One \( n \) Remains to Settle the Tree Conjecture

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Abstract

In the famous network creation game of Fabrikant et al. \cite{Fabrikant04} a set of agents play a game to build a connected graph. The \( n \) agents form the vertex set \( V \) of the graph and each vertex \( v \in V \) buys a set \( E_v \) of edges inducing a graph \( G = (V, \bigcup_{v \in V} E_v) \). The private objective of each vertex is to minimize the sum of its building cost (the cost of the edges it buys) plus its connection cost (the total distance from itself to every other vertex). Given a cost of \( \alpha \) for each individual edge, a long-standing conjecture, called the tree conjecture, states that if \( \alpha > n \) then every Nash equilibrium graph in the game is a spanning tree. After a plethora of work, it is known that the conjecture holds for any \( \alpha > 3n - 3 \). In this paper we prove the tree conjecture holds for \( \alpha > 2n \). This reduces by half the open range for \( \alpha \) with only \((n - 3, 2n)\) remaining in order to settle the conjecture.

1 Introduction

A foundational motivation for the field of algorithmic game theory was to understand the evolution and functionality of networks, specifically, the Internet; see Papadimitriou \cite{Papadimitriou91}. Of particular fascination concerned how the actions of self-motivated agents affected the structure of the World Wide Web and social networks more generally. An early attempt to study this conundrum was undertaken by Fabrikant et al. \cite{Fabrikant04} with their now classical network creation game. Despite its extreme simplicity, their model (detailed below) has become highly influential for two reasons. First, it inspired the development of a wide range of network formation models, \cite{Balogh08},\cite{Chen07}. Second, it has lead to one of the longest-standing open problems in algorithmic game theory, namely, the tree conjecture. The latter motivates this research. So let us begin by describing the model and the conjecture.

1.1 The Network Creation Game

Consider a set of vertices, \( V = \{1, 2, \ldots, n\} \), who attempt to construct a connected graph between themselves. To do this, each vertex (agent) can purchase individual edges for a fixed cost of \( \alpha \) each. Consequently, a strategy for vertex \( v \in V \) is a set of (incident) edges \( E_v \). Together the strategies of the agents forms a graph \( G = (V, E) \), where \( E = E_1 \cup E_2 \cup \cdots \cup E_n \). In the network creation game, the objective of each vertex is to minimize the sum of its building and its connection cost. The building cost for vertex \( v \) is \( \alpha \cdot |E_v| \), the cost of all the edges it buys. The connection cost is \( D(v) = \sum_{u:u \neq v} d_G(u,v) \), the sum of the distances in \( G \) of \( v \) to every other vertex, where \( d_G(u,v) = \infty \) if there is no path between \( u \) and \( v \). That is, the total cost to the vertex is \( c_v(E) = \alpha \cdot |E_v| + D(v) \).
Given the different objectives of the agents, we study Nash Equilibria (NE) in the network creation game. A Nash equilibrium graph is a graph $G = (V, E)$ in which no vertex $v$ can reduce its total cost by changing its strategy, that is, by altering the set of edges it personally buys, given the strategies of the other vertices remain fixed. Thus $E_v$ is a best response to $(E_u)_{u \neq v}$, for every vertex $v$. Observe that every Nash equilibrium graph must be a connected graph. Attention in the literature has focused on whether or not every Nash equilibrium graph is minimal, that is, a spanning tree.

1.2 The Tree Conjecture

The network creation game was designed by Fabrikant et al. [11]. They proved that any network equilibrium graph which forms a tree costs at most 5 times that of a star, i.e. the optimal network. They proposed that for $\alpha$ greater than some constant, every Nash equilibrium graph is a spanning tree. This was the original tree conjecture for network creation games.

In the subsequent twenty years, incremental progress has been made in determining the exact range of $\alpha$ for which all Nash equilibria are trees. Albers et al. [1] demonstrated that the conjecture holds for $\alpha \geq 12n \log n$. However, they also provided an counterexample to the original tree conjecture. Moreover, Mamageishvili et al. [13] proved the conjecture is false for $\alpha \leq n - 3$.

As a result, a revised conjecture took on the mantle of the tree conjecture, namely that every Nash equilibrium is a tree if $\alpha > n$. Mihalák and Schlegel [14] were the first to show that this tree conjecture holds for $\alpha \geq c \hat{n}$ for a large enough constant $c$, specifically, $c > 273$. Since then, the constant has been improved repeatedly, by Mamageishvili et al. [13], then Álvarez and Meseguer [3], followed by Bilò and Lenzner [6], and finally by Dippel and Vetta [10] who proved the result for $\alpha > 3n - 3$.

We remark that extensions and variations of the network creation game have also been studied; we refer the interested reader to [4, 7, 8, 9, 16, 17].

1.3 Our Contribution

In this paper we improve the range in which the tree conjecture is known to hold from $\alpha > 3n - 3$ [10] to $\alpha > 2n$:

Theorem 1. If $G$ is a Nash equilibrium graph for the network creation game $(n, \alpha)$ and $\alpha > 2n$, then $G$ is a tree.

Our high-level strategy to prove Theorem 1 is straightforward: we assume the existence of a biconnected component $H$ in a Nash equilibrium and then prove, via consideration of a collection of strategy deviations, that some vertex has a better strategy, provided $\alpha > 2n$. This contradicts the best response conditions and proves that every Nash equilibrium graph is a spanning tree.

Our approach applies some prior methodologies and combines them with some original tools. One important technique we exploit is that of min-cycles, introduced by Lenzner [12]. A min-cycle is a cycle in a graph with the property that, for every pair of vertices in the cycle, the cycle contains a shortest path between the pair. For certain values of $\alpha$, currently $\alpha > 2n - 3$, min-cycles are known to have the nice property that each vertex in the cycle buys exactly one edge of the cycle. This property has been leveraged in various ways [3, 6, 10] to show that no min-cycles can exist in a Nash equilibrium graph for large
enough $\alpha$. Importantly, the smallest cycle in any graph must be a min-cycle. Consequently, for large enough $\alpha$, there can be no smallest cycle and, thus, no cycle at all. This implies that, for such an $\alpha$, all Nash equilibrium graphs are trees.

The techniques we develop concern the analysis of the presupposed biconnected component $H$ in the Nash equilibrium. The existence of $H$ implies the existence of a special vertex, called $r$, which has the lowest connection cost amongst all vertices in $H$. That is, $D(r) \leq D(v)$ for all $v \in H$. Given $r$, we take a shortest path tree $T$ in $G$ rooted at $r$. The key is to exploit the structural properties inherent in $T$. To derive these properties, we design a new class of strategy deviations available to vertices in $H$. Because these deviations must be non-improving responses, they impose numerous beneficial restrictions on the edges in $H$. In particular, we can then show that the sum of degrees of vertices within $H$ exceeds 2 times the number of edges within $H$. This absurdity proves the Nash equilibrium graph is a tree.

1.4 Overview of Paper

This paper has three main sections. Section 2 consists of preliminaries and contains three things. First, we examine the structure of Nash equilibrium graphs which contain a hypothesized biconnected component $H$. Second, we discuss min-cycles, which appear in several previous papers, each of which contains a useful lemma that we take advantage of in this paper. Third, we introduce some new lemmas pertaining to $T$, the shortest path tree mentioned above.

Section 3 has two parts. The first presents a set of three deviation strategies. That is, three specific ways a vertex in a biconnected component might consider changing which edges it buys in order to decrease its personal cost. For a Nash equilibrium graph these alternate strategies cannot reduce the personal cost, and this allows us to derive deviation bounds for that vertex and the edges it buys. The second part uses these deviations bounds to prove claims about the structure of the graph. In particular, we show that vertices which buy edges outside $T$ cannot be too close to one another.

Finally, Section 4 presents two main lemmas. The first shows that, in a set $U$ of three vertices adjacent in $T$, all at different depths, where the lowest buys an edge outside $T$, only one vertex in $U$ can have degree two in $H$. The second lemma bounds the number of times such sets can intersect. These two lemmas are the core of the proof of Theorem 1. After proving the main result, the paper culminates with concluding remarks on the future of the tree conjecture.

2 Preliminaries

Recall our basic approach is to prove by contradiction the non-existence of a biconnected component in a Nash equilibrium graph. Accordingly, we begin in Section 2.1 by introducing the notation necessary to analyse biconnected components in network creation game graphs. To disprove the existence of a biconnected component it suffices to disprove the existence of a cycle. Of particular importance here is the concept of a min-cycle. So in Section 2.2 we present a review of min-cycles along with a corresponding set of fundamental lemmas which appear in [12, 6, 13]. Finally, in Section 2.3, we prove a collection of technical results concerning the shortest path tree $T$ that we will utilize throughout the rest of the paper. Lemma 9 is an especially useful tool for this method, and could prove valuable in future work on this problem.
2.1 Biconnected Components

Given a subgraph $W$ of a graph $G$, we let $d_W(v, u)$ denote the distance between $v$ and $u$ in $W$. In particular, $d(u, v) = d_G(u, v)$ is the distance from $v$ to $u$ in the whole graph. Let $D(v) = \sum_{w: w \neq v} d_G(u, v)$ be the connection cost for vertex $v$, the sum of the distances from $v$ to every other vertex.

In a Nash equilibrium graph $G$ containing biconnected components, we will refer to the largest biconnected component as $H$. For a vertex $v \in H$, we denote by $C(v)$ a smallest cycle containing $v$, breaking ties arbitrarily. Let $r \in H$ be a vertex whose connection cost is smallest amongst all the vertices in $H$. Once built, an edge $uv$ of $G$ can be traversed in either direction. Thus $G$ is an undirected graph. However, it will often be useful to view $G$ as a directed graph. Specifically, we may orient $uv$ is from $u$ to $v$ if the edge was bought by $u$ and orient it from $v$ to $u$ if it was bought by $v$.

Given $r$, we will make heavy use of the shortest path tree $T$ rooted at $r$. There may be multiple options for the choice of $T$. So we insist our choice of $T$ has the following property: For all shortest paths $P$ between $r$ and any vertex $u$ which is directed from $r$ to $u$, we have $P \subseteq T$. The proof of Lemma 3 shows that we will never have two shortest paths directed from $r$ to the same vertex $v$, so this choice of $T$ is well-defined. Next, given an edge $uv$ where $u$ is the parent of $v$ in $T$, we say that $uv$ is a down-edge of $T$ if the edge was bought by $u$; otherwise we say it is an up-edge of $T$. We denote by $T^\downarrow$ the set of down-edges and by $T^\uparrow$ the set of up-edges of $T$. Let $T(v)$ be the subtree rooted at $v$ in the tree $T$ rooted at $r$. If $uv \in T^\downarrow$ then we define $T(uv) = T(v)$. Otherwise, if $uv \notin T^\downarrow$ then we define $T(uv) = \emptyset$.

Finally, there remain two types of sets we need to define. The first type is the “$S$-sets”, which are sets of vertices. In a graph $G$, with a subgraph $W$, $S_W(v)$ is the set of vertices $w$ with the following property: a shortest path in $G$ from $w$ to $W$ contains $v$. Therefore $S_W(v)$ will always contain $v$, and it will contain no vertices of $W \setminus v$. $W$ can even be a single vertex $u$, i.e. $S_u(v)$ is the set of all vertices in $G$ with a shortest path to $u$ containing $v$. The second type is the “$X$-sets”, which are sets of edges. For largest biconnected component $H$ of $G$ and the shortest path tree $T$, the set $X_0$ is the set of all edges in $H \setminus T$, or out-edges. For all integers $i \geq 1$, $X_i$ contains the set $X_{i-1}$ as well as all edges $uv \in H \cap T^i$ bought by the parent $u$ and where the child $v$ buys an edge of $X_{i-1}$. The $X_i$ sets can be thought of as the set of all out-edges and all down-edges in directed paths of length $\leq i$ to a vertex which buys an out-edge. Sometimes it will be useful to include in the $X$-sets all the edges $uv \in H \cap T^i$, bought by child $v$ to parent $u$. To do this, we add superscript $+\oplus$ to the $X$ set. For instance $X_0^+ = X_0 \cup T^\uparrow$ is the set of all up-edges and out-edges, which happens to be the set of all edges $e$ with $T(e) = \emptyset$. Similarly, $X_i^+ = X_i \cup T^\uparrow$.

2.2 Min-Cycles

Recall, a min-cycle is a cycle in a graph with the property that, for every pair of vertices in the cycle, the cycle contains a shortest path between the pair. This concept, introduced by Lenzner [12], has proved very fruitful in the study of Nash equilibria in network creation games. In particular, we now present three useful lemmas concerning min-cycles that appear in [12, 6, 13], respectively. For completeness and because some of the ideas used are informative, we include short proofs of these three lemmas; similar proofs appeared in [10].

\begin{lemma}[[12]] \end{lemma}

The smallest cycle $C$ containing an edge $e$ is a min-cycle.
Proof. Consider the smallest cycle $C$ containing an edge $e$. Suppose for the sake of contradiction that there are two vertices $u, v \in C$ such $d_C(u, v) < d_C(u, v)$. Without loss of generality, suppose the shortest path between $u$ and $v$, labelled $P$, lies entirely outside $C$. Note that $C$ contains two paths between $u$ and $v$. Let $Q$ be the path of $C$ from $u$ to $v$ that contains $e$. Then $P \cup Q$ is a cycle containing $e$ that is strictly smaller than $C$, a contradiction. ◀

For an edge $e$ or vertex $u$ in a biconnected component $H$, we may write $C(e), C(u)$ to denote the smallest cycle containing edge $e$, vertex $u$ respectively, breaking ties arbitrarily. A nice property of min-cycles is that every vertex in the min-cycle buys exactly one edge of the cycle.

Lemma 3 ([6]). Let $\alpha > 2(n - 1)$. Every min-cycle in a Nash equilibrium graph $G$ is directed.

Proof. Let $C$ be a min-cycle that is not directed. Then there is a vertex $v$ that buys two edges of the cycle, say $vx$ and $vy$. Let $uw \in C$ be an edge furthest from $v$ (if $|C|$ is odd, there is a unique choice, otherwise choose one of two furthest edges). Let $u$ buy $uw$. We’ve chosen $uw$ so that both $u$ and $w$ have a shortest path to $v$ which does not contain $uw$, therefore every vertex in $G$ has a shortest path to $v$ without $uw$. Without loss of generality, let $d(u, y) < d(u, v)$. Then because of how we chose $uw$, both $v$ and $x$ have a shortest path to $u$ which does not contain $vx$, therefore every vertex in $G$ has a shortest path to $u$ without $vx$.

First, consider that if $u$ sells $uw$ and buys $uv$, no vertices become farther from $v$. It follows, by the Nash equilibrium conditions, that

$$D(u) \leq D(v) + n - 1$$

because $D(u)$ must be less than the cost to swap $uv$ for $uv$ and use the edge $uv$ in followed by the shortest path from $v$ to each of the $n - 1$ vertices.

On the other hand, $v$ can sell both $vx$ and $vy$ and instead buy the edge $vu$ without increasing the distance from $u$ to any other vertex. All of the vertices which used $vy$ in their shortest paths to $u$ become closer to $u$. It follows, by the Nash equilibrium conditions, that

$$D(v) \leq D(u) + n - 1 - \alpha$$

2.3 The Shortest Path Tree $T$

Let us now consider the shortest path tree $T$ rooted at the vertex $r$ with smallest connection cost amongst the vertices in the biconnected component $H$. In this section, we present some technical lemmas that give useful insights into the structure of this tree $T$. We begin by upper bounding the size of any subtree in $T$.

Lemma 6. If $v \neq r$ then $|T(v)| \leq \frac{n}{2}$.

Proof. Recall $T$ is the shortest path tree of the vertex $r$ with the smallest connection cost and $T(v)$ is the subtree of $T$ rooted at $v$. For any vertex $x \in T(v)$, it immediately follows that $d(v,x) = d(r,x) - d(v,r)$. On the other hand, for any vertex $x \notin T(v)$, the triangle inequality implies that $d(v,x) \leq d(r,x) + d(v,r)$. Thus
If, in addition,

Lemma 7. Let \( \alpha > 2n \). If \( u \) buys \( uv \in X_i \), for some \( i \leq 2 \), then all \( w \in T(uv) \) have a path to \( r \) of length \( \leq d(w, r) + 2i \) which does not contain \( u \).

\[ D(v) \leq D(r) - d(v, r) \cdot |T(v)| + d(v, r) \cdot (n - |T(v)|) \]
\[ = D(r) + d(v, r) \cdot (n - 2 \cdot |T(v)|) \]

But \( D(r) \leq D(v) \). Therefore \( d(v, r) \cdot (n - 2 \cdot |T(v)|) \geq 0 \) and rearranging gives \( |T(v)| \leq \frac{r}{2} \). ▲

The next two lemmas concern the properties of paths related to any vertex \( u \) that buys an edge \( uv \in X_i \), where \( i \leq 2 \).

**Lemma 7.** Let \( \alpha > 2n \). If \( u \) buys \( uv \in X_i \), for some \( i \leq 2 \), then all \( w \in T(uv) \) have a path to \( r \) of length \( \leq d(w, r) + 2i \) which does not contain \( u \).

**Proof.** If \( uv \in X_0 \) then, as \( uv \notin T^i \), we have \( |T(uv)| = 0 \). Thus the claim is trivially true.

So assume \( uv \notin X_0 \). Then \( uv \) is the first edge in a path of length \( i \) in \( T \) from \( u \) to a vertex \( x \) which buys \( xy \in H \setminus T \). Thus, there is a path \( P_0 \) in \( T \) of length \( \ell(P_0) = i - 1 \) from \( v \) to \( x \). Now let \( P_1 \) be the path from \( y \) to \( T \). Observe that \( P_1 \) does not contain \( u \); otherwise \( xy \) and \( T \) would define a cycle of length \( \leq 6 \) (because \( i \leq 2 \)), contradicting Corollary 5. Note also that \( P_1 \) has length \( \ell(P_1) \leq d(r, v) + i \) because \( T \) is a shortest path tree.

Next, let \( P_2 \) be the path from \( u \) to \( v \). Note \( P_2 \) has length \( d(w, r) - d(v, r) \) and also does not contain \( u \). Consequently \( P_2, P_0, xy, P_1 \) is a path from \( w \) to \( r \). This path has length at most

\[ \ell(P_2) + \ell(P_0) + 1 + \ell(P_1) \leq (d(w, r) - d(v, r)) + (i - 1) + 1 + (d(v, r) + i) \]
\[ = d(w, r) + 2i \]

Furthermore this path does not contain \( u \). ▲

**Lemma 8.** Let \( \alpha > 2n \). If \( u \) buys \( uv \in X_i \), for some \( i \leq 2 \), then \( d(u, r) \geq \left\lfloor \frac{|C(u)|}{2} \right\rfloor - i \).

**Proof.** So \( uv \in X_i \) is associated with an edge \( xy \in X_0 \), where \( xy = uv \) if \( uv \notin X_0 \). Let \( C \) be the cycle defined by \( T + xy \). Because \( T \) is a shortest path tree it cannot be the case that \( y \) is an ancestor of \( x \) in \( T \). Neither is \( y \in T(\alpha) \); if so, \( d(u, y) \leq 3 \) and therefore \( \ell(C) \leq 6 \), contradicting Corollary 5. Thus \( y \notin T(u) \) and, consequently, \( uv \in C \).

Let \( z \) be the lowest common ancestor of \( x \) and \( y \). Then \( C \) is a path from \( z \) to \( x \), plus the edge \( xy \), plus a path from \( y \) to \( z \). So it has length

\[ d(x, z) + 1 + d(y, z) \leq 2d(x, r) + 2 = 2d(u, r) + 2i + 2 \]

Observe that to achieve this maximum length, we must have \( d(x, z) = d(x, r) = d(y, r) - 1 \), and \( C \) must consist of two shortest paths from \( y \) to \( r \), meaning \( C \) will also contain \( r \).

Next consider the smallest cycle \( C(u) \) containing \( u \). Suppose for the sake of contradiction that \( d(u, r) \leq \left\lfloor \frac{|C(u)|}{2} \right\rfloor - i - 1 \). If \( |C(u)| \) is odd, then \( 2d(u, r) + 2i + 3 \leq |C(u)| \), which contradicts the choice of \( C(u) \). Therefore, \( |C(u)| \) is even and \( 2d(u, r) + 2 + 2i \leq |C(u)| \). However, \( |C| \leq 2d(u, r) + 2i + 2 \). Thus \( |C| = |C(u)| \) and therefore \( C \) is a min cycle, by Lemma 2. Furthermore, \( C \) is directed by Lemma 3. This is a contradiction because then \( xy \) is in a directed shortest path, and must be part of \( T \) and not an edge in \( X_0 \). Hence \( d(u, r) \geq \left\lfloor \frac{|C(u)|}{2} \right\rfloor - i \), as desired. ▲

We present one last technical lemma in this section. We remark that this lemma is of critical value in our analysis and, we believe, may be of importance in achieving future improvements.

**Lemma 9.** If \( uuvw \) is a directed path in \( H \) and \( \deg_H(v) = 2 \), then \( |S_H(v)| \geq \frac{\alpha}{2|C(v)| - 3} \).

If, in addition, \( uuvw \) is a directed path of down-edges in \( T \), then \( |S_H(v)| \geq |T(w)| \).
We now present a new class of deviation strategies. Specifically, in Section 3.1 we study three substantial observations concerning vertices that buy multiple edges of specific types in \( H \). Here the inequality holds as \(|S_H(v)| \geq |S_u(w)|\). This change must non-negative by the equilibrium condition. This implies \(|S_H(v)| \geq \frac{\alpha}{2|C(v)| - 3}\), as desired.

3 A Class of Deviation Strategies

We now present a new class of deviation strategies. Specifically, in Section 3.1 we study three related strategies that a vertex in the biconnected component may ponder. Concretely, we derive bounds on the change in cost to the vertex resulting from using these strategy changes. In particular, we will use the fact that, at a Nash equilibrium, these cost changes must be non-negative. Then, in Section 3.2, we apply these deviation strategies to make three related strategies that a vertex in the biconnected component may ponder. Concretely, the third deviation, pictured in Figure 2c, is slightly different than the first. The only change is that \( \alpha \) switches strategies. Further, if \( uvw \) is a path of down-edges in \( T \), then because \( u \) is the ancestor of \( w \) and \( T \) is a shortest path tree, \( T(w) \subseteq S_u(w) \). Thus \(|S_H(v)| \geq |T(w)|\), as desired.

3.1 Three Deviation Strategies

As stated, we now introduce a class of deviation strategies. The first, shown in Figure 2a, involves a vertex \( u \) selling edges of \( X_2 \). Lemma 7 guarantees that the graph is still connected without these edges. Now \( u \)'s cost will change: it saves \( \alpha \) for each edge sold, but its distance to many vertices may increase by varying amounts, requiring us to bound the new total distance to all vertices. The second deviation, depicted in Figure 2b, is very similar to the first. The only change is that \( u \) buys the edge \( ur \). This reduces the saving by \( \alpha \) for the extra edge bought, but it also reduces the bound on the increased distance to the other vertices considerably. Finally, the third deviation, pictured in Figure 2c, is slightly different than the second. The bound given is weaker, because \( u \) may sell edges of \( X_2^+ \), rather than just \( X_2 \). Selling the edge to \( u \)'s parent reduces some of the savings from the previous strategies.

\[ d(u, r) \cdot n - \sum_{\ell=0}^{d(u,r)-1} 2 \cdot |T(u)| - j \cdot \alpha + \sum_{k=1}^{j} (2t_k + 2d(u, r)) \cdot |T(uw_{ik})| \]  

(3)
Proof. We commence by bounding $D(u)$ before the changes, by comparing it to $D(r)$. By the triangle inequality, $u$'s distance to any vertex $v$ is at most $d(r, u) + d(r, v)$ because there is a path from $u$ to $r$ and a path from $r$ to $v$. This is the bound we use for vertices in $T(r) \setminus T(u)$. For other vertices this is obviously not the shortest path. For instance, for any vertex $v \in T(u)$, $d(u, v) = d(r, v) - d(r, u)$. Furthermore, for $l$ in $1 \leq l \leq d(u, r)$, the length of $u$'s shortest path to $v \in |T(u_l) \setminus T(u_{l-1})|$ differs from $r$'s by $(d(u, r) - 2l)$. Therefore we have the bound:

$$D(u) \leq D(r) - d(u, r) \cdot |T(u)| - \sum_{l=1}^{d(u, r)} (d(u, r) - 2l) \cdot |T(u_l) \setminus T(u_{l-1})|$$

Now $D(r) \leq D(u)$. Moreover, if we rearrange, we can more concisely write this as

$$0 \leq d(u, r) \cdot n - \sum_{l=0}^{d(u, r)-1} 2 \cdot |T(u_l)|$$

To complete the proof, consider what happens when $u$ sells $uv_1, \ldots, uv_j$. First, $u$ obviously saves $\alpha$ from it edge costs. Second, by Lemma 7, we know that all $w \in T(u_{i_k})$ have a path of length $2i_k + d(w, r)$ to $r$ that does not contain $u$. This leads to an increase in distance from $u$ to $w$ of at most

$$(d(u, r) + 2i_k + d(w, r)) - (d(w, r) - d(u, r)) = 2d(u, r) + 2i_k$$

This yields our desired bound on the change of cost for $u$:

$$0 \leq d(u, r) \cdot n - \sum_{l=0}^{d(u, r)-1} 2 \cdot |T(u_l)| - j \cdot \alpha + \sum_{k=1}^{j} (2i_k + 2d(u, r)) \cdot |T(u_{i_k})|$$

\[1\] Corollary 11. If $\alpha > 2n$, $u$ buys $uv_1 \in X_{i_1}, \ldots, uv_j \in X_{i_j}$ for $j \geq 1$, $i_k \leq 2$, for all $k, 1 \leq k \leq j$, $u = u_0, u_1, \ldots, u_{d(u, r)-1}, u_{d(u, r)} = r$ is the path from $u$ to $r$ in $T$, then the strategy change of $u$ selling $uv_1, \ldots, uv_j$ and buying $ur$ results in a change of cost of at most

$$n - |T_{u_{d(u, r)}}| - \sum_{l<d(u, r)-1} 2 \cdot |T(u_l)| - (j-1) \cdot \alpha + \sum_{k=1}^{j} (2i_k + d(u, r) + 1) \cdot |T(u_{i_k})|$$

where the second term is removed if $\frac{d(u, r)}{2}$ is not integral.
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**Proof.** This strategy change is identical to that of to Lemma 10 except $u$ also buys $ur$. This means that the new path from $u$ to $r$ is now length 1, not length $d(u, r)$. Hence we replace $d(u, r)\cdot n$ with $n$ in Equation (3). There are still vertices which are as close or closer to $u$ than to $r$. The vertices of $|T_{u,d(u,r)}(uv)|$, which only exists if $\frac{d(u,r)}{2}$ is integral, are at-least as close to $u$ as to $r$, therefore $u$ does not use the edge $ur$ on the path to any of the vertices in $|T_{u,d(u,r)}|$, thus we subtract $|T_{u,d(u,r)}|$ from Equation (3). For the remaining sets $T(u)$ for $l < \frac{d(u,r)}{2}$, each time $l$ decreases by 1, the vertices in the set are 1 closer to $u$ and 1 farther from $r$.

Finally, the last two terms of Equation (3) are affected as well. We now buy an additional edge, so we add $\alpha$, resulting in the $(j-1)\cdot \alpha$ term. Furthermore, as previously stated. The path from $u$ to $r$ now has length 1, meaning for all $w \in T(uw_i)$ the increase in distance from $u$ to $w$ is at most

$$(2d(u, r) + 2i_k) - (d(u, r) - 1) = 1 + d(u, r) + 2i_k$$

This yields our desired bound on the change of cost for $u$:

$$n - |T_{\frac{d(u,r)}{2}}| - \sum_{l < \frac{d(u,r)}{2}} 2\cdot |T(u)| - (j-1)\cdot \alpha + \sum_{k=1}^j (2i_k + d(u, r) + 1) \cdot |T(uw_i)|$$

**Corollary 12.** If $\alpha > 2n$, $u$ buys $uv_1 \in X_i^+, \ldots, uv_j \in X_i^+$ for $j \geq 1$, $i_k \leq 2$, for all $k, 1 \leq k \leq j$, then the strategy change of $u$ selling $uv_1, \ldots, uv_j$ and buying $ur$ results in a change of cost of at most

$$n - (j-1)\alpha - (d(u, r) + 1) \cdot |T(u)| + \sum_{k=1}^j (2i_k + d(u, r) + 1) \cdot |T(uw_i)|$$

**Proof.** This strategy change is nearly identical to Corollary 11, except now $u$ is potentially selling the edge to its parent in $T$. We are no longer certain that $u$ is closer to its former parent than $r$ is, nor any vertices along the original path from $u$ to $r$. Thus we lose the savings of all $T(u_i)$ except for $l = 0$, as $T(u_0) = T(u)$. ▲

### 3.2 Observations arising from the Deviation Strategies

We now apply these three deviation strategies to make three observations concerning vertices that buy multiple edges of specific types in $H$. In turn these observations will be instrumental in proving the main result in Section 4. The first observation simply states that no vertex can buy two edges in $X_i^+$.

**Observation 13.** No vertex $u \in H$ buys two edges $uv_1, uv_2 \in X_i^+$. 

**Proof.** Suppose $u \in H$ buys $uv_1, uv_2 \in X_i^+$. Observe $r$ cannot buy an edge in $X_i^+$, as this would imply the existence of a short cycle, contradicting Corollary 5; therefore $u \neq r$. Hence, Lemma 6 implies $|T(u)| \leq \frac{q^2}{\alpha}$. We now apply Corollary 12 to the strategy change where $u$ sells $uv_1, uv_2 \in X_i^+$ and buys $ur$. The change in cost for $u$ is at most

$$n - \alpha - (d(u, r) + 1) \cdot |T(u)| + \sum_{k=1}^2 (2i_k + d(u, r) + 1) \cdot |T(uw_i)|$$

$$= n - \alpha - (d(u, r) + 1) \cdot |T(u)| - |T(uw_1)| - |T(uw_2)| + 2(|T(uw_1)| + |T(uw_2)|)$$

$$\leq n - \alpha + 2 \cdot |T(u)|$$
We can apply Corollary 12 to the strategy change where

\[ \begin{align*}
  &
  \text{This contradicts the Nash equilibrium conditions.}
\end{align*} \]

This contradicts the Nash equilibrium conditions.

Next we observe that no vertex can buy three edges in \( X_2^+ \). Furthermore any vertex that buys two edges in \( X_2^+ \) must be the parent of a child at the root of a large subtree in \( T \).

**Observation 14.** No vertex \( u \in H \) buys three edges \( uv_1, uv_2, uv_3 \in X_2^+ \), and if \( u \) buys \( uv_1, uv_2 \in X_2^+ \), then \( |T(uv_1) \cup T(uv_2)| > \frac{n}{3} \).

**Proof.** Suppose \( u \in H \) buys \( uv_1, uv_2, uv_3 \in X_2^+ \). Again, \( r \) cannot buy edges in \( X_2 \), as this would contradict Corollary 5; so \( u \neq r \). Thus Lemma 6 implies \( |T(u)| \leq \frac{n}{3} \). Now we apply Corollary 12 to the strategy change where \( v \) sells \( v_1, v_2, v_3 \in X_2^+ \) and buys \( vr \). The change in cost for \( v \) is:

\[ \begin{align*}
  &
  n - 2\alpha - (d(u, r) + 1) \cdot |T(u)| + \sum_{k=1}^{2} (2i_k + d(u, r) + 1) \cdot |T(uv_k)| \\
  &=
  n - 2\alpha - (d(r, u) + 1) \cdot (|T(u)| - |T(uv_1)| - |T(uv_2)| - |T(uv_3)|) \\
  &+ 4(|T(uv_1)| + |T(uv_2)| + |T(uv_3)|) \\
  &\leq
  n - 2\alpha + 4 \cdot |T(u)| \\
  &\leq
  n - 2\alpha + 4 \cdot \frac{n}{2} \\
  &<
  3n - 2\alpha \\
  &<
  0
\end{align*} \]

This contradicts the Nash equilibrium conditions. Now suppose \( u \in H \) buys \( uv_1, uv_2 \in X_2 \). We can apply Corollary 12 to the strategy change where \( v \) sells \( uv_1, uv_2 \in X_2^+ \) and buys \( ur \). The change in cost for \( u \) is at most

\[ \begin{align*}
  &
  n - \alpha - (d(u, r) + 1) \cdot (|T(v)| - |T(uv_1)| - |T(uv_2)|) + 4 \cdot (|T(uv_1)| + |T(uv_2)|) \\
  &\leq
  n - \alpha + 4 \cdot |T(uv_1) \cup T(uv_2)| \\
  &<
  4 \cdot |T(uv_1) \cup T(uv_2)| - n
\end{align*} \]

This is non-negative by the Nash equilibrium conditions. Therefore \( |T(uv_1) \cup T(uv_2)| > \frac{n}{3} \), as desired.

Finally, we observe some properties that follow when a vertex buys two edges in \( X_2 \).

**Observation 15.** If \( u \) buys two edges \( uv_1, uv_2 \in X_2 \), then \( d(r, u) \geq 3 \). If, in addition, \( |T(uv_1) \cup T(uv_2)| > \frac{n}{3} \) then \( d(r, u) = 3 \).

**Proof.** By Corollary 5, \( r \) cannot buy edges of \( X_2 \), therefore \( u \neq r \). Further, by Lemma 6, \( |T(u)| \leq \frac{n}{3} \).

We now apply Lemma 10 with the strategy change where \( u \) sells \( uv_1, uv_2 \). Its cost changes by at most

\[ \begin{align*}
  &
  d(r, u) \cdot n - 2\alpha - 2d(r, u) \cdot |T(u)| + (2d(r, u) + 4) \cdot |T(uv_1) \cup T(uv_2)| \\
  &\leq
  d(r, u) \cdot n - 2\alpha + 4 \cdot |T(uv_1) \cup T(uv_2)| \\
  &\leq
  d(r, u) \cdot n - 2\alpha + 4 \cdot \frac{n}{2}
\end{align*} \]
One $n$ Remains to Settle the Tree Conjecture

\[ < d(r, u) \cdot n - 2(2n) + 2n \]
\[ = (d(r, u) - 2) \cdot n \]

If $d(r, u) \leq 2$ then this cost change is negative, a contradiction. Thus $d(r, u) \geq 3$.

Now suppose $|T(uv_1) \cup T(uv_2)| > \frac{n}{2}$. Without loss of generality, let $|T(uv_1)| \geq |T(uv_2)|$. Therefore $|T(uv_1)| > \frac{n}{2}$, meaning $T(uv_1) = T(v_1)$. As $uv_1 \in X_2$, we know $v_1$ buys an edge in $X_0$ or $X_1 \setminus X_0$. Suppose $v_1$ buys $v_1y \in X_0$. By Corollary 11, $v$ selling $v_1y$ and buying $v_1r$ changes $y$'s cost by

\[ \leq n - (d(v_1, r) + 1) \cdot |T(v_1)| - (d(v_1, r) - 1) \cdot |T(u) \setminus T(v_1)| \]
\[ \leq n - 2 \cdot |T(v_1)| - (d(v, r) - 1) \cdot |T(u)| \]
\[ < n - 2 \cdot \frac{n}{8} - (d(v, r) - 1) \cdot \frac{n}{4} \]
\[ = n - (d(u, r) + 1) \cdot \frac{n}{4} \]
\[ \leq n - (3 + 1) \cdot \frac{n}{4} \]
\[ = 0 \]

This contradicts the equilibrium condition, therefore $v_1$ buys $v_1w \in X_1 \setminus X_0$. $w$ buys $wy \in X_0$. By Corollary 11, $w$ selling $wy$ and buying $wr$ changes $w$'s cost by

\[ \leq n - (d(w, r) + 1) |T(w)| - (d(w, r) - 1) |T(v_1) \setminus T(w)| - (d(w, r) - 3) |T(u) \setminus T(v_1)| \]
\[ \leq n - 2 \cdot |T(v_1)| - (d(w, r) - 3) \cdot |T(u)| \]
\[ < n - 2 \cdot \frac{n}{8} - (d(w, r) - 3) \cdot \frac{n}{4} \]
\[ = \frac{3}{2} n - d(w, r) \cdot \frac{n}{4} \]
\[ = \frac{3}{2} n - (d(u, r) + 2) \cdot \frac{n}{4} \]
\[ = n - d(u, r) \cdot \frac{n}{4} \]

If $d(r, u) \geq 4$, this leads to a contradiction. Thus $d(r, u) = 3$, as desired.

\[ \text{4 The Tree Conjecture holds for } \alpha > 2n \]

We now have all the resources necessary to prove the tree conjecture holds if $\alpha > 2n$. To do this, we prove two final lemmas which together give the main result. The first of these lemmas investigates the path to the root $r$ from a vertex $u \in H$ that buys an out-edge.

\[ \text{Lemma 16. Let } \alpha > 2n. \text{ If } u_0 \text{ buys } u_0v \in X_0 \text{ and } u_0, u_1, ... u_{d(u, r) - 1}, u_{d(u, r)} = r \text{ is the path from } u_0 \text{ to } r \text{ in } T \text{ then at most one of } u_0, u_1, u_2 \text{ has } \deg_{H} = 2 \]

\[ \text{Proof. By Observation 13, no vertex buys two edges in } X_{1}^{+}. \text{ Thus } u_0 \text{ cannot buy } u_0u_1 \text{ as well as } u_0v. \text{ Therefore } u_1 \text{ buys } u_1u_0. \text{ Similarly, by Observation 13, } u_1 \text{ cannot buy } u_1u_2 \text{ as well as } u_1u_0. \text{ Therefore } u_2 \text{ buys } u_2u_1. \]

Now if $\deg_{H}(u_0) = 2$ then $S_{H}(u_0) = T(u_0)$, because $u_1$ is $u_0$'s parent and $v \notin T(u_0)$, thus $T(u_0) \cap H = \{u_0\}$. Similarly, if $\deg_{H}(u_1) = 2$ then $S_{H}(u_1) = T(u_1) \setminus T(u_0)$, and if $\deg_{H}(u_2) = 2$ then $S_{H}(u_2) = T(u_2) \setminus T(u_1)$.\]
Next by Lemma 8, we know that \(d(r, u_2) \geq \left\lceil \frac{|C(u_2)|}{2} \right\rceil - 2\), that \(d(r, u_1) \geq \left\lceil \frac{|C(u_1)|}{2} \right\rceil - 1\) and that \(d(r, u_0) \geq \left\lceil \frac{|C(u_0)|}{2} \right\rceil\).

Furthermore, if \(\deg_H(u_1) = 2\) then, by Lemma 9, \(|S_H(u_1)| \geq |T(u_0)|\). If \(\deg_H(u_2) = 2\) then, by Lemma 3, \(u_2u_1\) is in a directed cycle, which means \(u_3\) buys \(u_2u_2\). Thus, by Lemma 9, we have \(|S_H(u_2)| \geq |T(u_1)|\).

We proceed by case analysis. First, assume that \(\deg_H(u_2) = \deg_H(u_1) = 2\). Then, by Lemma 9, we have \(|S_H(u_1)| \geq \frac{\alpha}{2(|C(u_1)| - 3)}\) and \(|S_H(u_2)| \geq |T(u_1)|\). Thus \(|T(u_2)| = |S_H(u_2)| + |T(u_1)| \geq 2|T(u_1)| \geq 2|S_H(u_1)| \geq \frac{2\alpha}{2(|C(u_1)| - 3)}\). Now, by Lemma 6, \(|T(u_2)| \leq \frac{n}{2}\).

Therefore \(\frac{n}{2} \geq \frac{2\alpha}{2(|C(u_1)| - 3)}\). Rearranging gives \(|C(u_1)| \geq 8\). By Observation 15, \(d(u_1, r) \geq 3\).

Applying Corollary 11 when \(u_1\) sells \(u_1u_0\) and buys \(u_1r\), the change in cost to \(u_1\) is at most

\[
\begin{align*}
&n - |T_{d(u_2,r)}| - \sum_{l < \frac{d(u_2,r)}{2}} 2 \cdot |T(u_l)| - (j - 1) \cdot \alpha + \sum_{k=1}^{j} (2i_k + d(u, r) + 1) \cdot |T(u_{v_k})| \\
&= n - |T_{d(u_2,r)}| - \sum_{l < \frac{d(u_2,r)}{2}} 2 \cdot |T(u_l)| + (d(u_1, r) + 1 + 2) \cdot |T(u_1u_0)| \\
&= n - |T_{d(u_2,r)}| - \sum_{l < \frac{d(u_2,r)}{2}} 2 \cdot |T(u_l)| + (d(u_1, r) + 3) \cdot |T(u_0)| \\
&\leq n - |T_{d(u_2,r)}| - \sum_{l < \frac{d(u_2,r)}{2}} 2 \cdot |T(u_l)| + (2d(u_1, r)) \cdot |T(u_0)| \\
&\leq n - 2|T(u_1)| - (d(u_1, r) - 1) \cdot |T(u_2)| + (2d(u_1, r)) \cdot |T(u_0)| \\
&= n - (d(r, u_1) + 1) \cdot |T(u_1)| - (d(u_1, r) - 1) \cdot |T(u_2)| + (2d(u_1, r)) \cdot |T(u_0)| \\
&\leq n - (d(r, u_2) + 1) \cdot |T(u_1)| - (d(u_2, r) - 1) \cdot |T(u_2)| + (2d(u_2, r)) \cdot |T(u_0)| \\
&= n - 2d(r, u_2) \cdot |T(u_2)| + 2d(u_1, r) \cdot |T(u_0)| \\
&= n - 2d(r, u_1) \cdot |S_H(u_1)| \\
&\leq n - 2d(r, u_1) \cdot \frac{\alpha}{2(|C(u_1)| - 3)} \\
&\leq n - 2(|C(u_1)| - 3) \cdot \frac{\alpha}{2(|C(u_1)| - 3)} \\
&= n - \frac{\alpha}{2} \\
&< 0
\end{align*}
\]

Therefore it cannot be that \(\deg_H(u_2) = \deg_H(u_1) = 2\).

Second, suppose \(\deg_H(u_0) = \deg_H(u_2) = 2\). By Lemma 9, we have \(|S_H(u_0)| \geq \frac{\alpha}{2(|C(u_1)| - 3)}\) and \(|S_H(u_2)| \geq |T(u_1)|\). Because \(u_0v \in X_0\), it follows that \(T(v) \not\subseteq T(u_0)\). Thus \(T(u_0v) = \emptyset\).

Applying Corollary 11 when \(u_0\) sells \(u_0v\) and buys \(u_0r\), the change in cost to \(u_0\) is at most

\[
\begin{align*}
&n - |T_{d(u_2,r)}| - \sum_{l < \frac{d(u_2,r)}{2}} 2 \cdot |T(u_l)| - (j - 1) \cdot \alpha + \sum_{k=1}^{j} (2i_k + d(u, r) + 1) \cdot |T(u_{v_k})| \\
&= n - |T_{d(u_2,r)}| - \sum_{l < \frac{d(u_2,r)}{2}} 2 \cdot |T(u_l)| \\
&\leq n - 2|T(u_0)| - 2|T(u_1)| - (d(u_0, r) - 3) \cdot |T(u_2)|
\end{align*}
\]
Therefore it is not true that \( \deg_H(u_2) = \deg_H(u_0) = 2 \).

Third, suppose \( \deg_H(u_0) = \deg_H(u_1) = 2 \). By Lemma 9, we have \( |S_H(u_0)| \geq \frac{\alpha}{2(|C(u_1)| - 3)} \) and \( |S_H(u_1)| \geq |T(u_0)| \). Because \( uv \in X_0 \), it follows that \( T(v) \not\subseteq T(u_0) \). Thus \( T(u_0) v = 0 \).

Applying Corollary 11 when \( u_0 \) sells \( \alpha \) and buys \( \alpha \), the change in cost to \( u_0 \) is at most

\[
\begin{align*}
&n - 2(T(u_0) - (d(u_0, r) - 1) \cdot |T(u_1)| - (d(u_0, r) - 3) \cdot |T(u_0) \setminus T(u_1)| \\
n &\leq n - 2(T(u_0) - (d(u_0, r) - 1) \cdot |T(u_1)| \\
&= n - (d(u_0, r) + 1) \cdot |T(u_0)| - (d(u_0, r) - 1) \cdot |T(u_0) \setminus T(u_1)| \\
&\leq n - (d(u_0, r) + 1) \cdot |T(u_0)| - (d(u_0, r) - 1) \cdot |T(u_0)| \\
&\leq n - (2d(u_0, r)) \cdot |T(u_0)| \\
&= n - (2d(u_0, r)) \cdot |S_H(u_0)| \\
&\leq n - 2d(r, u_0) \cdot \frac{\alpha}{2(|C(u_0)| - 3)} \\
&\leq n - 2 \frac{|C(u_0)|}{2(|C(u_0)| - 3)} - 3 \cdot \frac{\alpha}{2(|C(u_0)| - 3)} \\
&= n - \frac{\alpha}{2} < 0
\end{align*}
\]

Therefore it is not true that \( \deg_H(u_1) = \deg_H(u_0) = 2 \). Thus, at most one of \( u_0, u_1, u_2 \) has \( \deg_H = 2 \), as desired.

We now apply a counting argument to upper bound the number of vertices in \( H \) that buy two edges in \( X_2 \).

**Lemma 17.** The number of vertices \( v \in H \) that buy two edges in \( X_2 \) is \( < \deg_H(r) \).

**Proof.** The only vertices that can buy two edges \( e_1, e_2 \) in \( X_2 \) are those with \( |T(e_1) \cup T(e_2)| > \frac{\alpha}{2} \).

By Observation 14. All of these vertices are distance 3 from \( r \), by Observation 15. The common ancestor in \( T \) of any pair of these vertices must be \( r \), otherwise there is a vertex.
\( u \neq r \) with \( |T(u)| > \frac{n}{2} \), contradicting Lemma 6. Furthermore, \( r \) cannot buy any edge \( e \) such that \( T(e) \) contains such a vertex \( v \), otherwise \( r \) could sell \( e \) and buy \( rv \), reducing its distance to \( T(v) \) by 2 and increasing its distance to \( T(e) \setminus T(v) \) by \( \leq 2 \). This gives a net increase in cost of \( \leq 2|T(v)| \leq 2|T(e) \setminus T(v)| \leq 2|T(e)| < \frac{4n}{2} - \frac{2n^2}{2} = 0 \), a contradiction. Therefore the number of vertices which buy two edges in \( X_2 \) is \( \leq \deg^{-H}(r) \). By Lemma 3, we must have \( \deg^++H(r) \geq 1 \), implying \( \deg^{-H}(r) < \deg^+H(r) \) and completing the proof. ▶

Putting everything together we can now prove the main result.

Proof of Theorem 1. Here we use combine all to prove that if \( G \) is a Nash equilibrium graph for the network creation game \((n, \alpha)\) and \( \alpha > 2n \), \( G \) is a tree.

This is a proof by contradiction based on the assumption that there exists a biconnected component \( H \) in \( G \) containing \( n_H \) vertices. Within \( H \), the sum of degrees of all vertices in \( H \) equals \( 2(n_H - 1) + 2 \sum_{v \in H, \deg_H(v) = i} i \). That is twice the number of edges in the spanning tree on \( H \) induced by \( T \) plus twice the number of out-edges in \( H \).

Let’s instead count the degrees in the following way:

\[
\sum_{i=2}^{n_H} \sum_{v \in H, \deg_H(v) = i} i = 2n_H + \sum_{i=3}^{n_H} \sum_{v \in H, \deg_H(v) = i} i - 2
= 2n_H + \deg_H(r) - 2 + \sum_{i=3}^{n_H} \sum_{v \in H \setminus r, \deg_H(v) = i} i - 2
\geq 2n_H + \deg_H(r) - 2 + 2|X_0| - \sum_{i=2}^{\frac{n_H}{2}} \sum_{v \in H \setminus v \text{ is in } i \text{ paths of } X_2 \text{ edges}} i - 1 \quad [\text{by Lemma 16}]
= 2n_H + \deg_H(r) - 2 + 2|X_0| - \sum_{v \in H \setminus v \text{ is in } 2 \text{ paths of } X_2 \text{ edges}} 1 \quad [\text{by Observation 14}]
> 2n_H + \deg_H(r) - 2 + 2|X_0| - \deg_H(r) \quad [\text{by Lemma 17}]
= 2(n_H - 1) + 2|X_0|
\]

This is a contradiction, implying that \( H \) does not exist for \( \alpha > 2n \). ▶

5 Conclusion

In this paper we proved the revised tree conjecture holds for \( \alpha > 2n \). Moreover, we have reached a natural limit in the quest to settle this conjecture. Specifically, for \( \alpha \in (n - 3, 2n] \), the range in which the conjecture is unsettled, we are no longer certain that directed cycles must be present in non-tree equilibrium graphs. To close this remaining gap, we believe min-cycles still have an important role to play but, to allow their usage, more precise analyses will be necessary. A potentially useful intermediate step would be to determine conditions that allow for an undirected min-cycle.
References


