Confluence of Conditional Rewriting Modulo

Salvador Lucas
DSIC & VRAIN, Universitat Politècnica de València, Spain

Abstract

We investigate confluence of rewriting with *Equational Generalized Term Rewriting Systems* $\mathcal{R}$, consisting of *Horn clauses*, some of them defining *conditional equations* $s = t \Leftarrow c$ and *rewriting rules* $\ell \to r \Leftarrow c$. In both cases, c is a sequence of *atoms*, possibly defined by using additional Horn clauses. Such systems include *Equational Term Rewriting Systems* and (join, oriented, and semi-equational) *Conditional Term Rewriting Systems*. A set of equations E defines an equivalence $=_E$ and quotient set $\mathcal{T}(\mathcal{F},\mathcal{X})/{=_E}$ of terms, where reductions $s \to_{\mathcal{R}/E} t$ using rules in $\mathcal{R}$ occur. For such systems, we obtain a *finite* set of conditional pairs π, which can be viewed as logical sentences, to prove and disprove confluence of $\to_{\mathcal{R}/E}$ by (dis)proving joinability of such conditional pairs π.

2012 ACM Subject Classification Theory of computation → Automated reasoning; Theory of computation → Logic and verification; Theory of computation → Equational logic and rewriting

Keywords and phrases Conditional rewriting, Confluence, Program analysis

Digital Object Identifier 10.4230/LIPIcs.CSL.2024.37

Funding *Salvador Lucas*: Supported by project PID2021-122830OB-C42 funded by MCIN/AEI/10.13039/501100011033 and by "ERDF A way of making Europe" and by the grant CIPROM/2022/6 funded by Generalitat Valenciana.

Acknowledgements I thank the anonymous referees for their helpful remarks.

1 Introduction

A sequence 0, $\mathsf{s}(0)$, $\mathsf{s}(\mathsf{s}(0))$ of numbers in Peano's notation is usually written as a *term* by using a "pairing" (binary) operator $+\!\!+$ as in $t_1 = (0 +\!\!+ \mathsf{s}(0)) +\!\!+ \mathsf{s}(\mathsf{s}(0))$ or $t_2 = 0 +\!\!+ (\mathsf{s}(0) +\!\!+ \mathsf{s}(\mathsf{s}(0)))$. This is necessary when computing with (variants of) Term Rewriting Systems (TRSs [1]). However, *multiple* presentations of the sequence are possible. We can overcome this if $+\!\!+$ is *associative*, i.e., the equation $xs +\!\!+ (ys +\!\!+ zs) = (xs +\!\!+ ys) +\!\!+ zs$ is satisfied for all terms xs, ys, and zs. Then, t_1 and t_2 are made *equivalent modulo associativity* and become members of an *equivalence class* $[t]$, consisting of all terms which are equivalent to t modulo associativity. Here, t can be t_1 or t_2, the specific choice being immaterial.

In general, if $\mathcal{T}(\mathcal{F},\mathcal{X})$ is the set of terms built from a signature $\mathcal{F}$ and variables in $\mathcal{X}$, a set of equations E on terms defines an equivalence $=_E$ and a partition $\mathcal{T}(\mathcal{F},\mathcal{X})/{=_E}$ of $\mathcal{T}(\mathcal{F},\mathcal{X})$ into equivalence classes. When additionally considering a set of rules $\mathcal{R}$, it is natural to view rewriting computations as transformations $[s]_E \to_{\mathcal{R}/E} [t]_E$ of *equivalence classes*. Here, $[s]_E \to_{\mathcal{R}/E} [t]_E$ (i.e., *rewriting modulo*) means that $s' \to_{\mathcal{R}} t'$ for some $s' \in [s]_E$ and $t' \in [t]_E$. We often just write $s \to_{\mathcal{R}/E} t$. In this paper we are interested in E-*confluence* of $\mathcal{R}$, i.e., the commutation of the following diagram:

$$\begin{tikzcd}
{[s]_E} \arrow[r, "*", "\mathcal{R}/E"'] \arrow[d, "*", "\mathcal{R}/E"'] & {[t]_E} \arrow[d, dotted, "*", "\mathcal{R}/E"'] \\
{[t']_E} \arrow[r, dotted, "*", "\mathcal{R}/E"'] & {[u]_E}
\end{tikzcd}$$

In [10], Jouannaud addressed the problem of proving E-confluence of *equational term rewriting*, where E and $\mathcal{R}$ consist of (unconditional) equations and rewrite rules, respectively. In this paper we consider *conditional rules* $\ell \to r \Leftarrow c$ and *conditional equations* $s = t \Leftarrow d$, where c and d are sequences of *atoms*, possibly defined by Horn clauses.

32nd EACSL Annual Conference on Computer Science Logic (CSL 2024).
Editors: Aniello Murano and Alexandra Silva; Article No. 37; pp. 37:1–37:21
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

▶ **Example 1.** The signature $\mathcal{F} = \{\mathsf{0}, \mathsf{s}, +\!\!+\}$ can be used to represent nonempty sequences of natural numbers in Peano's notation. A single number is considered a sequence as well.

$$
\begin{array}{rcll}
xs +\!\!+ (ys +\!\!+ zs) & = & (xs +\!\!+ ys) +\!\!+ zs & (1)\\
\mathsf{Nat}(\mathsf{0}) & & & (2)\\
\mathsf{Nat}(\mathsf{s}(n)) & \Leftarrow & \mathsf{Nat}(n) & (3)\\
x \approx y & \Leftarrow & x \to^* y & (4)
\end{array}
\qquad
\begin{array}{rcll}
\mathsf{0} + n & \to & n & (5)\\
\mathsf{s}(m) + n & \to & \mathsf{s}(m+n) & (6)\\
\mathsf{sum}(n) & \to & n \Leftarrow \mathsf{Nat}(n) & (7)\\
\mathsf{sum}(m +\!\!+ ns) & \to & m + n & \\
 & \Leftarrow & \mathsf{Nat}(m), \mathsf{sum}(ns) \approx n & (8)
\end{array}
$$

Predicate Nat defined by clauses (2) and (3) identifies an expression (without variables) as representing a *natural number*; clause (4) describes the interpretation of conditions $s \approx t$ as *reachability* in conditional rules like (8). The application of a rule like (8) to a term $\mathsf{sum}(t)$ is as follows: for each substitution σ, if (i) $t =_E \sigma(m +\!\!+ ns)$, (ii) $\mathsf{Nat}(\sigma(m))$ holds, and (iii) $\mathsf{sum}(\sigma(ns))$ rewrites to $\sigma(n)$, then we obtain $\sigma(m) + \sigma(n)$. Note that associativity of $+\!\!+$ is essential to obtain the expected functionality of sum as it permits the "reorganization" of t into t', i.e., $\sigma(m) +\!\!+ \sigma(ns)$, so that, for the first member $\sigma(m)$ of t', $\mathsf{Nat}(\sigma(m))$ holds.

For the analysis of E-confluence of ETRSs, *E-critical pairs* were considered [10, Definition 10]. Given unconditional (variable disjoint) rules $\ell \to r$ and $\ell' \to r'$, a nonvariable position $p \in \mathcal{P}os(\ell)$ and an E-unifier θ such that $\theta(\ell|_p) =_E \theta(\ell')$, an E-critical pair $\langle \theta(\ell)[\theta(r')]_p, \theta(r)\rangle$ is obtained. However, in sharp contrast with TRSs [13, 9], (i) there is no general *E-unification algorithm* and *"for each equational theory one must invent a special algorithm"* [22, page 74]. Furthermore, even for E-unifying terms, (ii) there can be several, even *infinitely many* E-unifiers θ which must be considered to obtain a *complete set of E-critical pairs* which can be used to check E-confluence of $\mathcal{R}$ [2, 20]. In order to improve this situation, we propose the use of *Logic-based Conditional Critical Pairs* instead.

▶ **Example 2** (Continuing Example 1)**.** Terms $\mathsf{sum}(n)$ and $\mathsf{sum}(m +\!\!+ ns)$ syntactically unify with *mgu* $\theta = \{n \mapsto m +\!\!+ ns\}$. However, there are infinitely many E-unifiers $\theta_{a,b,c} = \{n \mapsto (\mathsf{s}^a(\mathsf{0}) +\!\!+ \mathsf{s}^b(\mathsf{0})) +\!\!+ \mathsf{s}^c(\mathsf{0}), m \mapsto \mathsf{s}^a(\mathsf{0}), ns \mapsto (\mathsf{s}^b(\mathsf{0}) +\!\!+ \mathsf{s}^c(\mathsf{0}))\}$ for all $a, b, c \geq 0$ which *cannot* be seen as refinements $\tau \circ \theta$ of θ for some substitution τ (in the usual way). This leads to infinitely many (conditional) critical pairs for (7) and (8). Instead, a single *logic-based conditional critical pair* would represent them all:

$$\langle m' + n', n\rangle \quad \Leftarrow \quad \mathsf{sum}(n) = \mathsf{sum}(m' +\!\!+ ns'), \mathsf{Nat}(n), \mathsf{Nat}(m'), \mathsf{sum}(ns') \approx n' \qquad (9)$$

After some preliminary notions and notations (Section 2) and a summary of Jouannaud and Kirchner's results [11] we rely on (Section 3), the contributions of this paper are: (i) we introduce *Equational Generalized Term Rewriting Systems* (EGTRSs) $\mathcal{R}$ consisting of a set of *conditional equations E* and *conditional rules R* whose conditional parts are sequences of *atoms*, possibly defined by definite Horn clauses in a set H; then, (ii) rewriting computations (modulo) are described as *deduction* in a first-order theory obtained from E, H, and R (Section 4). After that, (iii) confluence of EGTRSs modulo is investigated by considering the structure of *peaks* that may lead to diverging computations. We distinguish between *rewriting* and *coherence* peaks and show that the first ones can be used to *disprove* confluence modulo (Sections 5 and 6). Also, (iv) we provide a *logic-based* definition of (conditional) critical pair which avoids the explicit computation of E-unifiers. We also show that other conditional pairs (namely, *conditional variable pairs* and *down conditional critical pairs*) are necessary to capture (non-)E-confluence of EGTRSs (Section 7). Finally, (v) we show that by using appropriate notions of joinability (modulo), such pairs permit to obtain proofs of E-confluence and non-E-confluence (Section 8). Section 9 discusses some related work. Section 10 concludes and points to some future work. For the sake of clarity, additional details about the analysis of confluence of $\mathcal{R}$ in Example 1 are supplied in Appendix A.

2 Preliminaries

In the following, *s.t.* means *such that* and *iff* means *if and only if*. We assume some familiarity with the basic notions of term rewriting [1, 19] and first-order logic [5, 18]. For the sake of readability, though, here we summarize the main notions and notations we use.

Abstract Reduction Relations. Given a binary relation $\mathsf{R} \subseteq A \times A$ on a set A, we often write $a \;\mathsf{R}\; b$ or $b \;\mathsf{R}^{-1}\; a$ instead of $(a, b) \in\mathsf{R}$. The *composition* of two relations R and R' is written $\mathsf{R} \circ \mathsf{R}'$ and defined as follows: for all $a, b \in A$, $a \;\mathsf{R} \circ \mathsf{R}'\; b$ iff there is $c \in A$ such that $a \;\mathsf{R}\; c$ and $c \;\mathsf{R}'\; b$. The *reflexive* closure of R is denoted by $\mathsf{R}^=$; the *transitive* closure of R is denoted by R^+; and the *reflexive and transitive* closure by R^*. An element $a \in A$ is R-*irreducible* (or just *irreducible* if no confusion arises) if there is no b such that $a \;\mathsf{R}\; b$. We say that $b \in A$ is R-reachable from $a \in A$ if $a \;\mathsf{R}^*\; b$. We say that $a, b \in \mathsf{R}$ are R-*joinable* if there is $c \in A$ such that $a \;\; \mathsf{R}^* \; c$ and $b \;\; \mathsf{R}^* \; c$. Also, $a, b \in\mathsf{R}$ are R-*convertible* if $a \; (\mathsf{R} \cup \mathsf{R}^{-1})^* \; b$. Given $a \in A$, if there is no infinite sequence $a = a_1 \;\; \mathsf{R} \;\; a_2 \;\; \mathsf{R} \;\; \cdots \;\; \mathsf{R} \;\; a_n \;\; \mathsf{R} \cdots$, then a is R-*terminating*; R is *terminating* if a is R-terminating for all $a \in A$. We say that R is (locally) *confluent* if, for all $a, b, c \in A$, if $a \;\; \mathsf{R}^* \; b$ and $a \;\; \mathsf{R}^* \; c$ (resp. $a \;\; \mathsf{R} \;\; b$ and $a \;\; \mathsf{R} \;\; c$), then b and c are R-joinable.

Signatures, Terms, Positions. In this paper, $\mathcal{X}$ denotes a countable set of *variables*. A *signature of symbols* is a set of *symbols* each with a fixed *arity*. We use $\mathcal{F}$ to denote a *signature of function symbols*, i.e., $\{f, g, \ldots\}$ whose arity is given by a mapping $ar : \mathcal{F} \to \mathbb{N}$. The set of terms built from $\mathcal{F}$ and $\mathcal{X}$ is $\mathcal{T}(\mathcal{F}, \mathcal{X})$. The set of variables occurring in t is $\mathcal{V}ar(t)$. Terms are viewed as labeled trees in the usual way. *Positions* p are represented by chains of positive natural numbers used to address subterms $t|_p$ of t. The *set of positions* of a term t is $\mathcal{P}os(t)$. The set of positions of a subterm s in t is denoted $\mathcal{P}os_s(t)$. The set of positions of non-variable symbols in t are denoted as $\mathcal{P}os_{\mathcal{F}}(t)$. Positions are ordered by the *prefix ordering* $\leq$ on sequences: given positions p, q, we write $p \leq q$ iff p is a prefix of q. If $p \not\leq q$ and $q \not\leq p$, we say that p and q are *disjoint* (written $p \parallel q$).

First-Order Logic. Here, Π denotes a signature of *predicate symbols*. First-order formulas are built using function symbols from $\mathcal{F}$, predicate symbols from Π, and variables from $\mathcal{X}$ in the usual way. In particular, *atomic formulas* A (often called *atoms* in the realm of automated theorem proving [23, page 2], but also in [12, pages 79 & 149]) are expressions $P(t_1, \ldots, t_n)$ where $P \in \Pi$ and $t_1, \ldots, t_n$ are terms; we often refer to P as $root(A)$.

A first-order theory (FO-theory for short) Th is a set of *sentences* (formulas whose variables are all *quantified*). In the following, given an FO-theory Th and a formula φ, $\mathsf{Th} \vdash \varphi$ means that φ is *deducible* from (or a *logical consequence* of) Th by using a correct and complete deduction procedure [5, 18]. A sequence $A_1, \ldots, A_n$ of atoms A_i, $1 \leq i \leq n$ is Th-*feasible* with respect to a theory Th (or just *feasible* if no confusion arises), if there is a substitution σ such that $\mathsf{Th} \vdash \sigma(A_i)$ holds for all $1 \leq i \leq n$; otherwise, it is *infeasible* [7].

3 Abstract analysis of confluence of rewriting modulo

Following [11, Section 2], in this section $t, t', \ldots$ refer to elements of a set A. Let $\vdash\!\dashv_{\mathsf{E}}$ be a *symmetric* relation on A and $\sim_{\mathsf{E}}$ be its reflexive and transitive closure: an *equivalence* relation often called E-*equality*. Let $\to_{\mathsf{R}}$ (R for short) be a binary relation on A. Given R and E, the relation $\to_{\mathsf{R}/\mathsf{E}}$ (R / E for short), is called *reduction (with* $\to_{\mathsf{R}}$*) modulo* $\sim_{\mathsf{E}}$ and *defined* as

$$\to_{\mathsf{R}/\mathsf{E}} \quad = \quad \sim_{\mathsf{E}} \circ \to_{\mathsf{R}} \circ \sim_{\mathsf{E}} \tag{10}$$

$$\begin{tikzcd}
t \arrow[rr, "\mathsf{R}"'] \arrow[d, "\mathsf{R^E}"'] & & t'' \arrow[d, dotted, "\mathsf{R^E}"', "*"] \\
t' \arrow[r, dotted, "*", "\mathsf{R^E}"'] & t'_1 \arrow[r, squiggly, dash, "\mathsf{E}"'] & t''_1
\end{tikzcd}$$

Local confluence modulo E *of* $\mathsf{R^E}$ *with* R

$$\begin{tikzcd}
t \arrow[r, "\mathsf{E}"', mapsto, -|] \arrow[d, "\mathsf{R^E}"'] & t'' \arrow[r, dotted, "\mathsf{R^E}"'] & t''_1 \arrow[d, dotted, "\mathsf{R^E}"', "*"] \\
t' \arrow[r, dotted, "*", "\mathsf{R^E}"'] & t'_1 \arrow[r, squiggly, dash, "\mathsf{E}"'] & t''_2
\end{tikzcd}$$

Local coherence modulo E *of* $\mathsf{R^E}$

Figure 1 Confluence and coherence properties: 3^{rd} and 5^{th} diagrams in [11, Figure 2.1].

▶ **Definition 3** (Confluence and termination of R modulo E)**.** *Let* R *and* E *be as above. Then,*

- R *is* confluent modulo E *(or* E-confluent*) iff for all* t*,* $t_1, t_2 \in A$*, if* $t \to^*_{\mathsf{R}/\mathsf{E}} t_1$ *and* $t \to^*_{\mathsf{R}/\mathsf{E}} t_2$*, then there are* t'_1 *and* t'_2 *such that* $t_1 \to^*_{\mathsf{R}/\mathsf{E}} t'_1$*,* $t_2 \to^*_{\mathsf{R}/\mathsf{E}} t'_2$ *and* $t'_1 \sim_{\mathsf{E}} t'_2$ *[11, Def. 1].*[1]
- R *is* terminating modulo E *(or* E-terminating*) iff* $\to_{\mathsf{R}/\mathsf{E}}$ *is terminating [11, p. 1158].*

Computing with R / E is difficult as it may involve searching inside an infinite E-equivalence class $[t]_{\mathsf{E}}$ for some t' on which a R-reduction step can be performed. Peterson and Stickel investigated this problem for TRSs $\mathcal{R}$ and equational theories E. They introduced a reduction relation on terms, usually denoted $\to_{\mathcal{R},E}$, which can be advantageously used for this purpose [20]. In their abstract setting, Jouannaud and Kirchner use a relation $\to_{\mathsf{R^E}}$ ($\mathsf{R^E}$ for short) satisfying the following *fundamental assumption* [11, page 1158]:

$$\mathsf{R} \subseteq \mathsf{R^E} \subseteq \mathsf{R} / \mathsf{E} \tag{11}$$

Then, confluence of R / E is investigated by means of appropriate properties of $\mathsf{R^E}$. As in [10, 11], we rely on the following related properties of (abstract) relations.

▶ **Definition 4.** *Consider* R, E, $\mathsf{R^E}$*, and* R / E *as above, and* $t_1, t_2 \in A$*. A pair* $\langle t_1, t_2\rangle$ *is*

1. R / E*-joinable* $(t_1 \downarrow_{\mathsf{R}/\mathsf{E}} t_2)$*, iff* $\exists\, t'_1, t'_2$ *s.t.* $t_1 \to^*_{\mathsf{R}/\mathsf{E}} t'_1$*,* $t_2 \to^*_{\mathsf{R}/\mathsf{E}} t'_2$*, and* $t'_1 \sim_{\mathsf{E}} t'_2$*.*[2]
2. $\mathsf{R^E}$*-joinable modulo* E*,* $(t_1 \downarrow_{\mathsf{R^E}} t_2)$*, iff* $\exists\, t'_1, t'_2$ *s.t.* $t_1 \to^*_{\mathsf{R^E}} t'_1$*,* $t_2 \to^*_{\mathsf{R^E}} t'_2$*, and* $t'_1 \sim_{\mathsf{E}} t'_2$ *[11, Def. 2].*
3. right-strict $\mathsf{R^E}$*-joinable modulo* E*,* $(t_1 \downarrow^{rs}_{\mathsf{R^E}} t_2)$*, iff* $\exists\, t'_1, t'_2$ *s.t.* $t_1 \to^*_{\mathsf{R^E}} t'_1$*,* $t_2 \to^+_{\mathsf{R^E}} t'_2$*, and* $t'_1 \sim_{\mathsf{E}} t'_2$*.*

▶ **Definition 5** (Abstract confluence and coherence)**.** *Consider* R, E, $\mathsf{R^E}$*, and* R / E *as above. According to [11, Definition 3] (see Figure 1),*

1. R *is* $\mathsf{R^E}$*-Church-Rosser modulo* E *iff for all* t *and* t'*,* $t\,(\vdash\!\dashv_{\mathsf{E}} \cup \to_{\mathsf{R}} \cup {}_{\mathsf{R}}\!\leftarrow)^*\, t'$ *implies* $t \downarrow_{\mathsf{R^E}} t'$*.*
2. $\mathsf{R^E}$ *is* locally confluent modulo E with R *iff for all* t*,* t'*, and* t''*, if* $t \to_{\mathsf{R^E}} t'$ *and* $t \to_{\mathsf{R}} t''$*, then* $t' \downarrow_{\mathsf{R^E}} t''$*.*
3. $\mathsf{R^E}$ *is* locally coherent modulo E *iff for all* t*,* t'*, and* t''*, if* $t \to_{\mathsf{R^E}} t'$ *and* $t \vdash\!\dashv_{\mathsf{E}} t''$*, then* $t' \downarrow^{rs}_{\mathsf{R^E}} t''$*.*

---

[1] Definition 1 in [11] does *not* use the last requirement $t'_1 \sim_{\mathsf{E}} t'_2$ as the authors assume t, t_1, and t_2 to be E-*equivalence classes* on A (i.e., $t, t_1, t_2 \in A/{\sim_{\mathsf{E}}}$) rather than $t, t_1, t_2 \in A$. In order to make the difference explicit, consider $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$, E be given by (the reflexive, transitive, and symmetric closure of) $\mathsf{b} \sim_{\mathsf{E}} \mathsf{c}$, and R be given by $\mathsf{a}\ \mathsf{R}\ \mathsf{b}$ and $\mathsf{a}\ \mathsf{R}\ \mathsf{c}$. Then, $\mathsf{a} \to_{\mathsf{R}/\mathsf{E}} \mathsf{b}$ and also $\mathsf{a} \to_{\mathsf{R}/\mathsf{E}} \mathsf{c}$, but b and c are $\to_{\mathsf{R}/\mathsf{E}}$-irreducible. And $\to^*_{\mathcal{R}/E} = \{(\mathsf{a}, \mathsf{a})(\mathsf{b}, \mathsf{b}), (\mathsf{c}, \mathsf{c}), (\mathsf{a}, \mathsf{b}), (\mathsf{a}, \mathsf{c})\}$. Thus, neither $\mathsf{b} \to^*_{\mathcal{R}/E} \mathsf{c}$ nor $\mathsf{c} \to^*_{\mathcal{R}/E} \mathsf{b}$, i.e., as a relation on A, $\to_{\mathsf{R}/\mathsf{E}}$ is *not* confluent. However, $\mathsf{b} \sim_{\mathsf{E}} \mathsf{c}$. As a relation on $A/{\sim_{\mathsf{E}}}$, $\to_{\mathsf{R}/\mathsf{E}}$ is confluent.

[2] Continuing footnote 1, requiring this last equivalence step can be essential to achieve "joinability". For instance, this is necessary for b and c in the example of the footnote to be R / E-joinable.

▶ **Proposition 6** ([11, p. 1160, bullet 1]). *If* R *is* R^{E}*-Church-Rosser modulo* E*, then* R *is* E*-confluent.*

▶ **Theorem 7** ([11, Theorem 5]). *If* R *is* E*-terminating, then* R *is* R^{E}*-Church-Rosser modulo* E *iff* R^{E} *is (i) locally confluent modulo* E *with* R *and (ii) locally coherent modulo* E*.*

Theorem 7 and Proposition 6, yield a *sufficient condition* for E-confluence of R.

▶ **Corollary 8.** *If* R *is* E*-terminating, then* R *is* E*-confluent if* R^{E} *is (i) locally confluent with* R *modulo* E *and (ii) locally coherent modulo* E*.*

In the following, we investigate how to deal with the *abstract* peaks displayed in Figure 1:

$$t' \;{}_{\mathsf{R}^{\mathsf{E}}}\!\leftarrow \quad t \quad \to_{\mathsf{R}} t'' \qquad (12) \qquad\qquad t' \;{}_{\mathsf{R}^{\mathsf{E}}}\!\leftarrow \quad t \quad \vdash\!\dashv_{\mathsf{E}} t'' \qquad (13)$$

that we call R-peaks (12) and E-peaks (13), as they share the same leftmost part, with R^{E}, but differ on the rightmost part, with R and E, respectively. In general, *non*-joinability of these peaks does *not* entail non-E-confluence of R (Corollary 8 is just a sufficient condition for E-confluence). However, we have:

▶ **Proposition 9.** *If* t' *and* t'' *in (12) are not* R/E*-joinable, then* R *is not* E*-confluent.*

Note that coherence peaks (13) are trivially R/E-joinable, as $t'' \to_{\mathsf{R}/\mathsf{E}} t'$.

4 Equational Generalized Term Rewriting Systems

The following definition introduces the kind of computational systems we consider here which can be viewed as an specialization of *Generalized Term Rewriting Systems* (GTRSs) introduced in [15] (see Section 9 for a more detailed comparison).

▶ **Definition 10** (Equational Generalized Term Rewriting Systems). *An* Equational Generalized Term Rewriting System (EGTRS) *is a tuple* $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ *where* $\mathcal{F}$ *is a signature of function symbols* Π *is a signature of predicate symbols with* $=, \to, \to^* \in \Pi$*, and, for* c *a sequence* $A_1, \ldots, A_n$ *of atomic formulas,*

- E *is a set of conditional equations* $s = t \Leftarrow c$*, for terms* s *and* t*;*
- H *is a set of definite Horn clauses* $A \Leftarrow c$ *where* $A = P(t_1, \ldots, t_n)$ *for some terms* $t_1, \ldots, t_n$*,* $n \geq 0$*, is such that* $P \notin \{=, \to, \to^*\}$*; and*
- R *is a set of conditional rules* $\ell \to r \Leftarrow c$ *for terms* $\ell \notin \mathcal{X}$ *and* r*.*

Note that $E \cup H \cup R$ *is a set of (definite) Horn clauses.*

Requiring $root(A) \notin \{=, \to, \to^*\}$ for all $A \Leftarrow c \in H$ ensures that computational predicates $=$, $\to$, and $\to^*$ are defined by E and R only (with an *auxiliary* use of H).

▶ Remark 11 (Conditions $s \approx t$ and their interpretation). In the literature about *Conditional TRSs* (CTRSs, see, e.g., [19, Chapter 7]), symbol $\approx$ is often used to specify conditions $s \approx t$ in rules having different *interpretations*: as *joinability*, *reachability*, etc. [19, Definition 7.1.3]. In EGTRSs, $\approx$ would be treated as a predicate in Π and the desired interpretation is *explicitly* obtained by including in H an appropriate set of clauses defining $\approx$. For instance, (4) in Example 1 interprets $\approx$ as *reachability*.

▶ **Notation 12.** *Equations* $s = t \Leftarrow c$ *in* E *are often* transformed *into rules by chosing a* left-to-right *or* right-to-left *orientation:* $\overrightarrow{E} = \{s \to t \Leftarrow c \mid s = t \Leftarrow c \in E\}$*, and* $\overleftarrow{E} = \{t \to s \Leftarrow c \mid s = t \Leftarrow c \in E\}$*. We let* $\overleftrightarrow{E} = \overrightarrow{E} \cup \overleftarrow{E}$*. Note that* $\overleftrightarrow{E}$ *may contain rules* $\lambda \to \rho \Leftarrow c$ *whose left-hand side* λ *is a* variable*. Let* $\mathcal{D}(\overleftrightarrow{E}) = \{root(\lambda) \mid \lambda \to \rho \Leftarrow d \in \overleftrightarrow{E}\}$*.*

Table 1 Generic sentences of the FO-theory of EGTRSs.

Label	Sentence
$(\mathrm{Rf})^{\bowtie}$	$(\forall x)\ x \bowtie x$
$(\mathrm{Tr})^{\bowtie}$	$(\forall x, y, z)\ x \bowtie y \wedge y \bowtie z \Rightarrow x \bowtie z$
$(\mathrm{Sy})^{\bowtie}$	$(\forall x, y)\ y \bowtie x \Rightarrow x \bowtie y$
$(\mathrm{Co})^{\bowtie}$	$(\forall x, y, z)\ x \bowtie y \wedge y \bowtie^* z \Rightarrow x \bowtie^* z$
$(\mathrm{Pr})^{\bowtie}_{f,i}$	$(\forall x_1, \ldots, x_k, y_i)\ x_i \bowtie y_i \Rightarrow f(x_1, \ldots, x_i, \ldots, x_k) \bowtie f(x_1, \ldots, y_i, \ldots, x_k)$
$(\mathrm{HC})_{A \Leftarrow A_1, \ldots, A_n}$	$(\forall \vec{x})\ A_1 \wedge \cdots \wedge A_n \Rightarrow A$
$(\mathrm{R,E})_{\ell \to r \Leftarrow A_1, \ldots, A_n}$	$(\forall x, \vec{x})\ x = \ell \wedge A_1 \wedge \cdots \wedge A_n \Rightarrow x \overset{ps}{\to} r$
$(\mathrm{R/E})$	$(\forall x, x', y, y')\ x = x' \wedge x' \to y' \wedge y' = y \Rightarrow x \overset{rm}{\to} y$

▶ **Definition 13.** *We say that $P \in \Pi$* depends on *R if $P \in \{\to, \to^*\}$ or there is $A \Leftarrow A_1, \ldots, A_n \in E \cup H$ with $root(A) = P$ such that $root(A_i)$ depends on R for some $1 \leq i \leq n$.*

In this paper, *computational relations* (e.g., $=_E$, $\to_{\mathcal{R}}$, $\to_{\mathcal{R},E}$, $\to_{\mathcal{R}/E}, \ldots$) induced by an EGTRS $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ are defined by deduction of atoms $s = t$ (equality in E), $s \to t$ (one-step rewriting in the usual sense), $s \overset{ps}{\to} t$ (rewriting modulo à la Peterson & Stickel), $s \overset{rm}{\to} t$ (rewriting modulo), etc., in some FO-theory. We *extend* Π with $\overset{ps}{\to}$, $\overset{rm}{\to}$, etc., and also $\approx_{ps}$, $\approx_{rm}$ (as they depend on the previous predicates). Our FO-theories are obtained from the generic sentences in Table 1, where:

- Sentences $(\mathrm{Rf})^{\bowtie}$, $(\mathrm{Tr})^{\bowtie}$, and $(\mathrm{Sy})^{\bowtie}$, which are parametric on a binary relation $\bowtie$, express *reflexivity*, *transitivity*, and *symmetry* of $\bowtie$, respectively;
- $(\mathrm{Co})^{\bowtie}$ expresses *compatibility* of one-step and many-step reduction with $\bowtie$;
- for each k-ary function symbol f, $1 \leq i \leq k$, and $x_1, \ldots, x_k$ and y_i distinct variables, $(\mathrm{Pr})^{\bowtie}_{f,i}$ *propagates* an $\bowtie$-step to the i-th immediate subterm of an f-rooted term;
- $(\mathrm{HC})_\alpha$ presents a clause $\alpha : A \Leftarrow A_1, \ldots, A_n$, with variables $\vec{x}$ as a sentence.
- $(\mathrm{R,E})_\alpha$ defines a Peterson & Stickel rewriting step $s \to_{\mathcal{R},E} t$ (at the root) using rule $\alpha : \ell \to r \Leftarrow c$ with variables $\vec{x}$. Here, $x \notin \vec{x}$.
- $(\mathrm{R/E})$ defines reduction modulo $\to_{\mathcal{R}/E}$ in the usual way.

The following example illustrates the differences between (i) rewriting with $\to_{\mathcal{R}}$ (where a term t is rewritten if $t|_p = \sigma(\ell)$ for some position p in t, rule $\ell \to r \Leftarrow c$ in $\mathcal{R}$, and substitution σ such that $\sigma(c)$ holds), (ii) rewriting modulo $\to_{\mathcal{R}/E}$ (where a term t is rewritten if it is E-equivalent to another term t' to which a rewrite rule applies as above), and (iii) rewriting *à la Peterson & Stickel* $\to_{\mathcal{R},E}$ (where a term t is rewritten if some subterm $t|_p$ is E-equivalent to an instance $\sigma(\ell)$ of the left-hand side ℓ of a rewrite rule $\ell \to r \Leftarrow c$ and $\sigma(c)$ holds).

▶ **Example 14.** Consider $E = \{(14), (15)\}$ and $\mathcal{R} = (\mathcal{F}, R)$ with $R = \{(16), (17)\}$, where

$$\mathsf{b} = \mathsf{f(a)} \quad (14) \qquad \mathsf{c} \to \mathsf{d} \quad (16)$$
$$\mathsf{a} = \mathsf{c} \quad (15) \qquad \mathsf{b} \to \mathsf{d} \quad (17)$$

Then, $\mathsf{f(a)}$ is $\to_{\mathcal{R}}$-irreducible. However, $\mathsf{f(a)} \to_{\mathcal{R},E} \mathsf{f(d)}$ because $\mathsf{a} =_E \underline{\mathsf{c}} \to_{\mathcal{R}} \mathsf{d}$. Furthermore, $\mathsf{f(a)} \to_{\mathcal{R}/E} \mathsf{f(d)}$ because $\mathsf{f(a)} =_E \mathsf{f(\underline{c})} \to_{\mathcal{R}} \mathsf{f(d)}$. However, $\mathsf{f(a)} \to_{\mathcal{R}/E} \mathsf{d}$ because $\mathsf{f(a)} =_E \underline{\mathsf{b}} \to_{\mathcal{R}} \mathsf{d}$, but $\mathsf{f(a)} \not\to_{\mathcal{R},E} \mathsf{d}$ because $\mathsf{f(a)}$ is not E-equivalent to the left-hand side of any rule.

Now, consider the following *parametric* theories with parameters $\mathbb{S}$ (referring to a signature), $\mathbb{E}$ (a set of equations), and $\mathbb{R}$ (a set of rules):

$$
\begin{array}{rcl}
\mathsf{Th}_{\mathrm{Eq}}[\mathbb{S},\mathbb{E}] & = & \{(\mathrm{Rf})^{=},(\mathrm{Sy})^{=},(\mathrm{Tr})^{=}\}\} \cup \{(\mathrm{Pr})^{=}_{f,i} \mid f \in \mathbb{S}, 1 \le i \le ar(f)\} \cup \{(\mathrm{HC})_e \mid e \in \mathbb{E}\} \\
\mathsf{Th}_{\mathrm{R}}[\mathbb{S},\mathbb{R}] & = & \{(\mathrm{Rf})^{\to^*},(\mathrm{Co})^{\to}\} \cup \{(\mathrm{Pr})^{\to}_{f,i} \mid f \in \mathbb{S}, 1 \le i \le ar(f)\} \cup \{(\mathrm{HC})_\alpha \mid \alpha \in \mathbb{R}\}\} \\
\mathsf{Th}_{\mathrm{R,M}}[\mathbb{S},\mathbb{R}] & = & \{(\mathrm{Rf})^{\xrightarrow{ps}^*},(\mathrm{Co})^{\overset{ps}{\to}}\} \cup \{(\mathrm{Pr})^{\overset{ps}{\to}}_{f,i} \mid f \in \mathbb{S}, 1 \le i \le ar(f)\} \cup \{(\mathrm{R{,}E})_\alpha \mid \alpha \in \mathbb{R}\} \\
\mathsf{Th}_{\mathrm{R/M}}[\mathbb{S},\mathbb{R}] & = & \{(\mathrm{Rf})^{\xrightarrow{rm}^*},(\mathrm{Co})^{\overset{rm}{\to}},\} \cup \{(\mathrm{Pr})^{\to}_{f,i} \mid f \in \mathbb{S}, 1 \le i \le ar(f)\} \cup \{(\mathrm{HC})_\alpha \mid \alpha \in \mathbb{R}\}\} \cup \{(\mathrm{R/E})\}
\end{array}
$$

Note that, in $\mathsf{Th}_{\mathrm{R/M}}[\mathbb{S},\mathbb{R}]$ propagation sentences are given for $\to$ rather than for $\overset{rm}{\to}$; this is consistent with the usual definition of rewriting modulo implemented by (R/E).

Since rules $\alpha : \ell \to r \Leftarrow c$ in R are used to specify *different* computational relations (see Definition 17), conditions $s \approx t \in c$ have different *interpretations* depending on the targetted relation: the meaning of $\approx$ is based on predicate $\to$ if α is used to describe the usual (conditional) rewrite relation $\to_{\mathcal{R}}$; however, $\approx$ should be treated using $\overset{ps}{\to}$ if α is used to describe Peterson & Stickel's rewriting modulo $\to_{\mathcal{R},E}$; and $\approx$ should be treated using $\overset{rm}{\to}$ if α is used to describe $\to_{\mathcal{R}/E}$. A simple way to deal with this situation is the following.

▶ **Definition 15.** *Let $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ be an EGTRS.*

- *H^{ps} (resp. H^{rm}) is obtained from H by replacing in each $A \Leftarrow A_1, \ldots, A_n \in H$ all occurrences of $\approx$, $\to$ and $\to^*$ by $\approx_{ps}$, $\overset{ps}{\to}$ and $\xrightarrow{ps}{}^*$ (resp. $\approx_{rm}$, $\overset{rm}{\to}$, and $\xrightarrow{rm}{}^*$).*
- *R^{ps} (R^{rm}) is obtained from R by replacing all occurrences of $\approx$ in each $\ell \to r \Leftarrow c \in R$ by $\approx_{ps}$ ($\approx_{rm}$).*

▶ **Example 16.** For $\mathcal{R}$ in Example 1, H^{rm} and R^{rm} are (H^{ps} and R^{ps} are similar):

$$
\begin{array}{rclr}
\mathsf{Nat}(0) & & & (18) \\
\mathsf{Nat}(\mathsf{s}(n)) & \Leftarrow & \mathsf{Nat}(n) & (19) \\
x \approx_{rm} y & \Leftarrow & x \xrightarrow{rm}{}^* y & (20)
\end{array}
\qquad
\begin{array}{rclr}
0 + n & \to & n & (21) \\
\mathsf{s}(m) + n & \to & \mathsf{s}(m+n) & (22) \\
\mathsf{sum}(n) & \to & n \Leftarrow \mathsf{Nat}(n) & (23) \\
\mathsf{sum}(m +\!\!+\, ns) & \to & m + n \Leftarrow \mathsf{Nat}(m), \mathsf{sum}(ns) \approx_{rm} n & (24)
\end{array}
$$

Given an EGTRS $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ whose equality predicate $=$ does *not* depend on R,[3] the following theories are obtained:

$$
\begin{array}{rcl}
\mathsf{Th}_E & = & \mathsf{Th}_{\mathrm{Eq}}[\mathcal{F},E] \cup \{(\mathrm{HC})_\alpha \mid \alpha \in H\} \\
\mathsf{Th}_{\mathcal{R}} & = & \mathsf{Th}_{\mathrm{Eq}}[\mathcal{F},E] \cup \mathsf{Th}_{\mathrm{R}}[\mathcal{F},R] \cup \{(\mathrm{HC})_\alpha \mid \alpha \in H\} \\
\mathsf{Th}_{\mathcal{R},E} & = & \mathsf{Th}_{\mathrm{Eq}}[\mathcal{F},E] \cup \mathsf{Th}_{\mathrm{R,M}}[\mathcal{F},R^{ps}] \cup \{(\mathrm{HC})_\alpha \mid \alpha \in H^{ps}\} \\
\mathsf{Th}_{\mathcal{R}/E} & = & \mathsf{Th}_{\mathrm{Eq}}[\mathcal{F},E] \cup \mathsf{Th}_{\mathrm{R/M}}[\mathcal{F},R^{rm}] \cup \{(\mathrm{HC})_\alpha \mid \alpha \in H^{rm}\}
\end{array}
$$

These theories are used to define the expected computational relations as follows.

▶ **Definition 17.** *Let $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ be an EGTRS and $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$.*

- *We write $s =_E t$ (resp. $s \to_{\overset{\leftrightarrow}{E}} t$) iff $\mathsf{Th}_E \vdash s = t$ (resp. $\mathsf{Th}_R[\mathcal{F}, \overset{\leftrightarrow}{E}] \vdash s \to t$) holds.*
- *We write $s \to_{\mathcal{R}} t$ (resp. $s \to_{\mathcal{R},E} t$ and $s \to_{\mathcal{R}/E} t$) iff $\mathsf{Th}_{\mathcal{R}} \vdash s \to t$ (resp. $\mathsf{Th}_{\mathcal{R},E} \vdash s \overset{ps}{\to} t$ and $\mathsf{Th}_{\mathcal{R}/E} \vdash s \overset{rm}{\to} t$) holds. Similarly for $s \to^*_{\mathcal{R}} t$ (resp. $s \to^*_{\mathcal{R},E} t$ and $s \to^*_{\mathcal{R}/E} t$).*

▶ **Definition 18** (Confluence and termination modulo of an EGTRS). *Let $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ be an EGTRS. We say that*

- *$\mathcal{R}$ is* confluent modulo E *(or E-*confluent*) iff for all terms t, t_1, and t_2, if $t \to^*_{\mathcal{R}/E} t_1$ and $t \to^*_{\mathcal{R}/E} t_2$, then there are t'_1 and t'_2 such that $t_1 \to^*_{\mathcal{R}/E} t'_1$, $t_2 \to^*_{\mathcal{R}/E} t'_2$ and $t'_1 =_E t'_2$.*
- *$\mathcal{R}$ is* terminating modulo E *(or E-*terminating*) iff $\to_{\mathcal{R}/E}$ is terminating.*

[3] Requiring that $=$ does not depend on R implies that the "meaning" of equational atoms $s = t$ does *not* depend on the *rules* in R.

Table 2 Abstract notions in Section 3 applied to EGTRSs.

Abstract reduction:	$\vdash\!\dashv_{\mathsf{E}}$	$\sim_{\mathsf{E}}$	$\to_{\mathsf{R}}$	$\to_{\mathsf{R}^{\mathsf{E}}}$	$\to_{\mathsf{R}/\mathsf{E}}$
Application to EGTRSs:	$\to_{\overleftrightarrow{E}}$	$=_E$	$\to_{\mathcal{R}^{rm}}$	$\to_{\mathcal{R}^{rm},E}$	$\to_{\mathcal{R}/E}$

Note that (i) $\to_{\overleftrightarrow{E}}$ is *symmetric* by definition of $\overleftrightarrow{E}$; (ii) $=_E$ is an equivalence due to $(\mathrm{Rf})^=$ (reflexivity), $(\mathrm{Sy})^=$ (symmetry), and $(\mathrm{Tr})^=$ (transitivity), all included in Th_E; and (iii) $=_E$ is the reflexive and transitive closure of $\to_{\overleftrightarrow{E}}$. Unfortunately, the relationship between $\to_{\mathcal{R}}$, $=_E$, and $\to_{\mathcal{R}/E}$, is *not* as required. In particular, $\to_{\mathcal{R}/E} = (=_E \circ \to_{\mathcal{R}} \circ =_E)$ does *not* hold.

▶ **Example 19.** Consider the following EGTRS

$$\begin{array}{rcl@{\qquad}r@{\qquad}rcl@{\qquad}r} \mathsf{a} & = & \mathsf{b} & (25) & \mathsf{a} & \to & \mathsf{c} & (27)\\ x \approx y & \Leftarrow & x \to^* y & (26) & \mathsf{a} & \to & \mathsf{d} \Leftarrow \mathsf{b} \approx \mathsf{c} & (28) \end{array}$$

We have $\mathsf{a} \to_{\mathcal{R}} \mathsf{c}$; but (28) is $\mathsf{Th}_{\mathcal{F},\mathcal{R}}$-infeasible, hence $\mathsf{a} \not\to_{\mathcal{R}} \mathsf{d}$. However, $\mathsf{a} \to_{\mathcal{R}/E} \mathsf{d}$, as $\mathsf{b} =_E \mathsf{a} \to_{\mathcal{R}} \mathsf{c}$, i.e., $\mathsf{b} \to_{\mathcal{R}/E} \mathsf{c}$ and (28) can be used. Thus, $\to_{\mathcal{R}} = \{(\mathsf{a},\mathsf{c})\}$, $(=_E \circ \to_{\mathcal{R}} \circ =_E) = \{(\mathsf{a},\mathsf{c}),(\mathsf{b},\mathsf{c})\}$, and $\to_{\mathcal{R}/E} = \{(\mathsf{a},\mathsf{c}),(\mathsf{b},\mathsf{c}),(\mathsf{a},\mathsf{d}),(\mathsf{b},\mathsf{d})\}$, i.e., $\to_{\mathcal{R}/E} \neq (=_E \circ \to_{\mathcal{R}} \circ =_E)$.

▶ Remark 20 (Rewriting modulo and rewriting in conditional systems). Example 19 shows a *mismatch* between the definition of $\to_{\mathcal{R}/E}$ for an EGTRS $\mathcal{R}$ (Definition 17) and the *abstract* definition (10), usually understood for ETRSs. For EGTRSs (and already for CTRSs), the connection $\to_{\mathcal{R}/E} = (=_E \circ \to_{\mathcal{R}} \circ =_E)$ is *lost*. This is because the conditions in rules are treated using, e.g., $\to^*_{\mathcal{R}}$ to obtain $\to_{\mathcal{R}}$ (see, e.g., [19, Definition 7.1.4]), whereas computations with $\to_{\mathcal{R}/E}$ evaluate conditions using $\to^*_{\mathcal{R}/E}$ instead, see, e.g., [4, page 819].

We overcome this problem as follows.

▶ **Definition 21** (CR-theory of an EGTRS). *Let $\mathcal{R} = (\mathcal{F},\Pi,E,H,R)$ be an EGTRS. The CR-theory of $\mathcal{R}$ is*

$$\overline{\mathcal{R}}^{\mathrm{CR}} = \mathsf{Th}_{Eq}[\mathcal{F},E] \cup \mathsf{Th}_{R}[\mathcal{F},R^{rm}] \cup \mathsf{Th}_{R,M}[\mathcal{F},R^{rm}] \cup \mathsf{Th}_{R/M}[\mathcal{F},R^{rm}] \cup \{(HC)_\alpha \mid \alpha \in H^{rm}\}$$

Then, $\to_{\mathcal{R}^{rm}}$ and $\to_{\mathcal{R}^{rm},E}$ (and also $\to^*_{\mathcal{R}^{rm}}$ and $\to^*_{\mathcal{R}^{rm},E}$) are defined as follows:

$$s \to_{\mathcal{R}^{rm}} t \;\Leftrightarrow\; \overline{\mathcal{R}}^{\mathrm{CR}} \vdash s \to t \qquad \textit{and} \qquad s \to_{\mathcal{R}^{rm},E} t \;\Leftrightarrow\; \overline{\mathcal{R}}^{\mathrm{CR}} \vdash s \stackrel{ps}{\to} t$$

In contrast to $\to_{\mathcal{R}}$ and $\to_{\mathcal{R},E}$, to obtain $\to_{\mathcal{R}^{rm}}$ and $\to_{\mathcal{R}^{rm},E}$ conditions in rules are evaluated using $\to_{\mathcal{R}/E}$ (instead of $\to_{\mathcal{R}}$ and $\to_{\mathcal{R},E}$). Now, the requirements (10) and (11) are fulfilled.

▶ **Proposition 22.** *Let $\mathcal{R} = (\mathcal{F},\Pi,E,H,R)$ be an EGTRS. Then, $\to_{\mathcal{R}/E} = (=_E \circ \to_{\mathcal{R}^{rm}} \circ =_E)$. Also, $\to_{\mathcal{R}^{rm}} \subseteq \to_{\mathcal{R}^{rm},E} \subseteq \to_{\mathcal{R}/E}$.*

▶ Remark 23 (Use of Jouannaud & Kirchner's abstract framework). By the first statement in Proposition 22, E-confluence of EGTRSs (Definition 18) and E-confluence of $\to_{\mathcal{R}^{rm}}$ (as an abstract relation on terms, Definition 3) *coincide*. This enables the use of the results of Section 3 to analyze E-confluence of EGTRSs.

The results in Section 3 apply to EGTRSs according to the correspondence in Table 2. As a consequence of Proposition 6 and Theorem 7 we have the following.

▶ **Corollary 24.** *Let $\mathcal{R} = (\mathcal{F},\Pi,E,H,R)$ be an EGTRS. If $\mathcal{R}$ is E-terminating and $\to_{\mathcal{R}^{rm},E}$ is locally confluent modulo E with $\to_{\mathcal{R}^{rm}}$ and locally coherent modulo E, then $\mathcal{R}$ is E-confluent.*

The following result is essential in the analysis of peaks in Sections 5 and 6.

▶ **Proposition 25.** *Let $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ be a EGTRS and $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$.*

- *If $s \to_{\mathcal{R}^{rm}} t$, then there is $p \in \mathcal{P}os(s)$ and $\ell \to r \Leftarrow c \in R^{rm}$ such that (i) $s|_p = \sigma(\ell)$ for some substitution σ, (ii) for all $A \in c$, $\overline{\mathcal{R}}^{\mathrm{CR}} \vdash \sigma(A)$ holds, and (iii) $t = s[\sigma(r)]_p$.*
- *If $s \to_{\mathcal{R}^{rm},E} t$, then there is $p \in \mathcal{P}os(s)$ and $\ell \to r \Leftarrow c \in R^{rm}$ such that (i) $s|_p =_E \sigma(\ell)$ for some substitution σ, (ii) for all $A \in c$, $\overline{\mathcal{R}}^{\mathrm{CR}} \vdash \sigma(A)$ holds, and (iii) $t = s[\sigma(r)]_p$.*

5 Analysis of local confluence modulo E of $\to_{\mathcal{R}^{rm},E}$ with $\to_{\mathcal{R}^{rm}}$

Given an EGTRS $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ and *terms* s, t, and t', *rewriting peaks* (12) become:

$$t \;{}_{\mathcal{R}^{rm},E}\!\leftarrow s \to_{\mathcal{R}^{rm}} t' \tag{29}$$

▶ **Example 26.** Consider E and R ($=R^{rm}$) in Example 14 viewed as an EGTRS $\mathcal{R} = (\{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{f}\}, \{=, \to, \to^*\}, E, \emptyset, R)$. Since (i) $\mathsf{f}(\mathsf{c}) =_E \mathsf{f}(\mathsf{a}) =_E \underline{\mathsf{b}} \to_{(17)} \mathsf{d}$ and (ii) $\underline{\mathsf{c}} \to_{(16)} \mathsf{d}$, we have the following rewriting peak:

$$\mathsf{d} \;{}_{(17),E}\!\leftarrow \overleftarrow{\mathsf{f}(\underrightarrow{\mathsf{c}})} \to_{(16)} \mathsf{f}(\mathsf{d}) \tag{30}$$

The $\mathcal{R}^{rm}, E$-joinability of all peaks (29) characterizes local confluence modulo E of $\to_{\mathcal{R}^{rm},E}$ with $\to_{\mathcal{R}^{rm}}$, which provides an ingredient to prove E-confluence (see Corollary 24). By Proposition 25, there are positions $\overline{p}, \overline{p'} \in \mathcal{P}os(s)$, rules $\alpha : \ell \to r \Leftarrow c, \alpha' : \ell' \to r' \Leftarrow c' \in R^{rm}$ sharing no variable (rename if necessary), and substitution σ, such that (i) $s|_{\overline{p}} = w =_E \sigma(\ell)$ and $\sigma(c)$ hold; and (ii) $s|_{\overline{p'}} = \sigma(\ell')$ and $\sigma(c')$ hold, i.e., every rewriting peak is of the form

$$t = s[\sigma(r)]_{\overline{p}} \;{}_{\mathcal{R}^{rm},E}\!\leftarrow s[\underleftarrow{w}]_{\overline{p}} \;=\; s \;=\; s[\underrightarrow{\sigma(\ell')}]_{\overline{p'}} \to_{\mathcal{R}^{rm}} s[\sigma(r')]_{\overline{p'}} = t' \tag{31}$$

Depending on the relative location of $\overline{p}$ and $\overline{p'}$, different classes of peaks (31) are distinguished.

***Disjoint* rewriting peaks.** If $\overline{p}$ and $\overline{p'}$ in (31) are *disjoint*, i.e., $\overline{p} \parallel \overline{p'}$, then $s = s[w]_{\overline{p}}[\sigma'(\ell')]_{\overline{p'}} = s[\sigma'(\ell')]_{\overline{p'}}[w]_{\overline{p}}$. Accordingly, (31) can be written as follows:

$$t = s[\sigma(r)]_{\overline{p}}[\sigma(\ell')]_{\overline{p'}} \;{}_{\mathcal{R}^{rm},E}\!\leftarrow s[\underleftarrow{w}]_{\overline{p}}[\underrightarrow{\sigma(\ell')}]_{\overline{p'}} \to_{\mathcal{R}^{rm}} s[w]_{\overline{p}}[\sigma(r')]_{\overline{p'}} = t' \tag{32}$$

Now, $t = s[\sigma(r)]_{\overline{p}}[\underrightarrow{\sigma(\ell')}]_{\overline{p'}} \to_{\mathcal{R}^{rm}} s[\sigma(r)]_{\overline{p}}[\sigma(r')]_{\overline{p'}} \;{}_{\mathcal{R}^{rm},E}\!\leftarrow s[\underleftarrow{w}]_{\overline{p}}[\sigma(r')]_{\overline{p'}} = t'$, i.e., t and t' are $\mathcal{R}^{rm}, E$*-joinable*.

***Nested* rewriting peaks.** If $\overline{p'} = \overline{p}.p \in \mathcal{P}os(s)$ for $\overline{p}$ and $\overline{p'}$ in (31) and some position p, then (31) can be written in one of the following possibilities, according to the position where the $\to_{\mathcal{R}^{rm}}$-step applies (by abuse, we also use t and t').

1. In the first case, rewriting with $\to_{\mathcal{R}^{rm}}$ occurs *above or on* the $\to_{\mathcal{R}^{rm},E}$-step and we have
$$t = \sigma(\ell')[\sigma(r)]_p \;{}_{\mathcal{R}^{rm},E}\!\leftarrow \underrightarrow{\sigma(\ell')[\overleftarrow{w}]_p} \to_{\mathcal{R}^{rm}} \sigma(r') = t' \tag{33}$$
where $w =_E \sigma(\ell)$ and $p \geq \Lambda$. We call (33) an $\mathcal{R}^{rm}$*-up* peak. We distinguish two cases:
   - **a.** If $p \in \mathcal{P}os_{\mathcal{F}}(\ell')$, then $\sigma(\ell')|_p = \sigma(\ell'|_p) = w =_E \sigma(\ell)$, i.e., ℓ and $\ell'|_p$ E-unify and we say that (33) is an *E-critical $\mathcal{R}^{rm}$-up peak*;
   - **b.** If $p \notin \mathcal{P}os_{\mathcal{F}}(\ell')$, there is $x \in \mathcal{V}ar(\ell')$ such that $\ell'|_q = x$ for some $q \leq p$ and (33) is a *variable $\mathcal{R}^{rm}$-up peak.*

2. In the second case, rewriting with $\to_{\mathcal{R}^{rm}}$ occurs *below* the $\to_{\mathcal{R}^{rm},E}$-step; we have

$$t = \sigma(r) \;\; {}_{\mathcal{R}^{rm},E}\!\leftarrow \;\; \overleftarrow{w[\underrightarrow{\sigma(\ell')}]_p} \;\; \to_{\mathcal{R}^{rm}} \;\; w[\sigma(r')]_p = t' \tag{34}$$

where $w = w[\sigma(\ell')]_p =_E \sigma(\ell)$ and $p > \Lambda$, as $p = \Lambda$ coincide with $p = \Lambda$ in (33). We call (34) an $\mathcal{R}^{rm}$-*down* peak. For instance, (30) is an $\mathcal{R}^{rm}$-*down* peak.

▶ Remark 27. If $E = \emptyset$, then $\to_{\mathcal{R}^{rm},E} = \to_{\mathcal{R}^{rm}}$, and $\mathcal{R}^{rm}$-up and $\mathcal{R}^{rm}$-down peaks boil down into a unique class of peaks, just distinguishing critical and variable peaks.

From the proof of [11, Theorem 16] (Case 4 in page 1171), for $\mathcal{R}^{rm}$-down peaks involving unconditional rules, we have:

▶ **Proposition 28.** *Let $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ be an EGTRS. If $\to_{\mathcal{R}^{rm},E}$ is locally coherent modulo E, then $\mathcal{R}^{rm}$-down peaks (34) such that α and α' are* unconditional rules *are $\mathcal{R}^{rm}, E$-joinable.*

6 Analysis of local coherence modulo E of $\to_{\mathcal{R}^{rm},E}$

Given an EGTRS $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ and terms s, t, t', *coherence peaks* (13) are of the form

$$t \;{}_{\mathcal{R}^{rm},E}\!\leftarrow s \to_{\overset{\leftrightarrow}{E}} t' \tag{35}$$

Given a term s, positions $\overline{p}, \overline{p'} \in \mathcal{P}os(s)$, an *oriented* equation $\lambda \to \rho \Leftarrow d \in \overset{\leftrightarrow}{E}$, a rule $\alpha : \ell \to r \Leftarrow c \in R^{rm}$ (sharing no variables), and substitution σ, such that (i) $s|_{\overline{p}} = w =_E \sigma(\ell)$ and $\sigma(c)$ hold; and (ii) $s|_{\overline{p'}} = \sigma(\lambda)$ and $\sigma(d)$ hold, every coherence peak (35) is of the form

$$t = s[\sigma(r)]_{\overline{p}} \;\; {}_{\mathcal{R}^{rm},E}\!\leftarrow \; s[\underleftarrow{w}]_{\overline{p}} \; = \; s \; = \; s[\underrightarrow{\sigma(\lambda)}]_{\overline{p'}} \; \to_{\overset{\leftrightarrow}{E}} \; s[\sigma(\rho)]_{\overline{p'}} = t' \tag{36}$$

***Disjoint* coherence peaks.** If $\overline{p}$ and $\overline{p'}$ in (36) are *disjoint*, then $s = s[w]_{\overline{p}}[\sigma(\lambda)]_{\overline{p'}} = s[\sigma(\lambda)]_{\overline{p'}}[w]_{\overline{p}}$ and we have:

$$t = s[\sigma(r)]_{\overline{p}}[\sigma(\lambda)]_{\overline{p'}} \;\; {}_{\overset{\leftrightarrow}{E}}\!\leftarrow \;\; s[\sigma(r)]_{\overline{p}}[\sigma(\rho)]_{\overline{p'}} \; {}_{\mathcal{R}^{rm},E}\!\leftarrow \; s[w]_{\overline{p}}[\sigma(\rho)]_{\overline{p'}} = t'$$

Since $t \; \to_{\overset{\leftrightarrow}{E}} \; s[\sigma(r)]_{\overline{p}}[\sigma(\rho)]_{\overline{p'}}$ implies $t \; =_E \; s[\sigma(r)]_{\overline{p}}[\sigma(\rho)]_{\overline{p'}}$, we conclude that t and t' are right-strict $\mathcal{R}^{rm}, E$-joinable.

***Nested* coherence peaks.** If $\overline{p}$ and $\overline{p'}$ in (36) are *not* disjoint, then (36) can be written in one of the following three ways (again, by abuse, we use t and t'):

$$t = \sigma(\lambda)[\sigma(r)]_p \;\; {}_{\mathcal{R}^{rm},E}\!\leftarrow \;\; \overleftarrow{\underrightarrow{\sigma(\lambda)}} \;\; \to_{\overset{\leftrightarrow}{E}} \; \sigma(\rho) = t' \tag{37}$$

$$t = \sigma(\lambda)[\sigma(r)]_p \;\; {}_{\mathcal{R}^{rm},E}\!\leftarrow \;\; \underrightarrow{\sigma(\lambda)[\overleftarrow{w}]_p} \;\; \to_{\overset{\leftrightarrow}{E}} \; \sigma(\rho) = t' \tag{38}$$

$$t = \; \sigma(r) \; {}_{\mathcal{R}^{rm},E}\!\leftarrow \;\; \overleftarrow{w[\underrightarrow{\sigma(\lambda)}]_p} \;\; \to_{\overset{\leftrightarrow}{E}} \; w[\sigma(\rho)]_p = t' \tag{39}$$

that we call E-root (37), E-up (38), and E-down (39) coherence peaks, depending on the application of the $\to_{\overset{\leftrightarrow}{E}}$-step. Note that $\sigma(\ell) =_E \sigma(\lambda)$ in (37), and $p > \Lambda$ in (38) and (39).

▶ **Proposition 29.** *Coherence peaks (37) and (39) are right-strict $\mathcal{R}^{rm}, E$-joinable.*

Now, we investigate finite representations of nested rewriting and coherence peaks.

7 Conditional pairs for proving E-confluence of EGTRSs

In the following, we deal with *general conditional pairs*, or just *conditional pairs*, as follows, see [15, Section 5]:

▶ **Definition 30** (Conditional pair). *A* conditional pair *is an expression* $\underbrace{\langle s,t\rangle}_{peak} \Leftarrow \underbrace{A_1,\ldots,A_n}_{conditional\ part}$, *where s and t are terms and $A_1,\ldots,A_n$ are atoms.*

▶ **Definition 31** (Joinability of conditional pairs). *Let $\mathcal{R}=(\mathcal{F},\Pi,E,H,E)$ be an EGTRS. A conditional pair $\pi:\langle s,t\rangle\Leftarrow c$ is $\mathcal{R}^{rm},E$-joinable (resp. right-strict $\mathcal{R}^{rm},E$-joinable, $\mathcal{R}/E$-*joinable*) iff for all substitutions σ, if $\overline{\mathcal{R}^{\rm CR}}\vdash\sigma(A)$ holds for all $A\in c$, then $\sigma(s)$ and $\sigma(t)$ are $\mathcal{R}^{rm},E$-joinable (resp. right-strict $\mathcal{R}^{rm},E$-joinable, $\mathcal{R}/E$-joinable).*

▶ **Definition 32** (Feasible conditional pair). *Let $\mathcal{R}$ be an EGTRS. A general conditional pair $\langle s,t\rangle\Leftarrow c$ is $\overline{\mathcal{R}^{\rm CR}}$-*feasible *(or just* feasible *if $\overline{\mathcal{R}^{\rm CR}}$ is clear from the context) if c is $\overline{\mathcal{R}^{\rm CR}}$-feasible.*

The following result is immediate from Definitions 31 and 32.

▶ **Proposition 33.** *Let $\mathcal{R}$ be an EGTRS. $\overline{\mathcal{R}^{\rm CR}}$-infeasible conditional pairs are $\mathcal{R}^{rm},E$-joinable (resp. right-strict $\mathcal{R}^{rm},E$-joinable, $\mathcal{R}/E$-joinable)*

We describe three families of conditional pairs which are useful to prove and disprove E-confluence.

7.1 Logic-based conditional critical pairs

These pairs capture $\mathcal{R}^{rm}$-up and E-up critical peaks.

▶ **Definition 34** (Logic-based conditional critical pair). *Let $\alpha:\ell\to r\Leftarrow c$ and $\alpha':\ell'\to r'\Leftarrow c'$ be two rules sharing no variables, together with a non-variable position $p\in\mathcal{P}os_{\mathcal{F}}(\ell)$. The* logic-based conditional critical pair *(LCCP for short) $\pi_{\alpha,p,\alpha'}$ of α at position p with α' is:*

$$\pi_{\alpha,p,\alpha'}:\langle\ell[r']_p,r\rangle\Leftarrow\ell|_p=\ell',c,c' \tag{40}$$

Our terminology "*logic-based conditional critical pair*" tries to avoid confusion with the *E-critical pairs* for ETRSs of [11, Definition 12] and also the *conditional critical pairs* for rewrite theories of [4, Definition 6] which make an *explicit* use of E-unifiers which we avoid by including the *atom* $\ell|_p=\ell'$ in the conditional part of (40). Given sets of rules U and V, we let $\mathsf{GLCCP}(U,V)=\{\pi_{\alpha,p,\alpha'}\mid\alpha:\ell\to r\Leftarrow c\in U,p\in\mathcal{P}os_{\mathcal{F}}(\ell),\alpha'\in V\}$. For $\mathcal{R}=(\mathcal{F},\Pi,E,H,R)$, we let

$$\mathsf{LCCP}(\mathcal{R})=\mathsf{GLCCP}(R^{rm},R^{rm})\quad\text{and}\quad\mathsf{LCCP}(E,\mathcal{R})=\mathsf{GLCCP}(\overleftrightarrow{E},R^{rm})$$

For GTRSs involving finite sets of equations and rules, both $\mathsf{LCCP}(\mathcal{R})$ and $\mathsf{LCCP}(E,\mathcal{R})$ are *finite*. The following result shows that they suffice to capture any possible critical peak.

▶ **Proposition 35** (Critical peaks and LCCPs). *Let $\mathcal{R}=(\mathcal{F},\Pi,E,H,R)$ be a EGTRS.*

- *Let $\alpha:\ell\to r\Leftarrow c,\alpha':\ell'\to r'\Leftarrow c'\in R^{rm}$, sharing no variable, induce $\mathcal{R}^{rm}$-up critical peaks (33) with $p\in\mathcal{P}os_{\mathcal{F}}(\ell')$ as in (33). Then, (33) is $\mathcal{R}^{rm},E$-joinable ($\mathcal{R}/E$-joinable), iff $\pi_{\alpha,p,\alpha'}\in\mathsf{LCCP}(\mathcal{R})$ is $\mathcal{R}^{rm},E$-joinable ($\mathcal{R}/E$-joinable).*
- *Let $\alpha:\ell\to r\Leftarrow c\in R^{rm}$ and $\beta:\lambda\to\rho\Leftarrow d\in\overleftrightarrow{E}$, sharing no variable, induce E-up critical peaks (38) with $p\in\mathcal{P}os_{\mathcal{F}}(\lambda)$ as in (38). Then, (38) is right-strict $\mathcal{R}^{rm},E$-joinable ($\mathcal{R}/E$-joinable) iff $\pi_{\alpha,p,\beta}\in\mathsf{LCCP}(E,\mathcal{R})$ is right-strict $\mathcal{R}^{rm},E$-joinable ($\mathcal{R}/E$-joinable).*

▶ **Example 36.** (Continuing Example 26) $\mathsf{LCCP}(\mathcal{R})$ consists of:

α	p	α'	LCCP	
(16)	Λ	(16)	$\langle \mathsf{d}, \mathsf{d}\rangle \Leftarrow \mathsf{c} = \mathsf{c}$	(41)
(16)	Λ	(17)	$\langle \mathsf{d}, \mathsf{d}\rangle \Leftarrow \mathsf{c} = \mathsf{b}$	(42)
(17)	Λ	(17)	$\langle \mathsf{d}, \mathsf{d}\rangle \Leftarrow \mathsf{b} = \mathsf{c}$	(43)

which are all trivially $\mathcal{R}^{rm}, E$-joinable. Note that $\pi_{(17),\Lambda,(16)}$ is *not* considered as it is $\mathcal{R}^{rm}, E$-joinable iff $\pi_{(16),\Lambda,(17)}$ is. Regarding $\mathsf{LCCP}(E, \mathcal{R})$, we have that $\pi_{\overleftarrow{(14)},1,(16)}$ i.e.,

$$\langle \mathsf{f}(\mathsf{d}), \mathsf{b}\rangle \Leftarrow \mathsf{a} = \mathsf{c}$$

is *not* right-strict $\mathcal{R}^{rm}, E$-joinable: although the conditional part $\mathsf{a} = \mathsf{c}$ holds, and $\mathsf{b} \to_{\mathcal{R}^{rm},E} \mathsf{d}$, $\langle \mathsf{f}(\mathsf{d}), \mathsf{d}\rangle$ is *not* $\mathcal{R}^{rm}, E$-joinable.

▶ **Example 37.** For $\mathcal{R}$ in Example 1, $\mathsf{LCCP}(\mathcal{R})$ consists of 22 LCCPs. The complete list and proofs of $\mathcal{R}^{rm}, E$-joinability are given in Appendix A. Representative examples are:

- $\pi_{(24),\Lambda,(24)}$ is

  $$\langle m'+n', m+n\rangle \Leftarrow \mathsf{sum}(m +\!\!+\, ns) = \mathsf{sum}(m' +\!\!+\, ns'), \mathsf{Nat}(m), \mathsf{sum}(ns) \approx_{rm} n, \mathsf{Nat}(m'), \mathsf{sum}(ns') \approx_{rm} n'$$

  If a substitution σ satisfies the conditional part, then $\sigma(m +\!\!+\, ns) =_E \sigma(m' +\!\!+\, ns')$, i.e., the sequences $\sigma(m +\!\!+\, ns)$ and $\sigma(m' +\!\!+\, ns')$ are identical except for the distribution of parenthesis. Furthermore, $\sigma(m)$ and $\sigma(m')$ are the same natural number in Peano's notation. Therefore, $\sigma(ns)$ and $\sigma(ns')$ are identical up to parenthization. Hence, $\mathsf{sum}(\sigma(ns))$ and $\mathsf{sum}(\sigma(ns'))$ can be reduced by $\to^*_{\mathcal{R}/E}$ to the same expression, i.e., we can assume that $\sigma(n)$ and $\sigma(n')$ coincide and thus that $\sigma(m+n)$ and $\sigma(m'+n')$ are $\mathcal{R}^{rm}, E$-joinable.
- $\pi_{(21),\Lambda,(22)}$ is $\langle \mathsf{s}(m'+n'), n\rangle \Leftarrow \mathsf{0} + n = \mathsf{s}(m') + n'$. Since the conditional part is clearly infeasible, $\pi_{(21),\Lambda,(22)}$ is trivially $\mathcal{R}^{rm}, E$-joinable.

$\mathsf{LCCP}(E, \mathcal{R})$ consists of 16 LCCPs. Every $\pi \in \mathsf{LCCP}(E, \mathcal{R})$ is *infeasible*: they include a condition $s = t$ where s is rooted with $+\!\!+$ and t is always rooted with a *different* symbol. For instance, $\pi_{\overrightarrow{(1)},\Lambda,(21)}$ is $\langle n, (xs +\!\!+\, ys) +\!\!+\, zs\rangle \Leftarrow xs +\!\!+\, (ys +\!\!+\, zs) = \mathsf{0} + n$. Thus, every $\pi \in \mathsf{LCCP}(E, \mathcal{R})$ is right-strict $\mathcal{R}^{rm}, E$-joinable, see Appendix A too. Hence, all pairs in $\mathsf{LCCP}(\mathcal{R})$ are $\mathcal{R}^{rm}, E$-joinable and all pairs in $\mathsf{LCCP}(E, \mathcal{R})$ are right-strict $\mathcal{R}^{rm}, E$-joinable.

7.2 Conditional variable pairs

These pairs capture $\mathcal{R}^{rm}$-up and E-up variable peaks.

▶ **Definition 38** (Parametric conditional variable pair). *Let $\alpha : s \to t \Leftarrow c$ be a rule where s can be a variable, $x \in \mathcal{V}ar(s)$, $p \in \mathcal{P}os_x(s)$, and $x' \notin \alpha$ be a fresh variable. Let $\bowtie$ be a binary relation on terms. Define a $\bowtie$-parametric* Conditional Variable Pair *(CVP) $\pi^{\bowtie}_{\alpha,x,p}$ as follows:*

$$\pi^{\bowtie}_{\alpha,x,p} : \langle s[x']_p, t\rangle \Leftarrow x \bowtie x', c \tag{44}$$

In the following, $\mathsf{CVP}(U, \bowtie)$ is the set of all $\bowtie$-parametric CVPs in a set of rules U. We let

$$\mathsf{CVP}(\mathcal{R}) = \mathsf{CVP}(R^{rm}, \xrightarrow{ps}) \quad \text{and} \quad \mathsf{CVP}(E) = \mathsf{CVP}(\overleftrightarrow{E}, \xrightarrow{ps})$$

▶ **Proposition 39** (Variable peaks and CVPs). *Let $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ be an EGTRS.*

- *Let $\alpha : \ell \to r \Leftarrow c, \alpha' : \ell' \to r' \Leftarrow c' \in R^{rm}$, sharing no variable, determine variable $\mathcal{R}^{rm}$-up peaks (33) with $p \notin \mathcal{P}os_{\mathcal{F}}(\ell')$ as in (33). Then, (33) is $\mathcal{R}^{rm}, E$-joinable ($\mathcal{R}/E$-joinable) iff $\pi^{\xrightarrow{ps}}_{\alpha',x,q} \in \mathsf{CVP}(\mathcal{R})$ (for some $x \in \mathcal{V}ar(\ell')$ and $q \in \mathcal{P}os_x(\ell')$ such that $q \leq p$) is $\mathcal{R}^{rm}, E$-joinable ($\mathcal{R}/E$-joinable).*

- *Let $\alpha : \ell \to r \Leftarrow c \in R^{rm}$ and $\beta : \lambda \to \rho \Leftarrow d \in \overset{\leftrightarrow}{E}$, sharing no variable, determine variable E-up peaks (38) with $p \notin \mathcal{P}os_{\mathcal{F}}(\lambda)$ as in (38). Then, (38) is right-strict $\mathcal{R}^{rm}, E$-joinable ($\mathcal{R}/E$-joinable) iff $\pi^{\overset{ps}{\to}}_{\beta,x,q} \in \mathsf{CVP}(E)$ (for some $q \in \mathcal{P}os_x(\lambda)$ and $x \in \mathcal{V}ar(\lambda)$ such that $q \leq p$) is right-strict $\mathcal{R}^{rm}, E$-joinable ($\mathcal{R}/E$-joinable).*

For unconditional rules, we have the following.

▶ **Proposition 40.** *Let $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ be an EGTRS and $\alpha : \lambda \to \rho$ be an* unconditional rule*, where λ can be a variable. Then, for all $x \in \mathcal{V}ar(\lambda)$ and $p \in \mathcal{P}os_x(\lambda)$, $\pi^{\overset{ps}{\to}}_{\alpha,x,p}$ is $\mathcal{R}^{rm}, E$-joinable. If $x \in \mathcal{V}ar(\rho)$, then π is right-strict joinable.*

Accordingly, in the following we dismiss from $\mathsf{CVP}(\mathcal{R})$ those CVPs obtained from unconditional rules in $\mathcal{R}$; and we also dismiss from $\mathsf{CVP}(E)$ those CVPs obtained from unconditional rules $\lambda \to \rho \in \overset{\leftrightarrow}{E}$ whose critical variable x occurs in ρ.

▶ **Example 41.** For $\mathcal{R}$ in Example 1, $\mathsf{CVP}(\mathcal{R})$ consists of the following:

α	var.	p	CVP	
(23)	n	1	$\langle \mathsf{sum}(n'), n \rangle \Leftarrow n \overset{ps}{\to} n', \mathsf{Nat}(n)$	(45)
(24)	m	1.1	$\langle \mathsf{sum}(m' +\!\!+\, ns), m + n \rangle \Leftarrow m \overset{ps}{\to} m', \mathsf{Nat}(m), \mathsf{sum}(ns) \approx_{rm} n$	(46)
(24)	ns	1.2	$\langle \mathsf{sum}(m +\!\!+\, ns'), m + n \rangle \Leftarrow ns \overset{ps}{\to} ns', \mathsf{Nat}(m), \mathsf{sum}(ns) \approx_{rm} n$	(47)

These CVPs are *infeasible*, as terms t satisfying $\mathsf{Nat}(t)$ are of the form $\mathsf{s}^p(0)$ for some $p \geq 0$ and hence $\to_{\mathcal{R}^{rm},E}$-irreducible. Hence they are $\mathcal{R}^{rm}, E$-joinable. The set $\mathsf{CVP}(E)$ is empty, as all variables in the left hand side of the unconditional rules $\overrightarrow{(1)}$ and $\overleftarrow{(1)}$ also occur in the corresponding right-hand side (Proposition 40).

For $\mathcal{R}$ in Example 26, $\mathsf{CVP}(\mathcal{R}) = \emptyset$.

7.3 Down conditional critical pairs

$\mathcal{R}^{rm}$-down peaks (34) combine possible *rule overlaps* (modulo) and the application of rules "below" a variable. Unfortunately, these two sources of divergence do not admit a neat separation (as done for $\mathcal{R}^{rm}$-up peaks) into "critical" and "variable" $\mathcal{R}^{rm}$-down peaks to be captured by means of LCCPs and CVPs. Alternatively, *down conditional critical pairs* capture these two (mingled) situations at once. First consider the predicate $\rhd^{\times}$ defining a strict *subterm* relation on *pairs* (s, t) of terms by the following clauses:

$(\mathrm{Sb})^{\rhd^{\times}}_{f,i}$	$(f(x_1, \ldots, x_i, \ldots, x_k), f(x_1, \ldots, x'_i, \ldots, x_k)) \rhd^{\times} (x_i, x'_i)$
$(\mathrm{Sb2})^{\rhd^{\times}}_{f,i}$	$(f(x_1, \ldots, x_i, \ldots, x_k), f(x_1, \ldots, x'_i, \ldots, x_k)) \rhd^{\times} (x, x') \Leftarrow (x_i, x'_i) \rhd^{\times} (x, x')$

Let $\mathsf{Th}_{\rhd^{\times}}(\mathcal{F}) = \{(\mathrm{HC})_{(\mathrm{Sb})^{\rhd^{\times}}_{f,i}}, (\mathrm{HC})_{(\mathrm{Sb2})^{\rhd^{\times}}_{f,i}} \mid f \in \mathcal{F}, 1 \leq i \leq ar(f)\}$.

▶ **Proposition 42.** *Let $\mathcal{F}$ be a signature and s, t, u, v be terms. Then, $\mathsf{Th}_{\rhd^{\times}}(\mathcal{F}) \vdash (s, t) \rhd^{\times} (u, v)$ holds iff there is a nonempty context $C[\,]$ such that $s = C[u]$ and $t = C[v]$.*

▶ **Definition 43** (Down conditional critical pairs)**.** *Let $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ be an EGTRS. Rules $\alpha : \ell \to r \Leftarrow c, \alpha' : \ell' \to r' \Leftarrow c' \in R^{rm}$ (sharing no variables) define a* Down Conditional Critical Pair *(DCCP for short) as follows:*

$$\pi_{\alpha,\alpha'} : \langle r, x' \rangle \Leftarrow x = \ell, (x, x') \rhd^{\times} (\ell', r'), c, c' \qquad (48)$$

where x and x' are fresh variables. The set of DCCPs of $\mathcal{R}$ is

$$\mathsf{DCCP}(\mathcal{R}) = \{\pi_{\alpha,\alpha'} \mid \alpha, \alpha' \in R^{rm}\}$$

▶ **Proposition 44** ($\mathcal{R}^{rm}$-peaks and DCCPs)**.** *Let $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ be an EGTRS. Let $\alpha : \ell \to r \Leftarrow c, \alpha' : \ell' \to r' \Leftarrow c' \in R^{rm}$, sharing no variable, determine $\mathcal{R}^{rm}$-down critical peaks (34). Then, (34) is $\mathcal{R}^{rm}, E$-joinable ($\mathcal{R}/E$-joinable) iff $\pi_{\alpha,\alpha'} \in \mathsf{DCCP}(\mathcal{R})$ is $\mathcal{R}^{rm}, E$-joinable ($\mathcal{R}/E$-joinable).*

▶ Remark 45 (Continuing Remark 27). If $E = \emptyset$ in an EGTRS $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$, then all peaks represented by $\mathsf{DCCP}(\mathcal{R})$ are captured by $\mathsf{LCCP}(\mathcal{R})$ and $\mathsf{CVP}(\mathcal{R})$.

▶ **Example 46.** For $\mathcal{R}$ in Example 26, $\mathsf{DCCP}(\mathcal{R})$ consists of the following DCCPs:

α	α'	DCCP			
(16)	(16)	$\langle \mathsf{d}, x' \rangle$	$\Leftarrow$	$x = \mathsf{c}, (x, x') \rhd^{\times} (\mathsf{c}, \mathsf{d})$	(49)
(16)	(17)	$\langle \mathsf{d}, x' \rangle$	$\Leftarrow$	$x = \mathsf{b}, (x, x') \rhd^{\times} (\mathsf{c}, \mathsf{d})$	(50)
(17)	(16)	$\langle \mathsf{d}, x' \rangle$	$\Leftarrow$	$x = \mathsf{c}, (x, x') \rhd^{\times} (\mathsf{b}, \mathsf{d})$	(51)
(17)	(17)	$\langle \mathsf{d}, x' \rangle$	$\Leftarrow$	$x = \mathsf{b}, (x, x') \rhd^{\times} (\mathsf{b}, \mathsf{d})$	(52)

As for (50), $\sigma = \{x \mapsto \mathsf{f}(\mathsf{c}), x' \mapsto \mathsf{f}(\mathsf{d})\}$ satisfies the conditional part as $\sigma(x) = \mathsf{f}(\mathsf{c}) =_E \mathsf{f}(\mathsf{a}) =_E \mathsf{b}$ and $(\mathsf{f}(\mathsf{c}), \mathsf{f}(\mathsf{d}) \rhd^{\times} (\mathsf{c}, \mathsf{d})$. However, d and $\mathsf{f}(\mathsf{d})$ are *not* $\mathcal{R}^{rm}, E$-joinable.

▶ **Proposition 47.** *Let $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ be an EGTRS and $\alpha : \ell \to r \Leftarrow c, \alpha' : \ell' \to r' \Leftarrow c' \in R^{rm}$ be such that $\ell = \ell[x_1, \ldots, x_n]$ is linear, $\mathcal{V}ar(\ell) = \{x_1, \ldots, x_n\}$, and for all terms t, if $t =_E \sigma(\ell)$ for some substitution σ satisfying c and c', then $t = \ell[t_1, \ldots, t_n]$ for some terms $t_1, \ldots, t_n$. If every LCCP $\pi_{\alpha,p,\alpha'}$ is $\mathcal{R}^{rm}, E$-joinable for all $p \in \mathcal{P}os_{\mathcal{F}}(\ell)$, and every CVP $\pi^{\overset{ps}{\to}}_{\alpha,x,q}$ is $\mathcal{R}^{rm}, E$-joinable for all $x \in \mathcal{V}ar(\ell)$ and $q \in \mathcal{P}os_x(\ell)$, then the DCCP $\pi_{\alpha,\alpha'}$ is $\mathcal{R}^{rm}, E$-joinable.*

By Propositions 28 and 44, dealing with EGTRS*s* $\mathcal{R}$ such that $\to_{\mathcal{R}^{rm},E}$ is locally coherent modulo E, we can dismiss DCCPs for *unconditional* rules α and α'.

▶ **Example 48.** For $\mathcal{R}$ in Example 1, $\mathsf{DCCP}(\mathcal{R})$ consists of 16 DCCPs (involving a conditional rule). The complete list (with all $\mathcal{R}^{rm}, E$-joinability proofs) is given in Appendix A. A representative example is $\pi_{(21),(24)}$, i.e.,

$$\langle n, x' \rangle \Leftarrow x = \mathsf{0} + n, (x, x') \rhd^{\times} (\mathsf{sum}(m' +\!\!+\, ns'), m' + n'), \mathsf{Nat}(m'), \mathsf{sum}(ns') \approx_{rm} n'$$

If σ satisfies the conditional part, then $\sigma(x) =_E \mathsf{0} + \sigma(n)$ holds. Since $+ \notin \mathcal{D}(\overset{\leftrightarrow}{E})$, it follows that $\sigma(x) = \mathsf{0} + \sigma(n)$. Thus, by Proposition 47 and since every $\pi \in \mathsf{LCCP}(\mathcal{R}) \cup \mathsf{CVP}(\mathcal{R})$ is $\mathcal{R}^{rm}, E$-joinable (Examples 37 and 41), $\pi_{(21),(24)}$ is $\mathcal{R}^{rm}, E$-joinable.

Example 46 shows that $\mathcal{R}^{rm}, E$-joinability of all $\pi \in \mathsf{LCCP}(\mathcal{R}) \cup \mathsf{CVP}(\mathcal{R})$ does *not* imply $\mathcal{R}^{rm}, E$-joinability of DCCPs in $\mathsf{DCCP}(\mathcal{R})$ unless the conditions in Proposition 47 are fulfilled.

8 Proving and disproving E-confluence

The following result shows how to prove and disprove E-confluence.

▶ **Theorem 49.** *Let $\mathcal{R} = (\mathcal{F}, \Pi, E, H, R)$ be an EGTRS.*

1. *$\to_{\mathcal{R}^{rm},E}$ is locally confluent modulo E with $\to_{\mathcal{R}^{rm}}$ iff every $\pi \in \mathsf{LCCP}(\mathcal{R}) \cup \mathsf{CVP}(\mathcal{R}) \cup \mathsf{DCCP}(\mathcal{R})$ is $\mathcal{R}^{rm}, E$-joinable.*
2. *$\to_{\mathcal{R}^{rm},E}$ is locally coherent modulo E iff every $\pi \in \mathsf{LCCP}(E, \mathcal{R}) \cup \mathsf{CVP}(E)$ is right-strict $\mathcal{R}^{rm}, E$-joinable.*
3. *If $\mathcal{R}$ is E-terminating, then $\mathcal{R}$ is $\to_{\mathcal{R}^{rm},E}$-Church-Rosser modulo E iff every $\pi \in \mathsf{LCCP}(\mathcal{R}) \cup \mathsf{CVP}(\mathcal{R}) \cup \mathsf{DCCP}(\mathcal{R})$ is $\mathcal{R}^{rm}, E$-joinable, and every $\pi \in \mathsf{LCCP}(E, \mathcal{R}) \cup \mathsf{CVP}(E)$ is right-strict $\mathcal{R}^{rm}, E$-joinable.*

4. *If $\mathcal{R}$ is E-terminating, every $\pi \in \mathsf{LCCP}(\mathcal{R}) \cup \mathsf{CVP}(\mathcal{R}) \cup \mathsf{DCCP}(\mathcal{R})$ is $\mathcal{R}^{rm}, E$-joinable, and every $\pi \in \mathsf{LCCP}(E, \mathcal{R}) \cup \mathsf{CVP}(E)$ is right-strict $\mathcal{R}^{rm}, E$-joinable, then $\mathcal{R}$ is E-confluent.*
5. *If there is $\pi \in \mathsf{LCCP}(\mathcal{R}) \cup \mathsf{CVP}(\mathcal{R}) \cup \mathsf{DCCP}(\mathcal{R})$ which is* not *$\mathcal{R}/E$-joinable, then $\mathcal{R}$ is not E-confluent.*

▶ **Example 50.** (Continuing Example 1) For $\mathcal{R}$ in Example 1, every $\pi \in \mathsf{LCCP}(\mathcal{R}) \cup \mathsf{CVP}(\mathcal{R}) \cup \mathsf{DCCP}(\mathcal{R})$ is $\mathcal{R}^{rm}, E$-joinable (see Examples 37, 41, and 48) and every $\pi \in \mathsf{LCCP}(E, \mathcal{R}) \cup \mathsf{CVP}(E)$ is right-strict $\mathcal{R}^{rm}, E$-joinable (see Examples 37 and 41). It is not difficult to see that $\mathcal{R}$ is E-terminating. Thus, by Theorem 49(4), $\mathcal{R}$ is E-confluent.

▶ **Example 51.** For $\mathcal{R}$ in Example 26, the DCCP (50) has been proved non-$\mathcal{R}/\mathcal{E}$-joinable in Example 46. By Theorem 49(5), $\mathcal{R}$ is *not* E-confluent. Note that all pairs in $\mathsf{LCCP}(\mathcal{R})$ are joinable (see Example 36) and $\mathsf{CVP}(\mathcal{R})$ is empty. Thus, $\mathsf{DCCP}(\mathcal{R})$ is the only set of conditional pairs that can be used to disprove E-confluence of $\mathcal{R}$.

9 Related work

GTRSs and EGTRSs. A Generalized Term Rewriting System (GTRS, [15, Definition 51]) is a tuple $\mathcal{R} = (\mathcal{F}, \Pi, \mu, H, R)$, where $\mathcal{F}$, Π, H and R are defined as above, and μ is a *replacement map* establishing which arguments $\mu(f)$ can be rewritten for each function symbol $f \in \mathcal{F}$ [14]. EGTRSs do not use replacement maps, which corresponds to "use" the so-called *top* replacement map $\mu_\top$ which permits all rewritings in all arguments of symbols. We have borrowed from [15, Definition 30] the notion of *conditional variable pair*, although we use it here in a slightly different way, as conditional pairs $\langle s, t\rangle \Leftarrow x \overset{ps}{\to} x', c$ where $\overset{ps}{\to}$ is interpreted as $\to_{\mathcal{R},E}$, but possible rewritings in c may correspond to $\to_{\mathcal{R}/E}$, $\to^*_{\mathcal{R}/E}$, etc. Conditional variable pairs of GTRSs $\mathcal{R}$ are written $\langle s, t\rangle \Leftarrow x \to x', c$, where $\to$ is the one-step rewrite relation $\to_\mathcal{R}$ of $\mathcal{R}$ and conditions in c are treated using $\to_\mathcal{R}$, $\to^*_\mathcal{R}$, etc.

Plaisted proposed quite a general notion of conditional rewrite systems where rules are viewed as clauses including (possibly many) negative literals [21]. In this respect, EGTRSs are particular cases of Plaisted's conditional rewrite systems. Plaisted also provides a complete specification of the logical theory which could be used (together with such rules) to obtain the desired reduction, see [21, page 217]. However, equational components are *not* allowed.

Jouannaud and Kirchner's main result for ETRSs [11, Theorem 16], cannot be used to *disprove* E-confluence of ETRSs. For instance, the proof of non-E-confluence of $\mathcal{R}$ in Example 26 would not be obtained. Actually, (30) in Example 26 is an $\mathcal{R}$-*down* rewriting peak. Such peaks are explictly *excluded* to obtain E-critical pairs in [10, 11][4]. Our *down conditional critical pairs* (DCCPs) fill this gap. On the other hand, Jouannaud and Kirchner's results for proving confluence of ETRSs modulo permit an application to relations R^E on terms like $\to_\mathcal{L} \cup \to_{\mathcal{N},E}$, where $\mathcal{L}$ and $\mathcal{N}$ are a partition of $\mathcal{R}$ where $\mathcal{L}$ includes left-linear rules only and $\mathcal{N}$ includes any other rules [11, Section 3.5]. The case considered here, $\to_{\mathcal{R},E}$, is a particular case of the previous one, where $\mathcal{L} = \emptyset$ and $\mathcal{N} = \mathcal{R}$. Under these conditions, [11, Theorem 16] treats proofs of E-confluence essentially as our Theorem 49.(4) (as $\mathsf{CVP}(\mathcal{R})$ and $\mathsf{DCCP}(\mathcal{R})$ can be dismissed according to the discussion above). The aforementioned more general treatment for EGTRSs is left as an interesting subject for future work.

[4] *"we do not consider the case where $\to_\mathcal{R}$ applies at an occurrence p and $\to_{\mathcal{R},\mathcal{E}}$ at the outermost occurrence Λ"* [11, below E-critical pairs lemma] (notation adapted).

Durán and Meseguer investigated E-confluence of *conditional rewrite theories* $\mathcal{R} = (\mathcal{F}, A, R)$ [4]. Such theories include CTRSs. Conditional equations $s = t \Leftarrow c$ can be specified, but they are treated as conditional rewrite rules (in R) by imposing some specific orientation (e.g., $s \to t \Leftarrow c$). Only unconditional equations $s = t$ (called *axioms*) which are *linear* and *regular* (i.e., $\mathcal{V}ar(s) = \mathcal{V}ar(t)$ [4, page 819]) are used in E (denoted A in [4]). The main result about E-confluence is [4, Theorem 2], which characterizes E-confluence by the joinability of the set of conditional E-critical pairs obtained from rules $\ell \to r \Leftarrow c$ and $\ell' \to r' \Leftarrow c'$ in $\mathcal{R}$ (which may include oriented conditional equations) by computing (if possible) the A-unifiers of $\ell|_p$ and ℓ' for some nonvariable position $p \in \mathcal{P}os_{\mathcal{F}}(\ell)$ [4, Definition 6]. However, a number of restrictions are imposed: (i) A is a set of linear and regular unconditional equations; (ii) R is strongly A-coherent (i.e., for all terms u, u', and v, if $u \to_{\mathcal{R}/A} v$ and $u =_A u'$, then $u' \to_{\mathcal{R},A} v'$ and $v =_A v'$ [4, page 819]); (iii) the rules in R are *strongly deterministic* [4, Definition 1]; (iv) $\mathcal{R}$ is *quasi-decreasing* [4, Definition 2]. Again, this result would *not* apply to disprove E-confluence of $\mathcal{R}$ in Example 26; note that E satisfies the requirements for axioms A in [4]. We have: $\mathsf{b} =_E \mathsf{f}(\underline{\mathsf{a}}) \to_{\mathcal{R}} \mathsf{f}(\mathsf{d})$, i.e., $\mathsf{b} \to_{\mathcal{R}/E} \mathsf{f}(\mathsf{d})$, but the only $\to_{\mathcal{R},E}$-step on b is $\mathsf{b} \to_{\mathcal{R},E} \mathsf{d}$, and $\mathsf{f}(\mathsf{d}) \neq_E \mathsf{d}$. Thus, $\mathcal{R}$ is not strongly E-coherent and [4, Theorem 2] does not apply. Also, the proof of E-confluence for $\mathcal{R}$ in Example 1 could not (in principle) be obtained from [4, Theorem 2]: since the set of E-unifiers for the left-hand sides of rules is infinite (Example 2), the joinability of infinitely many conditional E-critical pairs should be checked. Also, we do not require (i)-(iv) above in Theorem 49; only E-termination.

10 Conclusion and future work

We have introduced *Equational Generalized Term Rewriting Systems* (EGTRSs) consisting of *conditional rules* and *conditional equations* whose conditions are sequences of *atoms*, possibly defined by additional Horn clauses. Rewriting computations with EGTRSs $\mathcal{R}$ are described by *deduction* in appropriate FO-theories. We show that E-confluence of EGTRSs can be *proved* and *disproved* by checking (right-strict) $\mathcal{R}^{rm}, E$-joinability or non-$\mathcal{R}/E$-joinability of *finite sets* of conditional pairs of three kinds: *Logic-based Conditional Critical Pairs* (LCCPs), *Conditional Variable Pairs* (CVPs), and *Down Conditional Critical Pairs* (DCCPs). As far as we know, none of them had been used in proofs of E-confluence yet. The discussed examples suggest that the new techniques can be useful.

Future work. Much work remains to be done for a *practical* use of these new proposals. Theorem 49 heavily relies on checking (right-strict) $\mathcal{R}^{rm}, E$-joinability and non-$\mathcal{R}/E$-joinability of LCCPs, CVPs, and DCCPs, to obtain proofs of (non) E-confluence. We plan to improve our tool CONFident [8], which implements methods for the analysis of similar conditional pairs (see [15, 16]) and heavily relies on the (in)feasibility results developed in [7] and implemented in the tool infChecker. It also uses theorem provers like Prover9 [17] and model generators like Mace4 [17] and AGES [6] to implement these checkings. Overall, this approach has proved useful to prove confluence of variants of TRSs, see [8, Section 9] for an account. Unfortunately, our preliminary attempts to follow this methodology to prove E-confluence of EGTRSs in CONFident suggest that the use of the generic reasoning methods implemented in these tools is not powerful enough as to deal with the conditional pairs involved in E-confluence proofs. For instance, LCCPs avoid considering (infinitely many) conditional E-critical pairs. From a practical point of view, though (in particular, to obtain an efficient implementation), it is important to investigate the possibility of a *mixed* use of conditional

E-critical pairs together with LCCPs. For instance, if there is an E-unification algorithm and the set of (complete) E-unifiers is finite, then the corresponding (computable and finite set of) conditional E-critical pairs could be used instead of LCCPs. In spite of this, DCCPs and CVPs remain as main ingredients in proofs of (non-)E-confluence.

Finally, exploring the impact of our techniques in *first-order deduction modulo* see, e.g., [3] and the references therein, is another interesting subject of future work.

References

1 Franz Baader and Tobias Nipkow. *Term Rewriting and All That*. Cambridge University Press, 1998. doi:10.1017/CBO9781139172752.

2 Franz Baader and Wayne Snyder. Unification theory. In John Alan Robinson and Andrei Voronkov, editors, *Handbook of Automated Reasoning (in 2 volumes)*, pages 445–532. Elsevier and MIT Press, 2001. doi:10.1016/b978-044450813-3/50010-2.

3 Gilles Dowek. Automated theorem proving in first-order logic modulo: on the difference between type theory and set theory. *CoRR*, abs/2306.00498, 2023. doi:10.48550/ARXIV.2306.00498.

4 Francisco Durán and José Meseguer. On the church-rosser and coherence properties of conditional order-sorted rewrite theories. *J. Log. Algebraic Methods Program.*, 81(7-8):816–850, 2012. doi:10.1016/j.jlap.2011.12.004.

5 Melvin Fitting. *First-Order Logic and Automated Theorem Proving, Second Edition*. Graduate Texts in Computer Science. Springer, 1996. doi:10.1007/978-1-4612-2360-3.

6 Raúl Gutiérrez and Salvador Lucas. Automatic Generation of Logical Models with AGES. In Pascal Fontaine, editor, *Automated Deduction - CADE 27 - 27th International Conference on Automated Deduction, Proceedings*, volume 11716 of *Lecture Notes in Computer Science*, pages 287–299. Springer, 2019. doi:10.1007/978-3-030-29436-6_17.

7 Raúl Gutiérrez and Salvador Lucas. Automatically Proving and Disproving Feasibility Conditions. In Nicolas Peltier and Viorica Sofronie-Stokkermans, editors, *Automated Reasoning - 10th International Joint Conference, IJCAR 2020, Proceedings, Part II*, volume 12167 of *Lecture Notes in Computer Science*, pages 416–435. Springer, 2020. doi:10.1007/978-3-030-51054-1_27.

8 Raúl Gutiérrez, Miguel Vítores, and Salvador Lucas. Confluence Framework: Proving Confluence with CONFident. In Alicia Villanueva, editor, *Logic-Based Program Synthesis and Transformation - 32nd International Symposium, LOPSTR 2022, Proceedings*, volume 13474 of *Lecture Notes in Computer Science*, pages 24–43. Springer, 2022. doi:10.1007/978-3-031-16767-6_2.

9 Gérard P. Huet. Confluent Reductions: Abstract Properties and Applications to Term Rewriting Systems. *J. ACM*, 27(4):797–821, 1980. doi:10.1145/322217.322230.

10 Jean-Pierre Jouannaud. Confluent and coherent equational term rewriting systems: Application to proofs in abstract data types. In Giorgio Ausiello and Marco Protasi, editors, *CAAP'83, Trees in Algebra and Programming, 8th Colloquium, Proceedings*, volume 159 of *Lecture Notes in Computer Science*, pages 269–283. Springer, 1983. doi:10.1007/3-540-12727-5_16.

11 Jean-Pierre Jouannaud and Hélène Kirchner. Completion of a set of rules modulo a set of equations. *SIAM J. Comput.*, 15(4):1155–1194, 1986. doi:10.1137/0215084.

12 Stephen Cole Kleene. *Mathematical Logic*. Dover Publications, 2002. Originally published by John Wiley & Sons, 1967.

13 Donald E. Knuth and Peter E. Bendix. Simple Word Problems in Universal Algebra. In J. Leech, editor, *Computational Problems in Abstract Algebra*, pages 263–297. Pergamon Press, 1970.

14 Salvador Lucas. Context-sensitive Rewriting. *ACM Comput. Surv.*, 53(4):78:1–78:36, 2020. doi:10.1145/3397677.

15 Salvador Lucas. Local confluence of conditional and generalized term rewriting systems. *Journal of Logical and Algebraic Methods in Programming*, 136:paper 100926, pages 1–23, 2024. `doi:10.1016/j.jlamp.2023.100926`.

16 Salvador Lucas, Miguel Vítores, and Raúl Gutiérrez. Proving and disproving confluence of context-sensitive rewriting. *Journal of Logical and Algebraic Methods in Programming*, 126:100749, 2022. `doi:10.1016/j.jlamp.2022.100749`.

17 William McCune. Prover9 & Mace4. Technical report, University of New Mexico, 2005–2010. URL: `http://www.cs.unm.edu/~mccune/prover9/`.

18 Elliott Mendelson. *Introduction to mathematical logic (4. ed.)*. Chapman and Hall, 1997.

19 Enno Ohlebusch. *Advanced Topics in Term Rewriting*. Springer, 2002. `doi:10.1007/978-1-4757-3661-8`.

20 Gerald E. Peterson and Mark E. Stickel. Complete sets of reductions for some equational theories. *J. ACM*, 28(2):233–264, 1981. `doi:10.1145/322248.322251`.

21 David A. Plaisted. A Logic for Conditional Term Rewriting Systems. In Stéphane Kaplan and Jean-Pierre Jouannaud, editors, *Conditional Term Rewriting Systems, 1st International Workshop, Proceedings*, volume 308 of *Lecture Notes in Computer Science*, pages 212–227. Springer, 1987. `doi:10.1007/3-540-19242-5_16`.

22 Gordon D. Plotkin. Building-in equational theories. *Machine Intelligence*, 7:73–90, 1972.

23 John Alan Robinson. A review of automatic theorem proving. In J.T. Schwartz, editor, *Mathematical Aspects of Computer Science*, volume XIX of *Proceedings of Simposia in Applied Mathematics*, pages 1–18, 1967. `doi:10.1090/psapm/019`.

A Conditional pairs for the main running example

A.1 LCCP($\mathcal{R}$) for $\mathcal{R}$ in Example 1

The LCCPs in $\mathsf{LCCP}(\mathcal{R})$ for $\mathcal{R}$ in Example 1 are displayed in Figure 2. The following result is useful to analyze their joinability.

▶ **Proposition 52.** *Let $\mathcal{R}$ be an EGTRS and $\pi : \langle \ell[r']_p, r\rangle \Leftarrow \ell|_p = \ell', c, c' \in \mathsf{LCCP}(\mathcal{R})$. If $root(\ell|_p), root(\ell') \notin \mathcal{D}(\overleftrightarrow{E})$, and $root(\ell|_p) \neq root(\ell')$, then π is $\overline{\mathcal{R}^{\rm CR}}$-infeasible.*

Proof. By definition of LCCP, since $p \in \mathcal{P}os_{\mathcal{F}}(\ell)$ and $\alpha' \in R^{rm}$, we have that $\ell|_p, \ell' \notin \mathcal{X}$. Thus, for all substitutions σ satisfying $\ell|_p = \ell'$, we have $\sigma(\ell|_p) \to^*_{\overleftrightarrow{E}} \sigma(\ell')$. Since $root(\ell|_p) \neq root(\ell')$ and reductions with $\to_{\overleftrightarrow{E}}$ cannot ultimately lead to a root symbol $root(\sigma(\ell')) = root(\ell')$ in the sequence above, π is $\overline{\mathcal{R}^{\rm CR}}$-infeasible. ◀

Regarding $\mathcal{R}^{rm}, E$-joinability of these LCCPS,

- By Proposition 52, the LCCPs (54)–(60); (62)–(67); and (71)–(74) are $\overline{\mathcal{R}^{\rm CR}}$-infeasible, hence $\mathcal{R}^{rm}, E$-joinable.
- (69) is also infeasible as the satisfaction of $\mathsf{Nat}(n)$ by a substitution σ is possible only if $\sigma(n) = \mathsf{s}^p(\mathsf{0})$ for some $p \geq 0$; in this case, $\mathsf{sum}(\sigma(n)) = \mathsf{sum}(\sigma(m') \mathbin{+\!\!+} \sigma(ns'))$, i.e., $\mathsf{sum}(\mathsf{s}^p(\mathsf{0})) = \mathsf{sum}(\sigma(m') \mathbin{+\!\!+} \sigma(ns'))$ does *not* hold in $\overline{\mathcal{R}^{\rm CR}}$.
- The $\mathcal{R}^{rm}, E$-joinability of (70) is discussed in Example 37. The $\mathcal{R}^{rm}, E$-joinability of (53), (61), (68) and (70) is concluded in a similar way.

Thus, every $\pi \in \mathsf{LCCP}(\mathcal{R})$ is $\mathcal{R}^{rm}, E$-joinable.

α	p	α'	LCCP	
(21)	Λ	(21)	$\langle n', n\rangle \Leftarrow 0+n = 0+n'$	(53)
(21)	Λ	(22)	$\langle \mathsf{s}(m'+n'), n\rangle \Leftarrow 0+n = \mathsf{s}(m')+n'$	(54)
(21)	Λ	(23)	$\langle n', n\rangle \Leftarrow 0+n = \mathsf{sum}(n'), \mathsf{Nat}(n')$	(55)
(21)	Λ	(24)	$\langle m'+n', n\rangle \Leftarrow 0+n = \mathsf{sum}(m' +\!\!+\, ns'), \mathsf{Nat}(m'), \mathsf{sum}(ns') \approx_{rm} n'$	(56)
(21)	1	(21)	$\langle n'+n, n\rangle \Leftarrow 0 = 0+n$	(57)
(21)	1	(22)	$\langle \mathsf{s}(m'+n')+n, n\rangle \Leftarrow 0 = \mathsf{s}(m')+n'$	(58)
(21)	1	(23)	$\langle n'+n, n\rangle \Leftarrow 0 = \mathsf{sum}(n'), \mathsf{Nat}(n')$	(59)
(21)	1	(24)	$\langle (m'+n')+n, n\rangle \Leftarrow 0 = \mathsf{sum}(m' +\!\!+\, ns'), \mathsf{Nat}(m'), \mathsf{sum}(ns') \approx_{rm} n'$	(60)
(22)	Λ	(22)	$\langle \mathsf{s}(m'+n'), \mathsf{s}(m+n)\rangle \Leftarrow \mathsf{s}(m)+n = \mathsf{s}(m')+n'$	(61)
(22)	Λ	(23)	$\langle n', \mathsf{s}(m+n)\rangle \Leftarrow \mathsf{s}(m)+n = \mathsf{sum}(n'), \mathsf{Nat}(n')$	(62)
(22)	Λ	(24)	$\langle m'+n', \mathsf{s}(m+n)\rangle \Leftarrow \mathsf{s}(m)+n = \mathsf{sum}(m' +\!\!+\, ns'), \mathsf{Nat}(m'), \mathsf{sum}(ns')$	(63)
(22)	1	(21)	$\langle n'+n, \mathsf{s}(m+n)\rangle \Leftarrow \mathsf{s}(m) = 0+n'$	(64)
(22)	1	(22)	$\langle \mathsf{s}(m'+n')+n, \mathsf{s}(m+n)\rangle \Leftarrow \mathsf{s}(m) = \mathsf{s}(m')+n'$	(65)
(22)	1	(23)	$\langle n'+n, \mathsf{s}(m+n)\rangle \Leftarrow \mathsf{s}(m) = \mathsf{sum}(n'), \mathsf{Nat}(n')$	(66)
(22)	1	(24)	$\langle m'+n'+n, \mathsf{s}(m+n)\rangle \Leftarrow \mathsf{s}(m) = \mathsf{sum}(m' +\!\!+\, ns'), \mathsf{Nat}(m'), \mathsf{sum}(ns')$	(67)
(23)	Λ	(23)	$\langle n', n\rangle \Leftarrow \mathsf{sum}(n) = \mathsf{sum}(n'), \mathsf{Nat}(n), \mathsf{Nat}(n')$	(68)
(23)	Λ	(24)	$\langle m'+n', n\rangle \Leftarrow \mathsf{sum}(n) = \mathsf{sum}(m' +\!\!+\, ns'), \mathsf{Nat}(n), \mathsf{Nat}(m'), \mathsf{sum}(ns') \approx_{rm} n'$	(69)
(24)	Λ	(24)	$\langle m'+n', m+n\rangle \Leftarrow \mathsf{sum}(m +\!\!+\, ns) = \mathsf{sum}(m' +\!\!+\, ns'), \mathsf{Nat}(m),$ $\mathsf{sum}(ns) \approx_{rm} n, \mathsf{Nat}(m'), \mathsf{sum}(ns') \approx_{rm} n'$	(70)
(24)	1	(21)	$\langle \mathsf{sum}(n'), m+n\rangle \Leftarrow m +\!\!+\, ns = 0+n', \mathsf{Nat}(m), \mathsf{sum}(ns) \approx_{rm} n$	(71)
(24)	1	(22)	$\langle \mathsf{sum}(\mathsf{s}(m'+n')), m+n\rangle \Leftarrow m +\!\!+\, ns = \mathsf{s}(m')+n', \mathsf{Nat}(m), \mathsf{sum}(ns) \approx_{rm} n$	(72)
(24)	1	(23)	$\langle \mathsf{sum}(n'), m+n\rangle \Leftarrow m +\!\!+\, ns = \mathsf{sum}(n'), \mathsf{Nat}(m), \mathsf{sum}(ns) \approx_{rm} n, \mathsf{Nat}(n')$	(73)
(24)	1	(24)	$\langle \mathsf{sum}(m'+n'), m+n\rangle \Leftarrow m +\!\!+\, ns = \mathsf{sum}(m' +\!\!+\, ns'), \mathsf{Nat}(m),$ $\mathsf{sum}(ns) \approx_{rm} n, \mathsf{Nat}(m'), \mathsf{sum}(ns') \approx_{rm} n'$	(74)

Figure 2 LCCPs of $\mathcal{R}$ in Example 1.

A.2 $\mathsf{LCCP}(E,\mathcal{R})$ for $\mathcal{R}$ in Example 1

The LCCPs in $\mathsf{LCCP}(E,\mathcal{R})$ for $\mathcal{R}$ in Example 1 are displayed in Figure 3. The following result is useful to analyze their joinability. Here, we say that a set U of conditional rules is *collapsing* if there is a *feasible* rule in U whose right-hand side is a variable.

▶ **Proposition 53.** *Let $\mathcal{R}$ be an EGTRS and $\pi : \langle \ell[r']_p, r\rangle \Leftarrow \ell|_p = \ell', c, c' \in \mathsf{LCCP}(E,\mathcal{R})$. If $\overleftrightarrow{E}$ is* not *collapsing and $root(\ell') \notin \mathcal{D}(\overleftrightarrow{E})$, then π is $\overline{\mathcal{R}^{\text{CR}}}$-infeasible.*

Proof. By definition of LCCP, since $p \in \mathcal{P}os_{\mathcal{F}}(\ell)$ and $\alpha' \in R^{rm}$, we have that $\ell|_p, \ell' \notin \mathcal{X}$. Thus, for all substitutions σ satisfying $\ell|_p = \ell'$, we have $\sigma(\ell|_p) \to^*_{\overleftrightarrow{E}} \sigma(\ell')$. Since $\overleftrightarrow{E}$ is not collapsing and $root(\ell') \notin \mathcal{D}(\overleftrightarrow{E})$, reductions with $\to_{\overleftrightarrow{E}}$ cannot ultimately lead to a root symbol $root(\sigma(\ell')) = root(\ell') \notin \mathcal{D}(\overleftrightarrow{E})$ in the sequence above, π is $\overline{\mathcal{R}^{\text{CR}}}$-infeasible. ◀

The following example shows that non-collapsingness of $\overleftrightarrow{E}$ is necessary for Proposition 53 to hold.

▶ **Example 54.** Consider the following EGTRS

$$0 + x \;=\; x \tag{75}$$

$$\mathsf{f}(0) \;\to\; 0 \tag{76}$$

We have the following LCCP in $\mathsf{LCCP}(E,\mathcal{R})$:

$$\langle x, x\rangle \Leftarrow 0 + x = \mathsf{f}(0) \tag{77}$$

α	p	α'	LCCP	
$\overrightarrow{(1)}$	Λ	(21)	$\langle n, (xs +\!\!+ ys) +\!\!+ zs\rangle \Leftarrow xs +\!\!+ (ys +\!\!+ zs) = \mathsf{0} + n$	(78)
$\overrightarrow{(1)}$	Λ	(22)	$\langle \mathsf{s}(m+n), (xs +\!\!+ ys) +\!\!+ zs\rangle \Leftarrow xs +\!\!+ (ys +\!\!+ zs) = \mathsf{s}(m) + n$	(79)
$\overrightarrow{(1)}$	Λ	(23)	$\langle n, (xs +\!\!+ ys) +\!\!+ zs\rangle \Leftarrow xs +\!\!+ (ys +\!\!+ zs) = \mathsf{sum}(n), \mathsf{Nat}(n)$	(80)
$\overrightarrow{(1)}$	Λ	(24)	$\langle m+n, (xs +\!\!+ ys) +\!\!+ zs\rangle \Leftarrow xs +\!\!+ (ys +\!\!+ zs) = \mathsf{sum}(m +\!\!+ ns),$ $\mathsf{Nat}(m), \mathsf{sum}(ns) \approx_{rm} n$	(81)
$\overrightarrow{(1)}$	2	(21)	$\langle xs +\!\!+ n, (xs +\!\!+ ys) +\!\!+ zs\rangle \Leftarrow ys +\!\!+ zs = \mathsf{0} + n$	(82)
$\overrightarrow{(1)}$	2	(22)	$\langle xs +\!\!+ \mathsf{s}(m+n), (xs +\!\!+ ys) +\!\!+ zs\rangle \Leftarrow ys +\!\!+ zs = \mathsf{s}(m) + n$	(83)
$\overrightarrow{(1)}$	2	(23)	$\langle xs +\!\!+ n, (xs +\!\!+ ys) +\!\!+ zs\rangle \Leftarrow ys +\!\!+ zs = \mathsf{sum}(n), \mathsf{Nat}(n)$	(84)
$\overrightarrow{(1)}$	2	(24)	$\langle xs +\!\!+ (m+n), (xs +\!\!+ ys) +\!\!+ zs\rangle \Leftarrow ys +\!\!+ zs = \mathsf{sum}(m +\!\!+ ns), \mathsf{Nat}(m),$ $\mathsf{sum}(ns) \approx_{rm} n$	(85)
$\overleftarrow{(1)}$	Λ	(21)	$\langle n, xs +\!\!+ (ys +\!\!+ zs)\rangle \Leftarrow (xs +\!\!+ ys) +\!\!+ zs = \mathsf{0} + n$	(86)
$\overleftarrow{(1)}$	Λ	(22)	$\langle \mathsf{s}(m+n), xs +\!\!+ (ys +\!\!+ zs)\rangle \Leftarrow (xs +\!\!+ ys) +\!\!+ zs = \mathsf{s}(m) + n$	(87)
$\overleftarrow{(1)}$	Λ	(23)	$\langle n, xs +\!\!+ (ys +\!\!+ zs)\rangle \Leftarrow (xs +\!\!+ ys) +\!\!+ zs = \mathsf{sum}(n), \mathsf{Nat}(n)$	(88)
$\overleftarrow{(1)}$	Λ	(24)	$\langle m+n, xs +\!\!+ (ys +\!\!+ zs)\rangle \Leftarrow (xs +\!\!+ ys) +\!\!+ zs = \mathsf{sum}(m +\!\!+ ns), \mathsf{Nat}(m),$ $\mathsf{sum}(ns) \approx_{rm} n$	(89)
$\overleftarrow{(1)}$	1	(21)	$\langle n +\!\!+ zs, xs +\!\!+ (ys +\!\!+ zs)\rangle \Leftarrow xs +\!\!+ ys = \mathsf{0} + n$	(90)
$\overrightarrow{(1)}$	1	(22)	$\langle \mathsf{s}(m+n) +\!\!+ zs, xs +\!\!+ (ys +\!\!+ zs)\rangle \Leftarrow xs +\!\!+ ys = \mathsf{s}(m) + n$	(91)
$\overleftarrow{(1)}$	1	(23)	$\langle n +\!\!+ zs, xs +\!\!+ (ys +\!\!+ zs)\rangle \Leftarrow xs +\!\!+ ys = \mathsf{sum}(n), \mathsf{Nat}(n)$	(92)
$\overleftarrow{(1)}$	1	(24)	$\langle (m+n) +\!\!+ zs, xs +\!\!+ (ys +\!\!+ zs)\rangle \Leftarrow xs +\!\!+ ys = \mathsf{sum}(m +\!\!+ ns), \mathsf{Nat}(m)$ $\mathsf{sum}(ns) \approx_{rm} n$	(93)

Figure 3 LCCPs of E and $\mathcal{R}$ in Example 1.

Substitution $\sigma = \{x \mapsto \mathsf{f}(\mathsf{0})\}$ satisfies the conditional part of (77), i.e., it is $\overline{\mathcal{R}^{\text{CR}}}$-feasible.

By Proposition 53, the LCCPs in Figure 3 are all $\overline{\mathcal{R}^{\text{CR}}}$-infeasible, hence right-strict $\mathcal{R}^{rm}, E$-joinable. Thus, every $\pi \in \mathsf{LCCP}(E, \mathcal{R})$ is right-strict $\mathcal{R}^{rm}, E$-joinable.

A.3 DCCP($\mathcal{R}$) for $\mathcal{R}$ in Example 1

The DCCPs in $\mathsf{DCCP}(\mathcal{R})$ for $\mathcal{R}$ in Example 1 are displayed in Figure 4.

α	α'	DCCP	
(21)	(23)	$\langle n, x'\rangle \Leftarrow x = \mathsf{0} + n, (x, x') \rhd^{\times} (\mathsf{sum}(n'), n'), \mathsf{Nat}(n')$	(94)
(21)	(24)	$\langle n, x'\rangle \Leftarrow x = \mathsf{0} + n, (x, x') \rhd^{\times} (\mathsf{sum}(m' +\!\!+ ns'), m' + n'), \mathsf{Nat}(m'),$ $\mathsf{sum}(ns') \approx_{rm} n'$	(95)
(22)	(23)	$\langle \mathsf{s}(m+n), x'\rangle \Leftarrow x = \mathsf{s}(m) + n, (x, x') \rhd^{\times} (\mathsf{sum}(n'), n'), \mathsf{Nat}(n')$	(96)
(22)	(24)	$\langle \mathsf{s}(m+n), x'\rangle \Leftarrow x = \mathsf{s}(m) + n, (x, x') \rhd^{\times} (\mathsf{sum}(m' +\!\!+ ns'), m' + n'), \mathsf{Nat}(m'),$ $\mathsf{sum}(ns') \approx_{rm} n'$	(97)
(23)	(21)	$\langle n, x'\rangle \Leftarrow x = \mathsf{sum}(n), (x, x') \rhd^{\times} (\mathsf{0} + n', n'), \mathsf{Nat}(n)$	(98)
(23)	(22)	$\langle n, x'\rangle \Leftarrow x = \mathsf{sum}(n), (x, x') \rhd^{\times} (\mathsf{s}(m') + n', \mathsf{s}(m' + n')), \mathsf{Nat}(n)$	(99)
(23)	(23)	$\langle n, x'\rangle \Leftarrow x = \mathsf{sum}(n), (x, x') \rhd^{\times} (\mathsf{sum}(n'), n'), \mathsf{Nat}(n), \mathsf{Nat}(n')$	(100)
(23)	(24)	$\langle n, x'\rangle \Leftarrow x = \mathsf{sum}(n), (x, x') \rhd^{\times} (\mathsf{sum}(m' +\!\!+ ns'), m' + n'), \mathsf{Nat}(n), \mathsf{Nat}(m'),$ $\mathsf{sum}(ns') \approx_{rm} n'$	(101)
(24)	(21)	$\langle m+n, x'\rangle \Leftarrow x = \mathsf{sum}(m +\!\!+ ns), (x, x') \rhd^{\times} (\mathsf{0} + n', n'), \mathsf{Nat}(m),$ $\mathsf{sum}(ns) \approx_{rm} n$	(102)
(24)	(22)	$\langle m+n, x'\rangle \Leftarrow x = \mathsf{sum}(m +\!\!+ ns), (x, x') \rhd^{\times} (\mathsf{s}(m') + n', \mathsf{s}(m' + n')),$ $\mathsf{Nat}(m), \mathsf{sum}(ns) \approx_{rm} n$	(103)
(24)	(23)	$\langle m+n, x'\rangle \Leftarrow x = \mathsf{sum}(m +\!\!+ ns), (x, x') \rhd^{\times} (\mathsf{sum}(n'), n'), \mathsf{Nat}(m),$ $\mathsf{sum}(ns) \approx_{rm} n, \mathsf{Nat}(n')$	(104)
(24)	(24)	$\langle m+n, x'\rangle \Leftarrow x = \mathsf{sum}(m +\!\!+ ns), (x, x') \rhd^{\times} (\mathsf{sum}(m' +\!\!+ ns'), m' + n'),$ $\mathsf{Nat}(m), \mathsf{sum}(ns) \approx_{rm} n, \mathsf{Nat}(m'), \mathsf{sum}(ns') \approx_{rm} n'$	(105)

Figure 4 DCCPs for $\mathcal{R}$ in Example 1.

- By Proposition 47, since we have proven above that every $\pi \in \mathsf{LCCP}(\mathcal{R}) \cup \mathsf{CVP}(\mathcal{R})$ is $\mathcal{R}^{rm}, E$-joinable, the DCCPs (94)–(101) are $\mathcal{R}^{rm}, E$-joinable.
- Regarding (102)–(105), notice that all these DCCPs contain the following conditions in the conditional part: (i) $x = \mathsf{sum}(m +\!\!+\, ns)$ and (ii) $\mathsf{Nat}(m)$. Therefore, for all substitutions σ satisfying them, $\sigma(m) = \mathsf{s}^p(\mathsf{0})$ for some $p \geq 0$ and $\sigma(x) = \mathsf{sum}(\mathsf{s}^p(\mathsf{0}) +\!\!+\, \sigma(ns))$. Therefore, Proposition 47 can be applied to conclude $\mathcal{R}^{rm}, E$-joinability of all of them.

Therefore, every $\pi \in \mathsf{DCCP}(\mathcal{R})$ is $\mathcal{R}^{rm}, E$-joinable.