategies* $(\sigma_\mathsf{A}, \sigma_\mathsf{B}) \in \Sigma_\mathsf{A}(\mathcal{F}) \times \Sigma_\mathsf{B}(\mathcal{F})$*, their outcome in* $\mathcal{F}$ *is* $\mathsf{out}_\mathcal{F}(\sigma_\mathsf{A}, \sigma_\mathsf{B}) := \sum_{a \in \mathsf{Act}_\mathsf{A}} \sum_{b \in \mathsf{Act}_\mathsf{B}} \sigma_\mathsf{A}(a) \cdot \sigma_\mathsf{B}(b) \cdot \varrho(a,b) \in [0,1]$*.*

Let $\sigma_\mathsf{A} \in \Sigma_\mathsf{A}(\mathcal{F})$ *be a Player-*A *strategy. Its* value *is* $\mathsf{val}_\mathcal{F}(\sigma_\mathsf{A}) := \inf_{\sigma_\mathsf{B} \in \Sigma_\mathsf{B}(\mathcal{F})} \mathsf{out}_\mathcal{F}(\sigma_\mathsf{A}, \sigma_\mathsf{B})$*; and dually for Player* B*. When* $\sup_{\sigma_\mathsf{A} \in \Sigma_\mathsf{A}(\mathcal{F})} \mathsf{val}_\mathcal{F}(\sigma_\mathsf{A}) = \inf_{\sigma_\mathsf{B} \in \Sigma_\mathsf{B}(\mathcal{F})} \mathsf{val}_\mathcal{F}(\sigma_\mathsf{B})$*, it defines the* value *of the game* $\mathcal{F}$*, denoted* $\mathsf{val}_\mathcal{F}$*. If* $\mathsf{val}_\mathcal{F}(\sigma_\mathsf{A}) = \mathsf{val}_\mathcal{F}$*, the strategy* σ_A *is said to be* optimal *for Player* A*. This is defined analogously for Player* B*.*

Since the sets of actions are finite, Von Neumann's minimax theorem [14] ensures the existence of a value and of optimal strategies for both players in any game in normal form.

In the following, strategies in game forms will be called GF-strategies in order not to confuse them with strategies in concurrent games (on graphs).

3 Concurrent games

3.1 Concurrent arenas and games

▶ **Definition 4** (Finite stochastic concurrent arena). *A finite concurrent arena $\mathcal{C}$ is a tuple $\langle Q, \mathsf{F}\rangle$ where Q is a non-empty finite set of states and $\mathsf{F} : Q \to \mathsf{Form}(\mathsf{D})$ maps each state to its induced game form, which describes the interaction of the players at this state.*

In the following, the arena $\mathcal{C}$ will refer to a tuple $\langle Q, \mathsf{F}\rangle$, unless otherwise stated. In this paper, we focus on (max) parity objectives: given a coloring function, the goal of Player A is that the maximum of the colors visited infinitely often is even.

▶ **Definition 5** (Parity game). *Let $\mathsf{col} : Q \to \mathbb{N}$ be a coloring function. It induces the parity objective $W(\mathsf{col}) \subseteq Q^\omega$ defined by $W(\mathsf{col}) := \{\rho \in Q^\omega \mid \max(\mathsf{col}(\rho)_\infty) \text{ is even}\}$ where $\mathsf{col}(\rho)_\infty := \{k \in \mathbb{N} \mid \forall i \in \mathbb{N}, \exists j \geq i, \mathsf{col}(\rho)_j = k\} \neq \emptyset$ denotes the set of colors seen infinitely often along ρ. A parity game $\mathcal{G} = \langle \mathcal{C}, \mathsf{col}\rangle$ is a pair formed of a concurrent arena $\mathcal{C}$ and a coloring function $\mathsf{col} : Q \to \mathbb{N}$.*

We fix a parity game $\mathcal{G} = \langle \mathcal{C}, \mathsf{col}\rangle$ for the rest of this section. In such a game, strategies map the history of the game (i.e. the finite sequence of states visited so far) to a GF-strategy in the game form corresponding to the current state of the game.

▶ **Definition 6** (Strategies). *A strategy for Player A is a function $\mathsf{s_A} : \bigcup_{q \in Q}(Q^* \cdot q \to \Sigma_\mathsf{A}(\mathsf{F}(q)))$. It is positional if for all $q \in Q$, there is a GF-strategy $\sigma_\mathsf{A}^q \in \Sigma_\mathsf{A}(\mathsf{F}(q))$ such that, for all $\pi = \rho \cdot q \in Q^+$: $\mathsf{s_A}(\pi) = \sigma_A^q$. In that case, the strategy $\mathsf{s_A}$ is said to be defined by $(\sigma_\mathsf{A}^q)_{q \in Q}$. We denote by $\mathsf{S}_\mathcal{C}^\mathsf{A}$ and $\mathsf{PS}_\mathcal{C}^\mathsf{A}$ the set of all strategies and positional strategies respectively in arena $\mathcal{C}$ for Player A. A strategy $\mathsf{s_A}$ is deterministic if for all $\rho \in Q^+$, $\mathsf{s_A}(\rho)$ is deterministic. The definitions are analogous for Player B.*

Unlike deterministic games with deterministic strategies, the outcome of a game, given two strategies (one for each Player), is not a single play but rather a distribution over plays. To formalize this, we first define the probability to go from a state q to a state q' given two GF-strategies in a game form $\mathsf{F}(q)$.

▶ **Definition 7** (Probability transition). *Given states $q, q' \in Q$ and two strategies $(\sigma_\mathsf{A}, \sigma_\mathsf{B}) \in \Sigma_\mathsf{A}(\mathsf{F}(q)) \times \Sigma_\mathsf{B}(\mathsf{F}(q))$ the probability to go from q to q' if the players play, in q, σ_A and σ_B, is: $\mathbb{P}^{\sigma_\mathsf{A},\sigma_\mathsf{B}}(q, q') := \mathsf{out}_{\langle \mathsf{F}(q), \mathbb{1}_{q'}\rangle}(\sigma_\mathsf{A}, \sigma_\mathsf{B})$, where $\mathbb{1}_{q'} : Q \to [0, 1]$ is the indicator function such that, for all $q'' \in Q$, we have $\mathbb{1}_{q'}(q'') = 1$ if and only if $q'' = q'$.*

We now define the probability of occurrence of finite paths, and consequently of any Borel set, given a strategy per player.

▶ **Definition 8** (Probability distribution given two strategies). *Let $(\mathsf{s_A}, \mathsf{s_B}) \in \mathsf{S}_\mathcal{C}^\mathsf{A} \times \mathsf{S}_\mathcal{C}^\mathsf{B}$ be two arbitrary strategies. We denote by $\mathbb{P}^{\mathsf{s_A},\mathsf{s_B}} : Q^+ \to \mathcal{D}(Q)$ the function giving the probability distribution over the next state of the arena given the sequence of states already seen. That is, for all finite path $\pi = \pi_0 \ldots \pi_n \in Q^+$ and $q \in Q$, we have: $\mathbb{P}^{\mathsf{s_A},\mathsf{s_B}}(\pi)[q] := \mathbb{P}^{\mathsf{s_A}(\pi),\mathsf{s_B}(\pi)}(\pi_n, q)$. The probability of a finite path $\pi = \pi_0 \cdots \pi_n \in Q^+$ from a state $q_0 \in Q$ with the pair of strategies $(\mathsf{s_A}, \mathsf{s_B})$ is then equal to $\mathbb{P}^{\mathcal{C},q_0}_{\mathsf{s_A},\mathsf{s_B}}(\pi) := \Pi_{i=0}^{n-1} \mathbb{P}^{\mathsf{s_A},\mathsf{s_B}}(\pi_{\leq i})[\pi_{i+1}]$ if $\pi_0 = q_0$ and 0 otherwise. The probability of a cylinder set $\mathsf{Cyl}(\pi) := \{\pi \cdot \rho \mid \rho \in Q^\omega\}$ is $\mathbb{P}^{\mathcal{C},q_0}_{\mathsf{s_A},\mathsf{s_B}}[\mathsf{Cyl}(\pi)] := \mathbb{P}^{\mathsf{s_A},\mathsf{s_B}}(\pi)$ for any finite path $\pi \in Q^*$. This induces the probability measure over Borel sets in the usual way. We denote by $\mathbb{P}^{\mathcal{C},q_0}_{\mathsf{s_A},\mathsf{s_B}}$ this probability measure, mapping each Borel set to a value in $[0, 1]$.*

The values of strategies and of the game follow.

▶ **Definition 9** (Value of strategies and of the game). *Let $\mathsf{s_A} \in \mathsf{S}_\mathcal{C}^\mathsf{A}$ be a Player-*A *strategy. The vector $\chi_\mathcal{G}[\mathsf{s_A}] : Q \to [0,1]$ giving the value of the strategy $\mathsf{s_A}$ is such that, for all $q_0 \in Q$, we have $\chi_\mathcal{G}[\mathsf{s_A}](q_0) := \inf_{\mathsf{s_B} \in \mathsf{S}_\mathcal{C}^\mathsf{B}} \mathbb{P}^{\mathcal{C},q_0}_{\mathsf{s_A},\mathsf{s_B}}[W(\mathsf{col})]$. The vector $\chi_\mathcal{G}[\mathsf{A}] : Q \to [0,1]$ giving the value for Player* A *is such that, for all $q_0 \in Q$, we have $\chi_\mathcal{G}[\mathsf{A}](q_0) := \sup_{\mathsf{s_A} \in \mathsf{S}_\mathcal{C}^\mathsf{A}} \chi_\mathcal{G}[\mathsf{s_A}](q_0)$. The vector $\chi_\mathcal{G}[\mathsf{B}] : Q \to [0,1]$ giving the value of the game for Player* B *is defined symmetrically.*

By Martin's result on the determinacy of Blackwell games [13]: $\chi_\mathcal{G}[\mathsf{A}] = \chi_\mathcal{G}[\mathsf{B}]$, which defines the value *of the game: $\chi_\mathcal{G} := \chi_\mathcal{G}[\mathsf{A}] = \chi_\mathcal{G}[\mathsf{B}]$. A Player-*A *strategy $\mathsf{s_A}$ such that $\chi_\mathcal{G} := \chi_\mathcal{G}[\mathsf{s_A}]$ is* optimal *(and similarly for Player* B*).*

Note that optimal strategies may not exist in general; when they exist they can be arbitrarily complex; see the table page 2.

Finally, for convenience, we extend our formalism by considering stopping states with output values, i.e. states that, when visited, immediately stop the game and induce a specific value in $[0,1]$. The fact that the value of a stopping state q is set to be u is denoted $\mathsf{val}(q) \leftarrow u$. Stopping states can be encoded by simple gadgets in our formalism, they will be depicted as dashed states.

3.2 Markov chains and sufficient condition for optimality

In this subsection, we give a condition on positional strategies to be optimal in a parity game. First, we introduce the notions of Markov chain and bottom strongly connected component.

▶ **Definition 10** (Markov chain, BSCC). *A* Markov chain *$\mathcal{M}$ is a pair $\mathcal{M} = \langle Q, \mathbb{P} \rangle$ where Q is a finite set of states and $\mathbb{P} : Q \to \mathcal{D}(Q)$. A* bottom strongly connected component *(BSCC for short) $H \subseteq Q$ is a subset of states such that the underlying graph of H is strongly connected (w.r.t. edges given by positive probability transitions from states to states) and H cannot be exited: for all $q \in H$ and $q' \in Q$, $\mathbb{P}(q)(q') > 0$ implies $q' \in H$.*

Two positional strategies (one per player) in a concurrent arena not only induce a probability measure on infinite sequences of states, but also a Markov chain, whose graph is a subgraph of the arena. If we only fix a positional strategy for one of the players, we will consider the set of BSCCs that are compatible with that strategy in the following sense.

▶ **Definition 11** (Induced Markov chains and BSCCs compatible with a strategy). *Let* s *be a positional strategy for one of the players. For every positional and deterministic strategy* s' *for the other player, we denote by $\mathcal{M}_{\mathsf{s},\mathsf{s}'} = \langle Q, \mathbb{P}^{\mathsf{s},\mathsf{s}'} \rangle$ the Markov chain* induced *by* s *and* s'*, and by H_s the set of BSCCs* compatible *with* s*, i.e. the BSCCs of some Markov chain $\mathcal{M}_{\mathsf{s},\mathsf{s}'}$. A BSCC $H \in H_\mathsf{s}$ is* even-colored *if $\max \mathsf{col}[H]$ is even. Otherwise, it is* odd-colored.

We define three properties relating positional strategies and valuations of the states. A Player-A strategy dominates a valuation v if, regardless of what the other player plays, the value of every state is at most the expected value of its successors. Further, a Player-A strategy parity dominates the valuation v if in addition all the BSCCs compatible with it are even-colored. Finally, a Player-A strategy guarantees the valuation v if, from every state, the value of the strategy is at least the value of the states w.r.t. v. In particular, if a strategy guarantees the valuation $\chi_\mathcal{G} : Q \to [0,1]$, then it is optimal (by definition).

▶ **Definition 12** ((Parity) Domination, Guarantees). *Let $v : Q \to [0,1]$ be a valuation of the states. Let $\mathsf{s_A} \in \mathsf{PS}_\mathcal{C}^\mathsf{A}$ be a positional Player-*A *strategy. This strategy $\mathsf{s_A}$:*

- dominates v *if for all $q \in Q$, $v(q) \leq \mathsf{val}_{\langle \mathsf{F}(q), v \rangle}(\mathsf{s_A}(q))$;*

- parity dominates v *if it dominates* v *and all BSCCs* H *compatible with* $\mathsf{s_A}$ *s.t.* $\min v[H] > 0$*;*
- guarantees v *if for all* $q \in Q$*,* $v(q) \leq \chi_{\mathcal{G}}[\mathsf{s_A}](q)$*.*

The definitions are symmetrical for a Player-B *positional strategy* $\mathsf{s_B} \in \mathsf{PS}^{\mathsf{B}}_{\mathcal{C}}$*.*

As stated in Proposition 13 below, if a strategy parity dominates a valuation, then it also guarantees it.

▶ **Proposition 13.** *Let* $\mathsf{s_A} \in \mathsf{PS}^{\mathsf{A}}_{\mathcal{C}}$ *be a positional Player-*A *strategy and* $v : Q \to [0,1]$ *be a valuation. If* $\mathsf{s_A}$ *dominates* v*, then for all BSCC* $H \in H_{\mathsf{s_A}}$*, there is some* $v_H \in [0,1]$ *such that* $v[H] = \{v_H\}$*. If in addition* $\mathsf{s_A}$ *parity dominates* v*, it also guarantees* v*.*

In the remainder of this paper, we will be interested in showing that a strategy is optimal. We will do so by establishing that it parity dominates the valuation $\chi_{\mathcal{G}} : Q \to [0,1]$. The benefit of parity domination is that, compared to optimality, it specifies more explicitly how the strategy behaves in a game. It is for instance used in Proposition 28 where optimal Player-A strategies are obtained by gluing together pieces of strategies that parity dominates some valuations.

4 Local environment and local game

The goal of this section is to define small parity games with a single non-trivial local interaction, which will enlighten game forms that should be used in parity games if we require positional optimal strategies. We first consider what (parity) environments on a given set of outcomes are. Informally, these environments tell a game form how it should view its outcomes: either as stopping states, or as colored states. The formal definition is given below.

▶ **Definition 14** (Parity environment and its induced parity game)**.** *Let* O *be a non-empty finite set of outcomes. An* environment E *on* O *is a tuple* $E := \langle c, e, p\rangle$ *where* $c, e \in \mathbb{N}$ *with* $c \leq e$ *and* $p : \mathsf{O} \to \{q_{\mathsf{init}}\} \uplus [\![0, e]\!] \uplus [0,1]$ *maps each outcome to what will be states in small* $\mathcal{F}$*-games.*

The size *of* E *w.r.t. Player* A *(resp.* B*) is* $\mathsf{Sz_A}(E) := \mathsf{Even}(e) - c$ *(resp.* $\mathsf{Sz_B}(E) := \mathsf{Odd}(e) - c$*). We denote by* $\mathsf{Env}(\mathsf{O})$ *the set of all environments on* D*.*

We can then consider the games induced by such an environment (along with a game form). Informally, given a game form $\mathcal{F} \in \mathsf{Form}(\mathsf{O})$ and a parity environment $E = \langle c, e, p\rangle \in \mathsf{Env}(\mathsf{O})$, we consider the small parity arena $\mathcal{C}_Y$ induced by $Y := (\mathsf{O}, \mathcal{F}, E)$ defined as follows: there is a single central state q_{init} whose local interaction is given by $\mathcal{F}$. The outcomes of $\mathcal{F}$ lead in $\mathcal{C}_Y$ to states in $\{q_{\mathsf{init}}\} \uplus \{k_i \mid i \in [\![0, e]\!]\} \uplus [0,1]$, as prescribed by p. All states in $[0,1]$ are stopping states and all states in $\{k_i \mid i \in [\![0, e]\!]\}$ are trivial and loop back to q_{init}. The small parity game $\mathcal{G}_Y$ that we consider is then obtained from the arena $\mathcal{C}_Y$ by considering a coloring function col that maps q_{init} to c and every state k_i to i. These small games correspond to the $\mathcal{F}$-games in the introduction. This is formally defined below.

▶ **Definition 15** (Parity game induced by an environment)**.** *Consider a non-empty finite set of outcomes* O*, a game form* $\mathcal{F} \in \mathsf{Form}(\mathsf{O})$ *and an environment* $E = \langle c, e, p\rangle \in \mathsf{Env}(\mathsf{O})$*. Let* $Y := (\mathsf{O}, \mathcal{F}, E)$*. The local arena* $\mathcal{C}_Y = \langle Q, \mathsf{F}\rangle$ *induced by* Y *is such that:*

- $Q := \{q_{\mathsf{init}}\} \cup K_e \cup p_{[0,1]}$*, where* $K_e := \{k_i \mid i \in [\![0, e]\!]\}$ *and* $p_{[0,1]} = p[\mathsf{O}] \cap [0,1]$*;*
- *for all* $x \in p_{[0,1]}$*, we set the value of the stopping state* x *to be* x *itself:* $\mathsf{val}(u) \leftarrow u$*;*
- $\mathsf{F}(q_{\mathsf{init}}) := \mathcal{F}$ *(up to identifying integers in* $[\![0, e]\!]$ *and states in* $\{k_i \mid i \in [\![0, e]\!]\}$*), and for all* $i \in [\![0, e]\!]$*, we set* $\mathsf{F}(k_i)$ *to be a trivial game form with* q_{init} *as only possible outcome.*

$$\mathcal{F} := \begin{bmatrix} y & x \\ x & y \end{bmatrix}$$

- $E := \langle 2, 4, p \rangle$;
- $p(x) := 1/2 \in [0, 1]$;
- $p(y) := 3 \in [\![0, 4]\!]$.

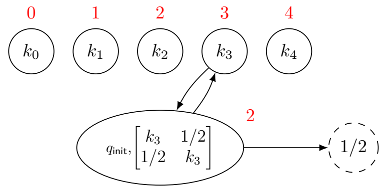

Figure 1 On the left, a game form with set of outcomes $\mathsf{O} := \{x, y\}$. In the middle, the description of an environment on O. On the right, the parity game $\mathcal{G}_{(\{x,y\},\mathcal{F},E)}$. Recall that the dashed state is a stopping state with value 1/2.

For all $u \in [0, 1]$, we denote by $v^u_Y : Q \to [0, 1]$ the valuation such that: $v^u_Y(q_{\mathsf{init}}) = v^u_Y(k_i) := u$ for all $i \in [\![0, e]\!]$ and $v^u_Y(x) := x$ for all $x \in p_{[0,1]}$.

Furthermore, for all Player-A GF-strategies $\sigma_{\mathsf{A}} \in \Sigma_{\mathsf{A}}(\mathcal{F})$, we denote by $\mathsf{s}^Y_{\mathsf{A}}(\sigma_{\mathsf{A}})$ the Player-A positional strategy defined by σ_{A} in the arena $\mathcal{C}_Y$.

The game $\mathcal{G}_Y$ is then equal to $\mathcal{G}_Y := \langle \mathcal{C}_Y, \mathsf{col} \rangle$ where $\mathsf{col}(q_{\mathsf{init}}) := c$ and for all $i \in [\![0, e]\!]$, we have $\mathsf{col}(k_i) := i$.

▶ **Example 16.** This definition is illustrated in Fig. 1 to the right. The colors of the non-stopping states are depicted in red next to the states. Furthermore, the edges from all k_i, for $i \neq 3$, leading back to q_{init} are not represented. ⌟

What we are interested in is the existence of positional optimal strategies for both players. In such games, these strategies are entirely defined by a GF-strategy in a game form $\mathcal{F}$.

▶ **Definition 17** (Optimal GF-strategies). *Given $E = \langle c, e, p \rangle \in \mathsf{Env}(\mathsf{O})$, and $Y := (\mathsf{O}, \mathcal{F}, E)$, a Player-$\mathsf{A}$ GF-strategy $\sigma_{\mathsf{A}} \in \Sigma_{\mathsf{A}}(\mathcal{F})$ is said to be* optimal *w.r.t. Y if the Player-A positional strategy $\mathsf{s}^Y_{\mathsf{A}}(\sigma_{\mathsf{A}})$ is optimal in $\mathcal{G}_Y$. The definition is analogous for Player B.*

Given a finite set of outcomes O, we can now define the game forms on O ensuring the existence of optimal strategies w.r.t. all environments.

▶ **Definition 18** (Optimizable game forms). *Given $\mathcal{F} \in \mathsf{Form}(\mathsf{O})$, $n \in \mathbb{N}$, and a player $\mathsf{C} \in \{\mathsf{A}, \mathsf{B}\}$, the game form $\mathcal{F}$ is said to be* positionally maximizable up to n *w.r.t. Player C if, for each environment $E \in \mathsf{Env}(\mathsf{O})$ with $\mathsf{Sz}_{\mathsf{C}}(E) \leq n$, there is an optimal GF-strategy for Player C w.r.t. $(\mathsf{O}, \mathcal{F}, E)$. When this holds for both players, $\mathcal{F}$ is said to be* positionally optimizable up to n. *If this holds for all $n \in \mathbb{N}$, $\mathcal{F}$ is simply said to be* positionally optimizable.

▶ Remark 19. Note first that there are some game forms that are not positionally maximizable w.r.t. any player up to 1. This is e.g. the case of the game form appearing in [2, Fig. 2].

Moreover, by definition, from a game form $\mathcal{F} \in \mathsf{Form}(\mathsf{D})$ that is not positionally optimizable up to some $n \in \mathbb{N}$, there exists an environment $E \in \mathsf{Env}(\mathsf{D})$ such that one player has no positional optimal strategy in the parity game $\mathcal{G}_{(\mathsf{D},\mathcal{F},E)}$, where the difference between $\mathsf{col}(q_{\mathsf{init}})$ and the maximum of the colors appearing in $\mathcal{G}_{(\mathsf{D},\mathcal{F},E)}$ is at most n.

▶ **Example 20.** In the game $\mathcal{G}_{(\mathsf{O},\mathcal{F},E)}$ on the right of Fig. 1, Player A has positional optimal strategies: it suffices to play both rows with positive probability. (This is similar for Player B.) As a side remark, the game form on the left of Fig. 1 is positionally optimizable. ⌟

In Lemma 21 below, we formulate more explicitly (using the notion of parity domination from Definition 12) what optimal GF-strategies are.

▶ **Lemma 21.** *Let $E = \langle c, e, p\rangle \in \mathsf{Env}(\mathsf{O})$ and $Y = (\mathsf{O}, \mathcal{F}, E)$. A Player-*$\mathsf{A}$ GF*-strategy $\sigma_{\mathsf{A}} \in \Sigma_{\mathsf{A}}(\mathcal{F})$ is* optimal *w.r.t. Y if and only if, letting $u := \chi_{\mathcal{G}_Y}(q_{\mathsf{init}})$, either (i) $u = 0$, or (ii) the positional Player-*A *strategy $\mathsf{s}^Y_{\mathsf{A}}(\sigma_{\mathsf{A}})$ parity dominates the valuation v^u_Y.*

Furthermore (ii) is equivalent to:

- *(1) the Player-*A *positional strategy $\mathsf{s}^Y_{\mathsf{A}}(\sigma_{\mathsf{A}})$ dominates the valuation v^u_Y; and*
- *(2) for all $b \in \mathsf{Act}_{\mathsf{B}}$, if the probability under (σ_{A}, b) to reach a stopping state is null, then $\max(\mathsf{Color}(\mathcal{F}, p, \sigma_{\mathsf{A}}, b) \cup \{c\})$ is even, where $\mathsf{Color}(\mathcal{F}, p, \sigma_{\mathsf{A}}, b) := \{i \in [\![0, e]\!] \mid \varrho(\sigma_{\mathsf{A}}, b)[p^{-1}[\{i\}]] > 0\}$ is the set of colors that can be seen with positive probability under (σ_{A}, b).*

This is symmetrical for Player B.

▶ Remark 22. This proposition states that for a Player-A GF-strategy σ_{A} to be optimal in a local game $\mathcal{G}_Y$ with positive value, it must be the case that for every Player-B action b: either there is a positive probability (w.r.t. (σ_{A}, b)) to exit q_{init} and the expected value of the stopping states visited is at least u; or the game loops on q_{init} with probability 1, and the maximum of the colors that can be seen with positive probability (w.r.t. (σ_{A}, b)) is even. In particular, if $c \leq \max \mathsf{Color}(\mathcal{F}, p, \sigma_{\mathsf{A}}, b)$ or if c is odd, then $\max \mathsf{Color}(\mathcal{F}, p, \sigma_{\mathsf{A}}, b)$ is even.

5 Local environment and global strategy

The goal of this section is to state and prove Theorem 25 below: the main theorem of this paper. This theorem states that it is possible to extract, for every state of a game and for each player, a local environment which summarizes the context of the state to the player, and tells her how to positionally play optimally.

For the remainder of this section, we fix a parity game $\mathcal{G} = \langle \mathcal{C}, \mathsf{col}\rangle$. In particular, the set of states Q is fixed. Before going any further, we give useful notations below.

▶ **Definition 23** (Value slice)**.** *For all subsets of states $S \subseteq Q$, we denote by $V_S := \{u \in [0, 1] \mid \exists q \in S,\ \chi_{\mathcal{G}}(q) = u\}$ the finite set of values of states in S. Furthermore, for all $u \in V_Q$, we let $Q_u := \{q \in Q \mid \chi_{\mathcal{G}}(q) = u\}$ be the set of states whose value is u: it is the u-*slice *of $\mathcal{G}$. Finally, for all $u \in V_Q$, we let $e_u :=$ $\mathsf{Even}(\mathsf{col}[Q_u])$ and $o_u :=$ $\mathsf{Odd}(\mathsf{col}[Q_u])$.*

We also introduce the notion of positional strategies generated by an environment function before stating Theorem 25: these are the positional strategies that play GF-strategies that are optimal in the (local) parity games induced by the environment function.

▶ **Definition 24** (Strategy generated by environment functions)**.** *For all environment functions $\mathsf{Ev} : Q \to \mathsf{Env}(Q)$, a Player-*$\mathsf{A}$ *positional strategy s_{A} is* generated *by Ev if for all $q \in Q$, the* GF*-strategy $\mathsf{s}_{\mathsf{A}}(q) \in \Sigma_{\mathsf{A}}(\mathsf{F}(q))$ is optimal w.r.t. $(\mathsf{O}, \mathsf{F}(q), \mathsf{Ev}(q))$ (and similarly for Player* B*).*

▶ **Theorem 25.** *Let $\mathcal{G} = \langle \mathcal{C}, \mathsf{col}\rangle$ be a parity game. Assume that for all states $q \in Q$, the game form $\mathsf{F}(q)$ is:*

- *positionally maximizable up to $e_{\chi_{\mathcal{G}}(q)} - \mathsf{col}(q)$ w.r.t. Player* A*; and*
- *positionally maximizable up to $o_{\chi_{\mathcal{G}}(q)} - \mathsf{col}(q)$ w.r.t. Player* B

Then, there is a function $\mathsf{Ev}_{\mathsf{A}} : Q \to \mathsf{Env}(Q)$ (resp. $\mathsf{Ev}_{\mathsf{B}} : Q \to \mathsf{Env}(Q)$) such that all Player-$\mathsf{A}$ *(resp. Player-*B*) positional strategies s_{A} (resp. s_{B}) generated by Ev_{A} (resp. Ev_{B}) are optimal in $\mathcal{G}$; and such Player-*A *(resp.* B*) positional strategies exist.*

▶ Remark 26. Given some $u \in V_Q$, one can realize that the requirement at states $q, q' \in Q_u$ changes depending on the color of q and q'. More specifically, if $\mathsf{col}(q) < \mathsf{col}(q')$, then the requirement at state q is at least as strong as the requirement at state q' since the game form

$\mathsf{F}(q)$ should behave well for environments of larger size than the game form $\mathsf{F}(q')$. As we shall see, by Proposition 50, the requirement at state q is actually strictly stronger than the requirement at state q'.

The remainder of this section is devoted to an explanation of the construction of the environment function Ev_A (the construction being similar for Player B). We first argue that we can restrict ourselves to a specific u-slice Q_u for some $u \in V_Q$.

▶ **Definition 27** (Game restricted to a u-slice)**.** *For all $u \in V_Q$, let $\mathcal{G}^u$ be the concurrent game obtained from $\mathcal{G}$ by making all states outside of Q_u stopping: for every $q \in Q \setminus Q_u$, we set $\mathsf{val}(q) \leftarrow \chi_\mathcal{G}(q)$. The states, game forms and coloring function on Q_u are left unchanged.*

Interestingly, a Player-A positional strategy optimal in $\mathcal{G}$ can be obtained by merging appropriate positional strategies s_A^u in the games $\mathcal{G}^u$ for all $u \in V_Q \setminus \{0\}$.

▶ **Proposition 28.** *For all $u \in V_Q \setminus \{0\}$, let s_A^u be a Player-A strategy that parity dominates the valuation $\chi_\mathcal{G}$ in $\mathcal{G}^u$. Then, the Player-A positional strategy s_A s.t. $\mathsf{s}_\mathsf{A}(q) := \mathsf{s}_\mathsf{A}^u(q)$ for all $u \in V_Q \setminus \{0\}$ and $q \in Q_u$ guarantees the valuation $\chi_\mathcal{G}$ in $\mathcal{G}$ (i.e. it is optimal).*

This justifies that, for the remainder of this section, we focus on a given u-slice Q_u for some positive $u \in (0, 1]$. We also let $e := e_u$ and $o := o_u$ for the remainder of this section.

5.1 Overview of the proof

In order to give an idea of the steps that we take to prove Theorem 25, let us first consider the very simple case of finite turn-based deterministic (i.e. where all probability distribution over successors states are deterministic) reachability games. In this setting, computing the area L_A from which Player A has a winning strategy can be done inductively. That is, initially we set $L_\mathsf{A} := T$ where T denotes the target that Player A wants to reach. Then, the inductive step is handled with a (deterministic) attractor: we add to L_A any Player-A state with a successor in L_A and any Player-B state with all successors in L_A. After finitely many steps, there is no more state to add in L_A: this exactly corresponds to the states from which Player A has a winning strategy.

Computing a single attractor is not merely enough to take into account the intricate behavior of parity objectives and the complexity of concurrent (and stochastic)interactions, which is what Theorem 25 deals with. Therefore, we are going to iteratively compute several layers of (virtual) colors, with a local update to change the (virtual) color (and therefore the layer it belongs to) of a state. This local update can be seen as an attractor except in a concurrent stochastic setting. Hence, when we update the (virtual) color of a state, we take into account the concurrent interaction of the players at each state along with the probability to see stopping states or states with different (virtual) colors. We define this local update in Subsection 5.3. Let us describe below the steps that we take to capture the behavior of the parity objective.

We compute layers of successive probabilistic attractors with leaks towards the stopping states. Although we compute a strategy, e.g., for Player A, we alternate players to build layers, then move the last non-empty layer into the closest layer with same parity, then backtrack the attractor computation from this layer downwards, and start over again the full attractor computation on the new layer structure. In a more concrete way, let us assume below that the highest color in the u-slice is 6. We proceed as follows:

1. Add the states colored with 4 to layer $\mathcal{L}^4$.
2. Recursively add to $\mathcal{L}^4$ (and give them virtual color 4) the states where Player A can guarantee that with positive probability (pp) either a leak towards stopping states occurs now with expected explicit value at least u ($\mathsf{Leak}_{\geq u}$), or with pp the next state is in $\mathcal{L}^4$.
3. Add the remaining states colored with 3 to layer $\mathcal{L}^3$.
4. Recursively add to $\mathcal{L}^3$ (and give them virtual color 3) the states where Player B can guarantee that either $\mathsf{Leak}_{<u}$ occurs now with pp, or the next state is surely not in $\mathcal{L}^4$ and with pp in $\mathcal{L}^3$.
5. Add the remaining states colored with 2 to layer $\mathcal{L}^2$.
6. Recursively add to $\mathcal{L}^2$ (and give them virtual color 2) the states where Player A can guarantee that either $\mathsf{Leak}_{\geq u}$ will occur with pp, or the maximal layer index of the next states seen with pp is 2 or 4.
7. And so on, for colors 1 and 0. The layers so far only give information about what can happen at finite horizon. For instance, from $\mathcal{L}^2$, Player A can guarantee that either $\mathsf{Leak}_{\geq u}$ will occur with pp, or the maximal color that will be seen with pp is in $\{2, 4\}$.
8. Now, if e.g. $\mathcal{L}^0 \neq \emptyset$, we merge $\mathcal{L}^0$ into $\mathcal{L}^2$ and we reset the states that are in layer $\mathcal{L}^1$. Similarly, if e.g. $\mathcal{L}^0 = \emptyset$ and $\mathcal{L}^1 \neq \emptyset$, we merge $\mathcal{L}^1$ into $\mathcal{L}^3$ and we reset the states that are in layer $\mathcal{L}^2$. This is, arguably, the most surprising step, we justify it in Ex. 42.
9. We then repeat the above attractor alternation from step 1. all over again, until all the states are eventually in $\mathcal{L}^4$, which is bound to happen as we shall prove.

The key property (namely *faithfulness*, defined below in Def. 38) that is growing throughout the above computation and will hold in the final layer $\mathcal{L}^4$ involves layer games: the $\mathcal{L}^n$-game is derived from the u-slice by abstracting each $\mathcal{L}^i$ with $i \neq n$ via one state k_i^n from which the player who dislikes the parity of n chooses any next state in $\mathcal{L}^n$, making it harder for the other to win. If $i > n$ then k_i^n is i-colored, else $(n-1)$-colored, also making it harder for the other to win. And states in $\mathcal{L}^n$ bear their true colors. See for instance Fig. 4. The $\mathcal{L}^n$-game is only seemingly harder to win: it is actually equivalently hard, but its useful properties are easier to prove.

The key growing property is as follows: between two merges, the attractor computation from the top layer down to $\mathcal{L}^n$ ensures that Player A has a positional strategy of value at least u in each $\mathcal{L}^i$-game for even $i \geq n$, and Player B less than u for odd $i \geq n$. In the very end, there is only one even layer with all states bearing their true colors, and no abstract states: the layer game equals the u-slice game. We have thus computed a positional optimal strategy.

Let us hint at how to show positional optimality in the $\mathcal{L}^n$-games when it holds: we break $\mathcal{L}^n$ each into one simple parity game built on $\mathsf{F}(q)$ per state q in $\mathcal{L}^n$, abstracting the other states in $\mathcal{L}^n$ into one. Our theorem assumption yields an optimal GF-strategy for Player A or B in the simple parity game. Gluing them does the job.

5.2 Extracting an environment function from a parity game

For the remainder of the section, we illustrate the definitions and lemmas on the game depicted in Fig. 2 and 3. We give the notations that we use to describe these examples below.

▶ **Example 29.** We explain the notations used to depict this game (it is in fact the same arena in both Fig. 2 and 3, with different coloring functions – real or virtual). On the sides in green are the slices $Q_0, Q_{1/4}, Q_{3/4}$ and Q_1 from left to right. We focus on the central slice $Q_{1/2}$. In $Q_{1/2}$, there are seven states, five of which (the square-shaped ones) are

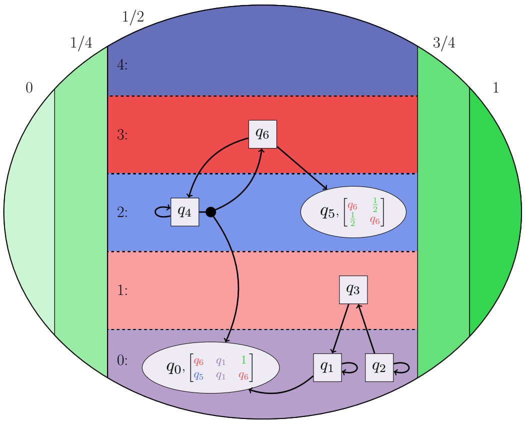

Figure 2 The depiction of a game restricted to the $1/2$-slice $Q_{1/2}$ with the initial coloring function col.

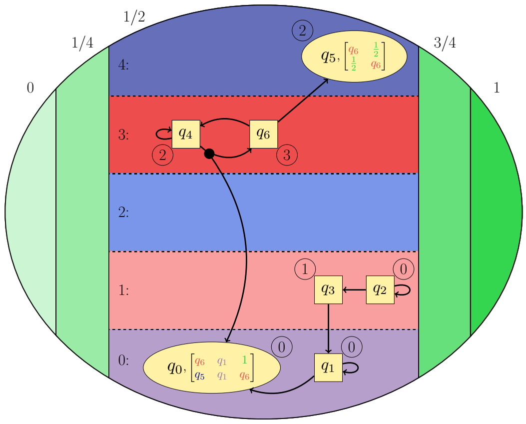

Figure 3 The same game restricted to $Q_{1/2}$ with a different coloring function vcol.

turn-based for Player B, that is, Player A has only one available action. On the other hand, the two circled-shaped states q_0 and q_5 are 'truly" concurrent in the sense that both players have several actions available. Furthermore, note that there is only one non-deterministic distribution function: from q_4, Player B may either loop on q_4 or go to with equal probability to q_0 and q_6. The other arrows lead to a single state and the outcomes of the game forms in q_0 or q_5 is a single state or a value: 1 or $1/2$. These formally refer to a (distribution over) stopping states outside of the $1/2$-slice $Q_{1/2}$. The horizontal layers depict the colors of the states. In Fig. 2, the coloring function considered is the initial one col whereas in Fig. 3 we have depicted a virtual coloring function vcol. For instance, $\mathsf{col}(q_6) = 3$ whereas $\mathsf{col}(q_5) = 2$. Similarly, $\mathsf{vcol}(q_6) = 3$ whereas state $\mathsf{vcol}(q_5) = 4$. Note that, in Fig. 3, the real colors (given by col) are reminded next to some states with circled numbers. Finally, note that $e := e_{1/2} = 4$. ⌟

Given a virtual coloring function (defining layers), we need to extract local environments from the parity game $\mathcal{G}$, which summarize how the Players see their neighboring states via the virtual coloring function. This is (partly) done in Def. 30 bia a successor function p.

▶ **Definition 30** (Successor function extracted from an arena and a virtual coloring function). *Given $S \subseteq Q_u$ and a virtual coloring function* $\mathsf{vcol} : Q_u \to [\![0, e]\!]$. *The function* $p_{S,\mathsf{vcol}} : S \to S \uplus [\![0, e]\!] \uplus V_{Q \setminus Q_u}$ *is such that, for all $d \in \mathsf{D}$:*

- *for all $q \in S$, $p_{S,\mathsf{vcol}}(q) := q \in Q$;*
- *for all $q \in Q \setminus Q_u$, $p_{S,\mathsf{vcol}}(q) := \chi_{\mathcal{G}}(q) \in [0, 1]$;*
- *for all $q \in S_u \setminus S$, $p_{S,\mathsf{vcol}}(q) := \mathsf{vcol}(q)$.*

Given a virtual coloring function $\mathsf{vcol} : Q_u \to [\![0, e]\!]$ and a color $n \in [\![0, e]\!]$, we can now extract a small parity game (the layer-games from Subsection 5.1) from $\mathcal{G}$ where the states with truly concurrent interactions are all in $\mathsf{vcol}^{-1}[n]$ (the interactions at these states is the same as in $\mathcal{G}$), the states in $Q \setminus Q_u$ are stopping states and the arena loops back to $\mathsf{vcol}^{-1}[n]$ when a state in $Q_u \setminus \mathsf{vcol}^{-1}[n]$ is seen. This is done in the next definition.

▶ **Definition 31** (Parity game extracted from the u-slice). *Consider a virtual coloring function* $\mathsf{vcol} : Q_u \to [\![0, e]\!]$ *and a color $n \in [\![0, e]\!]$. Let $\mathsf{C} \in \{\mathsf{A}, \mathsf{B}\}$ be a Player:* A *if n is odd and* B *if n is even. The arena $\mathcal{C}^n_{\mathsf{vcol}} = \langle Q', \mathsf{F}' \rangle$ along with the coloring function* $\mathsf{vcol}_n : Q' \to \mathbb{N}$ *are such that, denoting* $Q_n := \mathsf{vcol}^{-1}[n]$*:*

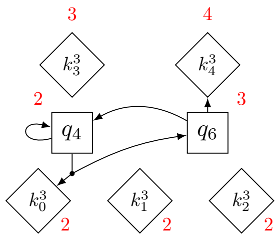

Figure 4 The game $\mathcal{L}^3_{\mathsf{vcol}}$. Exiting arrows from $k_0^3, k_1^3, k_2^3, k_3^3$ and k_4^3 are not depicted: they would all loop back to both q_4 and q_6.

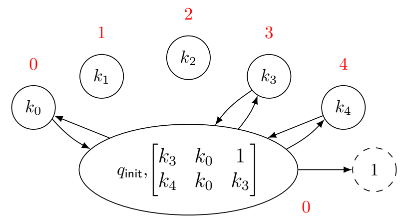

Figure 5 The game $\mathcal{G}^0_{q_0,\mathsf{vcol}}$ with vcol the coloring function depicted in Fig. 3.

- $Q' := Q_n \cup K^n \cup V_{Q \setminus Q_u}$ *where all* $x \in V_{Q \setminus Q_u}$ *are stopping states with* $\mathsf{val}(x) \leftarrow x$*;*
- *for all* $q \in Q_n$, $\mathsf{F}'(q) := \langle \mathsf{Act}^q_{\mathsf{A}}, \mathsf{Act}^q_{\mathsf{B}}, Q', \varrho_{p_{n,\mathsf{vcol}}} \rangle$ *where, for all* $\sigma_{\mathsf{A}} \in \mathsf{Act}^q_{\mathsf{A}}$, $\sigma_{\mathsf{B}} \in \mathsf{Act}^q_{\mathsf{B}}$, *and* $q \in Q$, *we have* $\varrho_{p_{n,\mathsf{vcol}}}(\sigma_{\mathsf{A}}, \sigma_{\mathsf{B}})(q) := \varrho(\sigma_{\mathsf{A}}, \sigma_{\mathsf{B}})[p^{-1}_{n,\mathsf{vcol}}[q]]$*;*
- *for all* $k \in K^n$, *we set* $\mathsf{F}'(k)$ *as a Player-*C *state whose outcomes are all the states in* Q_n*;*
- *for all* $q \in Q_n$, *we let* $\mathsf{vcol}_n(q) := \mathsf{col}(q)$ *and for all* $i \in [\![0, e]\!]$, *we have* $\mathsf{vcol}_n(k_i^n) := \max(i, n-1)$.

For $t \in [0, 1]$, *we define the valuation* $v^t_{n,\mathsf{vcol}} : Q' \to [0, 1]$*:* $v^t_{n,\mathsf{vcol}}[Q_n \cup K^n] := \{t\}$ *and for all* $x \in V_{Q \setminus Q_u}$, $v^t_{n,\mathsf{vcol}}(x) := x$.

The game $\mathcal{L}^n_{\mathsf{vcol}}$ *is then equal to* $\mathcal{L}^n_{\mathsf{vcol}} = \langle \mathcal{C}^n_{\mathsf{vcol}}, \mathsf{vcol}_n \rangle$.

The notation $\mathcal{L}^n_{\mathsf{vcol}}$ comes from the fact that the game is extracted for the n-colored layer w.r.t. the coloring function vcol. The idea behind Def. 31 is the following: the states of interest are those of Q_n, that is, those for which the virtual color given by vcol is n. Note however that the colors of these states in $\mathcal{L}^n_{\mathsf{vcol}}$ are given by the real coloring function col. On the other hand, for all $i \in [\![0, e]\!]$, the state k_i^n in $\mathcal{L}^n_{\mathsf{vcol}}$ correspond to the states in $\mathcal{G}^u$ colored with i w.r.t. vcol. In the case where n is even, as formally defined later in Def. 37, we will require that any Player-A positional strategy generated by a given environment has value at least u, in the game $\mathcal{L}^n_{\mathsf{vcol}}$, from all states in Q_n. However, all states k_i^n for $i \in [\![0, e]\!]$ are Player-B's, who can then choose to loop back to any state in Q_n. Therefore, given a Player-A positional strategy s_{A}, if the game cannot exit to any stopping state, for the strategy s_{A} not to have value 0, the game may loop on some k_i^n only at the condition that the highest color seen with positive probability is even. In addition, note that the color of the state k_i^n for $i \in [\![0, n-1]\!]$ is $n-1$ (which is odd). Hence, all other things being equal, the game is harder for Player A when $n = 4$ than when $n = 2$ or 0.

▶ **Example 32.** The game $\mathcal{L}^3_{\mathsf{vcol}}$ is partly depicted on the left of Fig. 4 (the virtual coloring function vcol being the one depicted in Fig. 3). The colors of the states in $\mathcal{L}^3_{\mathsf{vcol}}$ are depicted in red (for the central states q_4 and q_6, it is their real color in the original game). Although the arrows are not depicted, from all states $k_0^3, k_1^3, k_2^3, k_3^3$ and k_4^3, Player A can decide to which state among $\{q_4, q_6\}$ to loop back (since $n = 3$ is odd). In an even-colored layer, it would have been Player B to decide. The color of states k_i^3 is i if $i \geq 3$ or 2 if $i \leq 2$. ⌟

Given a virtual coloring function, we also associate a local environment to each state. This is crucial since this will allow us to properly define (in Definition 35 below) the probabilistic attractor with leaks towards the stopping states mentioned in Subsection 5.1.

▶ **Definition 33** (Induced local environment). *Given $q \in Q_u$, a virtual coloring function* $\mathsf{vcol} : Q_u \to [\![0, e]\!]$ *and* $n \in [\![0, e]\!]$, *the environment associated to state* q *w.r.t.* vcol *and* n *is* $E^n_{q,\mathsf{vcol}} := \langle \max(c_n, \mathsf{vcol}(q)), e, p_{\{q\},\mathsf{vcol}} \rangle$ *where* $c_n = n + 1$ *if* n *is odd and* $c_n := n - 1$ *if* n *is even. The corresponding (local) game* $\mathcal{G}_{(\mathsf{O},\mathsf{F}(q),E^n_{q,\mathsf{vcol}})}$ *is denoted* $\mathcal{G}^n_{q,\mathsf{vcol}}$, *and for all* $x \in [0, 1]$, *we set* $v^x_{q,\mathsf{vcol}} := v^x_{(\mathsf{O},\mathsf{F}(q),E^0_{q,\mathsf{vcol}})}$ *(see Def. 14).*

▶ **Example 34.** The game $\mathcal{G}^n_{q_5,\mathsf{col}}$ is depicted in Fig. 1 (right) for $n = 0, 1, 2$. However, if $n = 3$, the color of q_{init} would be 4, and if $n = 4$, it would be 3. The game $\mathcal{G}^n_{q_0,\mathsf{vcol}}$ is depicted in Fig. 5 for $n = 0$. However, if $n = 1$, the color of q_{init} would be 2, if $n = 2$, the color of would be 1, if $n = 3$, the color would be 4 and if $n = 4$ the color would be 3. ⌟

5.3 Local operator

We want to define a way to update a virtual coloring function vcol. This amounts to computing the probabilistic attractor with leaks towards the stopping states mentioned in Subsection 5.1. This is done via a local operator mapping a given state q to the best color k for which Player A can achieve the value u in the local parity game $\mathcal{G}^k_{q,\mathsf{vcol}}$. Note that "best" is to be understood considering an ordering compatible with the parity objective. Specifically, taking the point-of-view of Player A, any even number is better than any odd number, and when they increase, odd numbers get worse whereas even numbers get better. This induces a new total strict order relation $\prec_{\mathsf{par}}$ on $\mathbb{N}$ such that, for all $m, n \in \mathbb{N}$, we have $m \prec_{\mathsf{par}} n$ if: m is odd and n is even; or $m > n$ and m and n are odd; or $m < n$ and m and n are even.

▶ **Definition 35** (Local operator). *Let* $q \in Q_u$, *and* $\mathsf{vcol} : Q_u \to [\![0, e]\!]$ *a (possibly virtual) coloring function. The color* $\mathsf{NewCol}(q, \mathsf{vcol}) \in \mathbb{N}$ *induced by* vcol *at* q *is defined by:*

$$\mathsf{NewCol}(q, \mathsf{vcol}) := \max_{\prec_{\mathsf{par}}} \left\{ n \in [\![0, e]\!] \mid \chi_{\mathcal{G}^n_{q,\mathsf{vcol}}}(q_{\mathsf{init}}) \geq u \right\}$$

The meaning of a new virtual color n assigned to a state q via NewCol is the following: in the game $\langle \mathcal{C}, \mathsf{vcol} \rangle$, from state q and in at most one step, the highest color w.r.t. vcol seen with positive probability when both players play optimally is n (and no stopping state is seen).

Let us explain the choice of c_n in Def. 33. In a local environment parameterized by n, the integer n induces a shifted parity objective for Player A: her objective is that the maximal color seen infinitely often is at least n w.r.t. $\prec_{\mathsf{par}}$; in particular $n = 0$ induces the standard parity objective. The value c_n encodes that winning condition. For instance, if $n = 2$, assuming $\mathsf{vcol}(q) = 0$ for simplicity, then $c_n = 1$, which implies that seeing 0 infinitely often is not enough, but seeing 2 infinitely often is enough to win.

▶ **Example 36.** We first consider Fig. 1 and we compute $\mathsf{NewCol}(q_5, \mathsf{col})$ on the game on the right. We realize that, regardless of the color of state q_{init}, Player A can (positionally) play both rows with positive probability and ensure reaching (almost-surely) the stopping state $1/2$: for all $n \in [\![0, 4]\!]$, $\chi_{\mathcal{G}^n_{q_5,\mathsf{col}}}(q_{\mathsf{init}}) = 1/2$. Hence, $\mathsf{NewCol}(q_5, \mathsf{col}) = 4$.

We consider Fig. 5 and we compute $\mathsf{NewCol}(q_0, \mathsf{vcol})$. As mentioned in Ex. 34, the game $\mathcal{G}^4_{q_0,\mathsf{vcol}}$ corresponds to the game depicted in Fig. 5 except that q_{init} is colored with 3. One can realize that, with this choice (of coloring of the state q_{init}), if the highest color $i \in [\![0, 4]\!]$ such that k_i is seen infinitely often is such that $i \prec_{\mathsf{par}} 4$, then Player A loses. The value of this game is 0 as Player B can ensure looping on k_0 and q_{init} (by playing, positionally and deterministically, the middle column) thus ensuring that the highest color seen infinitely often is 3. Thus, $\mathsf{NewCol}(q_0, \mathsf{vcol}) \prec_{\mathsf{par}} 4$. In the game $\mathcal{G}^2_{q_0,\mathsf{vcol}}$, q_{init} is colored with 1. Again, with this choice (of coloring of the state q_{init}), if the highest color $i \in [\![0, 4]\!]$ such that k_i is seen infinitely is such that $i \prec_{\mathsf{par}} 2$, then Player A loses. The value of this game is also 0 as Player B can still play the middle column ensuring that the highest color seen infinitely often is 1. Thus, $\mathsf{NewCol}(q_0, \mathsf{vcol}) \prec_{\mathsf{par}} 2$. Consider now the game $\mathcal{G}^0_{q_0,\mathsf{vcol}}$, the one depicted in Fig. 5. The value of the state q_{init} is now 1. Indeed, if Player A plays the two rows with equal probability, one can see that this strategy parity dominates (see Def. 12) the valuation $v^1_{q_0,\mathsf{vcol}}$ (recall Def. 33). Indeed, the BSCCs compatible with this strategy are $\{q_{\mathsf{init}}, k_3, k_4\}$ and $\{q_{\mathsf{init}}, k_0\}$ and they are even-colored. Hence, by Proposition 13, $\chi_{\mathcal{G}^0_{q_1,\mathsf{col}}}(q_{\mathsf{init}}) = 1 \geq 1/2$ and $\mathsf{NewCol}(q_0, \mathsf{vcol}) \succeq_{\mathsf{par}} 0$. That is, $\mathsf{NewCol}(q_0, \mathsf{vcol}) = 0$. ⌟

5.4 Faithful coloring function

To prove Theorem 25, we iteratively build a virtual coloring function and a local environment function. We want to define the desirable property that the pair of coloring and environment functions should satisfy that will be preserved step by step. First, we need to define the notion of an environment function witnessing a color.

▶ **Definition 37** (Environment witnessing a color). *Let* $\mathsf{vcol} : Q_u \to [\![0, e]\!]$ *be a virtual coloring function,* $n \in [\![0, e]\!]$*, and* $\mathsf{Ev} : \mathsf{vcol}^{-1}[n] \to \mathsf{Env}(\mathsf{D})$ *be an environment function. Assuming that* n *is even (resp. odd), we say that the pair* $(\mathsf{vcol}, \mathsf{Ev})$ witnesses *the color* n *if for all* $q \in \mathsf{vcol}^{-1}[n]$, $\mathsf{Sz}_{\mathsf{A}}(\mathsf{Ev}(q)) \leq e - \mathsf{col}(q)$ *(resp.* $\mathsf{Sz}_{\mathsf{B}}(\mathsf{Ev}(q)) \leq o - \mathsf{col}(q)$*) and all positional strategies* s_{A} *(resp.* s_{B}*) generated by* Ev *(recall Def. 24) in the game* $\mathcal{L}^n_{\mathsf{vcol}}$ *parity dominate the valuation* $v^u_{n,\mathsf{vcol}}$ *(resp.* $v^{u'}_{n,\mathsf{vcol}}$ *for some* $u' < u$*), recall Def. 31.*

It means that, in the virtual games given by vcol, in the even layers, Player A can achieve at least what she should be able to achieve in this u-slice (i.e. the value of the states is at least u). Whereas, in the odd-colored layers, Player B can prevent Player A from achieving this.

We can now define the central notion of interest: for a pair of coloring function and environment function to be faithful (to what really happens in the parity game). We only give a definition of faitfhfulness that we can use in this paper, but note that in [4], we require additional properties for faithfulness.

▶ **Definition 38** (Faithful pair of coloring and environment functions). *Let* $\mathsf{vcol} : Q_u \to [\![0, e]\!]$ *be a virtual coloring function,* $n \in [\![0, e+1]\!]$*, and* $\mathsf{Ev} : Q_u \to \mathsf{Env}(\mathsf{D})$ *a partial environment function defined on* $\mathsf{vcol}^{-1}[[\![n, e]\!]]$*. We say that* $(\mathsf{vcol}, \mathsf{Ev})$ *is* faithful down to n *if:*

- *for all* $k \in [\![n, e]\!]$*, the pair* $(\mathsf{vcol}, \mathsf{Ev})$ *witnesses color* k*;*
- *for all* $q \in Q_u$*, if* $\mathsf{vcol}(q) < n$*, then* $\mathsf{col}(q) = \mathsf{vcol}(q)$ *and* $\mathsf{NewCol}(q, \mathsf{vcol}) < n$*;*

If $n = 0$*, we say that the pair* $(\mathsf{vcol}, \mathsf{Ev})$ *is* completely faithful.

A benefit of faithful environments and coloring functions lies in the proposition below: if all states are mapped w.r.t. the coloring function to e, then the environment function guarantees the value u in the whole u-slice Q_u.

▶ **Proposition 39.** *For a coloring function* $\mathsf{vcol} : Q_u \to [\![0, e]\!]$ *and an environment function* $\mathsf{Ev} : Q_u \to \mathsf{Env}(\mathsf{D})$*, assume that* $(\mathsf{vcol}, \mathsf{Ev})$ *is completely faithful and that* $\mathsf{vcol}[Q_u] = \{e\}$*. Then, all Player-*$\mathsf{A}$ *positional strategies generated by the environment function* Ev *parity dominate the valuation* $\chi_{\mathcal{G}}$ *in the game* $\mathcal{G}^u$*.*

Proof. This is direct from the definitions. Indeed, as $(\mathsf{vcol}, \mathsf{Ev})$ is completely faithful, it witnesses the color e: all Player-A positional strategies s_{A} generated by Ev in the game $\mathcal{L}^e_{\mathsf{vcol}}$ parity dominate the valuation $v^u_{e,\mathsf{vcol}}$. Since $\mathsf{vcol}[Q_u] = \{e\}$, the games $\mathcal{L}^e_{\mathsf{vcol}}$ and $\mathcal{G}^u$ are identical. Similarly, the valuation $v^u_{e,\mathsf{vcol}}$ is equal to the valuation $\chi_{\mathcal{G}}$ in the game $\mathcal{G}^u$. ◀

5.5 Computing a completely faithful pair

Given Prop. 39, our goal is to come up with a pair of an environment function and a coloring function completely faithful such that all states are colored with e. We first consider how to obtain a completely faithful pair from the initial coloring function and the empty environment function (which is faithful down to $e+1$): we proceed by building a new pair that is faithful down to $n-1$, given a pair $(\mathsf{vcol}, \mathsf{Ev})$ faithful down to some $n \in [\![1, e+1]\!]$.

To do so, let us be guided by the second property of faithfulness: to be faithful down to $n-1$, no state $q \in Q_u$ such that $\mathsf{vcol}(q) \leq n-2$ should be such that $\mathsf{NewCol}(q, \mathsf{vcol}) = n-1$. Hence, we adopt the following procedure $\mathsf{UpdateColEnv}$: we first associate an environment to all states whose color is already $n-1$. Then, for all states $q \in Q_u$ such that $\mathsf{NewCol}(q, \mathsf{vcol}) = n-1$, we change their colors to $n-1$ until no state $q \in Q_u$ with $\mathsf{vcol}(q) \leq n-2$ satisfies $\mathsf{NewCol}(q, \mathsf{vcol}) = n-1$. The environment associated to each such state q newly colored by $n-1$ is given by the coloring function vcol for which $\mathsf{NewCol}(q, \mathsf{vcol}) = n-1$ for the first time (crucially, this is done before the color of q is updated to $n-1$). We state as an informal lemma the property satisfied by the procedure described above.

▶ **Lemma 40.** *Let* $\mathsf{vcol} : Q_u \to [\![0, e]\!]$*,* $n \in [\![1, e+1]\!]$ *be a coloring function, and* $\mathsf{Ev} : Q_u \to \mathsf{Env}(\mathsf{D})$ *be a partial environment function defined on* $\mathsf{vcol}^{-1}[[\![n, e]\!]]$*. Assume that* $(\mathsf{vcol}, \mathsf{Ev})$ *is faithful down to* n*. Let* $(\mathsf{vcol}', \mathsf{Ev}') \leftarrow \mathsf{UpdateColEnv}(n-1, \mathsf{vcol}, \mathsf{Ev})$ *be the pair computed by the procedure described above. Then* $(\mathsf{vcol}', \mathsf{Ev}')$ *is faithful down to* $n-1$*.*

We illustrate below this lemma on examples.

▶ **Example 41.** We consider the example depicted in Fig. 2. The first step is to build a pair that is faithful down to $e = 4$. As mentioned in Ex. 36, we have $\mathsf{NewCol}(q_5, \mathsf{col}) = 4$. Hence, the color of this state is changed to 4 (we obtain a virtual coloring function vcol^{q_5}) and we set $\mathsf{Ev}(q_5) := E^4_{q_5, \mathsf{col}}$. Note that a Player-$\mathsf{A}$ GF-strategy σ_A is optimal in this environment if and only if it plays both rows with positive probability. Furthermore, note that, in the extracted game $\mathcal{L}^4_{\mathsf{vcol}^{q_5}}$, a Player-$\mathsf{A}$ positional strategy playing such a GF-strategy σ_A in q_5 parity dominates the valuation $v^{1/2}_{Q_4, \mathsf{col}^{q_5}}$. Hence, the pair $(\mathsf{vcol}^{q_5}, \mathsf{Ev})$ is faithful down to 4.

Consider now the layer 3. First, the state q_6 already has color 3, so it only remains to set its environment: $\mathsf{Ev}(q_6) := E^3_{q_6, \mathsf{vcol}^{q_5}}$. We then realize that $\mathsf{NewCol}(q_4, \mathsf{vcol}^{q_5}) = 3$. Indeed, q_4 is colored with 2 and may go with equal probability to a state colored with 0 and to a state colored with 3. The color of this state is therefore changed, thus obtaining a new virtual coloring function $\mathsf{vcol}^{q_5, q_6, q_4}$. We set its environment: $\mathsf{Ev}(q_4) := E^3_{q_4, \mathsf{vcol}^{q_5}}$. One can realize that the pair $(\mathsf{vcol}^{q_5, q_6, q_4}, \mathsf{Ev})$ witnesses the color 3: a positional Player-B strategy generated by this environment would be so that (i) from q_6, it goes to q_4 with probability 1 (to avoid k_4 that is colored with 4) and (ii) from q_5, it goes to q_6 with positive probability (to see the color 3 with positive probability). Such a strategy has value 0 in the game $\mathcal{L}^3_{\mathsf{vcol}^{q_5}} = \mathcal{L}^3_{\mathsf{vcol}}$ from Fig. 4, hence the pair $(\mathsf{vcol}^{q_5, q_6, q_4}, \mathsf{Ev})$ witnesses the color 3.

We illustrate on this step why the environment needs to be set before setting the new color and not after. That is, we explain why it would not be correct to set $\mathsf{Ev}(q_4) := E^3_{q_4, \mathsf{vcol}^{q_5, q_6, q_4}}$ instead of what we do above. In this environment, the state q_4 has color 3. Hence, looping with probability 1 on q_4 is an optimal GF-strategy for Player B w.r.t. $(\mathsf{D}, \mathsf{F}(q_4), \mathsf{Ev}(q_4))$. Then, the corresponding pair of coloring and environment functions would not witness the color 3. Indeed, a Player B strategy that loops with probability 1 on q_4 is generated by this environment, and it has value $1 \geq u$ (because the real color of this state is 2, and not 3).

This process is repeated down to 0. In Fig. 3, the depicted coloring function (with an appropriate environment function, not shown in Fig. 3) are in fact completely faithful (which is what the procedure $\mathsf{UpdateColEnv}$ would output on the coloring function of Fig. 2). ⌟

Proof sketch. We want to prove that the pair $(\mathsf{vcol}', \mathsf{Ev}')$ witnesses the color $n-1$ (the other condition for faithfulness is ensured by the construction). We consider the case where $n-1$ is even, the other case is similar (but one needs to take the point-of-view of Player B). Consider a Player-A positional strategy $\mathsf{s_A}$ generated by the environment function Ev' in the game $\mathcal{L}^{n-1}_{\mathsf{vcol}'}$. Let $Q_{n-1} := \mathsf{vcol}'^{-1}[n-1]$ and let $v := v^u_{n-1,\mathsf{vcol}'}$. For every $q \in Q_{n-1}$, let $Y_q := (\mathsf{D}, \mathsf{F}(q), \mathsf{Ev}'(q))$ be the local environment at state q and let $\mathsf{Ev}'(q) = \langle c_q, e, p_q \rangle$. From the characterization of Lemma 21 (item (ii.1)), by carefully analyzing the links between the local games $\mathcal{G}_{Y_q}$ for all $q \in Q_{n-1}$ and the game $\mathcal{L}^{n-1}_{\mathsf{vcol}'}$, we can show that the strategy $\mathsf{s_A}$ dominates the valuation v.

It remains to show that all BSCCs (that are not reduced to a stopping state and are) compatible with s_A are even-colored. Consider such a BSCC H and a Player-B deterministic positional strategy $\mathsf{s_B}$ which induces H. For every state $q \in H$, since no stopping state appears in H, it must be that the probability to reach a stopping state in $\mathcal{G}_{Y_q}$ w.r.t. (σ_A, b) is 0. For every state $q \in Q_{n-1}$, the coloring function vcol_q associated with environment $\mathsf{Ev}'(q)$ is such that $\mathsf{vcol}_q(q) \leq n-1$.[1] Hence, the color c_q is such that $c_q = \max(n-2, \mathsf{vcol}_q(q)) \leq n-1$. Now, assume that some state k_i is in H for some $i > n-1 \geq c_q$. In that case, as explained in Remark 22, the highest i such that k_i is in H must be even. Hence, H is even-colored. Assume now that no state k_i in H is such that $i > n-1$. In that case, if a state in H has color $n-1$ (like the state q_6 in Fig. 3 in the case where $n-1=3$), then $n-1$ is the highest color in H and H is even-colored. Consider the first state q whose color is now $n-1$ (w.r.t. vcol') but whose previous color was not $n-1$. In that case, we have $c_q = \max(n-2, \mathsf{vcol}_q(q)) = n-2$ is odd. Furthermore, the state q has changed its color because $\mathsf{NewCol}(q, \mathsf{vcol}_q) = n-1$. With Remark 19, since $\mathsf{s_A}(q)$ is optimal w.r.t. Y_q, it follows that there is a positive probability to reach, in the game $\mathcal{G}_{Y_q}$ the state k_{n-1}. In the game $\mathcal{L}^{n-1}_{\mathsf{vcol}'}$, this corresponds to a positive probability to reach a state $q' \in H$ colored with $n-1$ w.r.t. vcol_q (recall Def. 30). Since q is the first state to have changed its color, we can deduce that q' already had color $n-1$ w.r.t. vcol. Furthermore, one can show that q' is colored with $n-1$ w.r.t. the real coloring function col. Overall, in the game $\mathcal{L}^{n-1}_{\mathsf{vcol}'}$, with the GF-strategy $\mathsf{s_A}(q)$, there is a positive probability to reach in one step a state q' colored with $n-1$. Iteratively, we obtain that, considering the k-th state whose color is now $n-1$ (i.e. w.r.t. vcol') but whose initial color was not $n-1$, there is a positive probability to reach (in at most k steps) a state colored with $n-1$. Hence, the highest color appearing in H is $n-1$, which is even. We obtain that $\mathsf{s_A}$ parity dominates the valuation v. ◀

Applying iteratively this algorithm on all colors from e down to 0 starting with the initial coloring function induces a completely faithful pair $(\mathsf{vcol}, \mathsf{Ev})$. However, it may be the case that some states are not mapped to e, which does not allow us to apply Prop. 39. The question is then: from such a completely faithful configuration, how can one make some progress towards a situation where Prop. 39 can be applied?

▶ **Example 42.** Consider the coloring function of Fig. 3. As mentioned in Ex. 41, with an appropriate environment function (not shown in Fig. 3), we have a completely faithful pair. To gain some intuition on what should be done next, let us focus only on the states q_1, q_2, q_3. A simplified version is presented in Fig. 6 (with a slight modification: instead of going to q_0, q_1 loops on itself): the initial (and true) colors of the states are in circles next to them and their color w.r.t. the current virtual coloring function is written in red. In this game, it is

[1] This is because all states $q \in Q_{n-1}$ satisfy $\mathsf{col}(q) \leq n-1$. This is one of the additional conditions for faithfulness that we did mention, but that is used in the definition of faithfulness in [4].

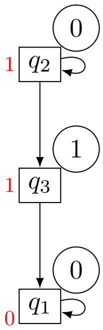

Figure 6 A (deterministic turn-based) game with only three states.

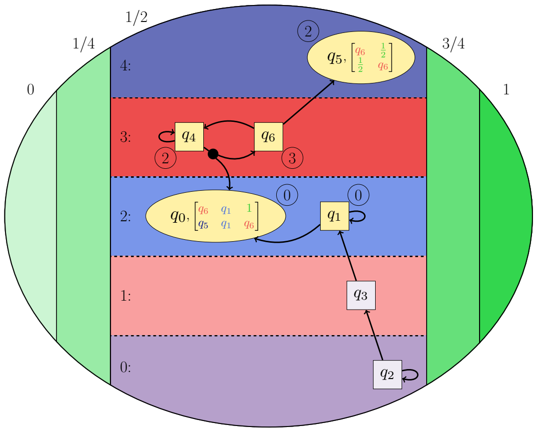

Figure 7 The same arena as in Fig. 2,3 but with a different coloring function.

obvious that Player A wins surely from q_2: indeed, either the game stays indefinitely in q_2, or it eventually reaches and settles in q_1.

The current virtual color 1 assigned to both q_2 and q_3 does not properly reflect the fact that if the game reaches q_3, even though Player B plays optimally according to the local game associated to q_2, it will end up looping in q_1, which will be losing for Player B. In a way, we would like to propagate the information that reaching q_1 is bad for Player B. Since 0 is the smallest color, there is no harm in increasing it to 2, the game from q_1 will be the same: it will be won by Player A by looping. Player B will now be able to know that going to q_1 is dangerous for her, which will be obtained by applying the previous iterative process.

In a more general concurrent game, the next step of the process when we have a completely faithful configuration not satisfying the assumptions of Proposition 39 consists in changing all the states with the least virtual color n to the color $n+2$. However, note that there is a (very important) second step: the colors of all states virtually colored with $n+1$ should be reset to their initial colors. The reason why can be seen again in Fig. 6. After the color of q_1 becomes 2, the color of q_3 will also become 2. However, if the color of the state q_2 is not reset, then it is not going to change since Player B can choose to loop to q_2 and see the color 1 for ever (in game $\mathcal{G}^0_{q_2,\mathsf{vcol}}$). That is, from Player B's perspective, looping on q_2 is winning, which is not what happens in the real game: the coloring function does not faithfully describes what happens in the game. The changes made to the coloring function vcol from Fig. 3 can be seen in Fig. 7. Note that the process of increasing the colors of some states by 2 can only be done with the *least* color (otherwise faithfulness will not be preserved). ⌟

The process IncLeast described in Ex. 42 can be summed up as follows: we increase the least virtually-colored layer n by 2 and we reset the environment and colors of the last but least virtually-colored layer. It ensures faithfulness down to $n+2$ if the initial pair is completely faithful, as informally stated below.

▶ **Lemma 43.** *Let* $\mathsf{vcol} : Q_u \to [\![0, e]\!]$, $\mathsf{Ev} : Q_u \to \mathsf{Env}(\mathsf{D})$ *be a coloring and an environment function. Let* $n := \min \mathsf{vcol}[Q]$. *Assume that* $n \leq e-2$ *and the pair* $(\mathsf{vcol}, \mathsf{Ev})$ *is completely faithful. If* $(\mathsf{vcol}', \mathsf{Ev}') \leftarrow \mathsf{IncLeast}(\mathsf{vcol}, \mathsf{Ev})$ *is the result of increasing the least-colored layer by 2 and resetting the environment of the last but least-colored layer as described above, then* $(\mathsf{vcol}', \mathsf{Ev}')$ *is faithful down to* $n+2$.

Proof sketch. Let $Q_n := \mathsf{vcol}^{-1}[n]$ and $Q_{n+2} := \mathsf{vcol}^{-1}[n+2]$. Let us argue that the pair $(\mathsf{vcol}'', \mathsf{Ev})$ obtained after increasing the least color by 2, before resetting the color and environment of the last but least-colored layer, witnesses the color $n+2$.

Consider a Player-A positional strategy $\mathsf{s_A}$ generated by the environment Ev in the game $\mathcal{L}^{n+2}_{\mathsf{vcol}''}$. Let $v := v^u_{n+1,\mathsf{vcol}''}$. Similarly to the case of Lemma 40, $\mathsf{s_A}$ dominates the valuation v. Consider a BSCC H compatible with $\mathsf{s_A}$. If $H \cap Q_{n+2} = \emptyset$, then H is even-colored. Indeed, $(\mathsf{vcol}, \mathsf{Ev})$ witnesses the color n and, in addition, the probability to go an $(n+1)$-colored state k_i^{n+2} in the game $\mathcal{L}^{n+2}_{\mathsf{vcol}''}$ is exactly the probability to go to an $(n+1)$-colored state k_i^n in the game $\mathcal{L}^n_{\mathsf{vcol}}$ (since n is the least color). Furthermore, H is also even-colored as soon as $H \cap Q_n = \emptyset$ since $(\mathsf{vcol}, \mathsf{Ev})$ witnesses the color $n+2$. Now, assume that none of these cases occur. Then, one can show that: either a state k_i is seen for some $i \geq n+2$, and H is even-colored; or, from some states in Q_{n+2}, there is a positive probability to exit Q_{n+2} and no state k_i is seen for $i \geq n+2$. Now, looking at what happens in game $\mathcal{L}^{n+2}_{\mathsf{vcol}}$, some states k_i are seen for $i \leq n+1$, and such states are colored with $n+1$. Hence, since $(\mathsf{vcol}, \mathsf{Ev})$ witnesses the color $n+2$, it must be that the highest color in H is $n+2$, which is even. Therefore it is also the case in the game $\mathcal{L}^{n+2}_{\mathsf{vcol}''}$. In all the cases, H is even-colored. ◀

As stated in Lemma 43, the update of colors described in Ex. 42 can be done only if, for a completely faithful pair, the least virtual color n appearing is at most $e-2$. If $n = e$, we are actually in the scope of Lemma 39 since in that case all states have virtual color e. However, there remains the case where we have $n = e-1$. In fact, this case cannot happen.

▶ **Lemma 44.** *Consider a coloring function* $\mathsf{vcol} : Q_u \to [\![0, e]\!]$*, an environment function* $\mathsf{Ev} : Q_u \to \mathsf{Env}(\mathsf{D})$*. Assume that* $(\mathsf{vcol}, \mathsf{Ev})$ *is completely faithful. Then, for* $C := \mathsf{vcol}[Q]$*, we have* $\min C \neq e-1$.

Proof sketch. Let $Q_{e-1} := \mathsf{vcol}^{-1}[e-1]$. Towards a contradiction, let $\mathsf{s_B}$ be a Player-B positional strategy generated by Ev in the game $\mathcal{L}^{e-1}_{\mathsf{vcol}}$. It parity dominates the valuation $v^{u'}_{e-1,\mathsf{vcol}}$ for some $u' < u$. Hence, all BSCCs compatible with $\mathsf{s_B}$ are odd-colored: they all stay in the layer Q_{e-1}. Indeed, since $e-1 = \min C$, exiting Q_{e-1} while staying in Q_u mean seeing $Q_e := \mathsf{vcol}^{-1}[e]$ with e even and the highest color in the game. Hence, either the game stays indefinitely in Q_{e-1} and Player B wins almost surely, or there is some positive probability to visit stopping states, and in that case their expected values is at most u'. Hence, in the game $\mathcal{G}^u$, the strategy $\mathsf{s_B}$ has values less than u from the states $Q_{e-1} \subseteq Q_u$, which is a contradiction. ◀

Finally, all these pieces are put together by iteratively applying $\mathsf{UpdateColEnv}$ until we obtain a completely faithful pair and applying $\mathsf{IncLeast}$ to a completely faithful pair to make some progress towards the completely faithful pair where all states are colored with e. The only remaining step is to prove the termination of this procedure. Consider the virtual coloring functions as vectors in $\mathbb{N}^{e+1}$ indicating the number of states mapped to each color. Then, one can realize that applying $\mathsf{UpdateColEnv}$ does not decrease these vectors for a lexicographic order (i.e. we first compare the number of states mapped to e, then the number of states mapped to $e-1$, etc). Furthermore, applying $\mathsf{IncLeast}$ increases these vectors for a lexicographic order. In addition, the maximum for this order is achieved when all states are colored with e. Hence, the procedure described above terminates in finitely many steps. We can now finalize the argument for proving Theorem 25.

Proof sketch of Theorem 25 for Player A. Pick $u \in V_\mathcal{G} \setminus \{0\}$. According to the previous discussion, there is a completely faithful pair of environment and coloring functions $(\mathsf{vcol}_u, \mathsf{Ev}^u_\mathsf{A})$ mapping each state in Q_u to e_u. Hence, by Proposition 39, all Player-A positional strategies generated by the environment function Ev^u_A parity dominate the valuation $\chi_\mathcal{G}$ in the game $\mathcal{G}^u$. Since we assume that all game forms appearing in Q_u are positionally maximizable up to $e_u - \mathsf{col}(q)$ w.r.t. Player A, such positional strategies generated by Ev^u_A do exist. Considering the environment function $\mathsf{Ev}_\mathsf{A} : Q \to \mathsf{Ev}(\mathsf{D})$ that merges all the environment functions $(\mathsf{Ev}^u_\mathsf{A})_{u \in V_\mathcal{G} \setminus \{0\}}$ together (and that is defined arbitrarily on Q_0), it follows by Proposition 28, that all Player-A positional strategies generated by that environment function Ev are optimal. ◀

6 Discussion on positionally optimizable game forms

As mentioned in the introduction, this work extends previous lines of research [1, 2, 3]. We discuss the more closely related work [3]. The goal in [3] was to characterize the game forms ensuring the existence of almost-optimal positional strategies for Büchi objectives (resp. optimal positional strategies for co-Büchi objectives). In both cases, there is a lift from local properties to global properties, similarly to what is done in this work. However, the techniques are quite different: in [3], the proofs involve the use of nested fixed points, as is done for computing values of graph games [9]. This establishes a link between local and global behaviors. However, this comes at the cost of having to handle local (i.e. at game form level) and global (i.e. at graph game level) fixed-points. With Büchi and co-Büchi objectives, there are only two nested fixed points. (Recall that they can be expressed as two-color parity objectives.) For general parity objectives, the number of fixed points would be linear in the number of involved colors. That is why, in this work, we decided to consider good local behaviors in a more abstract way, without considering how the values are effectively computed. That way, we handle arbitrarily many colors instead of only two without prohibitive complexification.

We conclude with a discussion on positionally optimizable game forms. First, we would like to emphasize that Theorem 25 along with Remark 19 give exactly the game forms that should be used in parity games to ensure the existence of positional optimal strategies for both players. Indeed:

- By Remark 19, given any game form that is not positionally optimizable, one can build a small parity game from it, like in Definition 14, where a player has no optimal strategy;
- By Theorem 25, if all the local interactions occurring in a concurrent parity game are positionally optimizable game forms, then both players have positional optimal strategies.

However, it has to be noted that there are concurrent parity games with non-positionally optimizable local interactions where both players have positional optimal strategies. This is e.g. the case of parity games without stopping states where all states have the same color.

Let us now give some properties that are ensured by positionally optimizable game forms. We first give some notations for positionally optimizable game forms (and the corresponding decision problems).

▶ **Definition 45.** *For $n \in \mathbb{N}$, we let $\mathsf{ParO}(n)$ be the set of all game forms positionally optimizable up to n and we let $\mathsf{IsO}(n)$ be the problem of deciding whether a game form is in $\mathsf{ParO}(n)$. Furthermore, we let $\mathsf{ParO} := \bigcap_{n \in \mathbb{N}} \mathsf{ParO}(n)$ and we denote by IsO the problem of deciding whether a game form is in ParO.*

First, let us introduce the notion of relevant environment, i.e. environment $E = \langle c, e, p \rangle$ such that $c \in \{0, 1\}$ and p takes all values in $[\![c, e]\!]$. They are fomally defined below.

▶ **Definition 46.** *For a set of outcomes* O*, an environment* $E = \langle c, e, p\rangle \in \mathsf{Env}$ *is* relevant *if* $p^{-1}[\{c-1\}] = p^{-1}[\{q_{\mathsf{init}}\}] = \emptyset$ *and for all* $i \in [\![c, e]\!]$*, there is some* $o \in \mathsf{O}$ *such that* $p(o) = i$*. The* size *of a relevant environment* E *is equal to* $\mathsf{Sz}(E) := e - c$*.*

The benefit of relevant environments appears below: a game form is positionally optimizable if and only if there are optimal GF-strategies w.r.t. all *relevant* environments.

▶ **Proposition 47.** *Consider a set of outcomes* O *and a game form* $\mathcal{F} \in \mathsf{Form}(\mathsf{O})$*. For all* $n \in \mathbb{N}$*, the game form* $\mathcal{F}$ *is* $\mathsf{ParO}(n)$ *if and only if, for all relevant environments* E *with* $\mathsf{Sz}(E) \leq n-1$*, for both players, there is an optimal* GF*-strategy w.r.t.* $(\mathsf{O}, \mathcal{F}, E)$*.*

Some positionally optimizable game forms. As mentioned above, given the distance to the highest color in the game, Theorem 25 and Remark 19 give exactly the game forms that should be used in parity games to ensure the existence of positional optimal strategies for both players. The game forms in ParO are the ones that can be used in all parity games, regardless of the number of colors involved, while ensuring the existence of positional optimal strategies (for both players). Such game forms do exist, e.g. turn-based game forms are positionally optimizable (straightforwardly). In fact, all *determined* [1] game forms are positionally optimizable. Furthermore, all game forms with at most two outcomes are in ParO.

▶ **Proposition 48.** *Consider a set of outcomes* O *and a game form* $\mathcal{F} = \langle \mathsf{Act_A}, \mathsf{Act_B}, \mathsf{O}, \varrho\rangle \in \mathsf{Form}(\mathsf{O})$*. Assume that* $|\mathsf{O}| \leq 2$ *or that* $\mathcal{F}$ *is turn-based or that* $\mathcal{F}$ *is determined, i.e. such that:*

- *for all* $(a, b) \in \mathsf{Act_A} \times \mathsf{Act_B}$*, we have* $\varrho(a, b)$ *deterministic;*
- *for all* $v : \mathsf{O} \to \{0, 1\}$*, there is either some* $a \in \mathsf{Act_A}$ *such that* $\varrho(a, \mathsf{Act_B}) \subseteq \{1\}$ *or there is either some* $b \in \mathsf{Act_B}$ *such that* $\varrho(\mathsf{Act_A}, b) \subseteq \{0\}$*.*

Then, $\mathcal{F}$ *is positionally optimizable.*

Decidability. Similarly to "maximizable" game forms designed for reachability games [2] and to "maximizable" game forms designed (co-)Büchi games [3], it is rather easy to get convinced that positionally optimizable game forms used in this paper can be defined in the first-order theory of the reals (FO-$\mathbb{R}$): the characterizations of Lemma 21 and the fact that it is sufficient to only consider relevant environment, as stated in Proposition 47, even allow us to place the $\mathsf{ParO}(n)$ (for all n) and the ParO problems in the $\forall\exists$-fragment of FO-$\mathbb{R}$.

▶ **Proposition 49.** *For all* $n \in \mathbb{N}$*, the problem* $\mathsf{IsO}(n)$ *is decidable. And so is* IsO*.*

Hierarchy. For all $n \in \mathbb{N}$, the game forms in $\mathsf{ParO}(n)$ are the ones to be used – to ensure the existence of positional optimal strategies for both players – in parity games at states where the gap between the color of the state and the maximum color in the game is at most n. Straightforwardly, $\mathsf{ParO}(n) \subseteq \mathsf{ParO}(n+1)$. In fact, this inclusion is strict for all $n \in \mathbb{N}$. This defines an infinite hierarchy of game forms.

▶ **Proposition 50.** *For all* $n \in \mathbb{N}$*, we have* $\mathsf{ParO}(n) \subsetneq \mathsf{ParO}(n+1)$*.*

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