# Ehrenfeucht–Fraïssé Games in Semiring Semantics

**Sophie Brinke**
RWTH Aachen University, Germany

**Erich Grädel**
RWTH Aachen University, Germany

**Lovro Mrkonjić**
RWTH Aachen University, Germany

## Abstract

Ehrenfeucht–Fraïssé games provide a fundamental method for proving elementary equivalence (and equivalence up to a certain quantifier rank) of relational structures. We investigate the soundness and completeness of this method in the more general context of semiring semantics. Motivated originally by provenance analysis of database queries, semiring semantics evaluates logical statements not just by true or false, but by values in some commutative semiring; this can provide much more detailed information, for instance concerning the combinations of atomic facts that imply the truth of a statement, or practical information about evaluation costs, confidence scores, access levels or the number of successful evaluation strategies. There is a wide variety of different semirings that are relevant for provenance analysis, and the applicability of classical logical methods in semiring semantics may strongly depend on the algebraic properties of the underlying semiring.

While Ehrenfeucht–Fraïssé games are sound and complete for logical equivalences in classical semantics, and thus on the Boolean semiring, this is in general not the case for other semirings. We provide a detailed analysis of the soundness and completeness of model comparison games on specific semirings, not just for classical Ehrenfeucht–Fraïssé games but also for other variants based on bijections or counting.

Finally we propose a new kind of games, called *homomorphism games*, based on the fact that there exist locally very different semiring interpretations that can be proved to be elementarily equivalent via separating sets of homomorphisms. We prove that these homomorphism games provide a sound and complete method for logical equivalences on finite lattice semirings.

**2012 ACM Subject Classification** Theory of Computation → Finite Model Theory

**Keywords and phrases** Semiring semantics, elementary equivalence, Ehrenfeucht–Fraïssé games

**Digital Object Identifier** 10.4230/LIPIcs.CSL.2024.19

**Related Version** *Full Version*: https://arxiv.org/abs/2308.04910 [4]

## 1 Introduction

*Semiring provenance* was proposed in 2007 in a seminal paper by Green, Karvounarakis, and Tannen [15]. It is based on the idea to annotate the atomic facts in a database by values in some commutative semiring, and to propagate these values through a database query, keeping track whether information is used alternatively (as in disjunctions or existential quantifications) or jointly (as in conjunctions or universal quantifications). Depending on the chosen semiring, the provenance valuation then gives practical information about a query, beyond its truth or falsity, for instance concerning the *confidence* that we may have in its truth, the *cost* of its evaluation, the number of successful evaluation strategies, and so on. Beyond such provenance evaluations in specific *application semirings*, more precise information is obtained by evaluations in *provenance semirings* of polynomials, which permit us to *track* which atomic facts are used (and how often) to compute the answer to the query.


32nd EACSL Annual Conference on Computer Science Logic (CSL 2024).
Editors: Aniello Murano and Alexandra Silva; Article No. 19; pp. 19:1–19:22
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

In databases, semiring provenance has been successfully applied to a number of different scenarios, such as conjunctive queries, positive relational algebra, datalog, nested relations, XML, SQL-aggregates, graph databases (see, e.g., the surveys [9, 16]), but for a long time, it had essentially been restricted to negation-free query languages; there has been no systematic tracking of *negative or absent information*, and for quite some time, this has been an obstacle for extending semiring provenance to other branches of logic in computer science.

A new approach to provenance analysis for languages with negation has been proposed in 2017 by Grädel and Tannen [13], based on transformations into negation normal form, quotient semirings of polynomials with dual indeterminates, and a close relationship to semiring valuations of games [14]. Since then, semiring provenance has been extended to a systematic investigation of *semiring semantics* for many logical systems, including first-order logic, modal logic, description logics, guarded logic and fixed-point logic [3, 5, 6, 7, 13] and also to a general method for strategy analysis in games [11, 14].

In classical semantics, a model $\mathfrak{A}$ of a formula assigns a Boolean value to each literal. $\mathcal{S}$-interpretations $\pi$, for a suitable semiring $\mathcal{S}$, generalise this by assigning a semiring value from $\mathcal{S}$ to each literal. We interpret a value of 0 as *false* and all other semiring values as *nuances of true*, or *true with additional information.* In this context, classical semantics corresponds to semiring semantics on the Boolean semiring $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$, the Viterbi semiring $\mathbb{V} = ([0,1], \max, \cdot, 0, 1)$ can model *confidence* scores, the tropical semiring $\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$ is used for cost analysis, and min-max-semirings $(A, \max, \min, a, b)$ for a totally ordered set $(A, <)$ can model different access levels. Other interesting semirings are the Łukasiewicz semiring $\mathbb{L}$, used in many-valued logic, and its dual $\mathbb{D}$, which we call the semiring of doubt. Provenance semirings of polynomials, such as $\mathbb{N}[X]$, *track* certain literals by mapping them to different indeterminates. The overall value of a formula is then a polynomial that describes precisely what combinations of literals imply the truth of the formula. There are other provenance semirings, obtained from $\mathbb{N}[X]$ by dropping coefficients and/or exponents or by absorption, to get semirings $\mathbb{B}[X], \mathrm{Trio}[X], \mathbb{W}[X], \mathbb{S}[X]$ and $\mathrm{PosBool}[X]$. They are less informative than $\mathbb{N}[X]$ (which is the free semiring generated by $X$), but have specific algebraic properties and admit simpler evaluation procedures. For applications to infinite universes, and for stronger logics than first-order logic, provenance semirings with more general objects than polynomials are needed, such as $\mathbb{N}^\infty[\![X]\!]$, the semirings of formal power series, and $\mathbb{S}^\infty[X]$, the semirings of generalised absorptive polynomials with potentially infinite exponents, which are fundamental for semiring semantics of fixed-point logics [7, 14].

The development of semiring semantics raises the question to what extent classical techniques and results of logic extend to semiring semantics, and how this depends on the algebraic properties of the underlying semiring, and this paper is part of a general research programme that explores such questions. In previous investigations, we have studied, for instance, the relationship between elementary equivalence and isomorphism for finite semiring interpretations and their definability up to isomorphism [12], 0-1 laws [10], and locality properties as given by the theorems of Gaifman and Hanf [2]. In all these studies, it has turned out that classical methods of mathematical logic can be extended to semiring semantics for certain semirings, but that they fail for others. Further, these questions are often surprisingly difficult: even quite simple facts of logic in the standard Boolean semantics become interesting research problems for semirings, and they often require completely new methods.

The objective of this paper is to study the applicability of Ehrenfeucht–Fraïssé games – and related model comparison games – as a method for proving elementary equivalence (i.e. indistinguishability by first-order sentences, denoted $\equiv$) and $m$-equivalence (i.e. indistinguishability by sentences of quantifier rank up to $m$, denoted $\equiv_m$) in semiring semantics.

Let us recall the classical Ehrenfeucht–Fraïssé Theorem[1] (see e.g. [8]).

▶ **Theorem 1** (Ehrenfeucht–Fraïssé). *Let $\tau$ be a finite relational vocabulary. For any two $\tau$-structures $\mathfrak{A}$ and $\mathfrak{B}$, and for all $m \in \mathbb{N}$, the following statements are equivalent:*
**(1)** $\mathfrak{A} \equiv_m \mathfrak{B}$*;*
**(2)** *Player II (Duplicator) has a winning strategy for the game $G_m(\mathfrak{A}, \mathfrak{B})$;*
**(3)** *There exists an $m$-back-and-forth system $(I_j)_{j \leq m}$ for $\mathfrak{A}$ and $\mathfrak{B}$;*
**(4)** $\mathfrak{B} \models \chi^m_{\mathfrak{A}}$*, where $\chi^m_{\mathfrak{A}}$ is the characteristic sentence of quantifier rank $m$ for $\mathfrak{A}$.*

In semiring semantics, the structures $\mathfrak{A}$ and $\mathfrak{B}$ are generalised to (model-defining) semiring interpretations $\pi_A$ and $\pi_B$ mapping instantiated $\tau$-literals into a semiring $\mathcal{S}$. The notions of $m$-equivalence, local isomorphisms, Ehrenfeucht–Fraïssé games, and back-and-forth systems all generalise in a straightforward way to $\mathcal{S}$-interpretations, for any semiring $\mathcal{S}$ (see Section 2). Also the observation that $m$-back-and-forth systems can be viewed as algebraic descriptions of winning strategies of Player II in $m$-turn Ehrenfeucht–Fraïssé games holds for arbitrary semiring interpretations, i.e. the equivalence (2) $\Leftrightarrow$ (3) holds for any semiring. The notion of characteristic sentences will be discussed later in Section 5. Our main concern is the relationship between (1) and (2), or equivalently (1) and (3). We shall have to consider both directions separately.

▶ **Definition 2.** Let $\mathcal{S}$ be an arbitrary commutative semiring. We say that
**(1)** *$G_m$ is sound for $\equiv_m$ on $\mathcal{S}$* if for any pair $\pi_A, \pi_B$ of model-defining $\mathcal{S}$-interpretations, the existence of a winning strategy of Player II for $G_m(\pi_A, \pi_B)$ implies that $\pi_A \equiv_m \pi_B$;
**(2)** *$G_m$ is complete for $\equiv_m$ on $\mathcal{S}$* if for any pair $\pi_A, \pi_B$ of model-defining $\mathcal{S}$-interpretations such that $\pi_A \equiv_m \pi_B$, Player II has a winning strategy for $G_m(\pi_A, \pi_B)$.

In this terminology, the Ehrenfeucht–Fraïssé Theorem says that for every $m$, $G_m$ is both sound and complete for $\equiv_m$ on the Boolean semiring. However, we shall prove that the Boolean semiring is the only semiring with this property, and for general semirings, the games $G_m$ need be neither sound nor complete. But there are also positive results, and the detailed study of soundness and completeness of Ehrenfeucht–Fraïssé games on semirings is quite interesting and diverse. For instance, we shall prove that $G_m$ is sound for $\equiv_m$ precisely on *fully idempotent* semirings (where both semiring operations are idempotent). Examples of fully idempotent semirings include all min-max semirings, lattice semirings, and the semirings $\text{PosBool}[X]$ of irredundant positive Boolean DNF-formulae. We shall then turn to more powerful games, which are more difficult to win for Duplicator, but if she wins, stronger results follow. In particular, we study the general Ehrenfeucht–Fraïssé game $G(\pi_A, \pi_B)$ where Spoiler can choose a number $m$, and then the game $G_m(\pi_A, \pi_B)$ is played. If, on a semiring $\mathcal{S}$, $G_m$ is sound for $\equiv_m$ for all $m$, then a winning strategy for Duplicator for $G(\pi_A, \pi_B)$ implies that $\pi_A \equiv_m \pi_B$ for all $m$, and hence $\pi_A \equiv \pi_B$. Thus, soundness of all games $G_m$ implies soundness of $G$. The converse is not true; there are semirings on which $G$ is sound for $\equiv$, although the games $G_m$ are unsound for $\equiv_m$. Trivially, $G$ is sound on semirings that do not admit interpretations with infinite universes due to the impossibility of infinite sums or products, such as $\mathbb{N}$ or the provenance semirings $\mathbb{B}[X], \mathbb{S}[X]$ and $\mathbb{N}[X]$. More interesting cases include semirings that are not idempotent, but where adding or multiplying any element repeatedly with itself stabilises after at most $n$ steps, or the semiring $\mathbb{N}^\infty = \mathbb{N} \cup \{\infty\}$. But there also exist a number of semirings on which the unrestricted

[1] Detailed definitions of all notions will be given in Section 2.

Ehrenfeucht–Fraïssé game $G$ is unsound for elementary equivalence, including the semirings $\mathbb{T}, \mathbb{V}, \mathbb{L}$ and $\mathbb{D}$. Further we shall consider *bijection* and *counting games*, which are variants of pebble games for bounded-variable logics with counting from [17, 18]. Actually the $m$-move bijection games $BG_m$ and counting games $CG_m$ are equivalent, and they turn out to be sound for $\equiv_m$ on *every* semiring. However, with few exceptions, such as the semirings $\mathbb{N}$ and $\mathbb{N}[X]$, they are not complete. We also study parametrised versions $CG^n_m$ of counting games.

On many semirings $\mathcal{S}$, the methods established in [12] permit us to construct elementarily equivalent $\mathcal{S}$-interpretations $\pi_A \equiv \pi_B$, although locally some elements of $\pi_A$ look different from all elements of $\pi_B$, so that Spoiler wins $G_m(\pi_A, \pi_B)$ for some small $m$. The game $G_m$ is then incomplete for $\equiv_m$, and the game $G$ is incomplete for $\equiv$. Since the games $CG^n_m$ and $BG_m$ are more difficult to win for Player II than $G_m$, they are incomplete as well. This approach successfully works for the semirings $\mathbb{V}, \mathbb{T}, \mathbb{L}, \mathbb{D}, \mathbb{N}^\infty, \mathbb{W}[X], \mathbb{S}[X], \mathbb{B}[X]$, and $\mathbb{S}^\infty[X]$.

The soundness and completeness results of these games are summarised in Figure 1.

| | Application semirings: | $\mathcal{S} \ncong \mathbb{B}$ fully idempotent | $\mathbb{T} \cong \mathbb{V}$ | $\mathbb{L} \cong \mathbb{D}$ | $\mathbb{N}$ | $\mathbb{N}^\infty$ |
|---|---|---|---|---|---|---|
| Soundness | $G_m$ for $\equiv_m$ | ✓ | ✗ | ✗ | ✗ | ✗ |
| | $CG^n_m$ for $\equiv_m$ | ✓ | ✗ | ✗ | ✗ | ✗ |
| | $BG_m$ for $\equiv_m$ | ✓ | ✓ | ✓ | ✓ | ✓ |
| | $G$ for $\equiv$ | ✓ | ✗ | ✗ | ✓ | ✓ |
| Completeness | $G_m$ for $\equiv_m$ | ✗ | ✗ | ✗ | ✓ | ✗ |
| | $CG^n_m$ for $\equiv_m$ | ✗ | ✗ | ✗ | ✓ | ✗ |
| | $BG_m$ for $\equiv_m$ | ✗ | ✗ | ✗ | ✓ | ✗ |
| | $G$ for $\equiv$ | ✗ | ✗ | ✗ | ✓ | ✗ |

| | Provenance semirings: | PosBool$[X]$ | $\mathbb{W}[X]$ | $\mathbb{S}[X], \mathbb{B}[X]$ | $\mathbb{N}[X]$ | $\mathbb{S}^\infty[X]$ |
|---|---|---|---|---|---|---|
| Soundness | $G_m$ for $\equiv_m$ | ✓ | ✗ | ✗ | ✗ | ✗ |
| | $CG^n_m$ for $\equiv_m$ | ✓ | ✓ | ✗ | ✗ | ✗ |
| | $BG_m$ for $\equiv_m$ | ✓ | ✓ | ✓ | ✓ | ✓ |
| | $G$ for $\equiv$ | ✓ | ✓ | ✓ | ✓ | ✓ |
| Completeness | $G_m$ for $\equiv_m$ | ✗ | ✗ | ✗ | ✓ | ✗ |
| | $CG^n_m$ for $\equiv_m$ | ✗ | ✗ | ✗ | ✓ | ✗ |
| | $BG_m$ for $\equiv_m$ | ✗ | ✗ | ✗ | ✓ | ✗ |
| | $G$ for $\equiv$ | ✗ | ✗ | ✗ | ✓ | ✗ |

**Figure 1** Due to full idempotence. Due to $n$-idempotence. Holds for any semiring. Follows from the finiteness of the universes. Cannot hold since elementary equivalence of finite interpretations does not imply isomorphism.

The proof that locally different $\mathcal{S}$-interpretations are nevertheless elementarily equivalent often proceeds via separating sets of homomorphisms. We use this method to propose a new kind of games, called *homomorphism games*, involving the selection of a homomorphism into the Boolean semiring, and a one-sided winning condition, due to the property that homomorphisms may transfer model-defining $\mathcal{S}$-interpretations into $\mathbb{B}$-interpretations that are no longer model-defining. We prove that these homomorphism games provide a sound and complete method for proving logical equivalences on finite lattice semirings.

## 2 Semiring semantics

We briefly summarise semiring semantics for first-order logic, as introduced in [13], and the resulting generalised notions of isomorphism and equivalence.

▶ **Definition 3** (Semiring). A *commutative semiring* is an algebraic structure $\mathcal{S} = (S, +, \cdot, 0, 1)$ with $0 \neq 1$, such that $(S, +, 0)$ and $(S, \cdot, 1)$ are commutative monoids, $\cdot$ distributes over $+$, and $0 \cdot s = s \cdot 0 = 0$.

A commutative semiring is *naturally ordered* (by addition) if $s \leq t :\Leftrightarrow \exists r(s + r = t)$ defines a partial order. In particular, this excludes rings. We only consider commutative and naturally ordered semirings and simply refer to them as *semirings*. A semiring $\mathcal{S}$ is *idempotent* if $s + s = s$ for each $s \in S$ and *multiplicatively idempotent* if $s \cdot s = s$ for all $s \in S$. If both properties are satisfied, we say that $\mathcal{S}$ is *fully idempotent*. Finally, $\mathcal{S}$ is *absorptive* if $s + st = s$ for all $s, t \in S$ or, equivalently, if multiplication is decreasing in $\mathcal{S}$, i.e. $st \leq s$ for $s, t \in S$ (equivalence is shown in [7]). Every absorptive semiring is idempotent.

**Application semirings.** There are several applications which can be modelled by semirings and provide useful practical information about the evaluation of a formula.

- A totally ordered set $(S, \leq)$ with least element $s$ and greatest element $t$ induces the *min-max semiring* $(S, \max, \min, s, t)$. It can be used to reason about access levels.
- The *tropical semiring* $\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$ provides the opportunity to annotate basic facts with a cost which has to be paid for accessing them and realise a cost analysis.
- The *Viterbi semiring* $\mathbb{V} = ([0, 1]_\mathbb{R}, \max, \cdot, 0, 1)$, which is in fact isomorphic to $\mathbb{T}$ via $y \mapsto -\ln y$, can be used for reasoning about confidence.
- An alternative semiring for this is the *Łukasiewicz semiring* $\mathbb{L} = ([0, 1]_\mathbb{R}, \max, \odot, 0, 1)$, where multiplication is given by $s \odot t = \max(s + t - 1, 0)$. It is isomorphic to the semiring of doubt $\mathbb{D} = ([0, 1]_\mathbb{R}, \min, \oplus, 1, 0)$ with $s \oplus t = \min(s + t, 1)$.
- The *natural semiring* $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ is used to count the number of evaluation strategies proving that a sentence is satisfied. It is also important for bag semantics in databases.

**Provenance semirings.** Provenance semirings of polynomials provide information on which combinations of literals imply the truth of a formula. The universal provenance semiring is the semiring $\mathbb{N}[X]$ of multivariate polynomials with indeterminates from $X$ and coefficients from $\mathbb{N}$. Other provenance semirings are obtained as quotient semirings of $\mathbb{N}[X]$ induced by congruences for (full) idempotence and absorption. The resulting provenance values are less informative, but their computation is more efficient.

- By dropping coefficients from $\mathbb{N}[X]$, we get the free idempotent semiring $\mathbb{B}[X]$ whose elements are finite sets of monomials. It is the quotient induced by $x + x \sim x$.
- If, in addition, exponents are dropped, we obtain the Why-semiring $\mathbb{W}[X]$ of finite sums of monomials that are linear in each argument.
- The free absorptive semiring $\mathbb{S}[X]$ consists of $0, 1$ and all antichains of monomials with respect to the absorption order $\succcurlyeq$. A monomial $m_1$ absorbs $m_2$, denoted $m_1 \succcurlyeq m_2$, if it has smaller exponents, i.e. $m_2 = m \cdot m_1$ for some monomial $m$.
- Finally, the lattice semiring $\mathrm{PosBool}[X]$ freely generated by the set $X$ arises from $\mathbb{S}[X]$ by collapsing exponents.

For a given finite relational vocabulary $\tau$, we denote by $\mathrm{Lit}_n(\tau)$ the set of literals $R\bar{x}$ and $\neg R\bar{x}$ where $R \in \tau$ and $\bar{x}$ is a tuple of variables from $\{x_1, \ldots, x_n\}$. The set $\mathrm{Lit}_A(\tau)$ refers to literals $R\bar{a}$ and $\neg R\bar{a}$ that are instantiated with elements from a universe $A$.

▶ **Definition 4** ($\mathcal{S}$-interpretation). Given a semiring $\mathcal{S}$, a mapping $\pi\colon \mathrm{Lit}_A(\tau) \to \mathcal{S}$ is an *$\mathcal{S}$-interpretation (of vocabulary $\tau$ and universe $A$)*. We say that $\mathcal{S}$ is *model-defining* if exactly one of the values $\pi(L)$ and $\pi(\overline{L})$ is zero for any pair of complementary literals $L, \overline{L} \in \mathrm{Lit}_A(\tau)$.

An $\mathcal{S}$-interpretation $\pi\colon \mathrm{Lit}_A(\tau) \to \mathcal{S}$ inductively extends to valuations $\pi[\![\varphi(\bar{a})]\!]$ of instantiated first-order formulae $\varphi(\bar{x})$ in negation normal form. Equalities are interpreted by their truth value, that is $\pi[\![a = a]\!] := 1$ and $\pi[\![a = b]\!] := 0$ for $a \neq b$ (and analogously for inequalities). Based on that, the semantics of disjunctions and existential quantifiers is defined via sums, while conjunctions and universal quantifiers are interpreted as products.

$$\pi[\![\psi(\bar{a}) \vee \vartheta(\bar{a})]\!] := \pi[\![\psi(\bar{a})]\!] + \pi[\![\vartheta(\bar{a})]\!] \qquad \pi[\![\psi(\bar{a}) \wedge \vartheta(\bar{a})]\!] := \pi[\![\psi(\bar{a})]\!] \cdot \pi[\![\vartheta(\bar{a})]\!]$$

$$\pi[\![\exists x \psi(\bar{a}, x)]\!] := \sum_{a \in A} \pi[\![\psi(\bar{a}, a)]\!] \qquad \pi[\![\forall x \psi(\bar{a}, x)]\!] := \prod_{a \in A} \pi[\![\psi(\bar{a}, a)]\!]$$

▶ **Lemma 5** (Fundamental Property). *Let $\pi\colon \mathrm{Lit}_A(\tau) \to \mathcal{S}$ be an $\mathcal{S}$-interpretation and $h\colon \mathcal{S} \to \mathcal{T}$ be a semiring homomorphism. Then, $(h \circ \pi)$ is a $\mathcal{T}$-interpretation and it holds that $h(\pi[\![\varphi(\bar{a})]\!]) = (h \circ \pi)[\![\varphi(\bar{a})]\!]$ for all $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ and instantiations $\bar{a} \subseteq A$.*

Basic model theoretic concepts such as equivalence and isomorphism naturally generalise to semiring semantics and yield more fine-grained notions. Given a mapping $\sigma\colon A \to B$ and some $L \in \mathrm{Lit}_A(\tau)$, we denote by $\sigma(L)$ the $\tau$-literal over $B$ which arises from $L$ by replacing each occurrence of $a \in A$ with $\sigma(a) \in B$.

▶ **Definition 6** (Isomorphism). $\mathcal{S}$-interpretations $\pi_A\colon \mathrm{Lit}_A(\tau) \to \mathcal{S}$ and $\pi_B\colon \mathrm{Lit}_B(\tau) \to \mathcal{S}$ are *isomorphic*, denoted as $\pi_A \cong \pi_B$, if there is a bijection $\sigma\colon A \to B$ such that $\pi_A(L) = \pi_B(\sigma(L))$ for all $L \in \mathrm{Lit}_A(\tau)$. A mapping $\sigma\colon \bar{a} \mapsto \bar{b}$ is a *local isomorphism* between $\pi_A$ and $\pi_B$ if it is an isomorphism between the subinterpretations $\pi_A|_{\mathrm{Lit}_{\bar{a}}(\tau)}$ and $\pi_B|_{\mathrm{Lit}_{\bar{b}}(\tau)}$.

▶ **Definition 7** (Elementary equivalence). Two $\mathcal{S}$-interpretations $\pi_A\colon \mathrm{Lit}_A(\tau) \to \mathcal{S}$ and $\pi_B\colon \mathrm{Lit}_B(\tau) \to \mathcal{S}$ with elements $\bar{a} \in A^n$ and $\bar{b} \in B^n$ are *elementarily equivalent*, denoted $(\pi_A, \bar{a}) \equiv (\pi_B, \bar{b})$, if $\pi_A[\![\varphi(\bar{a})]\!] = \pi_B[\![\varphi(\bar{b})]\!]$ for all $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$. They are *$m$-equivalent*, denoted $(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$, if the above holds for all $\varphi(\bar{x})$ with $\mathrm{qr}(\varphi(\bar{x})) \leq m$ where $\mathrm{qr}(\varphi(\bar{x}))$ refers to the quantifier rank of $\varphi(\bar{x})$.

As in classical semantics, isomorphic $\mathcal{S}$-interpretations are elementarily equivalent. The converse concerning finite $\mathcal{S}$-interpretations, however, marks an important difference to Boolean semantics; it fails for a number of semirings, including all min-max semirings with at least three elements, while it still holds on other semirings such as $\mathbb{T}, \mathbb{V}, \mathbb{N}$ and $\mathbb{N}[X]$ (see [12]).

## 3 $m$-turn Ehrenfeucht–Fraïssé games

Given that the notion of local isomorphisms extends in a straightforward way from structures to semiring interpretations, we also obtain Ehrenfeucht–Fraïssé games $G_m(\pi_A, \pi_B)$ played on $\mathcal{S}$-interpretations $\pi_A$, $\pi_B$: In the $i$-th turn, Spoiler chooses some element $a_i \in A$ or $b_i \in B$, and Duplicator answers with an element in the other $\mathcal{S}$-interpretation; the play then continues with the subgame $G_{m-i}(\pi_A, a_1, \dots, a_i, \pi_B, b_1, \dots, b_i)$. After $m$ moves, tuples $\bar{a} = (a_1, \dots, a_m)$ in $A$ and $\bar{b} = (b_1, \dots, b_m)$ in $B$ have been selected, and Duplicator wins the play if $\sigma\colon \bar{a} \mapsto \bar{b}$ is a local isomorphism.

However, while classical structures $\mathfrak{A}$ and $\mathfrak{B}$ are separated by a formula $\exists x\psi(x)$ or $\forall x\psi(x)$ if, and only if, there is some $a \in A$ (or $b \in B$) such that for all $b \in B$ (or $a \in A$, respectively) the formula $\psi(x)$ separates $(\mathfrak{A}, a)$ from $(\mathfrak{B}, b)$, neither of the implications translates to semiring semantics. This is illustrated by very simple semiring interpretations[2] and causes both soundness and completeness of $G_m(\pi_A, \pi_B)$ for $\equiv_m$ to fail in general.

$(\mathbb{N}, +, \cdot, 0, 1)$

$\pi_A$:

| $A$ | $R$ | $\neg R$ |
|---|---|---|
| $a_1$ | 1 | 0 |
| $a_2$ | 1 | 0 |
| $a_3$ | 2 | 0 |

$\pi_B$:

| $B$ | $R$ | $\neg R$ |
|---|---|---|
| $b_1$ | 1 | 0 |
| $b_2$ | 2 | 0 |
| $b_3$ | 2 | 0 |

$$\pi_A[\![\exists x Rx]\!] = 4 \neq 5 = \pi_B[\![\exists x Rx]\!]$$

$(\{0,1,2,3,4\}, \max, \min, 0, 4)$

$\pi_A$:

| $A$ | $R$ | $\neg R$ |
|---|---|---|
| $a_1$ | 1 | 0 |
| $a_2$ | 2 | 0 |
| $a_3$ | 4 | 0 |

$\pi_B$:

| $B$ | $R$ | $\neg R$ |
|---|---|---|
| $b_1$ | 1 | 0 |
| $b_2$ | 3 | 0 |
| $b_3$ | 4 | 0 |

$$\pi_A[\![\exists x Rx]\!] = 4 = \pi_B[\![\exists x Rx]\!]$$
$$\pi_A[\![\forall x Rx]\!] = 1 = \pi_B[\![\forall x Rx]\!]$$

This suggests that the direct adaptation of the game rules poses problems and raises the question on which semirings the game $G_m$ is sound, and on which it is complete for $\equiv_m$. In particular, we aim to relate this to the algebraic properties of the underlying semiring.

## 3.1 Soundness of the games and counting in semirings

The fact that quantifiers in classical semantics do not capture counting is one of the central limitations of the expressive power of first-order logic. However, in semiring semantics, this is more complicated: Given a formula $\psi(x)$ and some $s \in \mathcal{S}$, the number of $a \in A$ such that $\pi[\![\psi(a)]\!] = s$ may affect both $\pi[\![\exists x\psi(x)]\!]$ and $\pi[\![\forall x\psi(x)]\!]$. Only in fully idempotent semirings unequal sums or products can be attributed to differing sets of summands or factors, which causes full idempotence to be a necessary and sufficient condition for the soundness of $G_m$.

▶ **Theorem 8.** *The games $G_m$ are sound for $\equiv_m$ on a semiring $\mathcal{S}$ and all $m \in \mathbb{N}$ if, and only if, $\mathcal{S}$ is fully idempotent.*

**Proof.** ($\Leftarrow$): Suppose that $\mathcal{S}$ is fully idempotent. Based on a separating formula $\varphi(\bar{x}) \in \mathrm{FO}(\tau)$ with $\pi_A[\![\varphi(\bar{a})]\!] \neq \pi_B[\![\varphi(\bar{b})]\!]$ and $\mathrm{qr}(\varphi(\bar{x})) \leq m$ where $\bar{a} \in A^n$ and $\bar{b} \in B^n$, we construct a winning strategy for Spoiler in the game $G_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ by induction. We only consider the cases $\varphi(\bar{x}) = Qx\psi(\bar{x}, x)$ with $Q \in \{\exists, \forall\}$ where $\mathrm{qr}(\varphi(\bar{x})) \leq m$. It holds that

$$\pi_A[\![\exists x\psi(\bar{a}, x)]\!] = \sum_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] \neq \sum_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!] = \pi_B[\![\exists x\psi(\bar{b}, x)]\!] \text{ or}$$
$$\pi_A[\![\forall x\psi(\bar{a}, x)]\!] = \prod_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] \neq \prod_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!] = \pi_B[\![\forall x\psi(\bar{b}, x)]\!].$$

Both cases imply $\{\pi_A[\![\psi(\bar{a}, a)]\!] : a \in A\} \neq \{\pi_B[\![\psi(\bar{b}, b)]\!] : b \in B\}$ due to full idempotence. Spoiler wins the game $G_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ by choosing some element $a \in A$ or $b \in B$ witnessing this inequality. For all possible answers $b \in B$ or $a \in A$, respectively, it holds that $\pi_A[\![\psi(\bar{a}, a)]\!] \neq \pi_B[\![\psi(\bar{b}, b)]\!]$. Applying the induction hypothesis yields that Spoiler has a winning strategy for the remaining game $G_{m-1}(\pi_A, \bar{a}, a, \pi_B, \bar{b}, b)$ as $\mathrm{qr}(\psi(\bar{x}, x)) \leq m - 1$.

[2] We describe semiring interpretations over a monadic vocabulary by tables, whose rows are indexed by elements of the universe, and columns by the predicate symbols and their negations, such that the entry for row $a$ and column $P$ has the semiring value of the literal $Pa$.

(⇒): If $\mathcal{S}$ is not fully idempotent, there is some $s \in \mathcal{S}$ such that $s + s \neq s$ or $s \cdot s \neq s$. Clearly, Duplicator wins $G_1(\pi_A, \pi_B)$ on the following $\mathcal{S}$-interpretations, while $\pi_A \not\equiv_1 \pi_B$ due to $\pi_A[\![\exists x Rx]\!] = s + s \neq s = \pi_B[\![\exists x Rx]\!]$ or $\pi_A[\![\forall x Rx]\!] = s \cdot s \neq s = \pi_B[\![\forall x Rx]\!]$.

$\pi_A$ :

| $A$ | $R$ | $\neg R$ |
|---|---|---|
| $a_1$ | $s$ | 0 |
| $a_2$ | $s$ | 0 |

$\pi_B$ :

| $B$ | $R$ | $\neg R$ |
|---|---|---|
| $b$ | $s$ | 0 |

◀

This result motivates the consideration of more powerful games such as $m$-turn bijection games, a variant of the pebble games which, on finite classical structures, characterise $m$-equivalence in FO with counting quantifiers [17].

▶ **Definition 9.** The game $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$ uses the same positions and winning condition as $G_m(\pi_A, \bar{a}, \pi_B, \bar{b})$, but in each round Duplicator has to provide a bijection $h \colon A \to B$. If such a bijection does not exist, i.e. $|A| \neq |B|$, Spoiler wins immediately. Otherwise, Spoiler chooses some $a \in A$ and the pair $(a, h(a))$ is added to the current position.

In contrast to the classical Ehrenfeucht–Fraïssé game, this modification ensures soundness without requiring full idempotence of the underlying semiring.

▶ **Theorem 10.** *For every $m \in \mathbb{N}$, the game $BG_m$ is sound for $\equiv_m$ on every semiring $\mathcal{S}$.*

**Proof.** Suppose that $\varphi(\bar{x}) = Qx\psi(\bar{x}, x)$ with $Q \in \{\exists, \forall\}$ and $\mathrm{qr}(\varphi(\bar{x})) = m$ separates $(\pi_A, \bar{a})$ from $(\pi_B, \bar{b})$. For any bijection $h \colon A \to B$ Duplicator may choose in the game $BG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$, there must be some $a_h \in A$ such that $\pi_A[\![\psi(\bar{a}, a_h)]\!] \neq \pi_B[\![\psi(\bar{b}, h(a_h))]\!]$ since otherwise $\sum_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] = \sum_{a \in A} \pi_B[\![\psi(\bar{b}, h(a))]\!] = \sum_{b \in B} \pi_B[\![\psi(\bar{b}, b)]\!]$ and analogously for products. By choosing $a_h$, Spoiler wins the game by induction. ◀

While demanding a bijection from Duplicator does ensure the soundness of $BG_m$, it is often at the expense of completeness. This is due to the fact that different multiplicities of a semiring value in two interpretations do not necessarily imply separability by a first-order sentence. In particular, this is the case for fully idempotent semirings, on which the games $G_m$ are already sound, but the resulting issues concern other semirings as well. We illustrate this on the semiring $\mathbb{W}[x, y]$, where the precise numbers of occurrences of single semiring values may differ in their effect on the separability of the resulting interpretations, as shown below.

| $A$ | $R$ | $\neg R$ |
|---|---|---|
| $a_1$ | $x + y$ | 0 |

$\not\equiv_1$

| $B$ | $R$ | $\neg R$ |
|---|---|---|
| $b_1$ | $x + y$ | 0 |
| $b_2$ | $x + y$ | 0 |

$\equiv_1$

| $C$ | $R$ | $\neg R$ |
|---|---|---|
| $c_1$ | $x + y$ | 0 |
| $c_2$ | $x + y$ | 0 |
| $c_3$ | $x + y$ | 0 |

We observe that the semirings $\mathbb{W}[X]$, while not being fully idempotent, for instance due to $(x + y)(x + y) = x + xy + y$, satisfy a weaker idempotence condition.

▶ **Definition 11.** Let $n \in \mathbb{N}$. A semiring $\mathcal{S}$ is *$n$-idempotent* if $\sum_{i \in I} s = \sum_{j \in J} s$ and $\prod_{i \in I} s = \prod_{j \in J} s$ for all $s \in \mathcal{S}$ and all index sets $I, J$ such that $|I| \geq n$ and $|J| \geq n$.

It can easily be verified that $\mathbb{W}[X]$ is $|X|$-idempotent as monomials can be seen as sets of variables, and multiplication corresponds to their union. For such semirings, we want to replace the requirement for Duplicator to provide a bijection by a weaker requirement that still maintains soundness. For this, we use counting games, introduced by Immermann and Lander [18], which are equivalent to bijection games, but admit a parametrisation by the size of the sets that are picked in each turn.

▶ **Definition 12.** Let $n \in \mathbb{N}$. In each turn of the game $CG_m^n(\pi_A, \bar{a}, \pi_B, \bar{b})$, Spoiler chooses a set $X \subseteq A$ or $X \subseteq B$ with $|X| \leq n$ and Duplicator has to react with a subset $Y$ of the other universe such that $|X| = |Y|$. Afterwards, Spoiler picks some $y \in Y$, Duplicator must respond with some element $x \in X$ and the pair $(x, y)$, or $(y, x)$, is added to the current position. As before, the winning condition is given by local isomorphism.

Note that the game $CG_m^1$ corresponds to the classical Ehrenfeucht–Fraïssé game $G_m$ and 1-idempotence coincides with full idempotence. Theorem 8 can be generalised as follows.

▶ **Theorem 13.** *The game $CG_m^n$ is sound for $\equiv_m$ exactly on $n$-idempotent semirings $\mathcal{S}$.*

## 3.2 Completeness and incompleteness

As opposed to a Boolean quantifier or a move in an Ehrenfeucht–Fraïssé game, a quantifier in semiring semantics does not pick out a specific element of the universe. Instead, it induces a sum or product over all elements. As a consequence, completeness of the $m$-turn Ehrenfeucht–Fraïssé game, and thus also completeness of other variants of model comparison games, fail in general. In particular, this applies to semirings on which elementary equivalence of finite interpretations does not imply isomorphism. Indeed, on any pair of finite non-isomorphic semiring interpretations, Spoiler wins $G_m$ for sufficiently large $m$ by picking all elements in the larger universe, or in any universe if both have the same cardinality. A particular example, presented in [12], of non-isomorphic but elementarily equivalent $\mathcal{S}$-interpretations $\pi_A^{s,t}$ and $\pi_B^{s,t}$ for arbitrary elements $s, t$ of a fully idempotent semiring $\mathcal{S}$ is the following:

$\pi_A^{s,t}$ :

| $A$ | $R_1$ | $R_2$ | $\neg R_1$ | $\neg R_2$ |
|---|---|---|---|---|
| $a_1$ | 0 | $t$ | $s$ | 0 |
| $a_2$ | $s$ | 0 | 0 | $t$ |
| $a_3$ | $t$ | $s$ | 0 | 0 |
| $a_4$ | 0 | 0 | $t$ | $s$ |

$\pi_B^{s,t}$ :

| $B$ | $R_1$ | $R_2$ | $\neg R_1$ | $\neg R_2$ |
|---|---|---|---|---|
| $b_1$ | $t$ | 0 | 0 | $s$ |
| $b_2$ | 0 | $s$ | $t$ | 0 |
| $b_3$ | $s$ | $t$ | 0 | 0 |
| $b_4$ | 0 | 0 | $s$ | $t$ |

For any $s, t \in \mathcal{S}$, we have that $\pi_A^{s,t} \equiv \pi_B^{s,t}$ [12, Theorem 13], but obviously, Spoiler even wins the game $G_1(\pi_A^{s,t}, \pi_B^{s,t})$ for distinct and non-zero values $s, t \in \mathcal{S}$. Thus, completeness of $G_m$ for $\equiv_m$ and full idempotence are mutually exclusive on semirings with at least three elements, while soundness requires full idempotence, which entails the following result.

▶ **Theorem 14.** *If, for all $m \in \mathbb{N}$, the game $G_m$ is sound and complete for $\equiv_m$ on $\mathcal{S}$, then $\mathcal{S}$ is isomorphic to $\mathbb{B}$.*

Several further semirings, such as $\mathbb{L}, \mathbb{W}[X], \mathbb{S}[X]$ or $\mathbb{B}[X]$, admit pairs of finite interpretations that are non-isomorphic but elementarily equivalent, which immediately disproves completeness of $G_m$ for $\equiv_m$ on those semirings (see Figure 1). Moreover, even on semirings such as $\mathbb{T}$, $\mathbb{N}$ and $\mathbb{N}[X]$, where it is known that elementary equivalence does coincide with isomorphism on finite interpretations [12], $G_m$ is not necessarily complete. As a counterexample on the tropical semiring $\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$, consider the following $\mathbb{T}$-interpretations.

$\pi_A$ :

| $A$ | $R$ | $\neg R$ |
|---|---|---|
| $a_0$ | 0 | $\infty$ |
| $a_1$ | 1 | $\infty$ |
| $a_2$ | 1 | $\infty$ |

$\pi_B$ :

| $B$ | $R$ | $\neg R$ |
|---|---|---|
| $b_0$ | 0 | $\infty$ |
| $b_1$ | 2 | $\infty$ |

Clearly, Spoiler already wins $G_1(\pi_A, \pi_B)$, but we can show that $\pi_A \equiv_1 \pi_B$, thus the game $G_1$ is incomplete for $\equiv_1$ on $\mathbb{T}$. The 1-equivalence immediately follows from the following criterion.

▶ **Proposition 15.** *Two $\mathbb{T}$-interpretations $\pi_A, \pi_B$ over vocabulary $\tau = \{R\}$ consisting of a single unary relation symbol are 1-equivalent if*
**(1)** $\pi_A(\neg Ra) = \pi_B(\neg Rb) = \infty$ *for all $a \in A$ and $b \in B$,*
**(2)** $\min_{a\in A} \pi_A(Ra) = \min_{b\in B} \pi_B(Rb)$ *and*
**(3)** $\sum_{a\in A} \pi_A(Ra) = \sum_{b\in B} \pi_B(Rb)$.

On the other side, in contrast to the classical $m$-turn Ehrenfeucht–Fraïssé game, there are semirings other than $\mathbb{B}$ on which the $m$-turn bijection game is both sound and complete.

▶ **Theorem 16.** *For every $m \in \mathbb{N}$, the bijection game $BG_m$ is sound and complete for $\equiv_m$ on $\mathbb{N}$ and $\mathbb{N}[X]$.*

## 4 Characterising elementary equivalence

The Ehrenfeucht–Fraïssé theorem also provides a game-theoretic characterisation of elementary equivalence via the game $G(\mathfrak{A}, \mathfrak{B})$, where Spoiler chooses the number of turns at the beginning of each play. We now discuss soundness and completeness of $G$ for $\equiv$ on semirings. For classical structures, soundness and completeness of $G$ for $\equiv$ is equivalent to soundness and completeness of $G_m$ for $\equiv_m$, for all $m$, but this is in general not the case on semirings.

For the study of the game $G$, interpretations on infinite universes are of particular interest. This especially applies to soundness, which is trivial in the finite case since a winning strategy for Duplicator already implies isomorphism on finite interpretations. Semiring semantics for infinite interpretations requires sum and product operators on infinite families $(s_i)_{i\in I} \subseteq S$ of semiring elements. There are certain semirings such as $\mathbb{N}, \mathbb{N}[X], \mathbb{B}[X]$ and $\mathbb{S}[X]$ which do not admit a reasonable definition of such infinitary operations, and we thus have to restrict ourselves to finite universes. Otherwise, we make use of the natural order and interpret infinite sums according to $\sum_{i\in I} s_i := \sup\{\sum_{i\in I'} s_i | I' \subseteq I \text{ finite}\}$. For infinitary products we distinguish the case of absorptive semirings, where multiplication is decreasing and we thus interpret the product as the *infimum* of the finite subproducts, and the cases, such as $\mathbb{N}^\infty$ or $\mathbb{W}[X]$, where multiplication is increasing and we replace infima by suprema. Previous results such as the soundness of $G_m$ on fully idempotent semirings straightforwardly extend to infinite interpretations by transferring semiring properties such as full idempotence to the infinitary operations.

Soundness of $G$ for $\equiv$ holds whenever Spoiler wins $G(\pi_A, \pi_B)$ for all first-order separable interpretations $\pi_A$ and $\pi_B$. Thus, the following question is essential: Given $\pi_A$, $\pi_B$ and a separating sentence $\psi$, is the required number of turns for Spoiler to win $G(\pi_A, \pi_B)$ bounded in advance? On fully idempotent semirings, $m := \mathrm{qr}(\psi)$ turns suffice for Spoiler to win $G(\pi_A, \pi_B)$ since $G_m$ is sound for $\equiv_m$, which immediately implies soundness of $G$ on all fully idempotent semirings. However, full idempotence is not a necessary condition, soundness of $G$ is still preserved on many semirings that admit a weaker bound than $m$. For instance, on any $n$-idempotent semiring for some $n \in \mathbb{N}$, $n \cdot m$ turns suffice to ensure Spoiler's victory.

▶ **Proposition 17.** *Let $\mathcal{S}$ be $n$-idempotent for some $n \in \mathbb{N}$. For any $\mathcal{S}$-interpretations $\pi_A$ and $\pi_B$ it holds that $\pi_A \equiv_m \pi_B$ if Duplicator wins the game $G_{nm}(\pi_A, \pi_B)$. In particular, the game $G$ is sound for $\equiv$ on $\mathcal{S}$.*

This follows from soundness of $n$-counting games as stated in Theorem 13. If $\pi_A \not\equiv_m \pi_B$, Spoiler wins $G_{nm}(\pi_A, \pi_B)$ by adapting his winning strategy for $CG^n_m(\pi_A, \pi_B)$: Instead of drawing $n$-element sets, he draws $n$ elements one by one. Note that the bound $n \cdot m$ does not depend on $\pi_A$ and $\pi_B$ at all but only on the quantifier rank $m$ and the semiring.

However, other semirings, such as $\mathbb{N}^\infty$, may not admit an inherent bound $t(m) \in \mathbb{N}$ such that a winning strategy of Duplicator for $G_{t(m)}(\pi_A, \pi_B)$ always implies $\pi_A \equiv_m \pi_B$. To demonstrate this, consider a pair of sets ($\mathbb{N}^\infty$-interpretations with empty vocabulary) with $t(m)$ and $t(m)+1$ elements, respectively. Clearly, Duplicator wins on those sets for up to $t(m)$ turns, but the sentence $\exists x(x = x)$ with quantifier rank 1 suffices to separate them.

In order to prove that the game $G$ is still sound for $\equiv$ on $\mathbb{N}^\infty$, it is crucial to observe that two separable interpretations $\pi_A, \pi_B$ with $\pi_A[\![\psi]\!] \neq \pi_B[\![\psi]\!]$ admit a parameter $k$ that induces an upper bound on the number of moves required by Spoiler to win $G(\pi_A, \pi_B)$. On $\mathbb{N}^\infty$, this parameter is easily obtained by observing that $\pi_A[\![\psi]\!]$ or $\pi_B[\![\psi]\!]$ is finite (see [4] for a proof).

▶ **Theorem 18.** *Let $\pi_A$ and $\pi_B$ be $\mathbb{N}^\infty$-interpretations with elements $\bar{a} \in A^n$, $\bar{b} \in B^n$ and $k \geq 1$. If there is a separating formula $\varphi(\bar{x})$ with $\operatorname{qr}(\varphi(\bar{x})) \leq m$ such that $\pi_A[\![\varphi(\bar{a})]\!] < k$ or $\pi_B[\![\varphi(\bar{b})]\!] < k$, then Spoiler wins $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$.*

It turns out that a similar approach is applicable to the semiring $\mathbb{S}^\infty[X]$, which extends the semiring $\mathbb{S}[X]$ of absorptive polynomials to allow infinite exponents (and thus infinite products), albeit the derivation of a suitable parameter is more involved. Recall that a monomial $m$ *absorbs* a monomial $m'$ if the exponents for all $x \in X$, denoted by $m(x)$ and $m'(x)$ respectively, satisfy $m(x) \leq m'(x)$, and that absorptive polynomials only retain absorption-dominant monomials. We say that a monomial $m$ *separates* polynomials $p$ and $q$ if $m \in p$ and $m$ is not absorbed by any monomial from $q$.

These concepts can be extended to any subset $Y \subseteq X$: $m$ *$Y$-absorbs* $m'$ iff $m(x) \leq m'(x)$ for $x \in Y$, and it is *$Y$-separating* for $p$ and $q$ if it is contained in one of the polynomials but not $Y$-absorbed by any of the monomials from the other polynomial. Finally, we can parametrise monomials $m$ by adding their exponents $e_Y(m) := \sum_{x \in Y} m(x)$ for all the variables $x \in Y$. Now, we can extract a finite parameter from any pair of distinct polynomials $p, q$ as follows.

▶ **Lemma 19.** *For any two distinct polynomials $p, q \in \mathbb{S}^\infty[X]$, there is a set $Y \subseteq X$ and a $Y$-separating monomial $m$ such that the parameter $e_Y(m)$ is finite.*

**Proof.** It is known that $p \leq q$ holds if, and only if, every monomial in $p$ is absorbed by some monomial from $q$ (see [7]). Thus, there is a monomial $m$ in either $p$ or $q$ that is not absorbed by any monomial from the other polynomial. Otherwise, $p$ and $q$ would absorb each other, which would imply $p = q$. Pick $Y := \{x \in X \mid m(x) \neq \infty\}$. It follows that $e_Y(m)$ is finite and that $m$ is not $Y$-absorbed by any monomial from the other polynomial since any $m'$ that $Y$-absorbs $m$ would also absorb $m$ entirely. ◀

For example, $x^n y^\infty$ and $x^\infty y^\infty$ are $\{x\}$-separated by $m := x^n y^\infty$ with $e_{\{x\}}(x^n y^\infty) = n$. This property can be exploited to limit the number of turns required by Spoiler to win $G(\pi_A, \pi_B)$ on separable $\mathbb{S}^\infty[X]$-interpretations.

▶ **Theorem 20.** *Let $k \geq 1$ and $\pi_A, \pi_B$ be $\mathbb{S}^\infty[X]$-interpretations with elements $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$. If there is a separating formula $\varphi(\bar{x})$ with $\operatorname{qr}(\varphi(\bar{x})) \leq m$, a set $Y \subseteq X$ and a $Y$-separating monomial $m$ for $\pi_A[\![\varphi(\bar{a})]\!]$ and $\pi_B[\![\varphi(\bar{b})]\!]$ such that $e_Y(m) < k$, then Spoiler wins $G_{km}(\pi_A, \bar{a}, \pi_B, \bar{b})$.*

**Proof.** We show the claim by structural induction on the separating formula $\varphi(\bar{x})$. Since $\pi_A$ and $\pi_B$ are interchangeable, we may assume w.l.o.g. that the $Y$-separating monomial $m$ is part of $\pi_A[\![\varphi(\bar{a})]\!]$. If $\varphi(\bar{x})$ is a literal, Spoiler wins immediately.

- If $\varphi(\bar{x}) = \varphi_1(\bar{x}) \vee \varphi_2(\bar{x})$, the $Y$-separating monomial $m$ must be part of $\pi_A[\![\varphi_i(\bar{a})]\!]$ for some $i \in \{1, 2\}$, but by definition, it cannot be $Y$-absorbed by any monomial in $\pi_B[\![\varphi_i(\bar{b})]\!]$. Thus, $m$ $Y$-separates $\pi_A[\![\varphi_i(\bar{a})]\!]$ from $\pi_B[\![\varphi_i(\bar{b})]\!]$ and the claim follows by induction.
- If $\varphi(\bar{x}) = \exists x \psi(\bar{x}, x)$, then $\pi_A[\![\varphi(\bar{a})]\!] = \sum_{a\in A} \pi_A[\![\psi(\bar{a}, a)]\!]$, and analogous to the previous case, we observe that $m$ is part of $\pi_A[\![\psi(\bar{a}, a)]\!]$ for some $a \in A$, but not $Y$-absorbed by any $\pi_B[\![\psi(\bar{b}, b)]\!]$ for $b \in B$. Thus, Spoiler can pick such an element $a \in A$ and win the remaining subgame by induction hypothesis.
- If $\varphi(\bar{x}) = \varphi_1(\bar{x}) \wedge \varphi_2(\bar{x})$, the $Y$-separating monomial $m = m_1 \cdot m_2$ is obtained by multiplying two monomials with $m_i \in \pi_A[\![\varphi_i(\bar{a})]\!]$ for $i \in \{1, 2\}$. There is at least one $i \in \{1, 2\}$ such that $m_i$ $Y$-separates $\pi_A[\![\varphi_i(\bar{a})]\!]$ from $\pi_B[\![\varphi_i(\bar{b})]\!]$. Otherwise, each $m_i$ would be $Y$-absorbed by some $m'_i \in \pi_B[\![\varphi_i(\bar{b})]\!]$, which would yield a contradiction since $m' = m'_1 \cdot m'_2 \in \pi_B[\![\varphi(\bar{b})]\!]$ would $Y$-absorb $m$. Clearly, $e_Y(m_i) \leq e_Y(m) < k$, hence Spoiler wins by invoking the induction hypothesis on the suitable subformula.
- If $\varphi(\bar{x}) = \forall x \psi(\bar{x}, x)$, then $\pi_A[\![\varphi(\bar{a})]\!] = \prod_{a\in A} \pi_A[\![\psi(\bar{a}, a)]\!]$. Decompose the monomial $m$ into $m = \prod_{a\in A} m_a$ such that $m_a \in \pi_A[\![\psi(\bar{a}, a)]\!]$ holds for all $a \in A$. It follows that $e_Y(m) = \sum_{a\in A} e_Y(m_a) < k$, thus $e_Y(m_a)$ is nonzero for $\ell < k$ elements $a_1, \dots, a_\ell \in A$ and zero otherwise. Spoiler picks those elements and Duplicator replies with $b_1, \dots, b_\ell$.
  - If there is any $1 \leq i \leq \ell$ such that $m_{a_i}$ is not $Y$-absorbed by any monomial in $\pi_B[\![\psi(\bar{b}, b_i)]\!]$, then $m_{a_i}$ $Y$-separates $\pi_A[\![\psi(\bar{a}, a_i)]\!]$ from $\pi_B[\![\psi(\bar{b}, b_i)]\!]$, and together with $e_Y(m_{a_i}) \leq e_Y(m) < k$, we can apply the induction hypothesis.
  - Otherwise, each $m_{a_i}$ is $Y$-absorbed by some $m_{b_i} \in \pi_B[\![\psi(\bar{b}, b_i)]\!]$. Since $\prod_{i=1}^{\ell} m_{b_i}$ $Y$-absorbs $m$, it is impossible that each $\pi_B[\![\psi(\bar{b}, b)]\!]$ for $b \in B \setminus \{b_1, \dots, b_\ell\}$ contains some monomial $m'$ with $e_Y(m') = 0$. Otherwise, those monomials would not contribute anything to the exponents of variables $x \in Y$, and their product together with $m_{b_1}, \dots, m_{b_\ell}$ would result in a monomial $m'' \in \pi_B[\![\varphi(\bar{b})]\!]$ that $Y$-absorbs $m$, contradicting the definition of $m$. Now, it only remains for Spoiler to pick some $b \in B \setminus \{b_1, \dots, b_\ell\}$ such that $\pi_B[\![\psi(\bar{b}, b)]\!]$ only contains monomials $m'$ with $e_Y(m') > 0$. Duplicator must answer $a \in A \setminus \{a_1, \dots, a_\ell\}$, but then $e_Y(m_a) = 0$, hence $m_a$ $Y$-separates $\pi_A[\![\psi(\bar{a}, a)]\!]$ from $\pi_B[\![\psi(\bar{b}, b)]\!]$ and we can apply the induction hypothesis. ◀

▶ **Corollary 21.** *The game $G$ is sound for $\equiv$ on the semirings $\mathbb{W}[X], \mathbb{N}^\infty$ and $\mathbb{S}^\infty[X]$.*

However, $G$ is unsound for some important semirings. We construct a counterexample for the soundness of $G$ on $\equiv$ in the tropical semiring (which is isomorphic to the Viterbi semiring $\mathbb{V}$) and transfer it to the isomorphic variant $\mathbb{D}$ of $\mathbb{L}$ by making sure that the valuations are in the interval $[0, 1]$, and that the separating formula does not evaluate to a semiring element greater than 1 in both interpretations. The main idea behind the construction is that, given a sequence $(s_i)_{\geq 1}$ of edge labels, Spoiler cannot distinguish an infinite star with exactly $i$ edges labelled with $s_i \in \mathbb{T}$ for each $i \in \mathbb{N}$ from an infinite star where $\min(i, m)$ edges are labelled with $s_i$ (see Figure 2). However, for an appropriate sequence of edge labels such star graphs with distinguished centre nodes can be separated in FO by summing up all edge labels using the formula $\psi(x) = \forall y(x = y \vee Exy)$.

▶ **Lemma 22.** *There is a sequence $(s_i)_{i\geq 1}$ of real numbers in $[0, 1]$ such that for each $m \in \mathbb{N}_{>0}$*

$$1 > \sum_{i\geq 1} i \cdot s_i > \sum_{i\geq 1} \min(i, m) \cdot s_i.$$

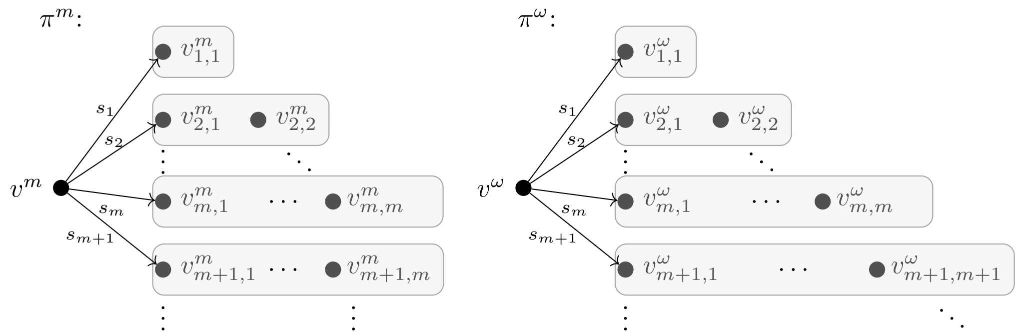


**Figure 2** Infinite star graphs used to construct a counterexample against the soundness of the game $G$ with respect to $\mathbb{T}$- and $\mathbb{D}$-interpretations. The grey boxes are meant to indicate $\pi^m[\![Ev^m v^m_{i,j}]\!] = \pi^\omega[\![Ev^m v^m_{i,j}]\!] = s_i$ for each $j$. Non-edges are assigned their Boolean truth value.

**Proof.** We prove the claim for $(s_i)_{i\geq 1}$ where $s_i := \frac{1}{i\cdot 2^{i+1}}$. Due to convergence of the geometrical series we obtain that $\sum_{i\geq 1} i\cdot s_i = 0.5$. Further,

$$\sum_{i\geq 1} i\cdot s_i = \sum_{i\geq 1}\min(i,m)\cdot s_i + \underbrace{\sum_{i>m}(m-i)\cdot s_i}_{>0} > \sum_{i\geq 1}\min(i,m)\cdot s_i,$$

which implies the claim. ◀

In order to ensure that Duplicator wins the game $G_m$ for each $m \in \mathbb{N}$ on single semiring interpretations $\pi$ and $\pi'$, we combine the star graphs $\pi^m$ for arbitrarily large $m$. The idea is to include in both $\pi$ and $\pi'$ the star graphs $\pi^m$ for each $m \in \mathbb{N}$ as disjoint subgraphs, and to add an additional copy of $\pi^\omega$ to $\pi'$ only. Using the sequence of edge labels satisfying $\sum_{i\geq 1} i\cdot s_i > \sum_{i\geq 1}\min(i,m)\cdot s_i$ for each $m \in \mathbb{N}_{>0}$ yields $\pi^\omega[\![\psi(v^\omega)]\!] > \pi^m[\![\psi(v^m)]\!]$, so the additional subgraph $\pi^\omega$ in $\pi'$ would not contribute to the valuation of the sentence $\exists x\psi(x)$. Hence, we add additional vertices to the star graphs $\pi^m$ in both $\pi$ and $\pi'$ which increase the sum over all outgoing edges and cause $\exists x\psi(x)$ to separate the resulting semiring interpretations.

▶ **Theorem 23.** *The game $G$ is not sound for $\equiv$ on $\mathbb{T}, \mathbb{D}, \mathbb{V}$ and $\mathbb{L}$.*

**Proof.** Let $\mathcal{S} \in \{\mathbb{T}, \mathbb{D}\}$ and $(s_i)_{i\geq 1}$ be defined by $s_i := \frac{1}{i\cdot 2^{i+1}}$. Further, let $s^m_\infty$ denote $\sum_{i\geq 1}\min(i,m)\cdot s_i$ for each $m \geq 1$. We inductively define a function $f\colon \mathbb{N}\setminus\{0\} \to \mathbb{N}\setminus\{0\}$ which determines the number of additional nodes that are added to the star graphs. Let $f(1)$ be the smallest number such that $s^1_\infty + f(1)\cdot s_1 > 0.5$. For $m > 1$, we define $f(m)$ as the minimum number yielding $s^m_\infty + f(m)\cdot s_m \geq s^{m-1}_\infty + f(m-1)\cdot s_{m-1}$. Since $0 < s_i < 1$ for all $i \geq 1$, $f$ is well-defined. Hence, we obtain a chain $s^1_\infty + f(1)\cdot s_1 \leq s^2_\infty + f(2)\cdot s_2 \leq \dots$ which is strictly upper bounded by 0.5. Based on $f$ and $(s_i)_{i\geq 1}$, we construct $\mathcal{S}$-interpretations $\pi$ and $\pi'$ over the vocabulary $\tau = \{E\}$ consisting of a binary relation symbol. The universes $V$ and $V'$ are composed as follows.

$$\begin{aligned} V &= \ \{v^m : m \geq 1\} \cup \{v^m_{i,j} : j \leq \min(i,m)\} \cup \{v^m_{m,m+j} : j \leq f(m)\} \\ V' &= V \ \cup \{v^\omega\} \cup \{v^\omega_{i,j} : j \leq i\} \end{aligned}$$

The valuations in $\pi$ and $\pi'$ are defined according to the following rules, which apply to all $m, n, i, j \in \mathbb{N}_{>0}$ with $m \neq n$ such that the respective nodes are contained in $V$ or $V'$.

- $\pi(Ev^m v^m_{i,j}) = \pi'(Ev^m v^m_{i,j}) = \pi'(Ev^\omega v^\omega_{i,j}) = s_i$
- $\pi(Ev^m v^n_{i,j}) = \pi'(Ev^m v^n_{i,j}) = \pi'(Ev^\omega v^m_{i,j}) = \pi'(Ev^m v^\omega_{i,j}) = 1$
- $\pi(Ev^m v^n) = \pi'(Ev^m v^n) = \pi'(Ev^\omega v^m) = \pi'(Ev^m v^\omega) = 1$

Further, the negations of the instantiated $\tau$-literals defined above are valuated with 0. All remaining unnegated $\tau$-literals over $V$ and $V'$ are valuated with 0 and their negations with 1. In both $\mathbb{T}$ and $\mathbb{D}$, we obtain the following valuations of the formula $\psi(x) = \forall y(x = y \vee Exy)$.

- $\pi[\![\psi(v^m_{i,j})]\!] = \pi'[\![\psi(v^m_{i,j})]\!] = \pi'[\![\psi(v^\omega_{i,j})]\!] = 0$
- $\pi[\![\psi(v^m)]\!] = \pi'[\![\psi(v^m)]\!] = s^m_\infty + f(m) \cdot s_m$
- $\pi'[\![\psi(v^\omega)]\!] = 0.5$

By construction of $f$, this implies $\pi_A[\![\exists x\psi(x)]\!] = s^1_\infty + f(1) \cdot s_1 > 0.5 = \pi_B[\![\exists x\psi(x)]\!]$, hence $\pi_A \not\equiv_2 \pi_B$. In order to construct a winning strategy for Duplicator in the game $G(\pi, \pi')$, let $V^n_0 = \{v^n\}$ and $V^n_i$ for $i \geq 1$ contain all elements $v^n_{i,j}$ in $V$. We consider the partition $\mathcal{P} := \{V^n_i : n \geq 1, i \geq 0\}$ of $V$ and $\mathcal{P}' := \mathcal{P} \cup \{V^\omega_i : i \geq 0\}$ of $V'$. Based on the number of turns $m$ Spoiler chooses in the game $G(\pi_A, \pi_B)$, we define a bijection $g_m : \mathcal{P} \to \mathcal{P}'$ as follows.

$$g_m(V^n_i) := \begin{cases} V^n_i, & n < m \\ V^\omega_i, & n = m \\ V^{n-1}_i, & n > m \end{cases}$$

Duplicator wins the game $G_m(\pi, \pi')$ by responding to any element in $V^n_i \subseteq V$ with an arbitrary element in $g_m(V^n_i)$ and every element in $V^n_i \subseteq V'$ with any element in $g^{-1}_m(V^n_i)$, merely making sure that (in)equalities with regard to the elements that have already been chosen are respected. This is possible because for each $V^n_i$ we have that $|V^n_i| = |g_m(V^n_i)|$ or that $|V^n_i| \geq m$ and $|g_m(V^n_i)| \geq m$. ◀

We now turn to the study of completeness. Analogous to $m$-turn Ehrenfeucht–Fraïssé games, the game $G$ cannot be complete for semirings where elementary equivalence and isomorphism of finite interpretations do not coincide since Duplicator clearly loses $G$ on non-isomorphic finite interpretations. In the remaining cases, $G$ must be complete with respect to finite interpretations because Spoiler winning the game implies non-isomorphism, but on finite interpretations, this already implies separability by a first-order formula.

▶ **Proposition 24.** *Let $\mathcal{S} \in \{\mathbb{T}, \mathbb{V}, \mathbb{N}, \mathbb{N}[X]\}$. If Spoiler wins $G(\pi_A, \pi_B)$ and $\pi_A$, $\pi_B$ are finite $\mathcal{S}$-interpretations, then $\pi_A \not\equiv \pi_B$. Thus, $G$ is complete for $\equiv$ on finite $\mathcal{S}$-interpretations.*

The question arises whether this completeness result can be lifted to infinite semiring interpretations. For the tropical semiring $\mathbb{T}$ we describe a counterexample which proves that $G$ is incomplete for $\equiv$ on $\mathbb{T}$ (and hence also on $\mathbb{V}$ due to $\mathbb{V} \cong \mathbb{T}$).

▶ **Theorem 25.** *There are $\mathbb{T}$-interpretations $\pi_A, \pi_B$ such that Spoiler wins $G_1(\pi_A, \pi_B)$ although $\pi_A \equiv \pi_B$. In particular, $G$ is incomplete for $\equiv$ on $\mathbb{T}$.*

**Proof.** Let $\pi_A$ and $\pi_B$ be $\mathbb{T}$-interpretations with just one unary predicate $R$ and universes $A := \{a_i : i \in \mathbb{N}\}$ and $B := \{b_i : i \in \mathbb{N}\}$, whose valuations are $\pi_A(Ra_i) = \pi_B(Rb_i) = 0$ if $i$ is even, while $\pi_A(Ra_i) = 1$ and $\pi_B(Rb_i) = 2$ for all odd $i$; since the interpretations are assumed to be model-defining this implies that $\pi_A(\neg Ra_i) = \pi_B(\neg Rb_i) = \infty$ for all $i \in \mathbb{N}$.

Clearly, Spoiler wins $G_1(\pi_A, \pi_B)$. To prove that $\pi_A \equiv \pi_B$, we first show that for each formula $\varphi(\bar{x})$ the valuations $\pi_A[\![\varphi(\bar{a})]\!]$ and $\pi_B[\![\varphi(\bar{b})]\!]$ can only take the values 0 and $\infty$ if the tuples $\bar{a}$ and $\bar{b}$ only consist of even elements $a_{2\ell}$ and $b_{2\ell}$. The reasoning is identical for both interpretations, so we just consider $\pi_A$, and proceed by induction on $\varphi(\bar{x})$. For literals the claim holds by definition and for conjunctions and disjunctions it follows since $\{0, \infty\}$ is closed under the operations min and $+$.

Consider $\varphi(\bar{x}) = \exists y \psi(\bar{x}, y)$. For all $a \in A$ with $\pi_A(Ra) = 0$, it follows by the induction hypothesis that $\pi_A[\![\psi(\bar{a}, a)]\!] \in \{0, \infty\}$. If there is some $a \in A$ such that $\pi_A[\![\psi(\bar{a}, a)]\!] = 0$, it immediately follows that $\pi_A[\![\varphi(\bar{a})]\!] = \inf_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] = 0$. Hence, it remains to show the claim for the case $\pi_A[\![\psi(\bar{a}, a)]\!] = \infty$ for all $a \in A$ with $\pi_A(Ra) = 0$. Fix some $c \in A$ that is not contained in $\bar{a}$ such that $\pi_A(Rc) = 0$. For each $a \in A$ with $\pi_A(Ra) = 1$ it holds, by monotonicity of the semiring operations, that $\pi_A[\![\psi(\bar{a}, a)]\!] \geq \pi_A[\![\psi(\bar{a}, c)]\!]$ with respect to the usual order on $\mathbb{R}_+^\infty$ (which is the inverse of the natural order on $\mathbb{T}$) and since $\pi_A[\![\psi(\bar{a}, c)]\!] = \infty$, we have that $\pi_A[\![\varphi(\bar{a})]\!] = \inf_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] = \infty$.

Finally, let $\varphi(\bar{x}) = \forall y \psi(\bar{x}, y)$. Again, for all $a \in A$ with $\pi_A(Ra) = 0$ it holds that $\pi_A[\![\psi(\bar{a}, a)]\!] \in \{0, \infty\}$ by induction hypothesis. If there is an $a \in A$ such that $\pi_A[\![\psi(\bar{a}, a)]\!] = \infty$, it immediately follows that $\pi_A[\![\varphi(\bar{a})]\!] = \sum_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] = \infty$. Therefore it remains to show the claim for the case that $\pi_A[\![\psi(\bar{a}, a)]\!] = 0$ for all $a \in A$ with $\pi_A(Ra) = 0$. We observe that for all $a, a' \in A$ that do not occur in $\bar{a}$ with $\pi_A(Ra) = \pi_A(Ra')$, it holds that $(\pi_A, \bar{a}, a) \cong (\pi_A, \bar{a}, a')$. Hence, if there was some $a \in A$ with $\pi_A(Ra) = 1$ such that $\pi_A[\![\psi(\bar{a}, a)]\!] = s$ for some $s > 0$, then $\pi_A[\![\psi(\bar{a}, a)]\!] = s$ would hold for all $a \in A$ with $\pi_A(Ra) = 1$, which implies $\pi_A[\![\varphi(\bar{a})]\!] = \sum_{a \in A} \pi_A[\![\psi(\bar{a}, a)]\!] = \infty$. Otherwise, we have that $\pi_A[\![\psi(\bar{a}, a)]\!] = 0$ for all $a \in A$, thus $\pi_A[\![\varphi(\bar{a})]\!] = 0$, which completes the induction.

In particular we have for every sentence $\varphi \in \mathrm{FO}(\{R\})$ that $\pi_A[\![\varphi]\!], \pi_B[\![\varphi]\!] \in \{0, \infty\}$. We claim that $\pi_A[\![\varphi]\!] = \pi_B[\![\varphi]\!]$. The function $h \colon \mathbb{T} \to \mathbb{T}$ defined by $s \mapsto 2s$ is an endomorphism on $\mathbb{T}$ that is compatible with the infinitary operations, and obviously, $(h \circ \pi_A) \cong \pi_B$. If $\pi_A[\![\varphi]\!] = 0$, then $\pi_B[\![\varphi]\!] = 2 \cdot 0 = 0$ due to the fundamental property. Otherwise, we have that $\pi_A[\![\varphi]\!] = \infty = 2 \cdot \infty = \pi_B[\![\varphi]\!]$. Hence $\pi_A \equiv \pi_B$. ◀

The natural semiring does not admit infinitary operations, so we consider its extension $\mathbb{N}^\infty$ instead. But on $\mathbb{N}^\infty$, counterexamples disproving completeness also exist, see [4].

▶ **Theorem 26.** *There are $\mathbb{N}^\infty$-interpretations $\pi_A$ and $\pi_B$ such that Spoiler wins $G_1(\pi_A, \pi_B)$ although $\pi_A \equiv \pi_B$. In particular, the game $G$ is incomplete for $\equiv$ on $\mathbb{N}^\infty$.*

Consequently, completeness of $G$ for $\equiv$ also fails on any semiring which extends $\mathbb{N}[X]$ and admits infinitary operations if it contains $\mathbb{N}^\infty$ as a subsemiring.

## 5 The homomorphism game

Finally, we propose a new kind of model comparison games referred to as *homomorphism games*. The idea is to reduce a given pair of $\mathcal{S}$-interpretations to $\mathbb{B}$-interpretations via homomorphisms. In general, the resulting $\mathbb{B}$-interpretations are no longer model-defining, which is why their $m$-equivalence is not captured by $G_m$. While soundness of $G_m$ for $\equiv_m$ on fully idempotent semirings $\mathcal{S}$ does not rely on the assumption that the $\mathcal{S}$-interpretations are model-defining, completeness for $\equiv_m$ even fails on $\mathbb{B}$ because a priori there is no connection between literals and their negations. This is illustrated by the $\mathbb{B}$-interpretations $\pi_A$ and $\pi_B$.

$\pi_A$ :

| $A$ | $R_1$ | $R_2$ | $\neg R_1$ | $\neg R_2$ |
|---|---|---|---|---|
| $a_0$ | 1 | 0 | 0 | 0 |
| $a_1$ | 0 | 0 | 0 | 0 |
| $a_2$ | 1 | 1 | 0 | 0 |
| $a_3$ | 0 | 0 | 0 | 0 |
| $a_4$ | 1 | 1 | 0 | 0 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

$\pi_B$ :

| $B$ | $R_1$ | $R_2$ | $\neg R_1$ | $\neg R_2$ |
|---|---|---|---|---|
| $b_0$ | 0 | 0 | 0 | 0 |
| $b_1$ | 1 | 1 | 0 | 0 |
| $b_2$ | 0 | 0 | 0 | 0 |
| $b_3$ | 1 | 1 | 0 | 0 |
| $b_4$ | 0 | 0 | 0 | 0 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

▶ **Proposition 27.** *On $\mathbb{B}$-interpretations that are not model-defining, the game $G$ is incomplete for $\equiv$ and the $m$-turn game $G_m$ is incomplete for $\equiv_m$ for any $m > 0$.*

**Proof.** Consider the interpretations $\pi_A$ and $\pi_B$. We can construct a bijection $\sigma_\leq : A \to B$ such that $\pi_B(\sigma_\leq(L)) \leq \pi_A(L)$ for all $L \in \mathrm{Lit}_A(\tau)$ by mapping $a_0$ to some $b_i$ with even $i$. Similarly, we can construct a bijection $\sigma_\geq : A \to B$ with $\pi_A(L) \leq \pi_B(\sigma_\geq(L))$ for all $L$ by mapping $a_0$ to some $b_i$ with odd $i$.

By structural induction, it follows that $\pi_B[\![\varphi(\sigma_\leq(\bar{a}))]\!] \leq \pi_A[\![\varphi(\bar{a})]\!] \leq \pi_B[\![\varphi(\sigma_\geq(\bar{a}))]\!]$ holds for all first-order formulae $\varphi(\bar{x})$ with $k$ free variables $\bar{x}$ and $\bar{a} \in A^k$. For sentences $\psi$, this yields $\pi_A[\![\psi]\!] = \pi_B[\![\psi]\!]$, hence we have $\pi_A \equiv \pi_B$. However, Spoiler already wins $G_1(\pi_A, \pi_B)$ by picking $a_0$, which proves the claim. ◀

Additionally, finite counterexamples showing the unsoundness of $G_m$ exist as well and can be constructed based on $\pi_A$ and $\pi_B$ by considering suitable subinterpretations of size $2m$ or $2m+1$, respectively (see [4]). Due to Proposition 27, we consider a one-sided variant of the Ehrenfeucht–Fraïssé game which yields a characterisation of $m$-equivalence for $\mathbb{B}$-interpretations without requiring them to be model-defining.

Let $\mathcal{S}$ be naturally ordered by $\leq$ and $\pi_A, \pi_B$ be two $\mathcal{S}$-interpretations. We say that $(\pi_A, \bar{a}) \leq (\pi_B, \bar{b})$ if for every literal $L(\bar{x})$ we have that $\pi_A(L(\bar{a})) \leq \pi_B(L(\bar{b}))$. Further, we say that $(\pi_A, \bar{a}) \preceq_m (\pi_B, \bar{b})$ if it holds that $\pi_A[\![\varphi(\bar{a})]\!] \leq \pi_B[\![\varphi(\bar{b})]\!]$ for any formula $\varphi(\bar{x})$ of quantifier rank at most $m$.

▶ **Definition 28.** The one-sided game $G_m^{\leq}(\pi_A, \pi_B)$ is played in the same way as $G_m(\pi_A, \pi_B)$, but the winning condition for Duplicator, assuming that the tuples $\bar{a}, \bar{b}$ were chosen after $m$ moves, is $(\pi_A, \bar{a}) \leq (\pi_B, \bar{b})$ instead of $(\pi_A, \bar{a}) \equiv_0 (\pi_B, \bar{b})$.

Using monotonicity of both semiring operations with respect to the natural order, we obtain the following soundness result, which can be proved analogously to Theorem 8.

▶ **Proposition 29.** *Let $\mathcal{S}$ be any fully idempotent semiring. Then $G_m^{\leq}$ is sound for $\preceq_m$ on $\mathcal{S}$.*

On $\mathbb{B}$, the one-sided game $G_m^{\leq}$ is also complete for $\preceq_m$ even for $\mathbb{B}$-interpretations that are not model-defining. To prove this, we inductively construct characteristic formulae $\chi^m_{\pi_A,\bar{a}}(\bar{x})$ analogous to the classical Ehrenfeucht–Fraïssé theorem, but we omit literals $\neg R\bar{x}$ in $\chi^0_{\pi_A,\bar{a}}(\bar{x})$ if $\pi_A(R\bar{a}) = 0$. Let $\varphi^{=}_{\bar{a}}(\bar{x})$ define the equalities and inequalities of the elements in $\bar{a}$.

$$\chi^0_{\pi_A,\bar{a}}(\bar{x}) := \varphi^{=}_{\bar{a}}(\bar{x}) \wedge \bigwedge\{L(\bar{x}) \in \mathrm{Lit}_n(\tau) : \pi_A(L(\bar{a})) = 1\}$$
$$\chi^{m+1}_{\pi_A,\bar{a}}(\bar{x}) := \bigwedge_{a \in A} \exists x\, \chi^m_{\pi_A,\bar{a},a}(\bar{x}, x) \wedge \forall x \bigvee_{a \in A} \chi^m_{\pi_A,\bar{a},a}(\bar{x}, x)$$

▶ **Theorem 30.** *For any two $\mathbb{B}$-interpretations $\pi_A$ and $\pi_B$ with elements $\bar{a} \in A^n$ and $\bar{b} \in B^n$ and any $m \in \mathbb{N}$, the following are equivalent:*
**(1)** *Duplicator wins $G_m^{\leq}(\pi_A, \bar{a}, \pi_B, \bar{b})$;*
**(2)** $\pi_B[\![\chi^m_{\pi_A,\bar{a}}(\bar{b})]\!] = 1$;
**(3)** $(\pi_A, \bar{a}) \preceq_m (\pi_B, \bar{b})$.

To construct homomorphism games based on the one-sided games $G_m^{\leq}$ on $\mathbb{B}$-interpretations, we make use of separating sets of homomorphisms, which were introduced in [12].

▶ **Definition 31.** Given semirings $\mathcal{S}$ and $\mathcal{S}'$, a set $H$ of homomorphisms from $\mathcal{S}$ to $\mathcal{S}'$ is called *separating* if for all $s, t \in S$ with $s \neq t$ there is some $h \in H$ with $h(s) \neq h(t)$.

For two given $\mathcal{S}$-interpretations $\pi_A$ and $\pi_B$ which are separable by some sentence $\psi$, we can think of the valuations $s \neq t$ of $\psi$ in $\pi_A$ and $\pi_B$, respectively, as witnesses for the separability of $\pi_A$ and $\pi_B$. Further, whenever there is a homomorphism $h$ such that $h(s) \neq h(t)$ and $(h \circ \pi_A) \equiv_m (h \circ \pi_B)$, we can exclude the pair $(s,t)$ as a candidate for witnessing $\pi_A \not\equiv_m \pi_B$ due to the fundamental property. Thus, separating sets of homomorphisms yield the following reduction technique.

▶ **Lemma 32.** *Let $\mathcal{S}$ and $\mathcal{S}'$ be semirings and $H$ a separating set of homomorphisms from $\mathcal{S}$ to $\mathcal{S}'$. Moreover let $\pi_A, \pi_B$ be $\mathcal{S}$-interpretations, $\bar{a} \in A^n$ and $\bar{b} \in B^n$. It holds that $(h \circ \pi_A, \bar{a}) \equiv_m (h \circ \pi_B, \bar{b})$ for all $h \in H$ if, and only if, $(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$.*

Based on a separating set $H$ of homomorphisms $h \colon \mathcal{S} \to \mathbb{B}$, the *homomorphism game* $HG_m(H, \pi_A, \pi_B)$ can be defined as follows. Spoiler first chooses some $h \in H$ and puts either $\pi_0 = h \circ \pi_A$ and $\pi_1 = h \circ \pi_B$, or the other way around, i.e. $\pi_0 = h \circ \pi_B$ and $\pi_1 = h \circ \pi_A$. Then the game $G_m^{\leq}(\pi_0, \pi_1)$ is played. Using the fact that $G_m^{\leq}$ is sound and complete for $\preceq_m$ even on $\mathbb{B}$-interpretations which are not model-defining, soundness and completeness of $HG_m$ for $\equiv_m$ can be stated as follows.

▶ **Theorem 33.** *Let $\mathcal{S}$ be a semiring with a separating set $H$ of homomorphisms into $\mathbb{B}$. Given $\mathcal{S}$-interpretations $\pi_A, \pi_B$ and $\bar{a} \in A^n, \bar{b} \in B^n$, the following are equivalent for $m \in \mathbb{N}$:*

**(1)** *Duplicator wins $HG_m(H, \pi_A, \bar{a}, \pi_B, \bar{b})$;*

**(2)** *$h(\pi_B[\![\chi^m_{h \circ \pi_A, \bar{a}}(\bar{b})]\!]) = h(\pi_A[\![\chi^m_{h \circ \pi_B, \bar{b}}(\bar{a})]\!]) = 1$ for each $h \in H$;*

**(3)** *$(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$.*

Motivated by Birkhoff's representation theorem [1], we can explicitly construct a separating set of homomorphisms from any finite lattice semiring (i.e. fully idempotent and absorptive semiring) into $\mathbb{B}$, and embed it into the rules of the homomorphism game. Indeed, every semiring for which there is a separating set of homomorphisms to $\mathbb{B}$ must be a lattice semiring since for every homomorphism $h \colon \mathcal{S} \to \mathbb{B}$ and $s, t \in S$, we have $h(s \cdot s) = h(s) \wedge h(s) = h(s)$ and $h(s + st) = h(s) \vee (h(s) \wedge h(t)) = h(s)$. Due to absorption, we assume that the infinitary operations of a lattice semiring are given by $\sum_{i \in I} s_i := \sup\{\sum_{i \in I'} s_i | I' \subseteq I \text{ finite}\}$ and $\prod_{i \in I} s_i := \inf\{\prod_{i \in I'} s_i | I' \subseteq I \text{ finite}\}$.

▶ **Definition 34.** Let $\mathcal{S}$ be a finite lattice semiring. A non-zero element $s \in \mathcal{S}$ is said to be *$+$-indecomposable* if for all $r, t \in \mathcal{S}$ with $r \neq s$ and $t \neq s$ it holds that $r + t \neq s$. We denote the set of non-zero $+$-indecomposable of elements in $\mathcal{S}$ as $idc(\mathcal{S})$.

In a min-max semiring, for instance, every non-zero element is $+$-indecomposable. By contrast, the $+$-indecomposable elements in PosBool$[X]$ correspond to the monomials.

▶ **Lemma 35.** *For each $s \in idc(\mathcal{S})$ the mapping $h_s \colon \mathcal{S} \to \mathbb{B}$ defined by*

$$h_s(t) = \begin{cases} 1, & t + s = t \\ 0, & \textit{otherwise} \end{cases}$$

*is a homomorphism from $\mathcal{S}$ into $\mathbb{B}$.*

**Proof.** Let $s \in idc(\mathcal{S})$ be non-zero and $+$-indecomposable.

**(1)** Since $0 + s = s \neq 0$, it holds that $h_s(0) = 0$. Further, we have that $1 + s = 1 + 1 \cdot s = 1$ due to absorption, hence $h_s(1) = 1$.

**(2)** In order to prove that $h_s(r+t) = h_s(r) + h_s(t)$ for all $r, t \in \mathcal{S}$, it remains to show that $s + (r+t) = r+t$ is equivalent to $s+r = r$ or $s+t = t$. If $s+(r+t) = r+t$, then with absorption and distributivity $sr + st = s(r+t) = s(s+r+t) = s + s(r+t) = s$. Since $s$ is $+$-indecomposable by assumption, this implies $sr = s$ or $st = s$. Suppose w.l.o.g. that $sr = s$ which yields $r = r + sr = r + s$. For the converse implication, assume that $r + s = r$ or $t + s = t$. Clearly, both implications immediately yield $s + (r+t) = r+t$.

**(3)** To prove $h_s(r \cdot t) = h_s(r) \cdot h_s(t)$, we show that $s + rt = rt$ is equivalent to $s + r = r$ and $s + t = t$. If $s + rt = rt$, we can infer that $s + r = s + (r + rt) = (s + rt) + r = rt + r = r$ and an analogous result for $t$. Conversely, suppose that $s + r = r$ and $s + t = t$. Then $rt = (s+r)(s+t) = s + (r \cdot t)$ follows by distributivity.

**(4)** Pertaining to the compatibility of $h_s$ with infinitary operations in $\mathcal{S}$, note that any infinite sum or product can be transformed into a finite sum or product due to full idempotence and the assumption that $\mathcal{S}$ is finite. Thus, the proof is already complete. ◀

Although we only consider the mappings $h_s$ for $+$-indecomposable $s$ to ensure that $h_s$ is a homomorphism, any two elements in $\mathcal{S}$ can be separated by some $h_s$.

▶ **Lemma 36.** *The set $\{h_s \colon s \in idc(\mathcal{S})\}$ is a separating set of homomorphisms from $\mathcal{S}$ to $\mathbb{B}$.*

**Proof.** For $t \in \mathcal{S}$ let $S_t = \{s \in idc(\mathcal{S}) \colon s + t = t\}$. Due to idempotence, we have that $t + \sum_{s \in S_t} s = t$. Since $\mathcal{S}$ is assumed to be finite, there must be a tuple $t_1, \dots, t_n \in idc(\mathcal{S})$ with $t_1 + \dots + t_n = t$. With idempotence, this implies $t + t_i = t$, which yields $t_i \in S_t$ for each $1 \le i \le n$. Hence, we have that $t + \sum_{s \in S_t} s = \sum_{1 \le i \le n} t_i + \sum_{s \in S_t} s = \sum_{s \in S_t} s$. Overall, we obtain $t = t + \sum_{s \in S_t} s = \sum_{s \in S_t} s$.

Let $r, t \in \mathcal{S}$ with $r \neq t$. Since $r = \sum_{s \in S_r} s$ and $t = \sum_{s \in S_t} s$, it must hold that $S_r \neq S_t$. Let $s$ be a witness for the inequality and assume w.l.o.g that $s \in S_r$. By definition of $S_r$, it holds that $s + r = r$, hence $h_s(r) = 1$. By contrast, $s \notin S_t$ yields $s + t \neq t$ and thus $h_s(t) = 0$. ◀

Now that we have an explicit construction a separating set of homomorphisms to $\mathbb{B}$ which applies to any finite lattice semiring, we can reformulate the homomorphism game as $HG_m(\pi_A, \pi_B)$ corresponding to $HG_m(H_{idc}, \pi_A, \pi_B)$ for finite lattice semirings as follows.

▶ **Definition 37.** At the beginning of each play in $HG_m(\pi_A, \pi_B)$, Spoiler chooses either $\pi_0 = \pi_A$ and $\pi_1 = \pi_B$ or vice versa, and some $s \in idc(\mathcal{S})$. In the $i$-th of $m$ rounds, Spoiler chooses some $a_i \in A$ or $b_i \in B$ and Duplicator has to respond with an element $a_i$ or $b_i$ in the other structure. Duplicator wins the play if for the chosen tuples $\bar{c}, \bar{d}$ and each $L(\bar{x}) \in \mathrm{Lit}_m(\tau)$ $\pi_0(L(\bar{c})) + s = \pi_0(L(\bar{c}))$ implies $\pi_1(L(\bar{d})) + s = \pi_1(L(\bar{d}))$.

The direct construction of the separating set of homomorphisms also allows an explicit formulation of characteristic formulae $\chi^{m,s}_{\pi_A,\bar{a}}(\bar{x})$ for each $s \in idc(\mathcal{S})$ corresponding to the $\mathbb{B}$-interpretations $h_s \circ \pi_A$. Again $\varphi^{=}_{\bar{a}}(\bar{x})$ characterises the equalities and inequalities of the elements in $\bar{a}$.

$$\chi^{0,s}_{\pi_A,\bar{a}}(x_1,\dots,x_n) := \varphi^{=}_{\bar{a}}(\bar{x}) \wedge \bigwedge \{L(\bar{x}) \in \mathrm{Lit}_n(\tau) \mid \pi_A(\bar{a}) + s = \pi_A(L(\bar{a}))\}$$
$$\chi^{m+1,s}_{\pi_A,\bar{a}}(x_1,\dots,x_n) := \bigwedge_{a \in A} \exists x\, \chi^{m,s}_{\pi_A,\bar{a},a}(\bar{x},x) \wedge \forall x \bigvee_{a \in A} \chi^{m,s}_{\pi_A,\bar{a},a}(\bar{x},x)$$

In terms of the set $H_{idc} = \{h_s \colon s \in idc(\mathcal{S})\}$, the correctness of the game $HG_m$ for finite lattice semirings can be stated as follows.

▶ **Theorem 38.** *The game $HG_m$ is sound and complete for $\equiv_m$ on every finite lattice semiring $\mathcal{S}$. More precisely, given any $\mathcal{S}$-interpretations $\pi_A, \pi_B$ and $\bar{a} \in A^n, \bar{b} \in B^n$ the following are equivalent for each $m \in \mathbb{N}$:*

**(1)** *Duplicator wins $HG_m(\pi_A, \bar{a}, \pi_B, \bar{b})$;*

**(2)** *For each $s \in idc(\mathcal{S})$, it holds that*

$$\pi_B[\![\chi^{m,s}_{\pi_A,\bar{a}}(\bar{b})]\!] + s = \pi_B[\![\chi^{m,s}_{\pi_A,\bar{a}}(\bar{b})]\!] \quad \textit{and} \quad \pi_A[\![\chi^{m,s}_{\pi_B,\bar{b}}(\bar{a})]\!] + s = \pi_A[\![\chi^{m,s}_{\pi_B,\bar{b}}(\bar{a})]\!];$$

**(3)** $(\pi_A, \bar{a}) \equiv_m (\pi_B, \bar{b})$.

While applying to arbitrary finite lattice semirings, the set $H_{idc}$ of homomorphisms is in general not sufficient to separate any two elements of an infinite lattice semiring. As an example, consider $\mathcal{S} = (\mathbb{Z}, +^{\mathcal{S}}, \cdot^{\mathcal{S}}, 0, 1)$ with $s +^{\mathcal{S}} t = \gcd(s,t)$ if $s \neq 0$ or $t \neq 0$, while $0 +^{\mathcal{S}} 0 = 0$ and $s \cdot^{\mathcal{S}} t = \mathrm{lcm}(s,t)$ for $s, t \in \mathbb{Z}$. For each $s \in \mathbb{Z}$, it holds that $\gcd(2s, 3s) = s$, so for $s \neq 0$ there are distinct $r$ and $t$ such that $s = r +^{\mathcal{S}} t$. By contrast, $\gcd(s,t) \neq 0$ for all $s, t \in \mathbb{Z} \setminus \{0\}$, hence $idc(\mathcal{S}) = \{0\}$, but $\{h_0\}$ is not a separating set of homomorphisms. Nevertheless, separating sets of homomorphisms into $\mathbb{B}$ also exist for infinite lattice semirings and can be constructed based on the prime ideals in $\mathcal{S}$. But in general, there does not have to be a separating set of *continuous* homomorphisms, which respect infinitary summation and multiplication in $\mathcal{S}$. Thus, the prime ideals in $\mathcal{S}$ yield a homomorphism game on finite $\mathcal{S}$-interpretations, while $\mathcal{S}$ itself might be infinite (see [4] for details).

▶ **Example 39.** We can use the homomorphism game to show that first-order logic with semiring semantics cannot express the following property on min-max-semirings with the monadic signature $\{Q, R\}$: *"For the majority of elements $e$ in the universe, $Qe$ has a greater value than $Re$."* To prove this, we use the following two $\mathcal{S}_4$-interpretations on the min-max-semiring $\mathcal{S}_4$ with four elements $\{0, 1, 2, 3\}$.

$\pi_A$ :

| $A$ | $Q$ | $R$ | $\neg Q$ | $\neg R$ |
|---|---|---|---|---|
| $a_1$ | 1 | 3 | 0 | 0 |
| $a_2$ | 2 | 1 | 0 | 0 |
| $a_3$ | 3 | 2 | 0 | 0 |

$\pi_B$ :

| $B$ | $Q$ | $R$ | $\neg Q$ | $\neg R$ |
|---|---|---|---|---|
| $b_1$ | 3 | 1 | 0 | 0 |
| $b_2$ | 1 | 2 | 0 | 0 |
| $b_3$ | 2 | 3 | 0 | 0 |

Clearly, $\pi_A$ has the desired property while $\pi_B$ does not. However, we can show with the homomorphism game $HG_m(\pi_A, \pi_B)$ that $\pi_A \equiv \pi_B$. First, we observe that every non-zero $s \in \mathcal{S}_4$ is $+$-indecomposable. The set $idc(\mathcal{S}_4)$ induces homomorphisms $h_{\geq i} \colon \mathcal{S}_4 \to \mathbb{B}$ for $i \in \{1, 2, 3\}$ such that $h_{\geq i}(j) = 1$ iff $j \geq i$. Hence, we essentially play the homomorphism game $HG_m(H, \pi_A, \pi_B)$ with the separating set of homomorphisms $H = \{h_{\geq 1}, h_{\geq 2}, h_{\geq 3}\}$. Now, it only remains to observe that applying any of these homomorphisms to $\pi_A$ and $\pi_B$ makes them isomorphic to each other, thus, Duplicator clearly has a winning strategy. This demonstrates the viability of homomorphism games as a proof method for inexpressibility results in semiring semantics.

## 6 Conclusion

We have provided a rather detailed study of soundness and completeness of Ehrenfeucht–Fraïssé games, and related model comparison games, for proving elementary equivalence and $m$-equivalence in semiring semantics. The general picture that emerges is quite diverse. While the $m$-move games $G_m$ are sound and complete for $\equiv_m$ only on the Boolean semiring, the games still provide a sound method on fully idempotent semirings, such as min-max semirings, lattice semirings, and the provenance semirings PosBool[$X$]. This permits to generalise certain classical results in logic, proved via Ehrenfeucht–Fraïssé games or back-and-forth systems, from Boolean structures to semiring interpretations in fully idempotent semirings. A particular example is the proof of a Hanf locality theorem for such semirings in [2]. For proving elementary equivalence, without restriction of the quantifier rank, Ehrenfeucht–Fraïssé games without a fixed number of moves provide a more powerful method, in the sense that it is sound on more semirings, including not only $\mathbb{N}$ and $\mathbb{N}^\infty$ but also the provenance semirings $\mathbb{W}[X]$, $\mathbb{B}[X], \mathbb{S}[X]$, $\mathbb{N}[X]$, and $\mathbb{S}^\infty[X]$. While in classical semantics, a separating sentence of quantifier rank $m$ leads to a winning strategy of Spoiler in at most $m$ moves, the situation in semirings may be more complicated, in the sense that a winning strategy of Spoiler which "simulates" a separating sentence may still exist, but may require a larger number of moves than given by the quantifier rank; as a consequence the unrestricted game $G$ may still provide a sound method for proving elementary equivalence, although the $m$-move games are unsound for $\equiv_m$.

The most straightforward application of Ehrenfeucht–Fraïssé games and other model comparison games are inexpressibility results, showing that a property $P$ is not expressible in a logic $L$. Classically, this is accomplished by constructing two structures, precisely one of which satisfies the property $P$, and then providing a winning strategy for Duplicator in an appropriate model comparison game on the two structures. This method only relies on the soundness of the model comparison game without requiring completeness. Hence, our soundness results enable us to lift inexpressibility results to semiring semantics for a significant class of semirings. Consider, for instance, a min-max-semiring $\mathcal{S}$ modelling access levels and $\mathcal{S}$-interpretations $\pi$ that annotate every edge of a graph with a required access level. Then there is no first-order formula $\varphi(x, y)$ such that $\pi[\![\varphi(v, w)]\!]$ evaluates to the minimal access level required to go from $v$ to $w$.

We have also studied bijection and counting games, and we have shown in particular, that $m$-move bijection games are sound for $\equiv_m$ on *all* semirings. We remark that these games have originally been invented in the form of $k$-pebble games for logics with counting. This means that rather than just selecting, in $m$ turns, two $m$-tuples, the games proceed by moving a fixed number of $k$ pairs of pebbles through the two structures in an a priori unrestricted number of moves. These games capture equivalences for formulae that may use at most $k$ variables which can, however, be quantified again and again. We have chosen here the simplified variants of $m$-move games rather than $k$-pebble games, to study the relationship with the classical Ehrenfeucht–Fraïssé games for $\equiv_m$. However, also the definition of $k$-pebble bijection and counting games extends in a straightforward way from classical structures to semiring interpretations and their soundness properties for $k$-variable equivalences are analogous to those of the $m$-move variants for $m$-equivalence. But clearly, the $k$-pebble variants of these games deserve further study, and this will be part of our future work on the subject. We conjecture that by lifting the well-known CFI-construction to semirings one can show that there is no semiring where first-order logic, and even fixed point logic, is strong enough to express all properties that are decidable in PTIME.

On the other side, it has turned out that all these model comparison games are incomplete for elementary equivalence and $m$-equivalence on most semirings, with the exceptions of $\mathbb{N}$ and $\mathbb{N}[X]$. Most of these incompleteness results rely on the construction of logically equivalent semiring interpretations on which, however, Spoiler wins the games in few moves. The proof of elementarily equivalence for such interpretations in general relies on separating sets of homomorphisms. Based on this technique, we have proposed a new kind of model comparison games, homomorphism games, which in fact are sound and complete for $m$-equivalence on finite lattice semirings. This also raises the question whether it is possible to develop further games that are sound and complete for more, or even all, semirings. An essential part of the homomorphism game is a one-sided version of the classical Ehrenfeucht–Fraïssé game, with a winning condition that is based on (weak) local homomorphisms rather than local isomorphisms, and which capture the notion that one interpretation never evaluates to strictly larger values than the other. This game itself is interesting also in many other contexts and will be further studied in future work.

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