Solving Discontinuous Initial Value Problems with Unique Solutions Is Equivalent to Computing over the Transfinite

Olivier Bournez
École polytechnique, LIX, Paris, France

Riccardo Gozzi
École polytechnique, LIX, Paris, France
University Paris-Est Créteil Val de Marne, LACL, Paris, France

Abstract

We study a precise class of dynamical systems that we call *solvable ordinary differential equations*. We prove that analog systems mathematically ruled by solvable ordinary differential equations can be used for transfinite computation, solving tasks such as the halting problem for Turing machines and any Turing jump of the halting problem in the hyperarithmetical hierarchy. We prove that the computational power of such analog systems is exactly the one of transfinite computations of the hyperarithmetical hierarchy.

It has been proved recently that polynomial ordinary differential equations correspond unexpectedly naturally to Turing machines. Our results show that the more general exhibited class of solvable ordinary differential equations corresponds, even unexpectedly, naturally to transfinite computations. From a wide philosophical point of view, our results contribute to state that the question of whether such analog systems can be used to solve untractable problems (both for complexity for polynomial systems and for computability for solvable systems) is provably related to the question of the relations between mathematical models, models of physics and our real world.

More technically, we study a precise class of dynamical systems: bounded initial value problems involving ordinary differential equations with a unique solution. We show that the solution of these systems can still be obtained analytically even in the presence of discontinuous dynamics once we carefully select the conditions that describe how discontinuities are distributed in the domain. We call the class of right-hand terms respecting these natural and simple conditions the class of *solvable* ordinary differential equations. We prove that there is a method for obtaining the solution of such systems based on transfinite recursion and taking at most a countable number of steps. We explain the relevance of these systems by providing several natural examples and showcasing the fact that these solutions can be used to perform limit computations and solve tasks such as the halting problem for Turing machines and any Turing jump of the halting problem in the hyperarithmetical hierarchy.

2012 ACM Subject Classification Mathematics of computing → Ordinary differential equations; Computer systems organization → Analog computers; Theory of computation; Theory of computation → Models of computation; Theory of computation → Computability

Keywords and phrases Analog models, computability, transfinite computations, dynamical systems

Digital Object Identifier 10.4230/LIPIcs.STACS.2024.20

Funding *Olivier Bournez*: Partially supported by ANR Project ∂IFFERENCE.
Riccardo Gozzi: Partially supported by ANR Project ∂IFFERENCE.

41st International Symposium on Theoretical Aspects of Computer Science (STACS 2024).
Editors: Olaf Beyersdorff, Mamadou Moustapha Kanté, Orna Kupferman, and Daniel Lokshtanov; Article No. 20; pp. 20:1–20:19
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

1 Introduction

It has been understood quite recently that it is possible to program with Ordinary Differential Equations (ODEs): for any discrete model such as Turing machines, it is possible to build a polynomial ODE that simulates its evolution. This possibility of programming with ODEs has already been exploited to obtain various results and solve various open problems. Examples include: characterization of computability and complexity classes using ODEs [22, 21, 30, 17]; proof of the existence of a universal (in the sense of Rubel) ODE [6]; proof of the strong Turing completeness of biochemical reactions [15], or more generally various statements about the completeness of reachability problems (e.g. PTIME-completeness of bounded reachability) for ODEs [29].

Most of these studies and conclusions originated as a side effect of attempts to relate the computational power of analog computational models to classical computability. Indeed all these results relate classical computations, such as computation with Turing machines, to polynomial ordinary differential equations. One fascinating question is to understand whether it could be possible to formulate stronger models than Turing machines using classes of ODEs that are more general than polynomial ODEs. The question is interesting both from a computability point of view (can we compute more?) and a complexity point of view (can we compute faster?). In general, this investigation has pertinence in the broader context of analog computation, or computation by various alternative models, and is not limited to the framework of ODEs.

▶ Remark 1 (A parallel that may help). If this helps, our discussions can be put in parallel with the context of quantum computations, often better known than the context of analog computations. Quantum computations are mathematically based on computation over the continuum using complex numbers. Quantum models can solve some problems faster than digital models: Grover's algorithm is corroborative evidence for such an argument. This seems to suggest that models of computations based on the continuum might have additional power compared to discrete ones. Is it true for the case of models based on polynomial ODEs? The answer is clearly no with polynomial ODEs if the point of view is computability. However, from the results of [30, 29], this is more subtle and related to lengths of solutions when considering time complexity. Is it true for the case of ODEs that are more general than polynomial ones? In this article, we prove that it is possible to solve undecidable problems with discontinuous (hence non-polynomial) ODEs: we prove that it is possible to simulate transfinite computations with some ODEs.

▶ Remark 2 (Is this "realistic"? Can considered results be used "in practice"?). It is important to realise that this relates to deep philosophical questions about the relations between mathematical models, physics and our real world. For example, are mathematical models of ODEs capturing the dynamics of our physical world? Are models of physics related to our physical world? We do not aim to discuss this. However, we point out that these questions are already present for any alternative model of computation. Using the above parallel, we mean: the statement about Grover's algorithm above is that the mathematical model, considered by Grover and others, based on quantum postulates, can solve a problem faster than by digital means. But then, today's question of constructing a quantum computer can be seen as whether this mathematical model can be implemented in our real world and hence whether this mathematical model is relevant. We try to avoid these questions as much as possible in the current article, and we stay at a mathematical model level. We however think that our statements help to discuss these issues and how various models relate to our physical world.

▶ Remark 3 (So what is our positioning). The point of the current article is to consider a mathematically well-founded natural notion of ordinary differential equation in a context where there is a (unique, hence unambiguous) solution. We then explore its computational power and relate it to models from computability. From a mathematical point of view, we prove that it corresponds exactly to transfinite computations.

Indeed, we identify such a robust class of ODEs as the class whose dynamic is ruled by functions that we call *solvable functions*. We prove that for such a class, the solution concept is well defined, and a transfinite procedure can solve these systems. To demonstrate that, we describe the transfinite procedure in detail revealing that the maximum number of transfinite steps needed is countable. This result is expressed clearly by our main result in Section 5. Moreover, for each countable ordinal $\alpha < \omega_1$ (or $\alpha < \omega_1^{CK}$ if effectivness is involved, where ω_1^{CK} is the first non-recursive ordinal), we show that is possible to construct examples of discontinuous IVPs with unique solution whose solution can be obtained only after α steps of our procedure. This suggests that, according to the spirit of the approach of [3], this class of IVPs can be used in order to simulate oracle machines deciding the α-jump of the empty set, fully populating the hierarchy of hyperarithmetical reals.

More on some historical accounts and related work

About Denjoy's totalization method for integration. It is clear that ODE solving and integrations are related since integrating is a particularly simple (restricted) case of ODE solving where the derivative is given explicitly. The question we solve has great similarities with a historical question in the context of mathematics about antidifferentiation and integration: does a method exist that can reconstruct a function f from its derivative f' in the most general setting? Unfortunately, two of the most well-known integration methods, the Riemann and Lebesgue integrals, are insufficient since they both require specific conditions on the derivative to work. Historically, Denjoy was the first to propose a concept of integral that extends the two and that is sufficiently general to solve the problem: starting from f', using some transfinite process, one can find back f for any derivative f'. He called the method describing his integral the *totalization* method [12]. The method was purely rooted on analysis and made use of transfinite iterations of operations such as taking limits and repeated Lebesgue integrations. Starting from the derivative f', the method can retrieve f within a maximum number of countably many transfinite steps.

Our method for solving ODEs can be related to the ideas of Denjoy, and our class of solvable ODEs have similar properties: solving such systems of ODEs can always be done at the price of a transfinite computational process. And as such ODEs can simulate any transfinite computation, they capture transfinite computations and relate transfinite computations over digital models to computations with analog models that use solvable dynamics.

ODEs as analog model of computation. From Shannon's model to (polynomial) ODEs. The idea of using ordinary differential equations (ODEs) as a computational model dates back to the original work of Claude Shannon. Shannon's theoretical interest was focused on defining a general model of computation that could describe the behaviour of integrator devices. He called the model GPAC, for general purpose analog computer. The key element immediately evident to Shannon in designing his model was that every function that can be produced as output of these machines is differential-algebraic [36]. It was therefore the first stone that led years later to interpret systems of polynomial ODEs as a proper analog computational model. The details of this evolution from algebraic differential equations to polynomial ODEs are articulate and technical: in [31], it has been demonstrated that Shannon's model was lacking of completeness and formality and hence required modifications. The authors of [20] solved these problems by restricting the connections allowed within the circuits described in the model. This modification naturally produced the interesting phenomenon of restricting the class of considered GPAC models, and the proof that its dynamics correspond precisely to solutions of polynomial ODEs.

What is known about polynomial ODEs: lower bounds. This was later proved to be equivalent to computable analysis, generating all computable functions over the reals [3]. This result was important to establish a practical bridge that could be crossed to pass from an analog model such as the GPAC to a discrete model such as the model of Turing machines. Specifically, the proofs included in [3] were based on the idea of simulating Turing machine computations by only using initial value problems (IVPs) constructed with polynomial ODEs. These particular results opened the doors for further investigations into the complexity of the GPAC model. It was subsequently discovered that the length of the solution of the ODEs involved was the right parameter to consider to measure complexity [4].

This correspondence between length and complexity can be effectively used to define a zoo of different complexity classes within the model, capturing for each of them a natural equivalence with discrete time-complexity classes such as FP or FEXP [4], [18]. The introduction of proper robustness conditions for the dynamical systems utilized to simulate Turing machines was then the last missing ingredient, which proved to be enough also to capture an equivalence with the polynomial-space-complexity class FPSPACE, as shown in [5]. The collection of all these results forms wide and substantial evidence of the fact that polynomial ODEs represent a valid paradigm for analog computation as well as showcasing their Turing completeness.

What is known about ODEs: upper bounds. Computing properties and solutions of ODEs. On the other side of the spectrum, various investigations have been conducted to outline computability and complexity properties of the more general operation of ODEs solving. The approach from this line of work is conceptually different from what we have discussed so far. Instead of producing a continuous version of an originally discrete computation, it wonders which classes of ODEs solutions can be solved algorithmically. Undecidability of related problems is quick to arise in this context even in the presence of computable data. For example, the authors of [19] proved that the boundedness of the domain of definition is undecidable even when only polynomial ODEs are considered. For the class of polynomial-time computable, Lipschitz continuous ODEs, it is known that the solution is computable since [28]. In this specific realm, a careful analysis of the complexity of this operation has been conducted in [25], where it is proved that the solution of such problems is indeed PSPACE-complete. Following the clue provided by the Lipschitz condition, it is reasonable to assume that the uniqueness of the solution of a given IVP is a necessary prerequisite for hoping to be able to compute it. It is not a sufficient condition, as there are cases of IVPs with a unique solution and computable data for which the unique solution is not computable, such as the one in [32]. The authors have further investigated the gap between necessary and sufficient conditions for these systems in [9], where they show that solutions of continuous ODEs with unique solutions are always computable. The algorithm formulated in [9] has been called *Ten Thousand Monkeys algorithm*, since it relies on a search method on the whole solution space by listing all finite sequences of open rational boxes.

Notations

We start the coming technical part by introducing the notation and the main definitions that will be used throughout this work. We use the standard notation $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{Q}$ for the set of natural, real and rational numbers respectively while $\mathbb{R}^+$ and $\mathbb{Q}^+$ represent the positive reals and the positive rationals. When making use of the norm operator, we always consider Euclidean norms. We refer to a compact domain of an Euclidean space as a nonempty, bounded, connected, closed subset of such space. Given a metric space X we indicate with the notation d_X the distance function in such space and with the notation $B_X(x, \delta)$ the open ball centred in $x \in X$ with radius $\delta > 0$. By default, we describe as open *rational ball* or as open *rational box* an open ball or box with rational parameters. Precisely, an open rational box B is a set of the form $(a_1, b_1) \times \ldots \times (a_r, b_r) \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ where $a_i, b_i \in \mathbb{Q}$ for $i = 1, \ldots, r$. We indicate with the notations $\mathrm{diam}(B)$ and $\mathrm{rad}(B)$ its diameter and its radius. Moreover, given a function $f : [a, b] \to \mathbb{R}^r$ for some $a, b \in \mathbb{R}, a < b$ and some $r \in \mathbb{N}$ we indicate with the notation $f' : [a, b] \to \mathbb{R}^r$ the derivative of such function, where the derivative on the extremes a and b is defined as the limit on the left and on the right respectively. Given a function $f : X \to Y$ and a set $K \subseteq X$ we indicate with the notation $f\restriction_K$ the restriction of the function f to the set K, i.e. $f\restriction_K$ is the function from K to Y defined as $f\restriction_K(x) = f(x)$. Given two topological spaces X and Y and a function $f : X \to Y$ we indicate with the notation D_f the set of discontinuity points of f on X. If A and B are two sets, we refer to the set difference operation using the symbol $A \setminus B$ and indicate with the notation $A + B$ the Minkowski sum of set A with set B. The expression $\mathrm{cl}(A)$ indicates the closure of A, $\emptyset$ stands for the empty set, while the notation ω_1 stands for the first uncountable ordinal number. Given a property of a function $f : X \to Y$, we say that this property is satisfied *almost everywhere* if the property is satisfied on $X \setminus D$, where D is a set with Lebesgue measure equal to zero.

2 IVPs with discontinuous ODEs

In this section we formally present the class of IVPs and dynamical systems considered, providing examples and motivations leading to the definition of our hypothesis over the right-hand term of the ODEs involved.

First, we recall the classical settings of initial value problems (IVPs) and ordinary differential equations (ODEs): Consider an interval $[a, b] \subset \mathbb{R}$, a compact domain $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$, a point $y_0 \in E$ and a function $f : E \to E$ such that the dynamical system:

$$\begin{cases} y'(t) = f(y(t)) \\ y(a) = y_0 \end{cases} \tag{1}$$

has one unique solution $y : [a, b] \to \mathbb{R}^r$ with $y([a, b]) \subset E$. Given y_0 and f, obtaining the solution in such a setting is called an *initial value problem*. The condition $y(a) = y_0$ (or, in short, just the point y_0) is referred to as the initial condition of the problem and function f is referred to as the right-hand term of the problem. Since the solution is uniquely defined in this case, we refer to function $y : [a, b] \to E$ satisfying Equation (1) as the solution of the problem.

Then, we discuss methods to solve initial value problems: In this particular setting, different ways exist to obtain the solution analytically when the right-hand term is continuous. Many of these methods, such as building Tonelli sequences, are often introduced for proving Peano's theorem related to the existence of the solution for IVPs with continuous right-hand terms and are based on the concept of defining sequences of continuous functions eventually converging to the solution. Once it is known that the solution is unique, every sequence considered in each of these methods can be shown to converge to the unique solution. Their argument generally relies on fixed point theorems on the function space; therefore, it is not constructive. An analysis based on these methods can also achieve computability for the solution, where computability has to be intended in the sense of computable analysis, as proved by the authors of [9] in the description of their (so-called) *ten thousand monkeys algorithm*. The idea of this algorithm is to exploit the hypothesis of unicity for enclosing the solution into covers of arbitrarily close rational boxes in E. Nonetheless, all these methods are correctly functioning as long as the right-hand term of the IVP is **continuous**.

It is very natural to relax continuity for the right-hand term of the IVP. This is motivated by the observation that we can easily have a discontinuous IVP with a unique solution as in the coming example.

▶ **Example 4** (A discontinuous IVP with a unique solution). As the simplest case of discontinuous IVP, we consider the following example: let $E = [-5, 5] \times [-15, 15]$ and define the function $f : E \to E$ as $f(x, z) = (1, 2x \sin \frac{1}{x} - \cos \frac{1}{x})$ if $x \neq 0$ and $f(0, z) = (1, 0)$. It is easy to see that f is a function of class Baire one, i.e. it is the pointwise limit of a sequence of continuous functions. Note also that in this case the set of discontinuity points of function f on E is the closed set $D_f = \{(0, z)$ for $z \in [-15, 15]\}$. Then consider the following IVP, with $y : [-2, 2] \to \mathbb{R}^2$ and $y_0 = (-3, 9 \sin(-\frac{1}{3}))$:

$$\begin{cases} y'(t) = f(y(t)) \\ y(-2) = y_0 \end{cases} \tag{2}$$

It is easy to verify that the solution of such a system is unique, and it is for the first component: $y_1(t) = t - 1$, and for the second component $y_2(t) = (t-1)^2 \sin(\frac{1}{t-1})$ for $t \neq 1$ and $y_2(1) = 0$. Therefore the solution $y : [-2, 2] \to \mathbb{R}^2$ is differentiable and can be expressed as the unique solution of the IVP above with right-hand side f discontinuous on E. Note that the only discontinuity of f encountered by the solution is the point $(0, 0)$, i.e. $D_f \cap y([-2, 2]) = (0, 0)$.

▶ Remark 5 (Such an ODE cannot be solved using numerical methods from literature). It is very important to stress that neither the ten thousand monkeys algorithm nor any (at least that we know) of the general well-known methods used in analysis for obtaining the solution of IVPs we know works when applied to the above example. The reason for it being the discontinuity of the right-hand term on a straight line in the domain and the fact that these methods assume continuity.

However, one would expect to be able to solve such an ODE, as its solution is very clear, and solving such an ODE is a very classical mathematical exercise or example found in most of the books about ODEs: see similar examples in [23].

The construction of this simple example (and of the solution of this classical exercise) is based on the well-known fact that the real function $f(x) = x^2 \sin(\frac{1}{x})$ if $x \neq 0$ and $f(0) = 0$ is differentiable over $[0, 1]$ and its derivative is bounded and discontinuous in 0. Moreover, we avoided some problems that arise for mono-dimensional ODEs with null derivative by introducing a *time* variable y_1 whose role is to prevent the system from stalling and ensure the unicity of the solution.

▶ Remark 6 (About literature & discontinuous ODEs). Several mathematical theories exist for discussing discontinuous ODEs: see for example [16, 1, 11]. It is important to realize that the concept of solution differs from one theory to the other (there is not a unique theory for discontinuous ODEs) and that the existence of a solution is often a non-trivial problem in all these theories. We are here, and in all the examples, in the case where we know that there is a solution and a unique solution, so in a case where there is no ambiguity and agreement about the solution concept. Furthermore, all the theories we know consider that equality almost everywhere in (1) is sufficient, mostly to be able to use Lebesgue integration. We indeed consider the notion of solution above: this must hold for all points. This is different in spirit to all these theories[1].

The concept behind such an example can be easily generalized. The following example provides an intuitive tool that can be deployed to construct a huge class of discontinuous IVPs with unique solutions.

▶ **Example 7** (Converting complex derivatives into complex IVPs)**.** Whenever we consider a differentiable function $g : [a, b] \to \mathbb{R}$ such that $g(a) = g_0$ and with derivative $g' : [a, b] \to \mathbb{R}$ we can obtain such function as a solution of an IVP of the type of (1) by constructing a system as the following:

$$\begin{cases} y_1'(t) = 1 \\ y_2'(t) = g'(y_1(t)) \end{cases} \qquad \begin{cases} y_1(a) = a \\ y_2(a) = g_0 \end{cases} \tag{3}$$

Getting to more and more complex examples. This consideration allows us to construct examples for which the set of discontinuity points of the right-hand term on the domain is more and more sophisticated.

For instance, we might consider a function whose set of discontinuity points is uncountable and nowhere dense. This case is considerably more complex to construct compared to the previous one, from a technical standpoint, but it is theoretically based on the same concept. Indeed, the idea is to use the discontinuous derivative seen in the previous case and copy it inside the Cantor set. This is done similarly to what happens when defining Volterra's function [7]. We first make use of the following statement.

▶ **Lemma 8.** *There exists a function $g : [0, 1] \to \mathbb{R}$ such that g is differentiable and its derivative g' is bounded and discontinuous on the Cantor set C.*

At this point, by using function g' just defined as the derivative involved with the construction of an IVP of the type of (3), we can construct an IVP with a right-hand term f with set of discontinuity points homeomorphic to the Cantor set.

More generally, the above technique makes it possible to introduce solutions that are more and more complicated depending on how discontinuous is the derivative used as function g' in (3). There are at least two possible directions in which the latter could be done.

First, since every uncountable closed subset of the Cantor set is homeomorphic to the Cantor set, it is possible to construct the differentiable function g in such a way that the restriction of its derivative to the Cantor set, i.e. $g' \restriction_C$ is discontinuous on an uncountable closed subset of C. This sets the basis for iterating the procedure any infinitely countable number of times due to homeomorphism.

Second, we can construct examples using the known possible complexity of a differentiable function. Several *differentiabiliy ranks* for measuring descriptive complexity of differentiable functions have been introduced in the literature. These ranks can be used for the purpose of

[1] And this also explains that we are closer to the question of Denjoy, which was asking about antidifferentiation of a derivative, observing that restricting to Lebesgue's integration was not solving the problem in the general case. Notice that ODE solving is a more general problem than integration, so we are also not exactly in the framework of Denjoy.

providing in a structured way more and more complex functions to play the role of function g in (3): indeed, several different differentiable ranks have been proposed in literature, such as the Kechris-Woodin rank or the Zalcwasser rank, which can be found in [26] and [38] respectively. The relations between all the existing notions is not yet fully clear, and [34] and [27] detail interesting comparisons. However, independently of which of these two routes is taken, the key concept behind the creation of highly elaborate examples is the same: one elementary discontinuous derivative (such as the one in Example 4) is used as a building block that is then rescaled and concatenated into smaller and smaller intervals converging to a new discontinuity point. It is then clear that the nature of the discontinuity on this point of the new function obtained this way would be strictly more complex than the nature of the discontinuities featured in the example used as an elementary block. Then, by using the function obtained this way as a new elementary block to rescale and concatenate, the process can continue in a fractal-like iterative fashion.

Therefore, our study establishes the premises for ranking discontinuous IVPs depending on the complexity required to solve them and foretells the development of a related hierarchy. Up to that point, all examples were mostly obtained from integrating various derivatives. However, ODE solving is a more general problem than integration, and more complicated examples can be constructed. We now go in particular to constructing an ODE that solves the halting problem of Turing machines[2].

3 Undecidability: solving the halting problem with discontinuous IVPs

We now show how these dynamical systems can be used for the purpose of obtaining precise undecidability results: given a Gödel enumeration of Turing machines, we define the halting problem as the problem of deciding the halting set $H = \{e : M_e(e) \downarrow\}$ where $M_e(e) \downarrow$ means that machine represented by natural e halts on input e. We consider a one-to-one total computable function over the naturals $h : \mathbb{N} \to \mathbb{N}$ that enumerates such a set. It is known that any such function enumerating a noncomputable set naturally generates a noncomputable real number [33]. The following definition expresses this:

▶ **Proposition 9.** *Let $h : \mathbb{N} \to \mathbb{N}$ be a one-to-one computable function such that $h(i) > 0$ for all $i \in \mathbb{N}$ and such that it enumerates a non-computable set A. Then the real number μ defined as:*

$$\mu = \sum_{i=0}^{\infty} 2^{-h(i)} \tag{4}$$

is noncomputable.

Note that in this way we always have $0 < \mu < 1$. We now describe a bidimensional dynamical system that generates the real number μ associated in the sense of the definition above to function h enumerating the halting problem.

The example mentioned above of the IVP that can assume value μ is illustrated by the following theorem:

▶ **Theorem 10.** *Let $E = [0,5] \times [0,5]$. There exists an IVP with unique solution $y : [0,5] \to E$, rational initial condition and right-hand term computable everywhere on E except a straight-line, with $y_2(5) = \mu$.*

[2] We believe this is not feasible using integration only.

▶ Remark 11. The spirit of the construction of such an example is inspired by the technique used in [19], where the solution of the IVP considered is stretched in a controlled manner so that it grows infinitely approaching a fixed noncomputable time. In our case instead, with the above example, we are replacing their indefinite growth with a dumped oscillation whose frequency increases as we approach the noncomputable target but whose absolute value decreases accordingly, yielding a finite convergence for the solution. This introduces many complications, and the fact that we want to guarantee the derivability of the solution is a true difficulty.

The sketch of the proof of the theorem above is the following. We first discretize time by introducing specific time slots in which both components of the solution, y_1 and y_2, have a well-defined behaviour. Specifically, we require the first component, which is negative, to increase by a factor of 2 in each of these time slots, converging to zero. Instead, the second component, which is positive, is required to incrementally converge to the real μ by adding to itself the quantity $2^{-h(i)}$ on the i-th time step. We need two components because we want the right-hand term to be computable outside its set of discontinuity points. This is achievable in this way since indeed it is possible to implement the correct derivative for y_1 in each time slot by only going through the enumeration described by function h while looking at the value of the second component y_2. Then, the existence and continuity of the solution is granted by designing an infinitely countable sequence of time slots that converges suitably.

We first define the function that represents the discretized time evolution of the dynamical system:

▶ **Definition 12.** *Let $h : \mathbb{N} \to \mathbb{N}$ be a one-to-one computable function such that $h(i) > 0$ for all $i \in \mathbb{N}$ and such that it enumerates the halting set H. Define the function $\tau : \mathbb{N} \to \mathbb{Q}$ to be the total computable function such that:*

$$\tau(i) = \begin{cases} 2^{-\frac{h(i)}{2}} & \text{if } h(i) < i \\ 2^{-\frac{i}{2}} & \text{if } h(i) \geq i \end{cases} \tag{5}$$

That means that $\tau^* = \sum_{i=0}^{\infty} \tau(i)$ is finite and $\tau^* < \mu + 2 + \sqrt{2} < 5$. This quantity τ^* represents the time required for the solution to reach the noncomputable value μ.

▶ Remark 13. The reason for measuring time steps with Definition 12, instead of directly exploiting the construction of μ via Proposition 9, is technical. The intuition behind it is that we want time to evolve slowly enough when compared to the increasing rate of the solution. This consideration takes care of the construction's main difficulty: the solution's differentiability at time τ^*, when the derivative is discontinuous.

Let us now proceed to analyze the behaviour of the solution y. For the first component y_1 we have a dynamic given by a function f_1 such that, for all $i \in \mathbb{N}$, if we have:

$$\begin{cases} y_1'(t) = f_1(y_1(t)) & \forall t \in [0, \tau(i)] \\ y_1(0) = -2^{-i} \end{cases} \tag{6}$$

then we have $y_1(\tau(i)) = -2^{-(i+1)}$. In other words, we require y_1 to be an increasing function such that at every time step $\tau(i)$ its value increases by a factor of 2, converging then to 0 as time converges to τ^*. Therefore, to make it continuous, we require $y_1(\tau^*) = 0$. Moreover, we define $y_1'(\tau^*) = 0$. Note also that to achieve this goal we require f_1 to be autonomous, with no explicit dependence on time. It is clear that if we construct y_1 this way, then its derivative will be discontinuous in τ^*.

For the second component y_2 we have a dynamic given by a function f_2 such that, for all $i \in \mathbb{N}$, if we have:

$$\begin{cases} y_2'(t) = f_2(y_1(t)) & \forall t \in [0, \tau(i)] \\ y_1(t) \in [-2^{-i}, -2^{-(i+1)}) & \forall t \in [0, \tau(i)] \\ y_2(0) = \sum_{m=0}^{i} 2^{-h(m)} \end{cases} \tag{7}$$

then we have $y_2(\tau(i)) = \sum_{m=0}^{i+1} 2^{-h(m)}$ (condition $*$). In other words, we require y_2 to be an increasing function such that at every time step $\tau(i)$ its value increases of the quantity $2^{-h(i+1)}$, converging then to μ as time converges to τ^*. Once again, we require $y_2(\tau^*) = 0$ and $y_2'(\tau^*) = 0$ and it is clear that if we construct y_2 this way then its derivative will be discontinuous in τ^*.

Such a solution y is indeed differentiable at time τ^* and the derivative of both components exists and equals zero at such time. Moreover, we have that $y_1(\tau^*) = 0$ and $y_2(\tau^*) = \mu$. At this point, forcing the dynamic to remain constant for the remaining time to obtain $y_2(5) = \mu$ is sufficient. This yields the desired outcome since the IVP has reached, at a computable time, a noncomputable value that encodes the halting problem. This ends the sketch of the proof of Theorem 10.

▶ Remark 14. Even if this is not fully formal: it is important to observe that the obtained function f remains very "simple" from a (possibly effective) descriptive point of view. Outside its straight-line of discontinuity, it is computable, and from the fact that Turing machines computations can be done using polynomial ODEs, we could even assume that it has a very simple form outside this straight-line: we basically only need to get condition (*).

We mean, it is important to realize that the fact that the solution is not computable does not come from the intrinsic uncomputability of the function f, nor from the form of the subset of discontinuities, but actually from the fact that it has a discontinuity and that the whole process and above construction is intrinsically forcing the solution to compute a limit.

Iterating limits. For a set A, let us call real number $\sum_{i \in A} 2^{-i}$ the real encoding of A. We just described a dynamics that maps some rational initial condition to the real μ encoding the halting set H. Writing A' for the jump of set A, H corresponds to $\emptyset'$.

It is possible to extend the previous construction to make it work for any set A (not only the empty set): starting from some initial condition corresponding to the real encoding of A, it eventually reaches the real encoding of A'.

We can climb the arithmetical hierarchy by iterating finitely many times this technique. Indeed, by repeating the ODE twice, we get a way to map a real encoding of A to a real encoding of A'', then A''', and so on.

We can even go up to higher levels. Indeed, for example, we can go up to A^ω: A^ω is the set of the pairs (n, w) such that word w is in the nth jump of A. Iterating the trick, we can reach the upper levels of the hyperarithmetical hierarchy. Given any recursive ordinal $\alpha < \omega_{CK}^1$, this provides a technique to map some real encoding of A in the initial condition to the encoding of A^α, where A^α is the αth jump: see [35] for the concepts from computability theory involved. Doing it in a recursive manner requires technically to deal with the encoding of ordinals, and in particular to deal with fixed point constructions as in [13, 37].

4 Transfinite analysis of complexity

4.1 The concept of solvable system

We propose the following definition.

▶ **Definition 15** (Solvable function). *Let $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ and $f : E \to \mathbb{R}^r$. We say that f is* solvable *if it is a function of class Baire one such that for every closed set $K \subseteq E$ the set of discontinuity points of the restriction $f \restriction_K$ is a closed set.*

It is important first to make the following observation:

▶ Remark 16. All the examples of dynamical systems discussed in Section 2 as well as the IVP introduced by Theorem 10 have a right-hand term that is a solvable function.

We say that a dynamical system or some ordinary differential equation is *solvable* when this holds: the right-hand term of the ordinary differential equation is a solvable function.

It is easy to see that the remarks still apply to the case of any of the more complicated examples that can be constructed based on the previous ones and on Theorem 10, once the techniques for building such examples are the ones mentioned at the end of Section 2.

The choice of the terminology *solvable* for these right-hand terms is made more clear by the coming Theorem 22. It says basically that solutions of such initial value problems can be solved, through a transfinite recursion process.

4.2 A ranking for solvable systems

Before getting to this, we introduce a ranking that allows us to quantify involved levels of discontinuities.

We have seen in previous sections that we can build examples of unique solutions of IVPs that are extremely complicated and that even simple examples can be used to obtain noncomputable reals. We now want to produce a more precise quantification of these statements.

▶ Remark 17 (Using differentiability ranks?). Following previous arguments, the most intuitive direction seems to derive such quantification directly from the examples and the differentiability ranks. Nonetheless, this approach does not suit our purpose, since it only characterizes a limited subclass of systems, i.e. the systems yielded by application of the trick introduced with (3). Despite being a relevant and insightful subclass, this approach "from below" fails to exhaust the generality of the problem. We instead propose an approach "from above" which builds from the commonalities of those examples and extends beyond them to a more complete analysis that is not just tailored on derivatives defined over the reals.

As a first step, we prepare the right setting for a transfinite classification of the right-hand terms of our systems. As it is clear by the examples illustrated in previous sections, such stratification should be based upon the degree of discontinuity for the right-hand term of the system. We can quantify this precisely by introducing the following definition.

▶ **Definition 18** (Sequence of f-removed sets on E). *Consider a compact domain $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ and a function $f : E \to \mathbb{R}^r$. Let $\{E_\alpha\}_{\alpha<\omega_1}$ be a transfinite sequence of sets and $\{f_\alpha\}_{\alpha<\omega_1}$ a transfinite sequence of functions such that $f_\alpha = f \restriction_{E_\alpha} : E_\alpha \to \mathbb{R}^r$ defined as following:*

- *Let $E_0 = E$*
- *For every $\alpha = \beta + 1$, let $E_\alpha = D_{f_\beta}$*
- *For every α limit ordinal, let $E_\alpha = \cap_\beta E_\beta$ with $\beta < \alpha$*

we call the sequence $\{E_\alpha\}_{\alpha<\omega_1}$ the sequence of f-removed sets *on E.*

We remark that since functions in the sequence $\{f_\alpha\}_{\alpha<\omega_1}$ above are allowed to be defined over disconnected sets, the notion of continuity in the above definition has to be intended with respect to the induced topology relative to E_α as a subset of $\mathbb{R}^r$. Moreover, note that it follows from the definition of the sequence that such sequence is decreasing, meaning that $E_\delta \subseteq E_\gamma$ if $\delta > \gamma$ for every E_δ and E_γ in the sequence.

▶ Remark 19 (Similar ranking for measuring discontinuities in literature). The definition of this sequence, or of slight variations of the same sequence, has already been considered in the literature. For instance, the author of [24] selects a version of this sequence where the closure of the sets is taken at each level and relates such sequence of functions with a bound for the topological complexity of any algorithm that computes them while using only comparisons and continuous arithmetic (and information) operations. Similarly, starting from the same transfinite sequence of functions applied to countably based Kolmogorov spaces, it is shown in [10] that a given function is at the α level of the hierarchy if and only if it is realizable through the α-jump of a representation.

Since computing the unique solution of continuous IVP is always possible, from the arguments of [9], being able to obtain analytically the unique solution of any given discontinuous IVP should be directly related to the amount of discontinuity for the right-hand term f, and consequently to the ordinal number of nonempty levels of the above sequence of f-removed sets. Moreover, we would like to obtain the solution within a countable number of steps. Hence, we want to pinpoint some sufficient conditions on f that permit us to restrict our attention to these well-behaved classes of discontinuous systems.

This ranking turns out to provide a way to rank the concept of *solvable* systems. Using the Cantor-Baire stationary principle, we can prove:

▶ **Theorem 20.** *Consider a closed domain $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ and a function $f : E \to \mathbb{R}^r$. If f is solvable, then there exists an ordinal $\alpha < \omega_1$ such that $E_\alpha = \emptyset$.*

Once we have singled out which conditions we must require for the right-hand term f, we can present the main tool used to converge to the solution of the IVP. In a similar fashion to the method designed by Denjoy, where the tool to be repeatedly applied was Lebesgue integration, we need to be able to apply such tool for each considered level of the sequence of f-removed sets on E until we finally reach the empty set. This is why we created a tool that can be defined for any countable ordinal in a uniform manner. We call the tool (α)Monkeys approach in honor of the ten thousand monkey algorithm from [9], since such an algorithm inspires the definition.

▶ **Definition 21** ((α)Monkeys approach). *Consider an interval $[a, b] \subset \mathbb{R}$, a domain $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ and a right-hand term $f : E \to \mathbb{R}^r$ for an ODE of the form of (1) with initial condition y_0. Let $\{E_\gamma\}_{\gamma<\omega_1}$ be the sequence of f-removed sets on E and let E_α be one set in the sequence for some $\alpha < \omega_1$. We call the* (α)Monkeys approach for (f, y_0) *the following method: consider all tuples of the form $(X_{i,\beta,j}, h_{i,\beta,j}, B_{i,\beta,j}, C_{i,\beta,j}, Y_{i,\beta,j})$ for $i = 0, \dots, l-1$, $\beta < \alpha$, $j = 1, \dots, m_{i,\beta}$, where $h_{i,\beta,j} \in \mathbb{Q}^+$, $l, m_i \in \mathbb{N}$ and $X_{i,\beta,j}$, $B_{i,\beta,j}$, $C_{i,\beta,j}$ and $Y_{i,\beta,j}$ are open rational boxes in E. A tuple is said to be valid if $y_0 \in \bigcup_{\beta,j} X_{0,\beta,j}$ and for all $i = 0, \dots, l-1$, $\beta < \alpha$, $j = 1, \dots, m_{i,\beta}$ we have:*

1. *Either $(B_{i,\beta,j} = \emptyset)$ or $(\mathrm{cl}(B_{i,\beta,j}) \cap E_\beta \neq \emptyset$ and $\mathrm{cl}(B_{i,\beta,j}) \cap E_{\beta+1} = \emptyset)$*
2. *$f \restriction_{E_\beta} (\mathrm{cl}(B_{i,\beta,j})) \subset C_{i,\beta,j}$;*
3. *$X_{i,\beta,j} \cup Y_{i,\beta,j} \subset B_{i,\beta,j}$;*
4. *$X_{i,\beta,j} + h_{i,\beta,j} C_{i,\beta,j} \subset Y_{i,\beta,j}$;*
5. *$\bigcup_{\beta,j} Y_{i,\beta,j} \subset \bigcup_{\beta,j} X_{i+1,\beta,j}$;*

We spend a few words explaining the rationale behind such a definition. Similarly to the case of the ten thousand monkey algorithm, this definition defines a search method within the space E where the solution of the IVP lives. The search is performed by considering tuples of the form $(X_{i,\beta,j}, h_{i,\beta,j}, B_{i,\beta,j}, C_{i,\beta,j}, Y_{i,\beta,j})$ which describe finite sequences of l open sets. Each of these tuples should be thought as an expression of its related finite sequence of l open sets that is $\{\bigcup_{\beta,j} X_{i,\beta,j}\}_{i=0,2,\ldots,l}$. Each of these open sets is the union of a transfinite collection of open rational boxes. Stating that one of these tuples is valid means two things: one, that the related sequence starts from a set that contains the initial condition, and two, that the sets in the sequence are concatenated correctly according to the five rules above. These rules are chosen so that the concatenation between the sets is dictated by the action of f, but only in a controlled fashion, i.e. in a manner that takes care of the portions of the domain where each restriction f_β is continuous, for all $\beta < \alpha$. This is clarified by the first item in the above list, whose direct consequence is that f_β is continuous on all rational boxes $B_{i,\beta,j}$. This consideration allows to interpret the sequences expressed by valid tuples as good candidates for possibly containing the solution. This sets the premises for the next section, where the method to obtain the solution of the IVP is finally presented.

5 Obtaining the solution: an analytical method

We now have all the elements needed to describe the transfinite method that obtains analytically the solution of the IVP considered and hence proves our main result.

The idea is, as in [9], to use a search method based on boxes covering the domain. By considering smaller and smaller radii for these boxes, we can derive a sequence of continuous piecewise linear functions that eventually converge to a solution. As we know that the solution is unique, it must converge to the solution.

However, compared to the authors of [9], we have to deal with possibly transfinitely many boxes, unlike their framework where everything remains finite. This requires some modifications (e.g. the domain must be bounded in the reasoning), and more technical care (e.g. for concatenating the boxes).

▶ **Theorem 22.** *Consider a closed interval, a compact domain $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ and a function $f : E \to E$ such that, given an initial condition, the IVP of the form of* (1) *with right-hand term f has a unique solution on the interval. If f is solvable, then we can obtain the solution analytically via transfinite recursion up to an ordinal α such that $\alpha < \omega_1$.*

Proof. Let $[a, b]$ be the closed interval such that $y : [a, b] \to E$ is the unique solution of the IVP with right-hand term f and initial condition $y_0 = y(a)$. Let $\{E_\gamma\}_{\gamma<\omega_1}$ be the sequence of f-removed sets on E. Since f is solvable, by means of Theorem 20 we know that there exists an $\alpha < \omega_1$ such that $E_\alpha = \emptyset$ and $E_\beta = \emptyset$ for all $\beta \geq \alpha$. Therefore by transfinite recursion up to α based on repeated application of f we can consider the whole sequence of f-removed sets on E. We now show how to obtain the solution $y([a, b])$. We first pick a $n \in \mathbb{N}$ and consider a valid tuple of the (α)Monkeys approach for (f, y_0) for this value of n. We consider a valid tuple with $(X_{i,\beta,j}, h_{i,\beta,j}, C_{i,\beta,j}, Y_{i,\beta,j})$ for $i = 0, \ldots, l-1$, $\beta < \alpha$, $j = 1, \ldots, m_{i,\beta}$ dependent on this fixed n. To do so, consider a set $\bigcup_{\beta,j} X_{0,\beta,j}$ such that $y_0 \in \bigcup_{\beta,j} X_{0,\beta,j}$. Then, for all sets $\bigcup_{\beta,j} X_{i,\beta,j}$ for all $i = 0, \ldots, l-1$ we can select each open rational box $X_{i,\beta,j}$ so that it satisfies either $(X_{i,\beta,j} = \emptyset)$ or $(X_{i,\beta,j} \cap E_\beta \neq \emptyset$ and $X_{i,\beta,j} \cap E_{\beta+1} = \emptyset)$ for all $i = 0, \ldots, l-1$, $\beta < \alpha$, $j = 1, \ldots, m_{i,\beta}$. Moreover, because every continuous function on a closed domain is uniformly continuous, we can define for all $i = 0, \ldots, l-1$, $\beta < \alpha$, $j = 1, \ldots, m_{i,\beta}$ the function $\delta_{i,\beta,j} : \mathbb{R}^+ \to \mathbb{R}^+$ to be a modulus of continuity of f_β on $\mathrm{cl}\,(X_{i,\beta,j}) \cap E_\beta$, i.e. a function such that $\|f_\beta(x) - f_\beta(z)\| < \delta_{i,\beta,j}(\|x - z\|)$ for all $x, z \in \mathrm{cl}\,(X_{i,\beta,j}) \cap E_\beta$, for all $i = 0, \dots, l-1$, $\beta < \alpha$, $j = 1, \dots, m_{i,\beta}$. By convention, for all $\epsilon > 0$, if there are no points $x, z \in \mathrm{cl}\,(X_{i,\beta,j}) \cap E_\beta$ such that $\|x - z\| < \epsilon$ then we define $\delta_{i,\beta,j}(\epsilon) = \epsilon$. Let us now call $K \in \mathbb{Q}^+$ a rational such that $\max_{x \in E} \|x\| < K$. At this point, by taking the partition sufficiently small, we can make sure to take each nonempty open rational box $X_{i,\beta,j}$ and rational $h_{i,\beta,j}$ such that $0 < \mathrm{rad}(X_{i,\beta,j}) < \delta_{i,\beta,j}(\frac{1}{2n}) - Kh_{i,\beta,j}$ and such that its $Kh_{i,\beta,j}$ neighborhood has no intersection with $E_{\beta+1}$ and has the same modulus of continuity $\delta_{i,\beta,j}$. Take then each one of these neighborhoods as the set $B_{i,\beta,j}$ for all $i = 0, \dots, l-1$, $\beta < \alpha$, $j = 1, \dots, m_{i,\beta}$. It follows that $\mathrm{rad}(B_{i,\beta,j}) < \mathrm{rad}(X_{i,\beta,j}) + Kh_{i,\beta,j} < \delta_{i,\beta,j}(\frac{1}{2n})$. We then choose open rational boxes $C_{i,\beta,j}$ such that they satisfy $f_\beta(\mathrm{cl}(B_{i,\beta,j})) \subset C_{i,\beta,j}$. Note that we can make the choice in a way that ensures $\mathrm{rad}\,(C_{i,\beta,j}) < \frac{1}{2n}$ by definition of the moduli of continuity. From these choices it follows that we can pick open rational boxes $Y_{i,\beta,j}$ satisfying $X_{i,\beta,j} + h_{i,\beta,j}C_{i,\beta,j} \subset Y_{i,\beta,j}$ and $\mathrm{rad}\,(Y_{i,\beta,j}) < \delta_{i,\beta,j}(\frac{1}{2n})$ for all $i = 0, \dots, l-1$, $\beta < \alpha$, $j = 1, \dots, m_{i,\beta}$. Finally, we consider as the set $\bigcup_{\beta,j} X_{i+1,\beta,j}$ a set such that $\bigcup_{\beta,j} Y_{i,\beta,j} \subset \bigcup_{\beta,j} X_{i+1,\beta,j}$ for all $i = 0, \dots, l-1$. It is clear that the tuple described this way is a valid tuple of the (α)Monkeys approach for (f, y_0).

Let us now define two sequences $\{h_{i,\beta(i),j(i)}\}_{i=0,\dots,l-1}$ and $\{t_i\}_{i=0,\dots,l}$ where $t_0 = a$ and $t_i = a + \sum_{k=0}^{i-1} h_{k,\beta(k),j(k)}$ for all $i = 1, \dots, l$ and a piecewise linear function $\eta_n : [a, t_l] \to E$ such that $\eta_n(a) = y_0$ and such that for all $i = 0, \dots, l-1$ we have $\eta_n(t_i) \in X_{i,\beta(i),j(i)}$ and $\eta_n(t) = \eta_n(t_i) + (t - t_i)c_{i,\beta(i),j(i)}$ for all $t_i < t \le t_{i+1}$, for some $c_{i,\beta(i),j(i)} \in C_{i,\beta(i),j(i)}$. Note that this function is well defined because $|t_{i+1} - t_i| = h_{i,\beta(i),j(i)}$ for all $i = 0, \dots, l-1$ and so it follows that $\eta_n(t) \in Y_{i,\beta(i),j(i)} \subset \bigcup_{\beta,j} X_{i+1,\beta,j}$ for all $t_i < t \le t_{i+1}$. In other words, we can always choose the sequences in such a way that function η_n is well defined. Moreover, note that $\eta_n'(t) = c_{i,\beta(i),j(i)} \in C_{i,\beta(i),j(i)}$ for all $t_i < t < t_{i+1}$, for all $i = 0, \dots, l-1$; note also that since $\eta_n(t) \in B_{i,\beta(i),j(i)}$ we have $f_{\beta(i)}(\eta_n(t)) \in C_{i,\beta(i),j(i)}$ for all $t_i < t < t_{i+1}$, for all $i = 0, \dots, l-1$. Therefore we have $\left\|\eta_n'(t) - f_{\beta(i)}(\eta_n(t))\right\| \le \mathrm{diam}\,(C_{i,\beta,j}) < \frac{1}{n}$ for all $t_i < t < t_{i+1}$ such that $\eta_n(t) \in E_{\beta(i)}$, for all $i = 0, \dots, l-1$.

Suppose we have just considered a valid tuple of the (α)Monkeys approach for (f, y_0) for a fixed value $\bar{n} \in \mathbb{N}$ following the above procedure and we have defined function $\eta_{\bar{n}} : [0, t_l] \to E$ in the way described. Let us indicate this t_l with the symbol T. It is clear that we can consider a new valid tuple and a new function $\eta_n : [0, T] \to E$ for each value of $n > \bar{n}$ while maintaining the same domain for each function. We can then consider a sequence of the functions $\{\eta_n\}_{n > \bar{n}}$ as defined above, where each function in the sequence is defined based on the valid tuple $(X_{i,\beta,j}, h_{i,\beta,j}, C_{i,\beta,j}, Y_{i,\beta,j})$ with $\mathrm{rad}\,(X_{i,\beta,j}) < \delta_{i,\beta,j}(\frac{1}{2n})$, $\mathrm{rad}\,(B_{i,\beta,j}) < \delta_{i,\beta,j}(\frac{1}{2n})$ and $\mathrm{rad}\,(C_{i,\beta,j}) < \frac{1}{2n}$ for all $n > \bar{n}$, $i = 0, \dots, l-1$, $\beta < \alpha$, $j = 1, \dots, m_{i,\beta}$ as described above. We want to show that such sequence $\{\eta_n\}_{n > \bar{n}}$ is uniformly bounded and equicontinuous. To prove that it is uniformly bounded we need to prove that there exists a constant $R \in \mathbb{R}^+$ such that $\|\eta_n(t)\| \le R$ for all $n > \bar{n}$, for all $a \le t \le T$. This is indeed trivial since $\eta_n(t) \in \bigcup_{i,\beta,j} B_{i,\beta,j}$ for all $n > \bar{n}$, for all $a \le t \le T$ and each open rational box $B_{i,\beta,j} \subset E$ for all $n > \bar{n}$, $i = 0, \dots, l-1$, $\beta < \alpha$, $1 \le j \le m_{i,\beta}$. For equicontinuity it is enough to prove that there exists a constant $M \in \mathbb{R}^+$ such that $\left\|\eta_n(\tilde{t}) - \eta_n(t)\right\| \le M \left|t - \tilde{t}\right|$ for all $n > \bar{n}$, for all $a \le t, \tilde{t} \le T$. The existence of M follows from the fact that the sequence is uniformly bounded together with the fact that $\|\eta_n'(t)\| < K$ for all $n > \bar{n}$, for almost all $a \le t \le T$. Therefore, since the sequence is uniformly bounded and equicontinuous, we can apply a well-known theorem in analysis (Ascoli's theorem, Theorem 28) in order to conclude that the sequence $\{\eta_n\}_{n > \bar{n}}$ has a subsequence $\{\eta_{n(u)}\}_{u > \bar{n}}$ that converges uniformly on $[a, T]$ to a function $\eta : [a, T] \to E$. Moreover, another known result for the differentiability of the limit of sequences (Theorem 29) tells us that function η is differentiable almost everywhere on

$[a, T]$. Note that by taking limit of $n \to \infty$ we have $l \to \infty$ and $\text{rad}(B_{i,\beta,j}), \text{rad}(C_{i,\beta,j}) \to 0$ and $h_{i,\beta,j} \to 0$ for all $i = 0, \ldots, l-1$, $\beta < \alpha$, $j = 1, \ldots, m_{i,\beta}$. Therefore the inequality $\|\eta_n'(t) - f_\beta(\eta_n(t))\| \leq \text{diam}(C_{i,\beta,j}) < \frac{1}{n}$ for all $t_i < t < t_{i+1}$ such that $\eta_n(t) \in E_{\beta(i)}$, for all $i = 0, \ldots, l-1$, leads to the equation $\eta'(t) = f(\eta(t))$ for almost all $t \in [a, T]$. Since $\eta(a) = y_0$, continuity and unicity of the solution of the IVP imply $\eta(t) = y(t)$ for all $t \in [a, T]$. Specifically, this means that any convergent subsequence converges to the same function, which is precisely the solution y of the IVP.

Finally, to obtain the solution over the whole domain $[a, b]$, it is sufficient to consider the initial valid tuple of the (α)Monkeys approach for (f, y_0) as described above but in such a way that, when defining the sequence of times $\{t_i\}_{i=0,\ldots,l}$ we have $t_l \geq b$ and then take $T = b$. This is always possible due to the definition of a valid tuple and the fact that f is bounded within E. ◀

The above statement, combined with the constructions from Section 3, hence proves that the computational power of solvable ODEs is the one of transfinite computations, up to some limit ordinal. We now discuss what this ordinal is, according to the adopted viewpoint (the issue is about which functions are considered definable in the above reasonings, and appears only for ordinals that would be countable but non-recursive, i.e. non-countable in the considered model of set theory).

▶ Remark 23 (On ω_1 vs ω_1^{CK}, Boldface view). The statement of Theorem 22 is formulated in an approach based on descriptive set theory, using the approach of the so-called boldface hierarchies. The description of the examples in Section 3 follows an approach that is closer to a computability theoretic point of view, that is to say using the approach of the so-called lightface hierarchies. From a boldface point of view, what the constructions of Section 3 say is that it is possible to reach the level of Baire's hierarchy using limits constructed in the spirit of the examples of this section. This can be done up to level ω_1 (non-included), the first uncountable ordinal. Combined with Theorem 22, the computational power of solvable ODEs corresponds to transfinite iterations of limits up to any ordinal less than ω_1.

▶ Remark 24 (On ω_1 vs ω_1^{CK}, Lightface view). From a lightface point of view, it makes sense to replace the hypothesis "of Class Baire one" (it is the pointwise limit of a sequence of continuous functions) in the definition of Solvable function (Definition 15) by the fact that it is the pointwise limit of a computable sequence of computable functions. All solvable examples that we considered are solvable in this new sense. Hyperarithmetic sets are known to correspond to sets that can be defined using transfinite induction up to ω_1^{CK}, which is the first non-recursive ordinal [2]. The above reasoning of the proof of Theorem 22 provides a way to obtain (define) the solution in the hyperarithmetical hierarchy via a transfinite recursion up to an ordinal α such that $\alpha < \omega_1^{CK}$: we are using arguments similar to the ones of [13]. We hence obtain that there is a precise correspondence between computations by the class of solvable systems and the hyperarithmetic hierarchy, as the hyperarithmetical hierarchy is known to correspond to transfinite recursion up to ordinal ω_1^{CK}: see [35, 2].

6 Conclusions and future work

We have discussed the properties of IVPs involving discontinuous ODEs that have a unique solution. This study has led us to the identification of a robust class of these systems, which we called *solvable*, for which the solution can always be obtained analytically by means of transfinite recursion up to a countable number of maximum steps. We have presented several examples of such systems and illustrated a technique that constructs examples of ever-increasing complexity. We have established that the solutions of solvable systems can be used, even in the simple case of a basic set of discontinuity points, to yield noncomputable values and solve the halting problem.

Due to the similarity of our approach to the method proposed by Denjoy for the problem of antidifferentiation, and in light of results and constructions illustrated in papers such as [13] and [37], we believe in having set the tables for a more in-depth light-face and bold-face analysis of the class of systems at hand, possibly leading to a rank and a hierarchy of these systems, as well to a classification of hyperarithmetical real numbers as targets reachable by solutions of discontinuous IVPs. More precisely: the integrability rank that inspired the modus operandi of our ranking is the Denjoy rank. The creation of such rank is directly based on a transfinite method.

- An analysis has been done for the Denjoy rank: an unpublished theorem from Ajtai, whose proof is included in [13], demonstrates that once a code for a derivative f is given as a computable sequence of computable functions converging pointwise to f, then the antiderivative F of f is Π^1_1 relative to f. This fact has direct implications related to the hierarchy of hyperarithmetical reals. The hyperarithmetical reals, or Δ^1_1, are defined as the reals x for which the set $\{r \in \mathbb{Q} : r < x\}$ is Δ^1_1. The implication mentioned above is expressed formally by a Theorem from [13], which proves that the hyperarithmetical reals are exactly those reals x such that $x = \int_0^1 f$ for some derivative f of which we know the code of.
- An alternative computability theoretic analysis of Denjoy's rank has been done in [37], relating levels of the hierarchy to levels of the arithmetical hierarchy in some precise manner, using a slightly alternative setting, on the way objects are encoded.

We believe that adapting similar analysis to the framework of solvable systems can be done using both views, and could lead to similar statements, with a more precise analysis of involved rankings, and of involved ordinals, given some class of functions or dynamics.

The latter, combined with papers such as [3] and [4] that describe simulations of discrete models of computations by analog models based on systems of ODEs, open the doors for identifying the model of solvable IVPs as an analog model for simulating transfinite computation, or as an alternative approach for presenting transfinite computations.

Notice that our discussions also pointed out classes of ordinary differential equations with solutions with levels of complications that we did not see discussed in any books about ordinary differential equations, in addition to many already existing counterexamples in literature. In particular, from arguments similar to [13], it follows that our results show that the totality of countable ordinals is necessary in any constructive process for solving an ODE in the general case and that for any countable ordinal, we can construct an example of solvable ODE of that difficulty.

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A Appendix: Useful definitions and results

We include in this appendix some definitions and theorems that can be used to integrate the main document for a deeper understanding of its arguments.

A.1 Ordinal numbers

We describe the process of *transfinite recursion* as the process that for each ordinal α associates with α an object that is described in terms of objects already associated with ordinals $\beta < \alpha$. Moreover, we use the expression *transfinite recursion up to* α if the process associates an object for all ordinals $\beta < \alpha$. We present the Cantor-Baire stationary principle [14], as expressed by the following theorem:

▶ **Theorem 25** (Cantor-Baire stationary principle). *Let* $\{E_\gamma\}_{\gamma<\omega_1}$ *be a transfinite sequence of closed subsets of* $\mathbb{R}^r$ *for some* $r \in \mathbb{N}$. *Suppose* $\{E_\gamma\}_{\gamma<\omega_1}$ *is decreasing; i.e.,* $E_\gamma \subseteq E_\beta$ *if* $\gamma \geq \beta$. *Then there exists* $\alpha < \omega_1$ *such that* $E_\beta = E_\alpha$ *for all* $\beta \geq \alpha$.

A.2 Sequences

Given a set X of elements and an index set Y we indicate sequences of elements from X with the notation $\{x_n\}_{n\in Y}$ where $x_n \in X$ for each $n \in Y$. If the index set is the set of natural numbers, we simply write $\{x_n\}_n$. Instead, if the index set is some ordinal number, we talk about *transfinite sequences*. Given a sequence $\{x_n\}_n$ for some set of elements X we indicate a subsequence of such sequence with the notation $\{x_{n(u)}\}_u$ where $n : \mathbb{N} \to \mathbb{N}$ is the function determining the elements of the subsequence considered. We now define *uniformly bounded* sequences of functions:

▶ **Definition 26** (Uniformly bounded). *Let $I \subset \mathbb{R}$, $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ and let $\{g_n\}_n : I \to E$ be a sequence of functions. We say that the sequence is* uniformly bounded *if there exists a constant $K > 0$ such that $\|g_n(t)\| \leq K$ for all $g_n \in \{g_n\}_n$ and for all $t \in I$.*

We then define *equicontinuous* sequences of functions:

▶ **Definition 27** (Equicontinuous). *Let $I \subset \mathbb{R}$, $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ and let $\{g_n\}_n : I \to E$ be a sequence of functions. We say that the sequence is* equicontinuous *if for any $\epsilon > 0$ there exists a $\delta_\epsilon > 0$ such that $\left\|g_n(t) - g_n(\tilde{t})\right\| \leq \epsilon$ whenever $\left|t - \tilde{t}\right| \leq \delta_\epsilon$ for all $g_n \in \{g_n\}_n$ and for all $t, \tilde{t} \in I$.*

For infinite, uniformly bounded, equicontinuous sequence of functions over the reals there exists a famous result due to Ascoli [8]:

▶ **Theorem 28** (Ascoli). *Let $I \subset \mathbb{R}$ be a bounded interval, $E \subset \mathbb{R}^r$ for some $r \in \mathbb{N}$ and let $\{g_n\}_n : I \to E$ be an infinite, uniformly bounded, equicontinuous sequence of functions. Then the sequence $\{g_n\}_n$ has a subsequence $\{g_{n(u)}\}_u$ that converges uniformly on I.*

It follows a theorem concerning differentiability of limits of converging sequences of functions:

▶ **Theorem 29.** *Let $\{f_n\}_n$ be sequence of functions from the closed interval $[a, b] \subset \mathbb{R}$ to $\mathbb{R}^r$ for some $r \in \mathbb{N}$ and pointwise converging to function f. Let $M > 0$ be such that $\left\|f_n(\tilde{t}) - f_n(t)\right\| \leq M \left|t - \tilde{t}\right|$ for all $n \in \mathbb{N}$, for all $a \leq t, \tilde{t} \leq b$. Then f is differentiable almost everywhere and $f(x) = \int_a^x f'(t)dt$ for all $\in [a, b]$.*

A.3 Functions of class Baire one

We define the *set of discontinuity points* of a given function:

▶ **Definition 30** (Set of discontinuity points). *Let f be a function $f : X \to Y$ where X and Y are two complete metric spaces. We define the* set of discontinuity points *(of f on X) as the the set:*

$$D_f = \{x \in X : \exists \epsilon > 0 : \forall \delta > 0 \; \exists y, z \in \; B_X(x, \delta) : d_Y(f(y), f(z)) > \epsilon\}$$

We define what it means for a given function to be *of class Baire one*:

▶ **Definition 31** (Baire one). *Let X, Y be two separable, complete metric spaces. A function $f : X \to Y$ is* of class Baire one *if it is a pointwise limit of a sequence of continuous functions, i.e. if there exists a sequence of continuous functions from X to Y, $\{f_m\}_m$, such that $\lim_{m \to \infty} f_m(x) = f(x)$ for all $x \in X$.*

An important property of functions of class Baire one is that the composition of a function of class Baire one with a continuous functions yields a function of class Baire one [39]. We now refresh a well known topological concept:

▶ **Definition 32** (Nowhere dense set). *Let X be a topological space and let S be a subset of X. We say that S is* nowhere dense *(in X) if its closure has empty interior.*