Realizability Models for Large Cardinals

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Abstract

Realizabilty is a branch of logic that aims at extracting the computational content of mathematical proofs by establishing a correspondence between proofs and programs. Invented by S.C. Kleene in the 1945 to develop a connection between intuitionism and Turing computable functions, realizability has evolved to include not only classical logic but even set theory, thanks to the work of J-L. Krivine. Krivine's work made possible to build realizability models for Zermelo-Frænkel set theory, ZF, assuming its consistency. Nevertheless, a large part of set theoretic research involves investigating further axioms that are known as *large cardinals axioms*; in this paper we focus on four large cardinals axioms: the axioms of (strongly) inaccessible cardinal, Mahlo cardinals, measurable cardinals and Reinhardt cardinals. We show how to build realizability models for each of these four axioms assuming their consistency relative to ZFC or ZF.

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1 Introduction

Realizability is an extension of the *proofs-as-programs* correspondence also known as the *Curry-Howard isomorphism*. In realizability, a theory (or a logical system) is interpreted in a model of computation by establishing a correspondence between formulae and programs in a way that is compatible with the rules of deduction. For instance, a realizer of an implication $A \to B$ is a program which, whenever applied to a realizer of A, returns a realizer of B. The origins of realizability date back to S.C. Kleene's work in constructive mathematics in 1945 [11]: Kleene's realizability formalized the intuitionistic view that proofs are algorithms (computable functions) by interpreting proofs in Heyting arithmetic as recursive functions. In the 90's, the work of T. Griffin [6] led to pass the barrier of intuitionistic logic and to extend the Curry-Howard correspondence to classical logic by using the λ_c-calculus, an extension of λ-calculus that formalizes computation in the programming language Scheme (for a presentation of the λ-calculus we refer to Barendregt's book [1], for a presentation of the λ_c-calculus we refer to [3]). J.-L. Krivine developed a method for realizing not only classical logic, but even Zermelo-Frænkel set theory, ZF (see [12] and [13]) using *realizability algebras* which are generalizations of the notion of Boolean algebra involving programs and stacks (realizability algebras will be presented in Section 2). Krivine's technique combines intuitionistic set theory, IZF, with a double negation translation of formulas. In this matter the work of H. Friedman [4] on IZF was crucial as it showed that IZF is equiconsistent with ZF and gave a natural way to interpret the classical theory within the intuitionistic one.

The method can also be seen as a generalization of the method of forcing in set theory. This is because every Boolean-valued model can be naturally interpreted as a realizability algebra; moreover, this interpretation is done in such a way that the two resulting models prove the same statements in some precise sense (we refer to section 19 of [19] for the details of this translation). Nevertheless, from a computational point of view, forcing models are not very informative since all realizers are interpreted as the same element (the bottom element of the boolean algebra).

For a long time it remained an open problem whether or not it was possible to realize the Axiom of Choice, AC; recent work of Krivine [16] shows that it is indeed possible to build a realizability model for AC (although, it remains unclear what would be an explicit realizer for AC in this model). Research in contemporary set theory is not limited to the axioms of ZF or ZFC (i.e. ZF plus AC), with an active area of research being the study of *large cardinals axioms*. These are strong axioms of infinity that assert the existence of uncountable cardinals with various closure properties. Large cardinals axioms can be ordered by their consistency strength and they all entail the existence of a *set* which satisfies all of the axioms of ZF. It follows by K. Gödel's second incompleteness theorem that the existence of large cardinals cannot be proven within ZF. Nevertheless, these axioms have many applications to various areas of mathematics and computer science (for a more detailed presentation of large cardinals we refer to A. Kanamori's book [10])

We address the problem of whether or not large cardinals axioms can be realized, and we focus on four major large cardinals notions: (strongly) *inaccessible cardinals*, *Mahlo cardinals*, *measurable cardinals* and *Reinhardt cardinals*. Inaccessible cardinals are uncountable cardinals that imply the existence of uncountable Grothendieck universes (see [24]). Mahlo cardinals imply strong reflection properties which have been used in type theories, such as Agda, to produce type universes which contain inductive-recursive types. The strongest known version of type theory for which there exists a constructive justification is a system of Martin-Löf type theory with a Mahlo universe, MLM, which was introduced by A. Setzer (see [22]). M. Rathjen showed that constructive set theory plus the axiom that asserts the existence of Mahlo cardinals has a canonical interpretation in Setzer's type theory (see [21]). A measurable cardinal is a cardinal κ for which there exists a non-trivial κ-additive, 0-1-valued measure on the power set of κ. As proved by A. Blass in [2], the existence of measurable cardinals is equivalent to the existence of an exact functor $F : \mathrm{Set} \to \mathrm{Set}$ that is not naturally isomorphic to the identity. Reinhardt cardinals generalize the notion of measurable cardinal and imply the existence of a non-trivial elementary embedding of the universe of sets into itself. The existence of Reinhardt cardinals is inconsistent with ZFC by Kunen's inconsistency theorem [17], so they are defined only in the context of ZF.

In this paper we show that these four large cardinals axioms can be realized within Krivine's framework. We consider for each of these large cardinal axioms φ an equivalent large cardinal axiom φ^* in the context of ZF, then we build realizability models for the theory "ZF plus φ^*" assuming its consistency. The work of H. Friedman and A. Ščedrov [5] provided a suitable formulation of large cardinals in intuitionistic set theory which made it possible to integrate these large cardinals notions into Krivine's machinery. We shall point out that not only do we build realizaiblity models for these large cardinals notions assuming their consistency with ZFC or ZF, but we prove that *any* realizability algebra of size less than the large cardinals considered preserves these large cardinals axioms.

The paper is structured as follows. We first introduce the technique for building realizability models for ZF in Sections 2–4. In Section 5 we illustrate the method of reish names, or *recursive* names, for transferring properties of sets and functions to realizability algebras. In Section 6 we discuss some useful relativization properties for transitive sets. In Section 7 we show how to preserve the axiom of inaccessible cardinals by realizability algebras. In Section 8, we show how to preserve the axiom of Mahlo cardinals by realizability algebras. Section 9 is devoted to realizability models for the second ordered set theory, GB. Finally, in Section 10 we show how to preserve by realizability algebras the axioms of measurable and Reinhardt cardinals.

2 Realizability algebras

In this section, we present *Realizability Algebras*, which are the main building blocks for the construction of realizability models for set theory. We shall being by briefly explaining the main intuition behind this construction. We start with a model of set theory and we will use programs and stacks to evaluate the potential truth and falsity values of set theoretic statements. For computational reasons we work with a non extensional version of set theory, called ZF_ε, that involve two membership relations: the usual one and a strict non-extensional relation. We will use the terms of the λ_c-calculus (a variant of λ-calculus that include as a term the operator *call-with-current-continuation*) to evaluate the truth value of formulas in the language of ZF_ε. On the other hand, we will use stacks, namely sequences of λ_c-terms, to evaluate the falsity values of such formulas. Truth values and falsity values will be related to each other, so that a λ_c-term is in the truth value of a formula (we say that it "realizes the formula"), if it is somehow "incompatible" with every stack in the falsity value of that formula. These definitions will respect certain logical constraints such as no stack can be in the falsity value of $\top$, and every stack is in the falsity value of $\bot$. Then we choose some privileged collection of λ_c-terms that we call *realizers* and we will show that the set of formulas that are realized by some realizer forms a consistent theory which includes ZF_ε and is closed under the rules of derivation of classical natural deduction. Finally, a realizability model will be a model of such a theory. Since ZF_ε is a conservative extension of ZF, such a model will induce a model of ZF.

The main ingredients of realizability algebras are λ_c*-terms*, *stacks* and *processes* which we define next. We will give the definition in full generality, in particular allowing for non-empty sets of *special instructions* and *stack bottoms*. These are customisable constants which can be added to our realizability algebras to ensure the models satisfies additional principles. For example, if the algebra is countable and contains the special instruction *quote* then one can prove Dependent Choice holds in the model. However, all the statements in this paper will be realized by terms of the λ_c-calculus. Therefore we will not need any special instructions but the arguments will still go through if they are present.

▶ **Definition 1.** *Let* V *be a model of* ZF *and let* A, B *be two sets in* V*:*

- *We let* $\Lambda^{open}_{A,B}$ *and* $\Pi_{A,B}$ *denote the elements of* V *defined by the following grammars, modulo* α*-equivalence. Their elements are called respectively* λ_c-terms *and* stacks*:*

  $\Lambda^{open}_{A,B}$ (λ_c**-terms**) :

$$
\begin{array}{rcll}
t, u & ::= & x & \textit{(variable; we choose a set of variables that is countable in } \mathrm{V}\textit{)}\\
 & | & tu & \textit{(application)}\\
 & | & \lambda x.t & \textit{(abstraction; } x \textit{ is a variable and } t \textit{ is a } \lambda_c\textit{-term)}\\
 & | & \mathrm{cc} & \textit{(call-with-current-continuation)}\\
 & | & \mathrm{k}_\pi & \textit{(continuation constants; } \pi \textit{ is a stack)}\\
 & | & \xi_\alpha & \textit{(special instructions; } \alpha \in A\textit{)}
\end{array}
$$

$\Pi_{A,B}$ (**Stacks**) :

$$\begin{array}{lll} \pi ::= & \omega_\beta & \textit{(stack bottoms; } \beta \in B\textit{)} \\ \mid & t.\pi & \textit{(t is a closed } \lambda_c\textit{-term and } \pi \textit{ is a stack)} \end{array}$$

- $\Lambda_{A,B} \in \mathrm{V}$ *denotes the set of all closed* λ_c*-terms,*
- $\mathcal{R}_{A,B} \in \mathrm{V}$ *denotes the set of all closed* λ_c*-terms that contain no occurrence of a continuation constant. Such terms are called* realizers.
- $\Lambda_{A,B} \star \Pi_{A,B} \in \mathrm{V}$ *denotes the cartesian product* $\Lambda_{A,B} \times \Pi_{A,B}$*. Its elements are called* processes. *We will write* $t \star \pi$ *for* $(t, \pi) \in \Lambda_{A,B} \star \Pi_{A,B}$.

Application on λ_c-terms is left associative ($tu_1u_2 \cdots u_n$ means $(\cdots((tu_1)u_2)\cdots)u_n$) and has higher priority than abstraction ($\lambda x.tu$ means $\lambda x.(tu)$). We define some rules of reduction on the set of processes through the notion of *evaluation*.

▶ **Definition 2.** *Let* V *be a model of* ZF *and let* A, B *be two sets in* V.

- $\prec_{A,B} \in \mathrm{V}$ *is called the* evaluation preorder *and denotes the smallest preorder on* $\Lambda_{A,B} \star \Pi_{A,B}$ *such that:*

$$\begin{array}{rcrl} tu \star \pi & \succ_{A,B} & t \star u.\pi & \textit{(push)} \\ \lambda x.t \star u.\pi & \succ_{A,B} & t[x := u] \star \pi & \textit{(grab)} \\ \mathrm{cc} \star t.\pi & \succ_{A,B} & t \star \mathrm{k}_\pi.\pi & \textit{(save)} \\ \mathrm{k}_{\pi'} \star t.\pi & \succ_{A,B} & t \star \pi' & \textit{(restore).} \end{array}$$

Note that there is no evaluation rule for the special instructions, thus $\prec_{A,B}$ treats the special instructions as inert constants; depending on the context we may define other evaluation relations with specific evaluation rules for the special instructions. If A and B can be well-ordered (which is always the case if V satisfies the Axiom of Choice), then the cardinality (from the point of view of V) of $\Lambda_{A,B}$, $\Pi_{A,B}$, $\mathcal{R}_{A,B}$ and $\Lambda_{A,B} \star \Pi_{A,B}$ is the maximum of the cardinality of A, the cardinality of B and $\aleph_0$.

▶ **Definition 3.** *Let* V *be a model of* ZF*. A* realizability algebra *in* V *is a tuple* $\mathcal{A} = (A, B, \perp\!\!\!\perp)$ *such that:*

- $\mathcal{A} \in \mathrm{V}$ *(i.e.* $A \in \mathrm{V}$, $B \in \mathrm{V}$ *and* $\perp\!\!\!\perp \in \mathrm{V}$*);*
- $\perp\!\!\!\perp$ *is a subset of* $\Lambda_{A,B} \star \Pi_{A,B}$ *that is a final segment for* $\succ_{A,B}$*, i.e. if* $t \star \pi \succ_{A,B} t' \star \pi'$ *and* $t' \star \pi' \in \perp\!\!\!\perp$*, then* $t \star \pi \in \perp\!\!\!\perp$*. It is called the* pole *of the realizability algebra.*

Given a model V of ZF, recall that the *Von Neumann hierarchy* is a collection of sets V_α indexed by ordinals and defined by transfinite recursion as follows: $\mathrm{V}_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}(\mathrm{V}_\beta)$. The Axiom of Foundation implies that $\mathrm{V} = \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{V}_\alpha$; thus every set belongs to some V_α and the *rank* ordinal of a set a, denoted $\mathrm{rk}(a)$, is the least ordinal α such that $a \in \mathrm{V}_\alpha$. We call the *footprint* of $\mathcal{A}$ the ordinal $\mathrm{fp}(\mathcal{A}) := \mathrm{rk}_\mathrm{V}(\mathcal{A}) + \omega_1^\mathrm{V}$ where ω_1^V is the least uncountable ordinal in V. We assume that the sets $\Lambda^{\mathrm{open}}_{A,B}$ and $\Pi_{A,B}$ were constructed in such a way that their ranks are strictly less than $\mathrm{fp}(\mathcal{A})$. When there is no ambiguity, we will drop the indices A, B and simply write Λ, Π, $\mathcal{R}$, etc.

3 The theory ZF_ε

In order to define a realizability model for classical set theory, we consider a non-extensional conservative extension of the usual set theory. This theory was originally formulated by Friedman in [4] in his proof that ZF is equiconsistent with IZF and notably contains two distinct membership relations: $\in$ which behaves like the standard membership relation, and ε which is a form of "*strong membership*".

Throughout this paper, we will work in first-order logic *without equality*: individuals languages may contain a symbol that happens to be written "=", but such a symbol has no special status. In particular, models are not required to interpret it by "meta" equality. In addition, we will assume that the only primitive logical constructions are $\rightarrow$, $\top$, $\perp$, and $\forall$; for $\vee$, $\wedge$, and $\exists$, we will use De Morgan's encoding. Thus:

- $\varphi \wedge \psi$ means $(\varphi \rightarrow \psi \rightarrow \perp) \rightarrow \perp$,
- $\varphi \vee \psi$ means $(\varphi \rightarrow \perp) \rightarrow (\psi \rightarrow \perp) \rightarrow \perp$,
- $\exists x\ \varphi(x)$ means $(\forall x\ (\varphi(x) \rightarrow \perp)) \rightarrow \perp$.

We will denote by $\mathcal{L}_\in$ the first-order language over the signature $\{\in, \simeq\}$ where $\in$ and $\simeq$ are binary relation symbols. The language of ZF_ε requires two distinct symbols for the membership relation, $\in$ and ε (the former will refer to the usual extensional membership relation, the latter will correspond to a strict non-extensional membership relation); however, for computational reasons it is better to take as primitives the negative versions of those symbols, thus the language of ZF_ε which is denoted $\mathcal{L}_\varepsilon$, is the first-order language over the signature $\{\notin, \subseteq, \not\varepsilon, \neq\}$, where all 4 symbols are binary relation symbols. It can be proven that $\neq$ is definable from $\not\varepsilon$ via the Leibniz equality and is therefore not necessary in the signature, however we include it here for practical purposes. $\mathrm{Fml}_\in$ and Fml_ε denote the collection of all formulas in $\mathcal{L}_\in$ and $\mathcal{L}_\varepsilon$ respectively. In the language $\mathcal{L}_\varepsilon$, we will use the following abbreviations:

Abreviation	Meaning	Abreviation	Meaning
$a \,\varepsilon\, b$	$a \not\varepsilon\, b \rightarrow \perp$	$a \simeq b$	$(a \subseteq b) \wedge (b \subseteq a)$
$a \in b$	$a \notin b \rightarrow \perp$	$\forall x \,\varepsilon\, a\ \varphi(x)$	$\forall x\ (x \,\varepsilon\, a \rightarrow \varphi(x))$
$a = b$	$a \neq b \rightarrow \perp$	$\exists x \,\varepsilon\, a\ \varphi(x)$	$(\forall x\ (\varphi(x) \rightarrow x \not\varepsilon\, a)) \rightarrow \perp$

In particular, by a slight abuse of notation, we will consider $\mathrm{Fml}_\in$ to be a subset of Fml_ε. ZF denotes the usual set theory, written in the language $\mathcal{L}_\in$, i.e. ZF is a subset of $\mathrm{Fml}_\in$, while ZF_ε denotes non-extensional set theory, as defined by Krivine, written in the language $\mathcal{L}_\varepsilon$ (i.e. ZF_ε is a subset of Fml_ε). In a nutshell, the axioms of ZF_ε state that:

- An equivalent presentation of the axioms of ZF *minus the Axiom of Extensionality* (essentially the double negation) are satisfied *over the signature* $\{\not\varepsilon, \neq\}$ (rather than $\{\in, \simeq\}$).
- $\in$ is the extensional collapse of ε: $x \in y$ *iff* there is $x' \,\varepsilon\, y$ such that $x \simeq x'$;
- $\subseteq$ is the extensional inclusion: $x \subseteq y$ *iff* for every $z \,\varepsilon\, x$, we have $z \in y$;
- $\simeq$ is extensional equivalence: two sets are $\simeq$-equal *iff* they have the same $\in$-elements.

For full details, including the list of the axioms of $\mathcal{L}_\varepsilon$, we refer the reader to [14]; see also Friedman's earlier account in [4]. As proven in [4], ZF_ε is a conservative extension of ZF:

▶ **Theorem 4.** *Let φ be a closed formula in $\mathcal{L}_\in$: φ is a consequence of* ZF *if and only if it is a consequence of* ZF_ε.

A proof of this fact can be found in [12]. For further details we refer to [19].

Whenever $\mathcal{L}$ is a first-order language that contains $\mathcal{L}_\varepsilon$, we will denote by $\mathrm{ZF}_\varepsilon^{\mathcal{L}}$ the theory obtained by taking ZF_ε and enriching all the axiom schemas to include the formulas of $\mathcal{L}$.

4 Construction of realizability models

Our construction of realizability models follows the presentation in [19]. Let V be a model of Zermelo-Frænkel set theory, ZF, and let $\mathcal{A} = (\Lambda, \Pi, \perp\!\!\!\perp)$ be a realizability algebra in V.

We define $\mathrm{N}^{\mathcal{A},\mathrm{V}} \subseteq \mathrm{V}$ as follows: for any ordinal $\alpha \in \mathrm{V}$, let $\mathrm{N}_\alpha^{\mathcal{A},\mathrm{V}} := \bigcup_{\beta<\alpha} \mathcal{P}(\mathrm{N}_\beta^{\mathcal{A},\mathrm{V}} \times \Pi)$, then let $\mathrm{N}^{\mathcal{A},\mathrm{V}} := \bigcup_{\alpha\in\mathrm{Ord}} \mathrm{N}_\alpha^{\mathcal{A},\mathrm{V}}$, where Ord denotes the class of ordinals in V. The elements of $\mathrm{N}^{\mathcal{A},\mathrm{V}}$ are called $(\mathcal{A},\mathrm{V})$-*names*. Note that for all α, $\mathrm{N}_\alpha^{\mathcal{A},\mathrm{V}} \in \mathrm{V}$, but $\mathrm{N}^{\mathcal{A},\mathrm{V}} \notin \mathrm{V}$ ($\mathrm{N}^{\mathcal{A},\mathrm{V}}$ is a *proper class* in V). We will generally drop the exponents and simply write N_α and N. Given an element $a \in \mathrm{N}$ we let $\mathrm{dom}(a) := \{b \mid \exists \pi \in \Pi\ (b,\pi) \in a\} \in \mathrm{V}$.

A function $f : \mathrm{N}^n \to \mathrm{N}$ is said to be $\mathcal{A}$-*definable* if there is a formula $\varphi(z, x_1, \ldots, x_n, y) \in \mathrm{Fml}_\varepsilon$ and $c \in \mathrm{V}_{\mathrm{fp}(\mathcal{A})}$ such that, for any $a_1, \ldots, a_n \in \mathrm{N}$ and $b \in \mathrm{V}$, $\mathrm{V} \models \varphi(c, a_1, \ldots, a_n, b)$ if and only if $b = f(a_1, \ldots, a_n)$. Let $\mathcal{L}_\varepsilon^{\mathcal{A}}$ be the language obtained from $\mathcal{L}_\varepsilon$ by adding for each $\mathcal{A}$-definable function $f : \mathrm{N}^n \to \mathrm{N}$ an n-ary function symbol "f".

The *realizability interpretation* of $\mathcal{L}_\varepsilon^{\mathcal{A}}$ in $\mathcal{A}$ consists of the following.

▶ **Definition 5.** *To each closed formula φ in $\mathcal{L}_\varepsilon^{\mathcal{A}}$ with parameters in* N*, we associate a* truth value $|\varphi| \subseteq \Lambda$ *and a* falsity value $\|\varphi\| \subseteq \Pi$*, they are defined jointly by induction on the complexity of φ:*

- $|\varphi| := \{t \in \Lambda \mid \forall \pi \in \|\varphi\|, t \star \pi \in \perp\!\!\!\perp\}$*;*
- $\|\top\| := \emptyset$ *and* $\|\bot\| := \Pi$*;*
- $\|a \notin b\| := \{\pi \in \Pi \mid (a,\pi) \in b\}$;
- $\|a \neq b\| := \|\top\|$ *if* $a \neq b$, $\|\bot\|$ *otherwise;*
- $\|a \not\in b\| := \bigcup_{c\in\mathrm{dom}(b)} \{t.t'.\pi \mid (c,\pi) \in b, t \in |a \subseteq c|, t' \in |c \subseteq a|\}$*;*
- $\|a \subseteq b\| := \bigcup_{c\in\mathrm{dom}(a)} \{t.\pi \mid (c,\pi) \in a, t \in |c \not\in b|\}$*;*
- $\|\psi \to \theta\| := \{t.\pi \mid t \in |\psi|, \pi \in \|\theta\|\}$*;*
- $\|\forall x\ \varphi(x)\| := \bigcup_{a\in\mathrm{N}} \|\varphi[a/x]\|$.

For atomic formulas, we identify the closed terms a and b with their valuations in N*. Formally,* $\|a \not\in b\|$ *and* $\|a \subseteq b\|$ *are defined by induction on the pair* $(\max(\mathrm{rk}_\mathrm{N}(a), \mathrm{rk}_\mathrm{N}(b)), \min(\mathrm{rk}_\mathrm{N}(a), \mathrm{rk}_\mathrm{N}(b)))$ *under the product order, where* $\mathrm{rk}_\mathrm{N}(c) := \min\{\alpha \mid c \in \mathrm{N}_{\alpha+1}\}$.

We say that a closed λ_c-term t realizes *a closed formula φ with parameters in* N *and write $t \Vdash \varphi$, whenever $t \in |\varphi|$.*

By standard set-theoretic arguments, for each formula $\varphi(\vec{x})$ in $\mathcal{L}_\varepsilon^{\mathcal{A}}$ (without parameters), there exist formulas $\varphi_\Pi(p, \vec{x})$ and $\varphi_\Lambda(p, \vec{x})$ in $\mathcal{L}_\in$ with parameters in $\mathrm{V}_{\mathrm{fp}(\mathcal{A})}$ such that for all sequences of sets $\vec{a} \in \mathrm{V}$, for all $\pi \in \Pi$ and $t \in \Lambda$,

$$\pi \in \|\varphi(\vec{a})\| \Leftrightarrow \mathrm{V} \models \varphi_\Pi(\pi, \vec{a}), \text{ and } t \in |\varphi(\vec{a})| \Leftrightarrow \mathrm{V} \models \varphi_\Lambda(t, \vec{a}).$$

Now, we would like to associate to $\mathcal{A}$ a "realizability theory" consisting of all closed formulas which are realized. However, for all $t \star \pi \in \perp\!\!\!\perp$, the λ_c-term $\mathrm{k}_\pi\, t$ realizes the formula $\bot$. Therefore, in order to obtain a realizability theory that is not automatically inconsistent, we will need to exclude terms of this shape; this is where the set $\mathcal{R}$ of *realizers* comes into play (i.e. the closed λ_c terms containing no continuation constant):

▶ **Definition 6.** *The* realizability theory of $(\mathcal{A}, \mathrm{V})$*, denoted by $T_{\mathcal{A},\mathrm{V}}$, is the set of all closed formulas, φ, of $\mathcal{L}_\varepsilon^{\mathcal{A}}$ with parameters in* N *such that there exists $t \in \mathcal{R}$ such that t realizes φ.*

The following facts are standard (see e.g. [14]):

- the realizability theory of $(\mathcal{A}, \mathrm{V})$ is closed under classical deduction, (i.e. if $\varphi \in T_{\mathcal{A},\mathrm{V}}$ and φ entails ψ in classical logic, then $\psi \in T_{\mathcal{A},\mathrm{V}}$);
- this theory is consistent if and only if for every $t \in \mathcal{R}$ there is a stack π such that $t \star \pi \notin \perp\!\!\!\perp$;
- this theory is generally not complete.

▶ **Theorem 7.** *Let* V *be a model of* ZF *and* $\mathcal{A}$ *a realizability algebra in* V*. The realizability theory of* $(\mathcal{A}, \mathrm{V})$ *contains* $\mathrm{ZF}_{\varepsilon}^{\mathcal{L}_{\varepsilon}^{\mathcal{A}}}$*. In particular, it contains* $\mathrm{ZF}_{\varepsilon}$*, and therefore* ZF*).*

We refer to [14] for a proof of this, or [19] for an alternative proof using the setup given. This justifies the following definition:

▶ **Definition 8.** *A* realizability model of $\mathrm{ZF}_{\varepsilon}$ *is a pair* $\mathcal{N} = (\mathrm{V}, \mathcal{A})$*, with* V *a model of* ZF *and* $\mathcal{A}$ *a realizability algebra in* V*. We write* $\mathcal{N} \Vdash \varphi$ *for "*$T_{\mathcal{A},\mathrm{V}}$ *contains* φ*".*

Sometimes, we will argue within models of the realizability theory $T_{\mathcal{A},\mathrm{V}}$ and by abuse of language we will call *realizability model* any model of $T_{\mathcal{A},\mathrm{V}}$.

5 Reish Names and Pairing

In this section we present the method of *reish names*, or *recursive* names, which is used for transferring properties of sets of the ground model to sets in a realizability model. Given a ground model set a, the *gimel of* a, $\gimel(a)$, is defined as $\gimel(a) \coloneqq a \times \Pi$, but $\daleth(a)$ shall instead apply this process recursively to all elements of a. This will have the benefit that $\daleth(a)$, and each one of its elements, is always an element of N.

▶ **Definition 9.** *For* $x \in \mathrm{V}$ *we define* $\daleth(x) \coloneqq \{(\daleth(y), \pi) \mid y \in x, \pi \in \Pi\}$.

This method will not in general give a straightforward interpretation of the ground model elements. For example, it is unclear if $\daleth(\omega)$ is an extensional name for the first limit ordinal in the ZF structure. However, it is a useful tool to transfer certain properties of sets of the ground model into the realizability model.

▶ **Proposition 10.** *If* $a \subseteq b$ *then* $\mathcal{N} \Vdash \daleth(a) \subseteq \daleth(b)$*.*

Proof. Let v_0 be a realizer such that $v_0 \Vdash \forall x(x \subseteq x)$ and set $v_1 \coloneqq \lambda f.(f(v_0))(v_0)$. It is easy to see that $v_1 \Vdash \daleth(a) \subseteq \daleth(b)$. ◀

▶ **Observation 11.** *If* $a \in b$ *then* $\mathbf{I} \Vdash \daleth(a) \,\varepsilon\, \daleth(b)$*, where* $\mathbf{I} = \lambda f.f$ *is the identity term. Thus, if* $a \in b$ *then* $\mathcal{N} \Vdash \daleth(a) \not\notin \daleth(b)$*.*

This construction then allows us to define a proper class of ordinals in a realizability model. For this we use the definition of ordinals as *transitive sets of transitive sets*, which can easily be seen to be equivalent to *a transitive set well-ordered by the* $\in$-relation.

▶ **Definition 12.** *(*$\mathrm{ZF}_{\varepsilon}$*) We say that a set* a *is a* ε-ordinal *if it is a* ε*-transitive set of* ε*-transitive sets. That is,* $\forall x \,\varepsilon\, a \,\forall y \,\varepsilon\, x\, (y \,\varepsilon\, a)$ *and* $\forall z \,\varepsilon\, a \,\forall x \,\varepsilon\, z \,\forall y \,\varepsilon\, x\, (y \,\varepsilon\, z)$*.*

Note that, over $\mathrm{ZF}_{\varepsilon}$, this definition is not equivalent to the definition of ordinals as transitive sets well-ordered by the ε-relation. As an example, consider the realizability model constructed at the end of [15] in which $\daleth(2)$ has size 4. In this case there are two ordinals, $a, b \,\varepsilon\, \daleth(2)$, such that $a \notin b$, $b \notin a$. However, $(a \cup \{a\}) \cup (b \cup \{b\})$ is an ε-ordinal on which the ε-relation does not linearly order the set.

▶ **Proposition 13.** *Suppose that* $\mathcal{N} = (\mathrm{N}, \not\varepsilon, \notin, \subseteq)$ *is a model of* $\mathrm{ZF}_{\varepsilon}$*. Then for any* $a \in \mathrm{N}$*:*

1. *If* a *is a* ε*-transitive set, then it is a* $\in$*-transitive set,*
2. *If* a *is a* ε*-ordinal, then it is a* $\in$*-ordinal.*

Proof. Suppose that a is a ε-transitive set and take $c \in b \in a$. Then there exists some $x \,\varepsilon\, a$ such that $x \simeq b$ and there exists some $y \,\varepsilon\, x$ such that $y \simeq c$. Since a is assumed to be ε-transitive, $y \,\varepsilon\, a$. Therefore $x \in a$ by definition of $\in$.

Next, suppose that a is a ε-ordinal. We have already shown that a is $\in$-transitive so it suffices to prove that every $b \in a$ is $\in$-transitive. So let $d \in c \in b \in a$. Then we can find $z \,\varepsilon\, y \,\varepsilon\, x \,\varepsilon\, a$ such that $x \simeq b$, $y \simeq c$ and $z \simeq d$. Since a is a ε-ordinal, $z \,\varepsilon\, x$ and therefore $d \in x$. Finally, $d \in x$ and $x \simeq b$ gives us $d \in b$, as required. ◀

▶ **Proposition 14.** *If δ is an ordinal in* V *then $\mathcal{N} \Vdash \daleth(\delta)$ is a ε-ordinal.*

Proof. Let δ be an ordinal in V. We show that $\daleth(\delta)$ is a ε-transitive set; the fact that it consists of ε-transitive sets will follow by a similar argument. To do this, we show that $\mathbf{I} \Vdash \forall x^{\daleth(\delta)} \forall y (y \notin \daleth(\delta) \to y \notin x)$. Fix $\beta \in \delta$, $c \in \mathrm{N}$, $t \Vdash c \notin \daleth(\delta)$ and $\pi \in \|c \notin \daleth(\beta)\|$. Now $\|c \notin \daleth(\beta)\| = \{\sigma \mid (c, \sigma) \in \daleth(\beta)\}$. Since this set is non-empty, it must be the case that $\|c \notin \daleth(\beta)\| = \Pi$ and $c = \daleth(\gamma)$ for some $\gamma \in \beta$. Therefore, $\|c \notin \daleth(\delta)\| = \|\daleth(\gamma) \notin \daleth(\delta)\| = \Pi$ hence $t \star \pi \in \perp\!\!\!\perp$, from which the result follows. ◀

We need a method to encode ordered pairs in the realizability structure; for this we introduce a function $\mathtt{op}$ satisfying $\mathcal{N} \Vdash$ "$\mathtt{op}(a, b)$ is the ordered pair of a and b" for any $a, b \in \mathrm{N}$. This definition is based on the *Wiener pairing function* which is $(a, b) = \{\{\{a\}, \emptyset\}, \{\{b\}\}\}$. Here 0 denotes the λ-term $\lambda x.\lambda y.y$ and 1 the λ-term $\lambda x.\lambda y.xy$.

▶ **Definition 15.** *For $a, b \in \mathrm{N}$, we define*
- *the* singleton *of a as the set* $\mathtt{sng}(a) := \{a\} \times \Pi$,
- *the* unordered pair *of a and b as the set* $\mathtt{up}(a, b) := \{(a, \underline{0}.\pi) \mid \pi \in \Pi\} \cup \{(b, \underline{1}.\pi) \mid \pi \in \Pi\}$,
- *the* ordered pair *of a and b as the set* $\mathtt{op}(a, b) := \mathtt{up}(\mathtt{up}(\mathtt{sng}(a), \daleth(0)), \mathtt{sng}(\mathtt{sng}(b)))$.

Note that the three functions $\mathtt{sng} : \mathrm{N} \to \mathrm{N}$, $\mathtt{up}, \mathtt{op} : \mathrm{N}^2 \to \mathrm{N}$ *are $\mathcal{A}$-definable.*

▶ **Theorem 16** ([19])**.** *The following are realizable in $\mathcal{N}$:*
- $\forall x_1 \forall x_2 \forall y_1 \forall y_2\, (\mathtt{op}(x_1, y_1) \simeq \mathtt{op}(x_2, y_2) \to (x_1 \simeq x_2 \wedge y_1 \simeq y_2))$.
- $\forall x_1 \forall x_2 \forall y_1 \forall y_2\, (x_1 \simeq x_2 \to (y_1 \simeq y_2 \to \mathtt{op}(x_1, y_1) \simeq \mathtt{op}(x_2, y_2)))$.

While we do not include a proof of this fact, we refer to [19] for all necessary details.

6 Relativization over transitive sets

In this setion we introduce a method to relativize formulas to certain objects in a realizability model. Relativization is a simple, but power, technique which provides a way to interpret a formula internally in a given transitive set or class. Given a formula φ and set M, the relativised formula φ^M is essentially constructed by replacing all unbounded quantifiers with quantifiers bounded by M, that is $\forall x$ becomes $\forall x \in M$.

Let $\mathcal{A}$ be a realizability algebra and construct the class of names, N. Given a transitive set M containing $\mathcal{A}$, we set $\mathrm{M}^{\mathcal{A}} := \{(a, \pi) \mid a \in \mathrm{M} \cap \mathrm{N},\ \pi \in \Pi\}$. Namely, $\mathrm{M}^{\mathcal{A}}$ is a name for the set of names that are in M and $\mathrm{dom}(\mathrm{M}^{\mathcal{A}}) = \mathrm{M} \cap \mathrm{N}$. We can then relativize to M any formula φ in the language of $\mathcal{L}_{\varepsilon}^{\mathcal{A}}$, which we denote by φ^{M}, by replacing universal quantifiers $\forall x$ by bounded quantifiers $\forall x^{\mathrm{M}^{\mathcal{A}}}$ defined in the usual way.

▶ **Definition 17.** *Suppose that* M *is a transitive set containing $\mathcal{A}$ and φ is a formula in* $\mathrm{Fml}_{\varepsilon}$. *We define* $\|\forall x^{\mathrm{M}^{\mathcal{A}}} \varphi(x)\| = \bigcup_{c \in \mathrm{dom}(\mathrm{M}^{\mathcal{A}})} \|\varphi(c)\|$.

It is easy to see that this restricted quantifier $\forall x^{\mathrm{M}^{\mathcal{A}}}$ corresponds to $\forall x \,\varepsilon\, \mathrm{M}^{\mathcal{A}}$.

▶ **Proposition 18.** *Suppose that* M *is a transitive set which contains* $\mathcal{A}$*. Then*
1. $\lambda f.\lambda g.gf \Vdash \forall x^{\mathrm{M}^{\mathcal{A}}} \varphi(x) \to \forall x(\neg\varphi(x) \to x \notin \mathrm{M}^{\mathcal{A}})$,
2. $\lambda f.\mathrm{cc}(\lambda k.fk) \Vdash \forall x(\neg\varphi(x) \to x \notin \mathrm{M}^{\mathcal{A}}) \to \forall x^{\mathrm{M}^{\mathcal{A}}} \varphi(x)$.

Proof. First, suppose that $t \Vdash \forall x^{\mathrm{M}^{\mathcal{A}}} \varphi(x)$, $s \Vdash \neg\varphi(b)$ for some $b \in \mathrm{N}$ and $\pi \in \|b \notin \mathrm{M}^{\mathcal{A}}\|$. Since $(\pi, b) \in \mathrm{M}^{\mathcal{A}}$ and M is transitive, we must have $b \in \mathrm{M} \cap \mathrm{N}$ and therefore $t \star \sigma \in \perp\!\!\!\perp$ for any $\sigma \in \|\varphi(b)\|$. It follows that $t \Vdash \varphi(b)$, hence $\lambda f.\lambda g.gf \star t.s.\pi \succ s \star t.\pi \in \perp\!\!\!\perp$, as required.

For the second claim, suppose that $t \Vdash \forall x(\neg\varphi(x) \to x \notin a)$ and $\pi \in \|\forall x^{\mathrm{M}^{\mathcal{A}}} \varphi(x)\|$. Fix $b \in \mathrm{M} \cap \mathrm{N}$ such that $\pi \in \|\varphi(b)\|$. We have

$$\|\forall x(\neg\varphi(x) \to x \notin \mathrm{M}^{\mathcal{A}})\| = \bigcup_{c \in \mathrm{N}} \|\neg\varphi(c) \to c \notin \mathrm{M}^{\mathcal{A}}\| = \bigcup_{c \in \mathrm{N}} \{s.\sigma \mid s \Vdash \neg\varphi(c), \sigma \in \|c \notin \mathrm{M}^{\mathcal{A}}\|\}.$$

Since $\pi \in \|\varphi(b)\|$, $\mathrm{k}_\pi \Vdash \neg\varphi(b)$. Moreover, $(b, \pi) \in \mathrm{M}^{\mathcal{A}}$ thus $\mathrm{k}_\pi.\pi \in \|\forall x(\neg\varphi(x) \to x \notin \mathrm{M}^{\mathcal{A}})\|$. It follows that $\lambda f.\mathrm{cc}(\lambda k.fk) \star t.\pi \succ \mathrm{cc} \star (\lambda k.tk).\pi \succ \lambda k.tk \star \mathrm{k}_\pi.\pi \succ t \star \mathrm{k}_\pi.\pi \in \perp\!\!\!\perp$. ◀

One can easily observe that if M is a transitive set containing $\mathcal{A}$, then $\mathrm{M}^{\mathcal{A}}$ is realized to be a ε-transitive set, and thus also a transitive set by Proposition 13.

▶ **Proposition 19.** *For every transitive set* M *which contains* $\mathcal{A}$*,* $\mathbf{I} \Vdash \forall x^{\mathrm{M}^{\mathcal{A}}} \forall y(y \notin \mathrm{M}^{\mathcal{A}} \to y \notin x)$*. Thus* $\mathcal{N}$ *realizes that* $M^{\mathcal{A}}$ *is a* ε*-transitive set.*

▶ **Theorem 20.** *Let* M *be a transitive class which contains* $\mathcal{A}$*, then for all sets* $a_1, \ldots, a_n$ *in* $\mathrm{M} \cap \mathrm{N}$*, and for every formula* $\varphi \in \mathrm{Fml}_\varepsilon$*,* $\|\varphi^{\mathrm{M}^{\mathcal{A}}}(a_1, \ldots, a_n)\| = \|\varphi(a_1, \ldots, a_n)\|^{\mathrm{M}}$*.*

Proof. We procede by induction on the formula, ignoring the parameters to simplify notation.

Let $\varphi(x, y) \equiv x \notin y$ and fix $a, b \in \mathrm{M} \cap \mathrm{N}$. Then we have $\|(a \notin b)^{\mathrm{M}^{\mathcal{A}}}\| = \|a \notin b\|^{\mathrm{V}} = \{\pi \in \Pi \mid (a, \pi) \in b\} = \|a \notin b\|^{\mathrm{M}}$, since M is a transitive class containing Π.

We will prove the cases $\varphi(x, y) \equiv x \notin y$ and $\varphi(x, y) \equiv x \subseteq y$ by simultaneous induction on the lexicographical order of the pair of ranks of a and b. So, fix $a, b \in \mathrm{M} \cap \mathrm{N}$. Then we have $\|(a \notin b)^{\mathrm{M}^{\mathcal{A}}}\| = \|a \notin b\|^{\mathrm{V}} = \bigcup_{c \in \mathrm{dom}(b)} \{t.t'.\pi \mid (c, \pi) \in b, t \Vdash a \subseteq c, t' \Vdash c \subseteq a\}$. Now, $t \Vdash c \subseteq a$ means $\forall \sigma \in \|c \subseteq a\| (t \star \sigma \in \perp\!\!\!\perp)$, by the induction hypothesis this is equivalent to $\forall \sigma \in \|c \subseteq a\|^{\mathrm{M}} (t \star \sigma \in \perp\!\!\!\perp)$ which corresponds to $(t \Vdash c \subseteq a)^{\mathrm{M}}$. Thus we have $\|(a \notin b)^{\mathrm{M}^{\mathcal{A}}}\| = \bigcup_{c \in \mathrm{dom}(b)} \{t.t'.\pi \mid (c, \pi) \in b, (t \Vdash a \subseteq c)^{\mathrm{M}}, (t' \Vdash c \subseteq a)^{\mathrm{M}}\} = \|a \notin b\|^{\mathrm{M}}$.

Similarly for the second case, by applying the induction hypothesis we have $\|(a \subseteq b)^{\mathrm{M}^{\mathcal{A}}}\| = \bigcup_{c \in \mathrm{dom}(a)} \{t.\pi \mid (c, \pi) \in a, (t \Vdash c \notin b)^{\mathrm{M}}\} = \|a \subseteq b\|^{\mathrm{M}}$.

The cases $\varphi \equiv \psi \to \chi$ and $\varphi \equiv \forall x\, \psi(x)$ follow easily from the induction hypothesis. ◀

We will apply these results in particular to the transitive sets of the Von Neumann hierarchy. We end this section by showing that if $\mathcal{A}$ is an element of V_γ then $\mathrm{V}_\gamma^{\mathcal{A}}$ is simply the construction of the names internally in V_γ.

▶ **Lemma 21.** *Let* γ *be a limit ordinal such that* $\mathcal{A} \in \mathrm{V}_\gamma$*. Then* $\mathrm{V}_\gamma^{\mathcal{A}} = \bigcup \mathrm{N}_\gamma$ *and* $\mathrm{N}_\gamma = (\mathrm{N})^{\mathrm{V}_\gamma}$*.*

Proof. First, observe that for every $a \in \mathrm{N}$, $\mathrm{rk}_{\mathrm{N}}(a) \leq \mathrm{rk}_{\mathrm{V}}(a) \leq \max\{\mathrm{rk}_{\mathrm{N}}(a), \mathrm{rk}_{\mathrm{V}}(\Pi)\} + 2$ where $\mathrm{rk}_{\mathrm{N}}(a)$ is the minimal α for which $a \in \mathrm{N}_\alpha$ and $\mathrm{rk}_{\mathrm{V}}(a)$, the minimal α for which $a \in \mathrm{V}_\alpha$, is the standard rank of a in V. From this it follows that $\mathrm{V}_\gamma \cap \mathrm{N} = \mathrm{N}_\gamma$ since γ is a limit ordinal and $\mathcal{A} \in \mathrm{V}_\gamma$, therefore $\mathrm{V}_\gamma^{\mathcal{A}} = \{(a, \pi) \mid a \in \mathrm{N}_\gamma, \pi \in \Pi\} = \bigcup \mathrm{N}_\gamma$.

We now prove inductively that $\mathrm{N}_\alpha = (\mathrm{N}_\alpha)^{\mathrm{V}_\gamma}$ for all $\alpha \leq \gamma$, starting with $\mathrm{N}_0 = \emptyset = (\mathrm{N}_0)^{\mathrm{V}_\gamma}$. So fix $\alpha \leq \gamma$ and suppose that the claim holds for all $\beta < \alpha$. Then

$$\mathrm{N}_\alpha = \bigcup_{\beta < \alpha} \mathcal{P}(\mathrm{N}_\beta \times \Pi) = \bigcup_{\beta < \alpha} \mathcal{P}^{\mathrm{V}_\gamma}((\mathrm{N}_\beta)^{\mathrm{V}_\gamma} \times \Pi) = (\mathrm{N}_\alpha)^{\mathrm{V}_\gamma}.$$ ◀

7 Realizing Inaccessibles

In this section, we assume the consistency of the theory "ZFC plus there is an inaccessible cardinal" and, from that, we show how to build a *realizability* model for the equiconsistent theory "ZF plus there is an inaccessible set".

We recall that over ZFC an uncountable cardinal κ is (strongly) inaccessible if it is a regular cardinal which is a strong limit, namely whenever $\alpha < \kappa$, $2^\alpha < \kappa$. However, in models where the Axiom of Choice fails this is no longer a satisfactory definition, for example if 2^ω is not well-ordered then no such (well-orderable) cardinals can possibly exist. Therefore, it is preferable to take an alternative definition.

From a structural point of view, the defining property of an inaccessible cardinal is that it provides a very robust model of set theory. Namely, if κ is inaccessible then V_κ is a Grothendieck Universe which contains ω. For our purposes we will take a slightly different, but equivalent, definition which is that a set will be *inaccessible* if it is a transitive model of full second-order ZF; this definition is motivated by [5, Definition 1].

▶ **Definition 22.** *We call a set z* inaccessible *if it satisfies the following:*
- *Transitivity:* $\forall u \in z\, \forall v \in u(v \in z)$.
- *Empty Set:* $\exists u \in z\, \forall v(v \notin u)$.
- *Pairing:* $\forall u \in z\, \forall v \in z\, \exists w \in z\, (u \in w \wedge v \in w)$.
- *Unions:* $\forall u \in z\, \exists v \in z\, \forall w\, (w \in v \leftrightarrow \exists x \in u(w \in x))$.
- *Infinity:* $\forall a \in z\, \exists u \in z\, (a \in u \wedge \forall v \in u \exists w \in u(v \in w))$.
- *Weak Power Set:* $\forall u \in z\, \exists v \in z\, \forall w\, \exists x \in v\, \forall y\, (y \in x \leftrightarrow (y \in u \wedge y \in w))$.
- *Second-order Collection:*

$$\forall a \in z \forall f (\forall x \in a\, \exists y \in z\, ((x, y) \in f) \to \exists b \in z\, \forall x \in a\, \exists y \in b\, ((x, y) \in f)).$$

The proof of the following proposition is standard and justifies calling such sets inaccessible.

▶ **Proposition 23.** *Over* ZFC *the following are equivalent:*
- *z is inaccessible,*
- *z is a Grothendieck Universe containing ω,*
- *$z = \mathrm{V}_\kappa$ for some inaccessible cardinal κ.*

Moreover, it is known that if V is a model of ZF with an inaccessible set z, then $z \cap \mathrm{L}$ is an inaccessible set in the constructible universe L, which is a model of ZFC. In fact, $z \cap \mathrm{L} = \mathrm{L}_\kappa$ where κ is an inaccessible cardinal in L. Therefore ZF with an inaccessible set is equiconsistent with ZFC plus an inaccessible cardinal.

We now consider any realizability algebra $\mathcal{A}$ in a model V of ZF (the ground model) with an inaccessible set z such that $\mathcal{A} \in z$. We shall give an appropriate translation of inaccessible sets to the language of ZF_ε, which we call ε-inaccessible sets. We shall then show that in any realizability model $\mathcal{N}$, $z^{\mathcal{A}}$ is a ε-inaccessible set and in the corresponding ZF structure $(\mathrm{N}, \in, \simeq)$, $z^{\mathcal{A}}$ is an inaccessible set.

▶ **Definition 24.** *In a model of $T_{\mathcal{A},\mathrm{V}}$, we call a set z ε-*inaccessible *if it satisfies the following:*
- *ε-Transitivity:* $\forall u\, \varepsilon\, z\, \forall v\, \varepsilon\, u(v\, \varepsilon\, z)$.
- *ε-Empty Set:* $\exists u\, \varepsilon\, z\, \forall v(v \not\varepsilon\, u)$.
- *ε-Pairing:* $\forall u\, \varepsilon\, z\, \forall v\, \varepsilon\, z\, \exists w\, \varepsilon\, z\, (u\, \varepsilon\, w \wedge v\, \varepsilon\, w)$.
- *ε-Unions:* $\forall u\, \varepsilon\, z\, \exists v\, \varepsilon\, z\, \forall w\, (w\, \varepsilon\, v \leftrightarrow \exists x\, \varepsilon\, u(w\, \varepsilon\, x))$.

- *ε-Infinity:* $\forall a \,\varepsilon\, z\, \exists u \,\varepsilon\, z\, (a \,\varepsilon\, u \wedge \forall v \,\varepsilon\, u \exists w \,\varepsilon\, u (v \,\varepsilon\, w))$.
- *ε-Weak Power Set:* $\forall u \,\varepsilon\, z\, \exists v \,\varepsilon\, z\, \forall w\, \exists x \,\varepsilon\, v\, \forall y\, (y \,\varepsilon\, x \leftrightarrow (y \,\varepsilon\, u \wedge y \,\varepsilon\, w))$.
- *ε-Second-order Collection:*

$$\forall a \,\varepsilon\, z \forall f (\forall x \,\varepsilon\, a\, \exists y \,\varepsilon\, z\, (\mathtt{op}(x,y) \,\varepsilon\, f) \to \exists b \,\varepsilon\, z\, \forall x \,\varepsilon\, a\, \exists y \,\varepsilon\, b\, (\mathtt{op}(x,y) \,\varepsilon\, f)).$$

▶ **Lemma 25.** *If z is an inaccessible set in V and $\mathcal{A} \in z$ is a realizability algebra, then for the corresponding realizability model $\mathcal{N} = (\mathrm{V}, \mathcal{A})$ we have $\mathcal{N} \Vdash z^{\mathcal{A}}$ is a ε-inaccessible set.*

Proof. Firstly, by Proposition 19 we have that $z^{\mathcal{A}}$ is a ε-transitive set. For all of the axioms except for Second-order Collection it suffices to verify that $\mathrm{dom}(z^{\mathcal{A}})$ is closed by certain relevant names. For this, we observe that since z is a transitive set which is closed under Weak Power Set and Second-order Collection, we have that $z = \mathrm{V}_{\mathrm{rank}(z)}$. Therefore, z is also closed under Separation and, by Lemma 21, $z^{\mathcal{A}} = (\mathrm{N})^z$.

For Empty Set the relevant name is $\emptyset$ which is in $z \cap \mathrm{N}$ be definition. Given $a, b \in z \cap \mathrm{N}$ the name for the pair is $\{a, b\} \times \Pi \in z \cap \mathrm{N}$. Given $a \in z \cap \mathrm{N}$ the name for the union is $\{(c, \sigma) \mid \exists (x, \pi) \in a\ (c, \sigma) \in x\} \in z \cap \mathrm{N}$. Given $a \in z \cap \mathrm{N}$ the name for the infinite set containing a is $\{(a^n, \pi) \mid n \in \omega, \pi \in \Pi\}$ where $a^0 := a$ and $a^{n+1} := \{a^n\} \times \Pi$. It is clear that all of these names are in $z \cap \mathrm{N}$. Finally, given $a \in z \cap \mathrm{N}$ the name for the Weak Power Set of a is $\mathcal{P}(\mathrm{dom}(a) \times \Pi) \times \Pi$.

It remains to prove the axiom of Second-order Collection. Fix $a \in z \cap \mathrm{N}$ and $f \in \mathrm{N}$. Since z is an inaccessible set, by Second-order Collection in z we can find a set $Y \in z$ such that

$$\forall (x, \pi) \in a \forall t \in \Lambda \exists y \in z \cap \mathrm{N} (t \Vdash \mathtt{op}(x,y) \,\varepsilon\, f) \to \forall (x, \pi) \in a \forall t \in \Lambda \exists y \in Y (t \Vdash \mathtt{op}(x,y) \,\varepsilon\, f).$$

Let $b := \{(y, \pi) \mid \exists t \in \Lambda\ \exists x\ ((x, \pi) \in a,\ t \Vdash \mathtt{op}(x,y) \,\varepsilon\, f,\ y \in Y)\} \in z \cap \mathrm{N}$. It will suffice to prove that for any $x \in \mathrm{N}$, $\|\forall y (\mathtt{op}(x,y) \,\varepsilon\, f \to x \notin a)\| \subseteq \|\forall y (\mathtt{op}(x,y) \,\varepsilon\, f \to x \notin b)\|$. For this, fix $t.\pi \in \|\forall y (\mathtt{op}(x,y) \,\varepsilon\, f \to x \notin a)\|$. Then we can fix some $c \in \mathrm{N}$ such that $t \Vdash \mathtt{op}(x,c) \,\varepsilon\, f$ and $(x, \pi) \in a$. By the definition of Y, this means that there exists a $c' \in Y$ such that $t \Vdash \mathtt{op}(x,c') \,\varepsilon\, f$ and $(x, \pi) \in a$ from which it follows that $(c', \pi) \in b$. Thus $t.\pi \in \|\forall y (\mathtt{op}(x,y) \,\varepsilon\, f \to x \notin b)\|$. ◀

▶ **Theorem 26.** *Let $\mathcal{N} = (\mathrm{V}, \mathcal{A})$ be a realizability model, then*
$\mathcal{N} \Vdash \forall z (z$ is an ε-inaccessible set $\to z$ is an inaccessible set$)$

Proof. We argue within a realizability model $\mathcal{N} = (\mathrm{N}, \not\varepsilon, \notin, \subseteq)$. Suppose that z is a ε-inaccessible set. We want to show that z is an inaccessible set in $(\mathrm{N}, \in, \simeq)$. It is easy to see that z satisfies every condition except for possibly Second-order Collection. In order to prove this axiom, note that Second-order Collection is equivalent to the statement

$$\forall a \in z \forall f\, \exists b \in z\, \forall x \in a\, (\exists y \in z\, ((x,y) \in f) \to \exists y \in b\, ((x,y) \in f)).$$

So fix $a \,\varepsilon\, z$ and f, we define $f' = \{\mathtt{op}(x,y) \mid x \,\varepsilon\, a, y \,\varepsilon\, z, \mathtt{op}(x,y) \in f\}$. Since z satisfies ε-Second-order Collection, we can find some $b \,\varepsilon\, z$ such that $\forall x \,\varepsilon\, a (\exists y \,\varepsilon\, z\, \mathtt{op}(x,y) \,\varepsilon\, f' \to \exists y \,\varepsilon\, b\, \mathtt{op}(x,y) \,\varepsilon\, f')$. We shall show that this same set b witnesses Second-order Collection for f in ZF. By Theorem 16, we know that for every x, y in the realizability model, $\mathtt{op}(x,y) \simeq (x,y)$. Fix $x \in a$ and suppose that $\exists y \in z ((x,y) \in f)$. Then, we can find $x' \,\varepsilon\, a$ and $y' \,\varepsilon\, z$ such that $x \simeq x'$ and $y \simeq y'$. Therefore $(x,y) \simeq \mathtt{op}(x',y')$, hence $\mathtt{op}(x',y') \,\varepsilon\, f'$. By definition of b, we can find some $y'' \,\varepsilon\, b$ such that $\mathtt{op}(x',y'') \,\varepsilon\, f'$, thus $(x',y'') \in f$ by definition of f'. Since $x \simeq x'$, we have $(x,y'') \simeq (x',y'')$, thus $(x,y'') \in f$ as required. ◀

▶ **Corollary 27.** *Let $\mathcal{N} = (\mathrm{V}, \mathcal{A})$ be a realizability model. Assume that there is an inaccessible set z in V such that $\mathcal{A} \in z$. Then $\mathcal{N} \Vdash \mathrm{ZF} +$ there exists an inaccessible set.*

▶ Remark 28. One should observe that the statement $z^{\mathcal{A}}$ is a ε-inaccessible set can be expressed by a single sentence. Therefore, given a realizability algebra $\mathcal{A}$, there exists a single realizer θ such that whenever z is an inaccessible set with $\mathcal{A} \in z$, $\theta \Vdash$ "$z^{\mathcal{A}}$ is a ε-inaccessible set".

8 Realizing Mahlo cardinals

In this section, we show that from the consistency of the theory "ZFC plus there is a Mahlo cardinal" we can build a *realizability* model for the equiconsistent theory "ZF plus there is a Mahlo set". Recall that κ is a Mahlo cardinal if $\{\alpha \in \kappa \mid \alpha \text{ is strongly inaccessible}\}$ is stationary in κ. However, as in the inaccessible case, it is beneficial to use the following, more structural, definition which was first formulated by Lévy in [18] and which is the version used by Friedman and Ščedrov [5, Definition 2].

▶ **Definition 29.** *A* Mahlo *set is an inaccessible set z such that for every $u \in z$ and for every binary relation R, there is an inaccessible set $v \in z$ such that*
1. $u \in v$,
2. R reflects to v, *which means that* $\forall x \in v(\exists y \in z\,(x, y) \in R \to \exists y \in v\,(x, y) \in R)$.

The proof of the following proposition is standard and justifies calling such sets Mahlo.

▶ **Proposition 30** (Lévy, [18, Theorem 3]). *Over* ZFC*, z is a Mahlo set iff $z = \mathrm{V}_\kappa$ for some Mahlo cardinal κ.*

Moreover, as with the inaccessible case, if V is a model of ZF with a Mahlo set z, then $z \cap \mathrm{L}$ remains a Mahlo set in L. Therefore ZF with a Mahlo set is equiconsistent with ZFC plus a Mahlo cardinal.

▶ **Definition 31.** *In a model of $T_{\mathcal{A},\mathrm{V}}$, we say that z is a ε-*Mahlo set *if z is a ε-inaccessible set and for every $u \,\varepsilon\, z$ and every binary relation R, there is a ε-inaccessible set $v \,\varepsilon\, z$ such that*
1. $u \,\varepsilon\, v$,
2. $\forall x \,\varepsilon\, v(\exists y \,\varepsilon\, z\ \mathtt{op}(x, y) \,\varepsilon\, R \to \exists y \,\varepsilon\, v\ \mathtt{op}(x, y) \,\varepsilon\, R)$.

▶ **Lemma 32.** *Let $\mathcal{N} = (\mathrm{V}, \mathcal{A})$ be a realizability model and suppose that z is a Mahlo set in* V *such that $\mathcal{A} \in z$, then $\mathcal{N} \Vdash z^{\mathcal{A}}$ is a ε-Mahlo set.*

Proof. By Remark 28 we know that whenever v is an inaccessible set such that $\mathcal{A} \in v$, $v^{\mathcal{A}}$ is realized to be a ε-inaccessible set by a realizer that does not depend on v. In particular, this means that $z^{\mathcal{A}}$ is realized to be a ε-inaccessible set. To realize that $z^{\mathcal{A}}$ is a ε-Mahlo set, we fix $a \in z \cap \mathrm{N}$ and $R \in \mathrm{N}$. First, we define $R' := \{((x, \pi), y) \mid (\mathtt{op}(x, y), \pi) \in R,\ y \in \mathrm{N}\}$. Since z is a Mahlo set in the ground model, we can find an inaccessible set v such that $a, \mathcal{A} \in v$ and R' reflects to v. The following hold:
1. $\mathbf{I} \Vdash v^{\mathcal{A}} \,\varepsilon\, z^{\mathcal{A}}$ and $\mathbf{I} \Vdash a \,\varepsilon\, v^{\mathcal{A}}$,
2. $v^{\mathcal{A}}$ is realized to be a ε-inaccessible set by a realizer that does not depend on v.

We want to realize that, in $\mathcal{N}$, R reflects to $v^{\mathcal{A}}$. To do this, it suffices to show that

$$\mathbf{I} \Vdash \forall x^{v^{\mathcal{A}}}(\forall y^{v^{\mathcal{A}}} \mathtt{op}(x, y) \notin R \to \forall y^{z^{\mathcal{A}}} \mathtt{op}(x, y) \notin R).$$

Fix t, π such that $t \Vdash \forall y^{v^{\mathcal{A}}} \mathtt{op}(x, y) \notin R$ and $\pi \in \|\forall y^{z^{\mathcal{A}}} \mathtt{op}(x, y) \notin R\|$. There is $y \in z \cap \mathrm{N}$ such that $(\mathtt{op}(x, y), \pi) \in R$, thus $((x, \pi), y) \in R'$. Since R' reflects to v there is $y' \in v$ such that $(\mathtt{op}(x, \pi), y') \in R'$. This means that $(\mathtt{op}(x, y'), \pi) \in R$ and $y' \in \mathrm{N}$. So $y' \in v \cap \mathrm{N}$, hence $\pi \in \|\forall y^{v^{\mathcal{A}}} \mathtt{op}(x, y) \notin R\|$ and $t \star \pi \in \perp\!\!\!\perp$. ◀

▶ **Theorem 33.** *Let $\mathcal{N} = (\mathrm{V}, \mathcal{A})$ be a realizability model, then* $\mathcal{N} \Vdash \forall z(z$ *is a* ε*-Mahlo set* $\to z$ *is a Mahlo set*$)$

Proof. We work within a realizability model. Suppose that z is a ε-Mahlo set. By Theorem 26 we know that z is an inaccessible set. Fix $u \in z$ and let R be a binary relation. Let $R_\varepsilon := \{\mathtt{op}(x, y) \mid x, y \,\varepsilon\, z, \mathtt{op}(x, y) \in R\}$. Since z is a ε-Mahlo set we can fix a ε-inaccessible (and hence inaccessible) set $v \,\varepsilon\, z$ such that $u \,\varepsilon\, v$ and R_ε reflects to v. By Theorem 16, we know that for every x, y in the realizability model, $\mathtt{op}(x, y) \simeq (x, y)$. Since $u \,\varepsilon\, v \,\varepsilon\, z$ we have $u \in v \in z$. For the final property, fix $x \in v$ and suppose that $(x, y) \in R$ for some $y \in z$. Next, take $x' \,\varepsilon\, v$ and $y' \,\varepsilon\, z$ such that $x \simeq x'$ and $y \simeq y'$. Then, by definition, $\mathtt{op}(x', y') \,\varepsilon\, R_\varepsilon$ so, since R_ε reflects to v, we can find some $y'' \,\varepsilon\, v$ for which $\mathtt{op}(x', y'') \,\varepsilon\, R_\varepsilon$. Unpacking the definition of R_ε this means that $(x', y'') \in R$. Therefore, since $x \simeq x'$ we have that there exists some $y'' \in v$ for which $(x, y'') \in R$, as required. ◀

▶ **Corollary 34.** *Let $\mathcal{N} = (\mathrm{V}, \mathcal{A})$ be a realizability model. Assume that there is a Mahlo set z in V such that $\mathcal{A} \in z$. Then $\mathcal{N} \Vdash \mathrm{ZF} +$ there exists a Mahlo set.*

9 Extending Realizability to Classes

Gödel-Bernays set theory (GB) is an extension of Zermelo-Frænkel set theory (ZF) with a built-in notion of *classes* – arbitrary collections of sets that may be too big to be sets themselves. In the section, we will show how to similarly extend ZF_ε to a theory GB_ε that supports classes, and we will show how to construct realizability models of GB_ε.

We will work in *two-sorted* first-order logic (without equality): one sort will represent *sets*, and the other, *classes*. We will use lowercase letters for set variables, and uppercase letters for class variables. Quantification over sets will be denoted by "$\forall^0$" and "$\exists^0$", and quantification over classes by "$\forall^1$" and "$\exists^1$" (though we may drop the exponents when there is no ambiguity).

Let $\mathcal{L}^2_\in$ denote the first-order language over the signature $\{\in^0, \in^1, \simeq\}$, where $\in^0$ and $\simeq$ are relation symbols of arity Set $\times$ Set, and $\in^1$ is a relation symbol of arity Set $\times$ Class. The reason why we need two versions of $\in$ is that both sets and classes can contain sets. Likewise, let $\mathcal{L}^2_\varepsilon$ denote the first-order language over the signature $\{\not\in^0, \not\in^1, \subseteq, \not\varepsilon^0, \not\varepsilon^1, \neq^0, \neq^1\}$, where $\not\in^0$, $\subseteq$, $\not\varepsilon^0$ and $\neq^0$ are relation symbols of arity Set $\times$ Set, $\not\varepsilon^1$ and $\not\in^1$ are relation symbols of arity Set $\times$ Class, and $\neq^1$ is a relation symbol of arity Class $\times$ Class. Since the context always makes it clear which "version" of a given relation symbol is being used, we will systematically drop these exponents and simply write $\not\in$ and $\not\varepsilon$.

The theory GB_ε is the theory over $\mathcal{L}^2_\varepsilon$ generated by the following axioms:

1. The axioms of ZF_ε, with the axioms schemas extended to all formulas of the language $\mathcal{L}^2_\varepsilon$ that contain *no quantifications over classes*.
2. Class Separation: $\forall^1 A\, \forall^0 b\, \exists^0 a\, \forall^0 x\, (x \,\varepsilon\, a \leftrightarrow x \,\varepsilon\, A \wedge x \,\varepsilon\, b)$.
3. Class Induction: $\forall^1 A\, \Big((\forall^0 x\, ((\forall^0 y \,\varepsilon\, x\; y \,\varepsilon\, A) \to x \,\varepsilon\, A)) \to \forall^0 z\, (z \,\varepsilon\, A)\Big)$.
4. Elementary Class Comprehension: $\forall^1 A\, \forall^0 u\, \exists^1 B\, \forall^0 x\, (x \,\varepsilon\, B \leftrightarrow \varphi(x, u, A))$ for every formula $\varphi(x, u, A)$ with no quantifications over classes.
5. Class Collection: $\forall^1 A\, \forall^0 u\, \forall^0 a\, \exists^0 b\, \forall^0 x \,\varepsilon\, a\, \Big((\exists^0 y\, \varphi(x, y, u, A)) \to (\exists^0 y \,\varepsilon\, b\, \varphi(x, y, u, A))\Big)$, for every formula $\varphi(x, y, u, A)$ with no quantifications over classes.
6. Definition of $\in^1$: $\forall^1 A\, \forall^0 x\, \Big(x \in A \leftrightarrow \exists^0 y\, (y \,\varepsilon\, A \wedge y \simeq x)\Big)$.

We refer the read to the end of Chapter 6 of [9] and Chapter 4 of [20] for more details on GB and second-order set theories in general. By a simple generalisation of the ZF case, we can see that the theory GB_ε is a conservative extension of the standard theory GB.

▶ **Theorem 35.** *Let φ be a closed formula in $\mathcal{L}^2_\in$, then $\mathrm{GB} \vdash \varphi$ if and only if $\mathrm{GB}_\varepsilon \vdash \varphi$.*

Whenever $\mathcal{L}^2$ is a first-order language that contains $\mathcal{L}^2_\varepsilon$, we will denote by $\mathrm{GB}_\varepsilon^{\mathcal{L}^2}$ the theory obtained by taking GB_ε and enriching all the axiom schemas to include all the formulas of $\mathcal{L}^2$ with no quantifications over classes.

Realizability Models with Classes

Let $(\mathrm{V}, \mathcal{C})$ be a model of GB and let $\mathcal{A} = (\Lambda, \Pi, \perp\!\!\!\perp)$ be a realizability algebra in V. Let:

- $\mathrm{N} \coloneqq \bigcup_{\alpha \in \mathrm{Ord}} \mathrm{N}_\alpha \in \mathcal{C}$, where $\mathrm{N}_\alpha \coloneqq \bigcup_{\beta<\alpha} \mathcal{P}(\mathrm{N}_\beta \times \Pi) \in \mathrm{V}$ as before,
- $\mathcal{D} \coloneqq \{X \in \mathcal{C} \mid X \subseteq \mathrm{N}\} \subseteq \mathcal{C}$.

As in Section 4, we let $\mathcal{L}^{\mathcal{A},2}_\varepsilon$ denote the language obtained by adding a function symbol f for each $\mathcal{A}$-definable function $f : \mathrm{N}^n \to \mathrm{N}$.

▶ **Definition 36.** *We extend Definition 5 to all formulas in $\mathcal{L}^{\mathcal{A},2}_\varepsilon$ with parameters in $(\mathrm{N}, \mathcal{D})$:*

- $\|a \not\varepsilon^1 B\| \coloneqq \{\pi \in \Pi \mid (a, \pi) \in^1 B\}$;
- $\|A \neq^1 B\| \coloneqq \|\top\|$ *if* $A \neq B$, $\|\bot\|$ *otherwise;*
- $\|a \notin^1 B\| \coloneqq \bigcup_{c \in \mathrm{dom}(B)} \{t.t'.\pi \mid (c, \pi) \in^1 B,\, t \in |a \subseteq c|,\, t' \in |c \subseteq a|\}$;
- $\|\forall^1 X\, \varphi(X)\| \coloneqq \bigcup_{A \in \mathcal{D}} \|\varphi[A/X]\|$.

▶ Remark 37. By standard set-theoretic arguments, for each formula $\varphi(x_1, \ldots, x_m, Y_1, \ldots Y_n)$ in $\mathcal{L}^2_\varepsilon$, there are formulas $\varphi_\Pi(p, x_1, \ldots, x_m, Y_1, \ldots Y_n)$ and $\varphi_\Lambda(p, x_1, \ldots, x_m, Y_1, \ldots Y_n)$ in $\mathcal{L}^2_\in$ with parameters in $\mathrm{V}_{\mathrm{fp}(\mathcal{A})}$ such that for all $a_1, \ldots, a_m \in \mathrm{V}$, all $B_1, \ldots, B_n \in D$, all $\pi \in \Pi$, and all $t \in \Lambda$, we have

$$\begin{array}{lll} \pi \in \|\varphi(a_1, \ldots, a_m, B_1, \ldots, B_n)\| & \textit{iff} & \mathrm{V} \models \varphi_\Pi(\pi, a_1, \ldots, a_m, B_1, \ldots, B_n) \\ \textit{and} \quad t \in |\varphi(a_1, \ldots, a_m, B_1, \ldots, B_n)| & \textit{iff} & \mathrm{V} \models \varphi_\Lambda(t, a_1, \ldots, a_m, B_1, \ldots, B_n). \end{array}$$

▶ **Definition 38.** *The* realizability theory *of $(\mathcal{A}, \mathrm{V}, \mathcal{C})$, denoted by $T_{\mathcal{A},\mathrm{V},\mathcal{C}}$, is the set of all closed formulas φ of $\mathcal{L}^{\mathcal{A},2}_\varepsilon$ with parameters in $(\mathrm{N}, \mathcal{D})$ such that there exists $t \in \mathcal{R}$ such that t realizes φ.*

▶ **Proposition 39.** *Let $(\mathrm{V}, \mathcal{C})$ be a model of GB and let $\mathcal{A} \in \mathrm{V}$ be a realizability algebra. The realizability theory of $(\mathcal{A}, \mathrm{V}, \mathcal{C})$ is closed under classical deduction and contains GB_ε.*

This justifies the following definition:

▶ **Definition 40.** *A* realizability model of GB_ε *is a tuple $\mathcal{N} = (\mathrm{V}, \mathcal{C}, \mathcal{A})$, with $(\mathrm{V}, \mathcal{C})$ a model of GB and $\mathcal{A}$ a realizability algebra in V. We write $\mathcal{N} \Vdash \varphi$ for "$T_{\mathcal{A},\mathrm{V},\mathcal{C}}$ contains φ".*

10 Realizing measurable and Reinhardt cardinals

In this section, we work with the theory GB and we build realizability models for measurable and Reinhardt cardinals. First, we will consider a model V of GB with Choice that contains a measurable cardinal and show how to *realize* the existence of a measurable cardinal. A cardinal κ is said to be measurable if there exists a non-principal, κ-complete ultrafilter over

κ. The reason we work with Choice in the ground model is to use the Łoś Theorem to define from the ultrafilter a class function $j\colon \mathrm{V} \to \mathrm{M}$ where M is a transitive inner model of ZFC, j is not the identity and for every formula $\varphi(x_1, \ldots, x_n)$ and sets $a_1, \ldots, a_n$ one has

$$\varphi(a_1, \ldots, a_n) \text{ if and only if } \mathrm{M} \models \varphi(j(a_1), \ldots, j(a_n)).$$

We call δ the *critical point* of an embedding j if $\forall\alpha \in \delta$, $j(\alpha) = \alpha$ and $j(\delta) > \delta$, that is to say δ is the first ordinal moved by j. It is well-known that κ is a measurable cardinal if and only if it is the critical point of such an elementary embedding.

What we shall realize is that the existence of such an embedding transfers nicely to realizability models and their corresponding extensional models of GB. The reason we work in a second-order theory is that formally the embedding $j\colon \mathrm{V} \to \mathrm{M}$ is a class function over our model. It is known that without Choice the Łoś Theorem need no longer hold and therefore the existence of an embedding might not be definable in a first-order way. On the other hand, it is easy to see that under ZF, if j is a non-trivial elementary embedding with critical point κ then there is a non-trivial κ-complete ultrafilter on κ defined by $U = \{X \subseteq \kappa \mid \kappa \in j(X)\}$. We refer the reader to [8] for more details on measurable cardinals without Choice.

When we are working in GB_ε it no longer needs to be the case that there is a unique critical point because we may have sets $a, b \in \mathrm{N}$ such that $\mathcal{N}$ believes a and b are ordinals with $a \simeq b$, the embedding will fix every element of both a and b while being non-trivial on them and yet $a \neq b$. Therefore, we shall instead refer to *a* critical point of some embedding. We shall show that there exists an embedding $j^\star$ and see that $\daleth(\kappa)$ is a critical point of $j^\star$. Then, when we restrict ourselves to the model of GB, the extensionality of $\in$ will give us the required uniqueness.

So let us fix such an embedding $j\colon \mathrm{V} \to \mathrm{M}$ and denote its critical point by κ. Let $\mathcal{A}$ be any realizability algebra such that $\mathcal{A} \in \mathrm{V}_\kappa$ (which implies $\mathrm{fp}(\mathcal{A}) < \kappa$). Then j will induce a non-trivial elementary embedding j^* of the realizability model into a transitive subclass of the realizability model that satisfies GB_ε where j^* is defined as

$$j^* = \{(\mathtt{op}(x, j(x)), \pi) \mid \pi \in \Pi\}. \tag{$\star$}$$

▶ **Definition 41.** *In a model of $T_{\mathcal{A},\mathrm{V},\mathcal{C}}$, we say that an ordinal a is* a critical point of j^* *if $\forall x \,\varepsilon\, a(\mathtt{op}(x, x) \,\varepsilon\, j^\star)$ and there exists some set b such that $\mathtt{op}(a, b) \,\varepsilon\, j^\star$ and $a \,\varepsilon\, b$.*

An important fact we will use is that the elementarity of j implies that $t \Vdash \varphi(a)$ if and only if $t \Vdash \varphi(j(a))$ (by Remark 37). In addition, since $\mathcal{A} \in \mathrm{V}_\kappa$, we have $\mathcal{P}(\Pi) \in \mathrm{M}$ so $\|\varphi\|^{\mathrm{M}} = \|\varphi\|$.

▶ **Proposition 42.** *For all formulas φ in $\mathcal{L}_\varepsilon^{\mathcal{A}}$ and all $a_1, \ldots, a_n \in \mathrm{N}$, $\|\varphi(a_1, \ldots, a_n)\| = \|\varphi(j(a_1), \ldots, j(a_n))\|$. Hence $\mathcal{N} \Vdash \varphi(a_1, \ldots, a_n) \leftrightarrow \varphi(j(a_1), \ldots, j(a_n))$.*

From this it is easy to see that $j^\star$ is realized to be a class elementary embedding.

▶ **Theorem 43.** *Suppose that $(\mathrm{V}, \mathcal{C})$ is a model of GB and $\mathcal{A} \in \mathrm{V}_\kappa$ where κ is the critical point of a non-trivial elementary embedding $j\colon \mathrm{V} \to \mathrm{M}$ for some class function j and transitive class M. Then $\mathcal{N}$ realizes every axiom of GB_ε plus:*

1. *$\mathcal{N} \Vdash j^\star$ is a class function compatible with $\simeq$, namely*

$$\mathcal{N} \Vdash \forall x_1 \forall x_2 \forall y_1 \forall y_2((x_1 \simeq x_2 \wedge \mathtt{op}(x_1, y_1) \,\varepsilon\, j^\star \wedge \mathtt{op}(x_2, y_2) \,\varepsilon\, j^\star) \to y_1 \simeq y_2).$$

2. *$\mathcal{N} \Vdash \forall x \forall y(\mathtt{op}(x, y) \,\varepsilon\, j^\star \to y \,\varepsilon\, \mathrm{M}^{\mathcal{A}}$,*
3. *$\mathcal{N} \Vdash \forall \alpha^{\daleth(\kappa)} \mathtt{op}(\alpha, \alpha) \,\varepsilon\, j^\star \wedge \exists b(\daleth(\kappa) \,\varepsilon\, b \wedge \mathtt{op}(\daleth(\kappa), b) \,\varepsilon\, j^\star$ (i.e.$\daleth(\kappa)$ is a critical point of $j^\star$),*
4. *For every formula $\varphi(x_1, \ldots, x_n)$ in Fml_ε we have*

$$\mathcal{N} \Vdash \forall a_1, \ldots, a_n \exists b_1, \ldots, b_n(\mathtt{op}(a_1, b_1) \,\varepsilon\, j^\star \wedge \cdots \wedge \mathtt{op}(a_1, b_1) \,\varepsilon\, j^\star \wedge (\varphi(\vec{a}) \leftrightarrow \varphi^{\mathrm{M}^{\mathcal{A}}}(\vec{b}))).$$

▶ **Theorem 44.** *Suppose that* $(\mathrm{V}, \mathcal{C})$ *is a model of* GB *plus Choice with a measurable cardinal* κ*, and* $\mathcal{A} \in \mathrm{V}_\kappa$ *is a realizability algebra. Let* $\mathcal{N} = (\mathrm{V}, \mathcal{A})$ *be the corresponding realizability model. Then,* $\mathcal{N} \Vdash \mathrm{GB} +$ *there exists a measurable cardinal.*

Proof. Let $j \colon \mathrm{V} \to \mathrm{M}$ be an elementary embedding with critical point κ witnessing that κ is a measurable cardinal and define $j^\star$ as in $(\star)$. By Theorem 35 and Proposition 39, we have $\mathcal{N} \Vdash \mathrm{GB}$. Theorem 43(1) proves that $j^\star$ is a function compatible with the extensional equality. Theorem 43(3) implies that $j^\star$ is non-trivial and $\urcorner(\kappa)$ is a critical point. Theorem 43(4) implies that $j^\star$ is elementary for formulas in the language of $\mathcal{L}_\varepsilon$ (and hence $\mathcal{L}_\in$). Therefore, we can realize that there is an elementary embedding $j^\star \colon \mathrm{N} \to \mathrm{M}^{\mathcal{A}}$ with critical point $\urcorner(\kappa)$. Finally, from this it follows that there is a non-principal $\urcorner(\kappa)$-complete ultrafilter on $\urcorner(\kappa)$ so $\urcorner(\kappa)$ is indeed a measurable cardinal. ◀

We end this section by showing that the above analysis naturally generalizes to larger cardinal notions involving the notion of elementary embedding, in particular Reinhardt cardinals. So suppose that $(\mathrm{V}, \mathcal{C})$ is a model of GB that contains a Reinhardt cardinal κ, this means that κ is the critical point of a non-trivial elementary embedding j of the universe into itself. It is important here that we work over GB because if we only consider Reinhardt cardinals in a purely first-order setting then any proper class must be definable by some formula. It is then possible to show that there are no definable elementary embedding of the universe into itself by work of Suzuki [23] or see [7] for an extended discussion on the metamathematics of dealing with Reinhardt cardinals.

▶ **Theorem 45.** *Suppose that* $(\mathrm{V}, \mathcal{C})$ *is a model of* GB *with a Reinhardt cardinal* κ*, and* $\mathcal{A} \in \mathrm{V}_\kappa$ *is a realizability algebra. Let* $\mathcal{N} = (\mathrm{V}, \mathcal{A})$ *be the corresponding realizability model, then* $\mathcal{N} \models \mathrm{GB} +$ *there exists a Reinhardt cardinal.*

Proof. Let $j \colon \mathrm{V} \to \mathrm{V}$ be an elementary embedding witnessing that κ is a Reinhardt cardinal, with j^* defined as in $(\star)$. By Theorem 35 and Proposition 39, we have $\mathcal{N} \Vdash \mathrm{GB}$. As before, from Theorem 43 one can realize that $j^\star$ is a non trivial function which is compatible with the extensional equality and has $\urcorner(\kappa)$ as a critical point. An easy generalization of Theorem 43(4) implies that j^* is elementary for GB formulas. Therefore, the existence of a Reinhardt cardinal is realized. ◀

11 Conclusion

We have shown how to realize the axioms of inaccessible, Mahlo, measurable and Reinhardt cardinals assuming their consistency relative to ZFC or ZF. We have reformulated each of these axioms in the context of ZF or GB and we have proven that the corresponding notions are preserved by any realizability algebra whose size is smaller than the large cardinals considered, in particular by any countable realizability algebra. Note that in each one of these four scenarios, no special instructions were needed, the axioms considered are realized by pure terms of the λ_c-calculus. This may be counterintuitive since large cardinals axioms are very strong axioms which entail the consistency of ZFC, yet our results show that despite their strength, large cardinals axioms do not add a computational content to the realizability machinery whenever the algebra is small enough (namely countable or of size smaller than the large cardinal considered). On the other hand, it remains an open problem to determine what would happen for a larger realizability algebra (equipotent with the large cardinal considered or larger). For example, in the case of forcing, if the forcing has size κ then it may collapse κ to be bijective with a smaller cardinal. Hence special instructions may be needed in order to prevent the large cardinal from collapsing and to preserve its properties.

There is opportunity for future work involving realizability models for large cardinals. The method presented to realize measurable and Reinhardt cardinals focused on preservation of non trivial elementary embeddings; this could be easily adapted to realize rank-into-rank embeddings, which correspond to some of the strongest known large cardinals axioms not known to be inconsistent in ZFC. Preservation of other large cardinal notions could be investigated such as Ramsey cardinals, weakly compact, strongly compact, supercompact cardinals and many others. Clearly, assuming larger cardinals we can get realizability models for those large cardinals notions: for instance if we assume the consistency of ZFC with a measurable cardinal, we can realize the existence of a measurable cardinal and in particular of a Ramsey cardinal, but it remains an open problem whether Ramsey cardinals can be *preserved* by realizability algebras, namely whether starting from a model of ZFC with a Ramsey cardinal κ such that $|\mathcal{A}| < \kappa$, one can realize the existence of a Ramsey cardinal in the corresponding realizability model.

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