Conditional Independence on Semiring Relations

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Abstract

Conditional independence plays a foundational role in database theory, probability theory, information theory, and graphical models. In databases, a notion similar to conditional independence, known as the (embedded) multivalued dependency, appears in database normalization. Many properties of conditional independence are shared across various domains, and to some extent these commonalities can be studied through a measure-theoretic approach. The present paper proposes an alternative approach via semiring relations, defined by extending database relations with tuple annotations from some commutative semiring. Integrating various interpretations of conditional independence in this context, we investigate how the choice of the underlying semiring impacts the corresponding axiomatic and decomposition properties. We specifically identify positivity and multiplicative cancellativity as the key semiring properties that enable extending results from the relational context to the broader semiring framework. Additionally, we explore the relationships between different conditional independence notions through model theory.

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1 Introduction

Conditional independence (CI) is an expression of the form $Y \perp\!\!\!\perp Z \mid X$, stating that Y and Z are conditionally independent given X. Common to its different interpretations is that conditional independence is a mark of redundancy. For instance, on a relation schema over attributes X, Y, Z, the multivalued dependency (MVD) $X \twoheadrightarrow Y$ can be viewed as the counterpart of the CI $Y \perp\!\!\!\perp Z \mid X$, expressing that a relation can be losslessly decomposed into its projections on X, Y and X, Z. The process of splitting the schema into smaller parts – in order to avoid data redundancy – is called normalization, and a database schema is in fourth normal form if every non-trivial MVD follows from some key. In probability theory, CIs over random variables give rise to factorizations of joint probability distributions into conditional distributions. Since the decomposed distributions can be represented more compactly, this allows more efficient reasoning about the random variables. In addition to these classical examples, conditional independence has applications in ordinal conditional functions [24], Dempster-Schaefer theory [8, 23], and possibility theory [29].

Since the notion of conditional independence has a relatively fixed meaning across various contexts, it is no coincidence that the central rules governing its behavior are universally shared. The semigraphoid axioms [21] state five basic rules that hold true for diverse interpretations of conditional independence. Initially conjectured to be complete by Pearl, Studený [25] proved incompleteness of these rules by discovering a new rule that is not derivable by the semigraphoid axioms, while being sound for probability distributions. Later he [26] proved that there cannot be any finite axiomatization for conditional independence, a fact that had been established earlier for embedded multivalued dependencies (EMVDs) [15].

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The *implication problem*, which is to determine whether some set of dependencies Σ logically implies a dependency τ, is in fact undecidable not only for EMVDs [14], but also for CIs in probability theory, as has been recently shown [18, 20].

In some partial cases, the semigraphoid axioms are known to be complete. A *saturated conditional independence* (SCI) is a CI that contains all the variables of the underlying joint distribution. The semigraphoid axioms are complete for the implication of arbitrary CIs by saturated ones under various semantics [12], and for the implication of CIs from a set of CIs encoded in the topology of a Bayesian network [10]. In databases, where SCIs correspond to MVDs, the implication problem for MVDs combined with functional dependencies (FDs) is well-known to have a finite axiomatization and a polynomial-time algorithm [4].

Moving beyond saturated CIs, the implication problem not only becomes undecidable, but also more sensitive to the underlying semantics. Studený [27] presents several example inference rules that involve non-saturated CIs and are sound in one setting while failing to be sound in others. For instance, the aforementioned rule[1] showing incompleteness of the semigraphoid axioms is not sound for database relations, but its soundness for probability distributions follows by a simple information-theoretic argument. FDs and MVDs can also be alternatively expressed in terms of information measures over a uniformly distributed database relation [19], and their implication problem has recently been connected to validity of information inequalities [16]. Galliani and Väänänen [9] associate relations with a so-called diversity measure to capture FDs and other data dependencies. These measure-theoretic approaches, however, fail to capture the semantics of the embedded multivalued dependency in full generality.

This paper examines K-relations as a unifying framework for conditional independence and other dependency concepts. Introduced in the seminal work [11], K-relations extend ordinary relations by tuple annotations from a commutative semiring K, providing a powerful abstraction for data provenance. While it is natural to consider propagation of tuple annotations through queries in this context, one can also ask how tuple annotations couple with data dependencies. Dependencies on K-relations have thus far received limited attention (see, e.g., [2, 3, 7]). Related to this work, Barlag et al. [3] define conditional independence for K-relations, and raise the question of how much the related axiomatic properties depend on the algebraic properties of K. Atserias and Kolaitis [2] study the relationship between local and global consistency for K-relations, introducing also many concepts that will be adopted in this paper. Although the authors do not consider conditional independence, they show that functional dependencies on K-relations entail lossless-join decompositions.

The following contributions are presented in this paper: First, we show that conditional independence for K-relations corresponds to lossless-join decompositions whenever K is positive and multiplicatively cancellative. Then, we provide a proof that, for any K exhibiting these characteristics, the semigraphoid axioms are sound for general CIs, and extend to a complete axiomatization of SCI+FD which is comparable to that of MVD+FD. This entails that database normalization techniques extend to K-relations whenever positivity and multiplicative cancellativity are assumed. To showcase potential applications, we illustrate through an example how the semiring perspective can lead to decompositions of data tables which appear non-decomposable when interpreted relationally. Lastly, we explore how K-relations and model theory can shed light into the interconnections among different CI semantics.

[1] This rule states that $A \perp\!\!\!\perp B \mid CD \wedge C \perp\!\!\!\perp D \mid A \wedge C \perp\!\!\!\perp D \mid B \wedge A \perp\!\!\!\perp B \mid \emptyset$ if and only if $C \perp\!\!\!\perp D \mid AB \wedge A \perp\!\!\!\perp B \mid C \wedge A \perp\!\!\!\perp B \mid D \wedge C \perp\!\!\!\perp D \mid \emptyset$. For probability distributions the rule follows by the non-negativity of conditional mutual information $I(Y; Z|C)$, and the fact that $I(Y; Z|X) = 0$ if and only if Y and Z are conditionally independent given X. For database relations the rule is not sound; see a counterexample in [27].

2 Semirings

We commence by recapitulating concepts related to semirings. A *semiring* is a tuple $K = (K, \oplus, \otimes, 0, 1)$, where $\oplus$ and $\otimes$ are binary operations on K, $(K, \oplus, 0)$ is a commutative monoid with identity element 0, $(K, \otimes, 1)$ is a monoid with identity element 1, $\otimes$ left and right distributes over $\oplus$, and $x \otimes 0 = 0 = 0 \otimes x$ for all $x \in K$. The semiring K is called *commutative* if $(K, \otimes, 1)$ is a commutative monoid. That is, semirings are rings which need not have additive inverses. As usual, we often write ab instead of $a \otimes b$. In this paper, we assume that every semiring is non-trivial ($0 \neq 1$) and commutative. The symbols $\oplus, \otimes, \bigoplus, \bigotimes$ are used in reference to specific semiring operations, and symbols $+, \cdot, \sum, \prod$ refer to ordinary arithmetic operations.

We list some example semirings that will be considered in this paper.

- The *Boolean semiring* $\mathbb{B} = (\mathbb{B}, \vee, \wedge, 0, 1)$ models logical truth and is formed from the two-element Boolean algebra. It is the simplest example of a semiring that is not a ring.
- The *probability semiring* $\mathbb{R}_{\geq 0} = (\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$ consists of the non-negative reals with standard addition and multiplication.
- The *semiring of natural numbers* $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ consists of natural numbers with their usual operations.
- The *tropical semiring* $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$ consists of the reals expanded with infinity and has min and standard addition respectively plugged in for addition and multiplication.
- The *Viterbi semiring* $\mathbb{V} = ([0, 1], \max, \cdot, 0, 1)$ associates the unit interval with maximum as addition and standard multiplication.

Other examples include the semiring of multivariate polynomials $\mathbb{N}[\boldsymbol{X}] = (\mathbb{N}[\boldsymbol{X}], +, \cdot, 0, 1)$ which is the free commutative semirings generated by the indeterminates in $\boldsymbol{X}$, and the *Lukasiewicz semiring* $\mathbb{L} = ([0, 1], \max, \cdot, 0, 1)$, used in multivalued logic, which endows the unit interval with max addition and multiplication $a \cdot b := \max(0, a + b - 1)$.

Let $\leq$ be a partial order. A binary operator $*$ is said to be *monotone under* $\leq$ if $a \leq b$ and $a' \leq b'$ implies $a * a' \leq b * b'$. If $* = \oplus$ (resp. $* = \otimes$), we call this property of $(K, \leq)$ *additive monotony* (resp. *multiplicative monotony*). A *partially ordered semiring* is a tuple $K = (K, \oplus, \otimes, 0, 1, \leq)$, where $(K, \oplus, \otimes, 0, 1)$ is a semiring, and $(K, \leq)$ is a partially ordered set satisfying additive and multiplicative monotony. Given a semiring $K = (K, \oplus, \otimes, 0, 1)$, define a binary relation $\leq_K$ on K as

$$a \leq_K b :\Leftrightarrow \exists c : a \oplus c = b. \tag{1}$$

This relation is a preorder, meaning it is reflexive and transitive. If $\leq_K$ is also antisymmetric, it is a partial order, called the *natural order* of K, and K is said to be *naturally ordered*. In this case, K endowed with its natural order is a partially ordered semiring. If additionally the natural order is *total*, i.e., $a \leq_K b$ or $b \leq_K a$ for all $a, b \in K$, we say that K is *naturally totally ordered*.

If a semiring K satisfies $ab = 0$ for some $a, b \in K$ where $a \neq 0 \neq b$, we say that K has *divisors of* 0. The semiring K is called $\oplus$*-positive* if $a \oplus b = 0$ implies that $a = b = 0$. If K is both $\oplus$-positive and has no divisors of 0, it is called *positive*. For example, the modulo two integer semiring $\mathbb{Z}_2$ is not positive since it is not $\oplus$-positive (even though it has no divisors of 0). Conversely, an example of a semiring with divisors of 0 is $\mathbb{Z}_4$. A semiring is called *additively* (resp. *multiplicatively*) *cancellative* if $a \oplus b = a \oplus c$ implies $b = c$ (resp. $ab = ac$ and $a \neq 0$ implies $b = c$). It is simply *cancellative* if it is both additively and multiplicatively cancellative. A semiring K in which each non-zero element has a multiplicative inverse is called a *semifield*. A semifield K in which each element has an additive inverse is a *field*.

In particular, note that the probability semiring $\mathbb{R}_{\geq 0}$, the semiring of natural numbers $\mathbb{N}$, the Boolean semiring $\mathbb{B}$, and the tropical semiring are positive, multiplicatively cancellative, and naturally ordered. Of these only the first two are also additively cancellative. This difference seems to be crucial for the behavior of conditional independence.

This section concludes with two lemmata. The first lemma is applied when examining the relationship between lossless-join decompositions and conditional independence (Theorem 10). The second lemma comes into play when comparing the CI implication problem for different semirings (Theorem 20). A formal definition of an *embedding* of a model into another model is located in Appendix A. The lemma proofs can be found in the arXiv version [13]. A field F endowed with a total order $\leq$ is a *totally ordered field* if $(F, \leq)$ satisfies additive monotony and *monotony of non-negative multiplication*: $a \geq 0$ and $b \geq 0$ implies $ab \geq 0$.

▶ **Lemma 1.** *Any positive multiplicatively cancellative semiring K embeds in a positive semifield F. Furthermore, if K is additively cancellative, then F is additively cancellative, and if K is a naturally totally ordered, then F is naturally totally ordered.*

▶ **Lemma 2.** *Any naturally totally ordered cancellative semiring embeds in a totally ordered field.*

3 K-relations

This section introduces ordinary relations as well as K-relations and their associated basic properties.

We use boldface letters to denote sets. For two sets $\boldsymbol{X}$ and $\boldsymbol{Y}$, we write $\boldsymbol{XY}$ to denote their union. If A is an individual element, we sometimes write A instead of $\{A\}$ to denote the singleton set consisting of A.

3.1 Relations

Fix disjoint countably infinite sets Var and Val of variables and values. Each variable $A \in \mathsf{Var}$ is associated with a subset of Val, called the *domain* of A and denoted $\mathrm{Dom}(A)$. Given a finite set of variables $\boldsymbol{X}$, an $\boldsymbol{X}$*-tuple* is a mapping $t : \boldsymbol{X} \to \mathsf{Val}$ such that $t(A) \in \mathrm{Dom}(A)$. We write $\mathrm{Tup}(\boldsymbol{X})$ for the set of all $\boldsymbol{X}$-tuples. Note that $\mathrm{Tup}(\emptyset)$ is a singleton set consisting of the empty tuple. For $\boldsymbol{Y} \subseteq \boldsymbol{X}$, the *projection* $t[\boldsymbol{Y}]$ of t on $\boldsymbol{Y}$ is the unique $\boldsymbol{Y}$-tuple that agrees with t on $\boldsymbol{X}$. In particular, $t[\emptyset]$ is always the empty tuple.

A *relation* R over $\boldsymbol{X}$ is a subset of $\mathrm{Tup}(\boldsymbol{X})$. The variable set $\boldsymbol{X}$ is also called the *(relation) schema of* R. We sometimes write $R(\boldsymbol{X})$ instead of R to emphasize that $\boldsymbol{X}$ is the schema of R. For $\boldsymbol{Y} \subseteq \boldsymbol{X}$, the *projection* of R on $\boldsymbol{Y}$, written $R[\boldsymbol{Y}]$, is the set of all projections $t[\boldsymbol{Y}]$ where $t \in R$. A *database* D is a finite collection of relations $\{R_1[\boldsymbol{X}_1], \ldots, R_n[\boldsymbol{X}_n]\}$. Unless stated otherwise, we assume that each relation is finite.

3.2 K-relations

Fix a semiring K, and let $\boldsymbol{X}$ be a set of variables. A K*-relation* over $\boldsymbol{X}$ is a function $R : \mathrm{Tup}(\boldsymbol{X}) \to K$. Again, the variable set $\boldsymbol{X}$ is called the *(relation) schema of* R, and we can write $R(\boldsymbol{X})$ instead of R to emphasize that $\boldsymbol{X}$ is the schema of R. If K is the Boolean semiring $\mathbb{B}$, the tuple annotation $R(t)$ characterizes an ordinary relation, and thus we will often in this paper identify $\mathbb{B}$-relations and relations. Note that a K-relation over $\emptyset$ associates the empty tuple with some value of K. The *support* $\mathrm{Supp}(R)$ of a K-relation R over $\boldsymbol{X}$ is the set $\{t \in \mathrm{Tup}(\boldsymbol{X}) \mid R(t) \neq 0\}$ of tuples associated with a non-zero value. We often write

R' for the support of R. The K-relation R is called *total* if for all $t \in \mathrm{Tup}(\boldsymbol{X})$ it holds that $R(t) \neq 0$, i.e., if $\mathrm{Supp}(R) = \mathrm{Tup}(\boldsymbol{X})$. It is called *normal* if $\bigoplus_{t\in\mathrm{Tup}(\boldsymbol{X})} R(t) = 1$. For $a \in K$, we write aR for the K-relation over $\boldsymbol{X}$ defined by $(aR)(t) = aR(t)$. For a $\boldsymbol{Y}$-tuple t, where $\boldsymbol{Y} \subseteq \boldsymbol{X}$, the *marginal* of R over t is defined as

$$R(t) := \bigoplus_{\substack{t'\in\mathrm{Tup}(\boldsymbol{X})\\ t'[\boldsymbol{Y}]=t}} R(t'). \tag{2}$$

We then write $R[\boldsymbol{Y}]$ for the relation over $\boldsymbol{Y}$, called the *marginal* of R on $\boldsymbol{Y}$, that consists of the marginals of R over all $\boldsymbol{Y}$-tuples. Note that the marginal $R[\emptyset]$ of R on the empty set is a function that maps the empty tuple to $\sum_{t\in\mathrm{Tup}(\boldsymbol{X})} R(t)$. In particular, if K is the Boolean semiring $\mathbb{B}$, the marginal of R on $\boldsymbol{Y}$ is the projection of R on $\boldsymbol{Y}$. In this paper, we assume that each relation is finite and non-empty, and likewise each K-relation is assumed to have a finite and non-empty support.

K-relations instantiated in different ways lead to familiar notions. For instance, a database relation can be viewed as $\mathbb{B}$-relation, and a probability distribution as a normal $\mathbb{R}_{\geq 0}$-relation. Alternatively, database relations can be transformed to K-relations by reinterpreting variables as tuple annotations.

▶ **Example 3.** Tab. 1 collects data about room prizes in a hotel. The table can be viewed as a standard database relation. Since Price is a function of Room, Date, and Persons, one can also interpret it as a K-relation Price(Room, Date, Persons) over some semiring K containing positive integers. In principle, other variables such as Room and Persons can also be turned into annotations.

Table 1 Price data for hotel rooms.

Room	Date	Persons	Price
double	2023-12-01	1	100
double	2023-12-01	2	120
double	2023-08-20	1	120
double	2023-08-20	2	140
twin	2023-08-20	1	110
twin	2023-08-20	2	120

3.3 Basic properties

Prior to delving into the concept of conditional independence, we here list some basic properties regarding projections and supports of K-relations. Lemmata 4 and 5 appear in [2], with the exception that there K is always assumed to be positive. Also the concept of a marginal in that paper is stated otherwise as in Eq. (2), except that there t' ranges over R' instead of $\mathrm{Tup}(X)$. Obviously the two versions lead to the same concept. To account for these slight modifications, we include the proofs of these two lemmata in the arXiv version [13].

▶ **Lemma 4.** *Let $R(\boldsymbol{X})$ be a K-relation, and let $\boldsymbol{Z} \subseteq \boldsymbol{Y} \subseteq \boldsymbol{X}$. The following statements hold:*

1. *Assuming K is $\oplus$-positive, for all $\boldsymbol{Y} \subseteq \boldsymbol{X}$ it holds that $R'[\boldsymbol{Y}] = R[\boldsymbol{Y}]'$.*
2. *For all $\boldsymbol{Z} \subseteq \boldsymbol{Y} \subseteq \boldsymbol{X}$ it holds that $R[\boldsymbol{Y}][\boldsymbol{Z}] = R[\boldsymbol{Z}]$.*

Two K-relations R and R' over a variable set $\boldsymbol{V}$ are said to be *equivalent (up to normalization)*, written $R \equiv R'$, if there are $a, b \in K \setminus \{0\}$ such that $aR = bR'$.

▶ **Lemma 5.** *Let K be a semiring, let $\boldsymbol{W}, \boldsymbol{V}$, $\boldsymbol{W} \subseteq \boldsymbol{V}$, be two variable sets, and let R, R', R'' be three K-relations over $\boldsymbol{V}$. Then,*

1. *$R \equiv R'$ implies $R[\boldsymbol{W}] \equiv R'[\boldsymbol{W}]$; and*
2. *if K has no divisors of zero, $R \equiv R'$ and $R' \equiv R''$ implies $R \equiv R''$.*

4 Conditional independence and decompositions

Regardless of the context, what we call conditional independence tends to describe essentially the same property. For a "system" consisting of three components $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, we might say that $\boldsymbol{Y}$ is conditionally independent of $\boldsymbol{Z}$ given $\boldsymbol{X}$ if $\boldsymbol{Y}$ does not reveal anything about $\boldsymbol{Z}$, once $\boldsymbol{X}$ has been fixed. This usually entails that the "system" can be decomposed to its "subsystems" over $\boldsymbol{X}, \boldsymbol{Y}$ and $\boldsymbol{X}, \boldsymbol{Z}$ without loss of information. In this section we consider a general semantics for conditional independence over K-relations, and show that under certain assumptions, this definition matches the above intuition.

▶ **Definition 6** (Conditional independence for K-relations [3]). *Let R be a K-relation over a variable set $\boldsymbol{V}$, and let $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be disjoint subsets of $\boldsymbol{V}$. An expression of the form $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$ is called a* conditional independence *(CI). We say that R* satisfies *$\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$, denoted $R \models \boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$, if for all $\boldsymbol{V}$-tuples t,*

$$R(t[\boldsymbol{XY}])R(t[\boldsymbol{XZ}]) = R(t[\boldsymbol{XYZ}])R(t[\boldsymbol{X}]). \tag{3}$$

Fix a relation schema $\boldsymbol{V}$ and three pairwise disjoint subsets $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \subseteq \boldsymbol{V}$. A *saturated conditional independence* (SCI) is a CI of the form $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$, where $\boldsymbol{XYZ} = \boldsymbol{V}$. Over $\mathbb{B}$-relations SCIs coincide with *multivalued dependencies* (MVDs), which are expressions of the form $\boldsymbol{X} \twoheadrightarrow \boldsymbol{Y}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ may overlap. A $\boldsymbol{V}$-relation R *satisfies* $\boldsymbol{X} \twoheadrightarrow \boldsymbol{Y}$, written $R \models \boldsymbol{X} \twoheadrightarrow \boldsymbol{Y}$, if for all two tuples $t, t' \in R$ such that $t[\boldsymbol{X}] = t'[\boldsymbol{X}]$ there exists a third tuple $t'' \in R$ such that $t''[\boldsymbol{XY}] = t'[\boldsymbol{XY}]$ and $t[\boldsymbol{V} \setminus \boldsymbol{XY}] = t'[\boldsymbol{V} \setminus \boldsymbol{XY}]$. An *embedded multivalued dependency* (EMVD) is an expression of the form $\boldsymbol{X} \twoheadrightarrow \boldsymbol{Y} \mid \boldsymbol{Z}$, where $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ may overlap. We say that R *satisfies* $\boldsymbol{X} \twoheadrightarrow \boldsymbol{Y} \mid \boldsymbol{Z}$, written $R \models \boldsymbol{X} \twoheadrightarrow \boldsymbol{Y} \mid \boldsymbol{Z}$, if the projection $R[\boldsymbol{XYZ}]$ satisfies the MVD $\boldsymbol{X} \twoheadrightarrow \boldsymbol{Y}$.

▶ **Example 7.** Returning to Example 3, we observe that the price function Price(Room, Date, Persons) exhibits certain types of dependencies between its arguments. The room prices vary depending on the date and the room type. Additionally, adding a second person incurs a price increase by a flat rate which is independent of the date but depends on the room type. This kind of independence can be captured by viewing the price function as a $\mathbb{T}$-relation, in which case it satisfies the SCI Date $\perp\!\!\!\perp$ Persons | Room. Suppose instead of a flat price increase, the addition of a second person incurs a 20% price increase for double rooms, and a 10% price increase for twin rooms. Then, interpreting Price(Room, Date, Persons) as a $\mathbb{R}_{\geq 0}$-relation, we again obtain Price $\models$ Date $\perp\!\!\!\perp$ Persons | Room. When Tab. 1 is viewed as an ordinary relation, it satisfies the EMVD Room $\twoheadrightarrow$ Date | Persons, while failing to satisfy any MVD.

Several conditional independence notions from the literature can be recovered through K-relations. For instance, beside EMVDs, the following examples were considered in [27] and can now be restated using the previous definition.

- For $K = \mathbb{R}_{\geq 0}$, the definition coincides with the concept of conditional independence in probability theory.
- For $K = \mathbb{T}$, the definition correponds to conditional independence over natural conditional functions. A *natural conditional function* is a mapping $f\colon \mathrm{Tup}(\boldsymbol{X}) \to \mathbb{N}$, where $\min_{t\in\mathrm{Tup}(\boldsymbol{X})} f(t) = 0$. The notion of conditional independence over such functions [27] coincides with Def. 6 over integral-valued, total, and normal $\mathbb{T}$-relations. Recall that for (min-plus) tropical semirings, addition is interpreted as minimum, and multiplication as the usual addition, meaning that its neutral element is 0.
- For $K = \mathbb{V}$, the definition correponds to conditional independence over possibility functions. A *possibility function* is a function $f\colon \mathrm{Tup}(\boldsymbol{X}) \to [0,1]$, where $\sum_{t\in\mathrm{Tup}(\boldsymbol{X})} f(t) = 1$. Such functions can be viewed as normal $\mathbb{V}$-relations, where $\mathbb{V}$ is the Viterbi semiring, in which case their notion of conditional independence [27] matches Def. 6.

In order to connect conditional independence over K-relations to decompositions, we next consider the concept of a join. An arguably reasonable expectation is that whenever a K-relation $T(ABC)$ satisfies a CI $A \perp\!\!\!\perp C \mid B$, then one should be able to retrieve T from its projections on AB and BC using the join. That is, T should be equivalent to the join of $T[AB]$ and $T[BC]$ up to normalization. In the relational context this is indeed the outcome once $A \perp\!\!\!\perp C \mid B$ is interpreted as the MVD $B \twoheadrightarrow A$, and the join $R \bowtie S$ of two relations $R(\boldsymbol{X})$ and $S(\boldsymbol{Y})$ is given in the usual way, i.e., as the relation consisting of those $\boldsymbol{XY}$-tuples t whose projections $t[\boldsymbol{X}]$ and $t[\boldsymbol{Y}]$ appear respectively in R and S. In the context of K-relations the join of $R(\boldsymbol{X})$ and $S(\boldsymbol{Y})$ is often defined via multiplication as the K-relation $R * S$ over $\boldsymbol{XY}$ where

$$(R * S)(t) = R(t[\boldsymbol{X}])S(t[\boldsymbol{Y}]) \tag{4}$$

(see, e.g., [11]). Substituting $K = \mathbb{B}$ in this definition now yields the standard relational join. Similarly, letting $K = \mathbb{N}$ we arrive at the bag join operation of SQL. However, as illustrated in the next example, this notion of a join falls short of our expectations.

▶ **Example 8.** Continuing our running example, the two top tables in Fig. 1 illustrate the projections of the $\mathbb{T}$-relation Price(Room, Date, Persons) on {Room, Date} and {Room, Persons}. The table in the bottom row is the multiplicative join (4) of the two projections. Note that in the tropical semiring the aforementioned projections are formed as minima of prices, while addition plays the role of multiplication in the join operation. We observe that the multiplicative join is not equivalent to the original price function. In particular, there is no uniform (tropical) scaling factor that returns us Price from Price([Room, Date]) $*$ Price([Room, Persons]).

Two K-relations $R(\boldsymbol{X})$ and $S(\boldsymbol{Y})$ are said to be *consistent* if there exists a third relation $T(\boldsymbol{XY})$ such that $T[\boldsymbol{X}] \equiv R$ and $T[\boldsymbol{Y}] \equiv S$. Atserias and Kolaitis [2] demonstrate that the multiplicative join does not always witness the consistency of two K-relations, a fact that can be also seen from our running example. Consequently, they introduce a novel join operation which we will now incorporate into our approach. Intuitively this notion of a join is an adaptation of the factorization of a probability distribution obtained from conditional independence. Suppose two random events A and C are independent given a third event B. The joint probability $P(A, B, C)$ can then be rewritten as $P(B)P(A \mid B)P(C \mid B) = P(A, B)P(B, C)/P(B)$. We may recognize that this equation is similar to the multiplicative join of two K-relations *conditioned* on their common part. In our example this corresponds to multiplying the multiplicative join Price([Room, Date]) $*$ Price([Room, Persons]) with the (tropical) multiplicative inverse of Price([Room]). We observe from Fig. 1 that this sequence of operations yields the initial price function depicted in Tab. 1 (even without re-scaling), in accordance with our expectations.

Price[Room, Date]

Room	Date	*Price*
double	2023-12-01	100
double	2023-08-20	120
twin	2023-08-20	110

Price[Room, Persons]

Room	Persons	*Price*
double	1	100
double	2	120
twin	1	110
twin	2	120

Price([Room, Date]) $*$ Price([Room, Persons])

Room	Date	Persons	*Price*
double	2023-12-01	1	200
double	2023-12-01	2	220
double	2023-08-20	1	220
double	2023-08-20	2	240
twin	2023-08-20	1	220
twin	2023-08-20	2	230

(Price[Room])$^{-1}$

Room	*Price*
double	-100
twin	-110

Figure 1 Decomposition of the price function.

We will now provide a precise definition of the join operation introduced in [2]. This definition matches the above intuitive description with one exception: Semirings generally lack multiplicative inverses, and therefore the conditioning on the common part of two K-relations is defined indirectly. For a K-relation $R(\boldsymbol{X})$, a subset $\boldsymbol{Z} \subseteq \boldsymbol{X}$, and a $\boldsymbol{Z}$-tuple u, define

$$c^*_{R,\boldsymbol{Z}} := \bigotimes_{v \in R[\boldsymbol{Z}]'} R(v) \quad \text{and} \quad c_R(u) := \bigotimes_{\substack{v \in R[\boldsymbol{Z}]' \\ v \neq u}} R(v),$$

with the convention that the empty product evaluates to 1, the neutral element of multiplication in K. We isolate the following simple property which is applied frequently in the sequel.

▶ **Proposition 9.** *Suppose K does not have divisors of zero. If $R(\boldsymbol{X})$ is a K-relation, $\boldsymbol{Z} \subseteq \boldsymbol{X}$, and u is a $\boldsymbol{Z}$-tuple, then $c^*_{R,\boldsymbol{Z}} \neq 0$ and $c_R(u) \neq \emptyset$*

If $R(\boldsymbol{X})$ and $S(\boldsymbol{Y})$ are two K-relations, the *join* $R \bowtie S$ of R and S is the K-relation over $\boldsymbol{XY}$ defined by

$$(R \bowtie S)(t) := R(t[\boldsymbol{X}])S(t[\boldsymbol{Y}])c_S(t[\boldsymbol{X} \cap \boldsymbol{Y}]). \tag{5}$$

If K is a semifield (i.e., it has multiplicative inverses), we may rewrite the join as

$$(R \bowtie S)(t) = \frac{c^*_{S,\boldsymbol{X}\cap\boldsymbol{Y}} R(t[\boldsymbol{X}])S(t[\boldsymbol{Y}])}{S(t[\boldsymbol{X} \cap \boldsymbol{Y}])}.$$

The definition of $R \bowtie S$ is not symmetric, and hence there may be occasions where commutativity fails, i.e., $R \bowtie S \neq S \bowtie R$. However, whenever R and S agree on the marginals on their shared variable set $\boldsymbol{X} \cap \boldsymbol{Y}$, commutativity holds by definition. In particular, Lemma 4 entails that the join of two projections $R[\boldsymbol{X}]$ and $R[\boldsymbol{Y}]$ of the same relation R is commutative.

The join operation (5) can also be described in terms of conditional independence and consistency. Suppose K is a semifield, and suppose $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ are pairwise disjoint. Let $S(\boldsymbol{XY})$ and $T(\boldsymbol{XZ})$ be two normal K-relations that are consistent. Since equivalence entails identity for normal K-relations over semifields K, this is tantamount to finding a normal K-relation $R(\boldsymbol{XYZ})$ such that

$$R[\boldsymbol{XY}] = S \text{ and } R[\boldsymbol{XZ}] = T. \tag{6}$$

In particular, Lemma 4 and Eq. (6) yield $S[\boldsymbol{X}] = T[\boldsymbol{X}]$, whereby $c^*_{S,\boldsymbol{X}} = c^*_{T,\boldsymbol{X}}$. We may now observe that $R = 1/c^*_{T,\boldsymbol{X}}(S \bowtie T)$ is the unique K-relation that satisfies (6) and the CI $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$. In the particular case where $K = \mathbb{R}_{\geq 0}$ – in which case S and T are two consistent probability distributions – we also know that $R = 1/c^*_{T,\boldsymbol{X}}(S \bowtie T)$ is the unique probability distribution that satisfies (6) and maximizes the entropy of $\boldsymbol{XYZ}$ (or alternatively, the conditional entropy of $\boldsymbol{YZ}$ given $\boldsymbol{X}$) [2].

Let R be a K-relation over $\boldsymbol{XY}$. The *decomposition of R along $\boldsymbol{X}$ and $\boldsymbol{Y}$* consists of its projections $R[\boldsymbol{X}]$ and $R[\boldsymbol{Y}]$ on $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively. Such a decomposition is called a *lossless-join decomposition* if $R[\boldsymbol{X}] \bowtie R[\boldsymbol{Y}] \equiv R$. This definition, which appears already in [2], generalizes the definition of a lossless-join decomposition in database relations. It turns out, as we will next show, that if K is positive and multiplicatively cancellative, conditional independence holds on a K-relation if and only if the corresponding decomposition is a lossless-join one.

▶ **Theorem 10** (Lossless-join decomposition). *Let K be a positive semiring, $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ pairwise disjoint sets of variables, and $R(\boldsymbol{XYZ})$ a K-relation. If R satisfies $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$, then the decomposition of R along $\boldsymbol{XY}$ and $\boldsymbol{XZ}$ is a lossless-join one. If K is additionally multiplicatively cancellative, then the converse direction holds.*

Proof. Assume R satisfies $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$. We need to show that $R[\boldsymbol{XY}] \bowtie R[\boldsymbol{XZ}] \equiv R$. Let t be an arbitrary tuple from $\mathrm{Tup}(\boldsymbol{XYZ})$. By assumption and Lemma 4 we obtain

$$\begin{aligned}
(R[\boldsymbol{XY}] \bowtie R[\boldsymbol{XZ}])(t) &= R(t[\boldsymbol{XY}])R(t[\boldsymbol{XZ}])c_{R[\boldsymbol{XZ}]}(t[\boldsymbol{X}]) \\
&= R(t[\boldsymbol{X}])R(t) \bigotimes_{\substack{v \in R[\boldsymbol{XZ}][\boldsymbol{X}]' \\ v \neq t[\boldsymbol{X}]}} R[\boldsymbol{XZ}](v) \\
&= R(t[\boldsymbol{X}])R(t) \bigotimes_{\substack{v \in R[\boldsymbol{X}]' \\ v \neq t[\boldsymbol{X}]}} R(v) \\
&= c^*_{R,\boldsymbol{X}} R(t),
\end{aligned}$$

where $c^*_{R,\boldsymbol{X}} \neq 0$ by Proposition 9. This proves that $R[\boldsymbol{XY}] \bowtie R[\boldsymbol{XZ}] \equiv R$.

For the converse direction, suppose $R[\boldsymbol{XY}] \bowtie R[\boldsymbol{XZ}] \equiv R$. Let $a, b \in K \setminus \{0\}$ be such that $aR = b(R[\boldsymbol{XY}] \bowtie R[\boldsymbol{XZ}])$. By Lemma 1, we may assume without loss of generality that K is a submodel of some positive semifield F. Hence $R[\boldsymbol{XY}] \bowtie R[\boldsymbol{XZ}] = cR$ for $c = ab^{-1} \in F$. We claim that $c = c^*_{R,\boldsymbol{X}}$. Since we assume a non-empty support for each K-relation, we may select a tuple t from R'. By Lemma 4 we have $t[\boldsymbol{X}] \in R[\boldsymbol{X}]'$, i.e., $R(t[\boldsymbol{X}]) \neq 0$. We can also deduce the following:

$$\begin{aligned}
cR(t[\boldsymbol{X}]) &= \bigoplus_{\substack{t' \in \mathrm{Tup}(\boldsymbol{XYZ}) \\ t'[\boldsymbol{X}]=t[\boldsymbol{X}]}} cR(t') = \bigoplus_{\substack{t' \in \mathrm{Tup}(\boldsymbol{XYZ}) \\ t'[\boldsymbol{X}]=t[\boldsymbol{X}]}} (R[\boldsymbol{XY}] \bowtie R[\boldsymbol{XZ}])(t') \\
&= \bigoplus_{\substack{t' \in \mathrm{Tup}(\boldsymbol{XYZ}) \\ t'[\boldsymbol{X}]=t[\boldsymbol{X}]}} R(t'[\boldsymbol{XY}])R(t'[\boldsymbol{XZ}])c_{R[\boldsymbol{XZ}]}(t'[\boldsymbol{X}]) \\
&= c_{R[\boldsymbol{XZ}]}(t'[\boldsymbol{X}]) \bigoplus_{\substack{t' \in \mathrm{Tup}(\boldsymbol{XY}) \\ t'[\boldsymbol{X}]=t[\boldsymbol{X}]}} R(t'[\boldsymbol{XY}]) \bigoplus_{\substack{t' \in \mathrm{Tup}(\boldsymbol{XZ}) \\ t'[\boldsymbol{X}]=t[\boldsymbol{X}]}} R(t'[\boldsymbol{XZ}]) \\
&= R(t[\boldsymbol{X}])R(t[\boldsymbol{X}]) \bigotimes_{\substack{v \in R[\boldsymbol{X}]' \\ v \neq t[\boldsymbol{X}]}} R(v) = c^*_{R,\boldsymbol{X}} R(t[\boldsymbol{X}]).
\end{aligned}$$

Multiplying (in F) by the inverse of $R(t[\boldsymbol{X}])$ then yields $c = c^*_{R,\boldsymbol{X}}$, proving our claim.

Since $R[\boldsymbol{XY}] \bowtie R[\boldsymbol{XZ}] = c^*_{R,\boldsymbol{X}}R$, we may apply the sequence of equations from the previous case to obtain that for all $t \in \mathrm{Tup}(\boldsymbol{XYZ})$,

$$R(t[\boldsymbol{XY}])R(t[\boldsymbol{XZ}])c_{R[\boldsymbol{XZ}]} = R(t[\boldsymbol{X}])R(t)c_{R[\boldsymbol{XZ}]}.$$

Since $c_{R[\boldsymbol{XZ}]}$ is non-zero by Proposition 9, it can be removed from both sides of the equation by multiplicative cancellativity. We conclude that R satisfies $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$. ◀

The preceding proof entails that over positive and multiplicatively cancellative semirings K, the satisfaction of $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$ by a K-relation $R(\boldsymbol{XYZ})$ holds if and only if $R[\boldsymbol{XY}] \bowtie R[\boldsymbol{XZ}] = c^*_{R,\boldsymbol{X}}R$. If K is additionally a semifield, then $R \models \boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$ exactly when we find two K-relations $S(\boldsymbol{XY})$ and $T(\boldsymbol{XZ})$ such that $R(t) = S(t[\boldsymbol{XY}])T(t[\boldsymbol{XZ}])$.

Our running example demonstrates that semiring interpretations can give rise to lossless-join decompositions which are unattainable under the relational interpretation.

▶ **Example 11.** Consider again Tab. 1 as a $\mathbb{T}$-relation Price(Room, Date, Persons). As we see from Fig. 1, this $\mathbb{T}$-relation decomposes along {Room, Date} and {Room, Persons}. In particular, Price[Room, Date] $\bowtie$ Price[Room, Persons] $= c^*_{\text{Price},\{\text{Room}\}}$Price where $c^*_{\text{Price},\{\text{Room}\}} = 210$. However, viewed as an ordinary relation R over {Price, Room, Date, Persons} this table is in sixth normal form, meaning that no decomposition along $X_1, \dots, X_n$ is a lossless-join one, unless X_i for some i is the full variable set. Specifically, for any $X \subsetneq$ {Price, Room, Date, Persons}, the projection of the tuple (double, 2023-08-20, 2, 120) on X is in $R[X]$, even though the tuple itself does not belong to R.

Examples of positive semirings which are not multiplicatively cancellative seem somewhat artificial. Consider $K = (\mathbb{N}_{>0}, \mathbb{Z}_2) \cup \{(0,0)\}$ with the semiring structure pointwise inherited from $\mathbb{N}$ and $\mathbb{Z}_2$. Note that K is positive but violates multiplicative cancellativity, as $(1,1) \otimes (1,0) = (1,0) \otimes (1,0)$, while $(1,1) \neq (1,0) \neq (0,0)$. Using K we can demonstrate that the assumption of multiplicative cancellativity cannot be dropped from the second statement of Lemma 10. We write $\boldsymbol{X} \perp\!\!\!\perp \boldsymbol{Y}$ for the *marginal independence* between $\boldsymbol{X}$ and $\boldsymbol{Y}$, defined as the CI $\boldsymbol{X} \perp\!\!\!\perp \boldsymbol{Y} \mid \emptyset$. Consider the K-relation R from Fig. 2. This K-relation does not satisfy $A \perp\!\!\!\perp B$: Choosing $t(A,B) = (0,0)$ we observe $R(t[A]) \otimes R(t[B]) = (2,1) \otimes (2,1) \neq (4,0) \otimes (1,0) = R(\emptyset) \otimes R(t[AB])$. On the other hand, we have $R \equiv R[A] \bowtie R[B]$ because $aR = b(R[A] \bowtie R[B])$, where $a = (4,0) \neq (0,0) \neq (1,0) = b$.

Similarly, the assumption of positivity is necessary for the first statement of Lemma 10. Suppose $a, b \in K \setminus \{0\}$ are such that $a \oplus b = 0$, and consider variables X, Y with domain $\{0,1\}$.

R

A	B	#
0	0	$(1,0)$
0	1	$(1,1)$
1	0	$(1,1)$
1	1	$(1,0)$

$R[C], C \in \{A, B\}$

C	#
0	$(2,1)$
1	$(2,1)$

$R[A] \bowtie R[B]$

A	B	#
0	0	$(4,1)$
0	1	$(4,1)$
1	0	$(4,1)$
1	1	$(4,1)$

Figure 2 Decomposition without independence.

(S1) Triviality: $\boldsymbol{Y} \perp\!\!\!\perp \emptyset \mid \boldsymbol{X}$.
(S2) Symmetry: $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$, then $\boldsymbol{Z} \perp\!\!\!\perp \boldsymbol{Y} \mid \boldsymbol{X}$.
(S3) Decomposition: $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{ZW} \mid \boldsymbol{X}$, then $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$.
(S4) Weak union: $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{ZW} \mid \boldsymbol{X}$, then $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{W} \mid \boldsymbol{XZ}$.
(S5) Contraction: $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$ and $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{W} \mid \boldsymbol{XZ}$, then $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{ZW} \mid \boldsymbol{X}$.
(G) Interaction: $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{XW}$ and $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{W} \mid \boldsymbol{XZ}$, then $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{ZW} \mid \boldsymbol{X}$.

Figure 3 Semigraphoid axioms (S1–S5) and graphoid axioms (S1–S5,G).

Then, the K-relation $R(XY)$ corresponding to the set $\{(0,0;a),(0,1;b),(1,0;b),(1,1;a)\}$ of triples $(t(X),t(Y);R(t))$ satisfies $X \perp\!\!\!\perp Y$, but the decomposition along X and Y is obviously not a lossless-join one. In fact, the definition of the marginal, Eq. (2), may not even be useful if K is not positive. For instance, a pure quantum state $|\psi\rangle_{XY}$ within a finite-dimensional composite Hilbert space $\mathcal{H}_{XY}$ can be conceived as a $\mathbb{C}$-relation $R(XY)$ over complex numbers $\mathbb{C}$. Its marginal with respect to $\mathcal{H}_X$ is however not obtained from Eq. (2), but through a partial trace of the relevant density matrix. The marginal state may not even be a $\mathbb{C}$-relation anymore, because it can be mixed, i.e., a probability distribution over pure states.

5 Axiomatic properties

The previous section identifies positivity and multiplicative cancellativity as the key semiring properties underlying the correspondence between conditional independence and lossless-join decompositions. The main observation of the present section will be that the same key semiring properties guarantee soundness and completeness of central axiomatic properties associated with CIs.

5.1 Semigraphoid axioms

The *semigraphoid axioms* [21] (the first five rules in Fig. 3) are a collection of fundamental conditional independence properties observed in various contexts, including database relations and probability distributions. The *graphoid axioms* are obtained by extending the semigraphoid axioms with the interaction rule (the last rule in Fig. 3). While not sound in general, the interaction rule is known to hold for probability distributions in which every probability is positive. We observe next that these results extend to K-relations whenever K is positive and multiplicatively cancellative; the interaction rule, in particular, is sound over total K-relations.

To offer context for Theorem 12 which is proven in Appendix B, recall from information theory the concept of *conditional mutual information*, which can be defined over sets of random variables $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ as $I(\boldsymbol{V};\boldsymbol{W} \mid \boldsymbol{U}) := H(\boldsymbol{UV}) + H(\boldsymbol{UW}) - H(\boldsymbol{U}) - H(\boldsymbol{UVW})$, where H is the Shannon entropy. The conditional mutual information $I(\boldsymbol{V};\boldsymbol{W} \mid \boldsymbol{U})$ is zero if and only if the CI $\boldsymbol{V} \perp\!\!\!\perp \boldsymbol{W} \mid \boldsymbol{U}$ holds in the underlying probability distribution. Now, consider the *chain rule*

$$I(\boldsymbol{Y};\boldsymbol{Z} \mid \boldsymbol{X}) + I(\boldsymbol{Y};\boldsymbol{W} \mid \boldsymbol{X}\boldsymbol{Z}) = I(\boldsymbol{Y};\boldsymbol{Z}\boldsymbol{W} \mid \boldsymbol{X})$$

of conditional mutual information. Since conditional mutual information is non-negative, the chain rule readily entails Decomposition, Weak union, and Contraction for probability distributions. In the semiring setting we cannot deduce these rules analogously, as there seems to be no general measure to capture conditional independence over K-relations. We can however use the measure-theoretic interpretation of conditional independence as a guide toward a proof. Consider, for instance, the contraction rule, which can be restated in the information context as follows: if

$$H(\boldsymbol{XY}) + H(\boldsymbol{XZ}) = H(\boldsymbol{X}) + H(\boldsymbol{XYZ}) \text{ and} \tag{7}$$

$$H(\boldsymbol{XYZ}) + H(\boldsymbol{XZW}) = H(\boldsymbol{XZ}) + H(\boldsymbol{XYZW}), \tag{8}$$

then

$$H(\boldsymbol{XY}) + H(\boldsymbol{XZW}) = H(\boldsymbol{X}) + H(\boldsymbol{XYZW}). \tag{9}$$

In particular, Eq. (9) is a consequence of subtracting $H(\boldsymbol{XZ}) + H(\boldsymbol{XYZ})$ from the combination of Eqs. (7) and (8). The soundness proof for K-relations has now the same general structure. Instantiations of Eq. (3) for the CIs appearing in the contraction rule are structurally similar to Eqs. (7), (8), and (9), with addition between entropies being replaced by multiplication within K. Instead of subtraction, one now applies multiplicative cancellativity to remove all superfluous terms from the combination of two equations. Additionally, one has to deal with those cases where the terms to be eliminated are zero, and multiplicative cancellativity cannot be applied.

▶ **Theorem 12.** *Triviality, Symmetry, and Decomposition are sound for K-relations. Weak union and Contraction are sound for K-relations where K is positive and multiplicatively cancellative. Interaction is sound for total K-relations where K is positive and multiplicatively cancellative.*

Having considered the graphoid axioms for K-relations, we next consider the interaction between conditional independence and functional dependence.

5.2 Functional dependencies

Given two sets of variables $\boldsymbol{X}$ and $\boldsymbol{Y}$, the expression $\boldsymbol{X} \to \boldsymbol{Y}$ is called a *functional dependency* (FD). A relation R *satisfies* $\boldsymbol{X} \to \boldsymbol{Y}$, denoted $R \models \boldsymbol{X} \to \boldsymbol{Y}$, if for all $t, t' \in R$, $t[\boldsymbol{X}] = t'[\boldsymbol{X}]$ implies $t[\boldsymbol{Y}] = t'[\boldsymbol{Y}]$. We extend this definition to K-relations R by stipulating that R *satisfies* an FD σ whenever its support R' satisfies σ.

The Armstrong axioms for FDs [1] comprise the first three rules in Fig. 4. These rules are sound and complete for database relations, and hence, by definition, for K-relations over any K. The last two rules are two combination rules for MVDs and FDs [4] rewritten in different syntax. To extend these rules K-relations, we again need positivity and multiplicative cancellativity. The following proposition is proven in the arXiv version [13].

▶ **Proposition 13.** *CI introduction is sound for all K-relations, where K is $\oplus$-positive. FD contraction is sound for all K-relations, where K is positive and multiplicatively cancellative.*

(FD1)	Triviality: if $\boldsymbol{Y} \subseteq \boldsymbol{X}$, then $\boldsymbol{X} \to \boldsymbol{Y}$.
(FD2)	Augmentation: if $\boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{XZ} \to \boldsymbol{YZ}$.
(FD3)	Transitivity: if $\boldsymbol{X} \to \boldsymbol{Y}$ and $\boldsymbol{Y} \to \boldsymbol{Z}$, then $\boldsymbol{X} \to \boldsymbol{Z}$.
(FD-CI1)	CI introduction: if $\boldsymbol{X} \to \boldsymbol{Y}$, then $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$.
(FD-CI2)	FD contraction: if $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$ and $\boldsymbol{XY} \to \boldsymbol{Z}$, then $\boldsymbol{X} \to \boldsymbol{Z}$.

Figure 4 Armstrong's axioms (FD1–FD3) and combination rules (FD–CI1,FD–CI2).

Soundness of CI introduction means that, for positive K, a functional dependency on a K-relation leads to a lossless-join decomposition. The next proposition, stating this fact, was proven originally in [2]. Alternatively, we now see that the proposition follows directly by Theorem 10 and Proposition 13.

▶ **Proposition 14** ([2]). *Let K be a positive semiring, $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ be pairwise disjoint sets of variables, and $R(\boldsymbol{XYZ})$ be a K-relation. If R satisfies $\boldsymbol{X} \to \boldsymbol{Y}$, then the decomposition of R along $\boldsymbol{XY}$ and $\boldsymbol{XZ}$ is a lossless-join one.*

We have now examined fundamental inference rules for CIs and FDs that have their origins in database theory and probability theory. The combination of these rules however is not – and cannot be – complete in either context. Specifically, over both finite relations and finite distributions, the implication problems for EMVDs/CIs are not even r.e. since the problems are known to be undecidable [14, 18, 20] and co-r.e. [17]. In the next section we restrict attention to saturated CIs which are known to exhibit favorable algorithmic and axiomatic properties.

5.3 Saturated conditional independence and functional dependence

We next show that SCI+FD enjoys a complete axiomatization that is shared by all positive and multiplicatively cancellative semirings K. This result readily entails that logical implication within the class SCI+FD does not depend on the chosen semiring K, provided that it has the fundamental properties mentioned above.

Given a set $\Sigma \cup \{\tau\}$ of dependencies, we say that Σ *implies* τ over relations (resp. K-relations), denoted $\Sigma \models \tau$ (resp. $\Sigma \models_K \tau$), if every relation (resp. K-relation) satisfying Σ satisfies τ. Let $\sigma \mapsto \sigma^*$ associate an SCI/CI with its corresponding MVD/EMVD. Extend this mapping to be the identity on FDs, and extend it to sets in the natural way: $\Sigma^* = \{\sigma^* \mid \sigma \in \Sigma\}$.

▶ **Theorem 15.** *Let K be a positive and multiplicatively cancellative semiring. Let $\Sigma \cup \{\tau\}$ be a set of SCIs and FDs. The following are equivalent:*

1. *τ can be derived from Σ using (S1-S5), (FD1-FD3), and (FD-CI1,FD-CI2).*
2. *Σ implies τ over K-relations.*
3. *Σ^* implies τ^* over relations consisting of two tuples.*
4. *Σ^* implies τ^* over relations.*

Proof. (1) ⇒ (2). This direction is immediate due to Theorem 12, Proposition 13, and soundness of the Armstrong axioms for ordinary relations. (2) ⇒ (3). Any two-tuple relation $R = \{t, t'\}$ can be transformed to a K-relation S such that the support S' is R, and $S(t) = S'(t) = 1$. It is straightforward to verify that R satisfies σ^* if and only if S satisfies σ, for all CIs and FDs σ. From this, the direction follows. (3) ⇒ (4). This direction has been proven in [22]. (4) $\Rightarrow$ (1). This direction follows from the fact that the system (S1-S5), (FD1-FD3), (FD-CI1,FD-CI2) mirrors the complete axiomatization of MVDs and FDs. We give an explicit proof in the arXiv version [13]. ◀

▶ **Corollary 16.** *Let K, K' be positive and multiplicatively cancellative semirings, and let $\Sigma \cup \{\tau\}$ be a set of SCIs and FDs. Then, Σ implies τ over K-relations if and only if Σ implies τ over K'-relations.*

Theorem 10 and Corollary 16 demonstrate that the decomposition properties arising from multivalued and functional dependencies hold invariably for all K-relations, given K is multiplicatively cancellative and positive. Standard database normalization methods thus extend to diverse contexts and may sometimes coincide with existing methods. The following example shows that relational normalizations can sometimes match the factorizations of probability distributions arising from Bayesian networks.

A relational database schema is a set of relation schemata, each associated with a set of constraints. It is in *fourth normal form* (4NF) if for any of its MVD constraints $\boldsymbol{X} \twoheadrightarrow \boldsymbol{Y}$, $\boldsymbol{X}$ is a superset of a key. A *Bayesian network* is a directed acyclic graph in which the nodes represent random variables and the directed edges probabilistic dependencies between variables. Each node is thus directly influenced by its parents in the graph. Conversely, each node indirectly influences its descendants by transitivity. The *local Markov property* states that once the parent nodes are known, the state of the current node does not reveal any additional information about the states of its non-descendants, i.e., each node is conditionally independent of its non-descendants given its parents.

▶ **Example 17.** Consider the Bayesian network in Fig. 5. The chain rule of probability distributions and the local Markov property implies that the joint distribution $P(A, B, C, D, E)$ has a factorization $P(A)P(B \mid A)P(C \mid A)P(D \mid BC)P(E \mid D)$.

The local Markov property produces three non-trivial CIs up to symmetry, rewritten as the following EMVDs $A \twoheadrightarrow B|C$, $BC \twoheadrightarrow A|D$, $D \twoheadrightarrow ABC|E$. Suppose our goal is to transform the unirelational database schema $\{ABCDE\}$ into 4NF, assuming absence of key constraints. Since the last EMVD is also an MVD, we first decompose $ABCDE$ along $ABCD$ and DE. Since the second EMVD is an MVD on $ABCD$, we continue by splitting $ABCD$ into ABC and BCD. To remove the last remaining MVD, we decompose ABC along AB and AC. The final schema $\{AB, AC, BCD, DE\}$ is free of MVDs, and thus in 4NF. Furthermore, the decomposition of P (as a $\mathbb{R}_{\geq 0}$-relation) along $\{AB, AC, BCD, DE\}$ reproduces the aforementioned factorization of P into conditional probabilities.

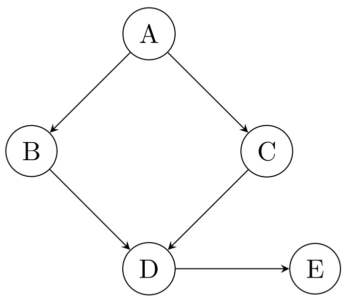

Figure 5 A simple Bayesian network.

6 Comparison of implication

As mentioned previously, implication for non-saturated CIs depends heavily on the underlying semantics. In this section, we examine the connections between different conditional independence semantics in relation to the semiring properties they rely on. Using model-theoretic arguments, we first show that $\Sigma \models_{\mathbb{R}_{\geq 0}} \tau$ implies $\Sigma \models_K \tau$, whenever K is cancellative and equipped with a natural total order.

Consider a CI of the form $\tau = \boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$, where $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ are disjoint subsets of a schema $\boldsymbol{V}$. Suppose the domain of each variable in $\boldsymbol{V}$ is finite. For each $\boldsymbol{V}$-tuple t, introduce a variable x_t. Denote by $\vec{x}_{\boldsymbol{V}}$ a sequence listing all variables x_t, $t \in \mathrm{Tup}(\boldsymbol{V})$. We associate τ and $\boldsymbol{V}$ with a quantifier-free first-order arithmetic formula

$$\phi_{\tau,\boldsymbol{V}} \coloneqq \bigwedge_{\substack{t_{\boldsymbol{X}} \in \mathrm{Tup}(\boldsymbol{X}) \\ t_{\boldsymbol{Y}} \in \mathrm{Tup}(\boldsymbol{Y}) \\ t_{\boldsymbol{Z}} \in \mathrm{Tup}(\boldsymbol{Z})}} \bigoplus_{\substack{t \in \mathrm{Tup}(\boldsymbol{XYZ}) \\ t[\boldsymbol{X}]=t_{\boldsymbol{X}}}} x_t \bigoplus_{\substack{t \in \mathrm{Tup}(\boldsymbol{XYZ}) \\ t[\boldsymbol{X}]=t_{\boldsymbol{X}} \\ t[\boldsymbol{Y}]=t_{\boldsymbol{Y}} \\ t[\boldsymbol{Z}]=t_{\boldsymbol{Z}}}} x_t = \bigoplus_{\substack{t \in \mathrm{Tup}(\boldsymbol{XYZ}) \\ t[\boldsymbol{X}]=t_{\boldsymbol{X}} \\ t[\boldsymbol{Y}]=t_{\boldsymbol{Y}}}} x_t \bigoplus_{\substack{t \in \mathrm{Tup}(\boldsymbol{XYZ}) \\ t[\boldsymbol{X}]=t_{\boldsymbol{X}} \\ t[\boldsymbol{Z}]=t_{\boldsymbol{Z}}}} x_t.$$

A formula ϕ is said to be *universal* if it is of the form $\forall x_1 \ldots \forall x_n \theta$, where θ is quantifier-free. All universal first-order properties of a model are preserved for its submodels (and any of their isomorphic copies) [6].

▶ **Proposition 18.** *Let $\mathcal{A}$ and $\mathcal{B}$ be models over a vocabulary τ, and let ϕ be a universal first-order sentence over τ. If $\mathcal{A}$ embeds in $\mathcal{B}$, then $\mathcal{A} \models \phi$ implies $\mathcal{B} \models \phi$.*

The following theorem lists some basic properties of real-closed fields [5, 6]. A field F is called *real* if it can be associated with an ordering $\leq$ such that $(F, \leq)$ becomes an ordered field. A *field* F' is an *extension* of a field F if $F \subseteq F'$, and the field operations of F are those inherited from F'. The extension is *proper* if F is a strict subset of F', and *algebraic* if every element in F' is a root of a non-zero polynomial with coefficients in F. A real field with no proper real algebraic extension is called *real closed*. For instance, the field of real numbers is real closed, whereas the field of rational numbers is real but not real closed. On the other hand, no finite or algebraically closed field is real. A real closed field has a unique ordering, which is definable by $a \leq b :\Leftrightarrow \exists c(a \oplus c^2 = b)$. An algebraic extension F' of an ordered field $(F, \leq)$ is called a *real closure* of F if F' is real closed, and its unique ordering extends that of F (i.e., the ordering is preserved under the inclusion map $F \hookrightarrow F'$). Two models $\mathcal{A}$ and $\mathcal{B}$ are *elementarily equivalent*, written $\mathcal{A} \equiv \mathcal{B}$, if they satisfy the same first-order sentences.

▶ **Theorem 19.**
- *Any totally ordered field $(F, \leq)$ has a real closure F'.*
- *If $(F, \leq)$ is a totally ordered field, and F_0 and F_1 are its real closures uniquely ordered by $\leq_0$ and $\leq_1$, there is an isomorphism between $(F_0, \leq_0)$ and $(F_1, \leq_1)$ which is identity on F.*
- *Any two real-closed fields F_0 and F_1 are elementarily equivalent.*

We can now prove the property that $\mathbb{R}_{\geq 0}$-implication entails K-implication, for any semiring K embedded in a cancellative and naturally totally ordered one.

▶ **Theorem 20.** *Let $\Sigma \cup \{\tau\}$ be a finite set of CIs, and suppose K embeds in a naturally totally ordered cancellative semiring. Then, $\Sigma \models_{\mathbb{R}_{\geq 0}} \tau$ implies $\Sigma \models_K \tau$.*

Proof. By Lemma 2, Theorem 19, and transitivity of the embedding relation, K embeds in a real-closed field F. Let $R(\boldsymbol{V})$ be a K-relation, where $\boldsymbol{V}$ is a set of variables that includes each variable appearing in $\Sigma \cup \{\tau\}$. We need to show that $R \models \Sigma$ implies $R \models \tau$. Since satisfaction of a CI by R does not depend on tuple values that do not appear in R, we may without loss of generality assume that the domain of each variable in $\boldsymbol{V}$ is finite. Then, assuming $\Sigma = \{\sigma_1, \dots, \sigma_n\}$, we may consider the universal first-order sentence

$$\psi_{\Sigma,\tau,\boldsymbol{V}} := \forall \vec{x}_{\boldsymbol{V}} (\vec{x}_{\boldsymbol{V}} \geq \vec{0} \wedge \phi_{\sigma_1,\boldsymbol{V}} \wedge \dots \wedge \phi_{\sigma_n,\boldsymbol{V}} \to \phi_{\tau,\boldsymbol{V}}),$$

where, given a sequence $\vec{x} = (x_1, \dots, x_l)$, we write $\vec{x} \geq \vec{0}$ as a shorthand for $x_1 \geq 0 \wedge \dots \wedge x_l \geq 0$. Since $\Sigma \models_{\mathbb{R}_{\geq 0}} \tau$ by hypothesis, $\psi_{\Sigma,\tau,\boldsymbol{V}}$ must be true for the field of reals $\mathbb{R}$ (equipped with its unique ordering). Since F and $\mathbb{R}$ are real-closed, they share the same first-order properties; in particular, F also satisfies $\psi_{\Sigma,\tau,\boldsymbol{V}}$. By Proposition 18, K likewise satisfies $\psi_{\Sigma,\tau,\boldsymbol{V}}$, and hence $R \models \Sigma$ implies $R \models \tau$. We conclude that $\Sigma \models_K \tau$. ◀

The preceding theorem readily entails that implication over $\mathbb{R}_{\geq 0}$ entails implication over $\mathbb{N}_{\geq 0}$ and $\mathbb{Q}_{\geq 0}$. Another example is $K = \mathbb{N} \times \mathbb{N}$ with pointwise addition and multiplication, and neutral elements $(0,0)$ and $(1,1)$. This semiring is not naturally totally ordered, because it contains incomparable elements, such as $(0,1)$ and $(1,0)$. However, it can be extended to $K \cup (\mathbb{Z} \times \mathbb{N}_{>0})$ which is naturally totally ordered and cancellative. For another example, consider the semiring $K = \mathbb{N}[X]$ of polynomials in X with coefficients from natural numbers. Since K contains incomparable elements, such as $X + 2$ and $2X + 1$, its natural order is not total. The extension of K with those polynomials of $\mathbb{Z}[X]$ in which the leading coefficient is positive produces a cancellative semiring whose natural order is total.

Let us then turn attention to the Boolean semiring $\mathbb{B}$ and its connections with other semirings. First we note that although both $\mathbb{R}_{\geq 0}$ and $\mathbb{B}$ are naturally totally ordered, only the first one is additively cancellative. In light of Theorem 20, this may help explain why there is no implication from $\Sigma \models_{\mathbb{R}_{\geq 0}} \tau$ to $\Sigma \models_{\mathbb{B}} \tau$. Another difference is that only the Boolean semiring is associated with an idempotent addition; an operation $*$ on K is said to be *idempotent* if $a * a = a$ for all $a \in K$. We observe that $\mathbb{B}$-implication entails K-implication, whenever K has an idempotent addition.

▶ **Proposition 21.** *Let K be a semiring associated with an idempotent addition. Let $\Sigma \cup \{\tau\}$ be a set of CIs. Then, $\Sigma \models_K \tau$ implies $\Sigma \models_{\mathbb{B}} \tau$.*

Proof. Recall that we consider only non-trivial semirings K, where $0 \neq 1$. Thus, any $\mathbb{B}$-relation R can be readily interpreted as a K-relation R'. The idempotence of addition guarantees that $R \models \sigma$ if and only if $R' \models \sigma$, for any CI σ. The statement of the lemma then follows. ◀

We leave it as an open question whether the statements of Theorem 20 and Proposition 21 hold also in the converse directions.

7 Conclusion

We have studied axiomatic and decomposition properties of conditional independence over K-relations. For positive and multiplicatively cancellative K, we showed that (i) conditional independence corresponds to lossless-join decompositions, (ii) the semigraphoid axioms of conditional independence are sound, and (iii) saturated conditional independence and functional dependence have a sound and complete axiom system, mirroring the sound and complete axiom system of MVDs and FDs. To demonstrate possible applications, we provided an example data table that admits a lossless-join decomposition only when one of its variables is reinterpreted as a semiring annotation. Finally, we considered a model-theoretic approach to study the relationships between different CI semantics.

The questions of the axiomatic characterization [15, 26, 28] and decidability [14, 20] of the CI implication problem have been answered in the negative in different frameworks. Having identified positivity and multiplicative cancellativity as the fundamental semiring properties for the notion of conditional independence, we may now ask whether these negative results extend to any K with these characteristics.

References

1 William W. Armstrong. Dependency Structures of Data Base Relationships. In *Proc. of IFIP World Computer Congress*, pages 580–583, 1974.

2 Albert Atserias and Phokion G. Kolaitis. Consistency, acyclicity, and positive semirings. In Alessandra Palmigiano and Mehrnoosh Sadrzadeh, editors, *Samson Abramsky on Logic and Structure in Computer Science and Beyond*, pages 623–668, Cham, 2023. Springer International Publishing.

3 Timon Barlag, Miika Hannula, Juha Kontinen, Nina Pardal, and Jonni Virtema. Unified foundations of team semantics via semirings. In *KR*, pages 75–85, 2023. doi:10.24963/kr.2023/8.

4 Catriel Beeri, Ronald Fagin, and John H. Howard. A complete axiomatization for functional and multivalued dependencies in database relations. In *SIGMOD Conference*, pages 47–61. ACM, 1977. doi:10.1145/509404.509414.

5 J. Bochnak, M. Coste, and M-F. Roy. *Real Algebraic Geometry*. Springer, 1998.

6 Chen C. Chang and H. Jerome Keisler. *Model theory, Third Edition*, volume 73 of *Studies in logic and the foundations of mathematics*. North-Holland, 1992.

7 Shumo Chu, Brendan Murphy, Jared Roesch, Alvin Cheung, and Dan Suciu. Axiomatic foundations and algorithms for deciding semantic equivalences of SQL queries. *Proc. VLDB Endow.*, 11(11):1482–1495, 2018. doi:10.14778/3236187.3236200.

8 A. P. Dempster. A generalization of bayesian inference. *Journal of the Royal Statistical Society. Series B (Methodological)*, 30(2):205–247, 1968.

9 Pietro Galliani and Jouko Väänänen. Diversity, dependence and independence. *Ann. Math. Artif. Intell.*, 90(2-3):211–233, 2022. doi:10.1007/s10472-021-09778-8.

10 Dan Geiger, Thomas Verma, and Judea Pearl. Identifying independence in bayesian networks. *Networks*, 20(5):507–534, 1990. doi:10.1002/net.3230200504.

11 Todd J. Green, Gregory Karvounarakis, and Val Tannen. Provenance semirings. In *PODS*, pages 31–40. ACM, 2007. doi:10.1145/1265530.1265535.

12 Marc Gyssens, Mathias Niepert, and Dirk Van Gucht. On the completeness of the semigraphoid axioms for deriving arbitrary from saturated conditional independence statements. *Inf. Process. Lett.*, 114(11):628–633, 2014. doi:10.1016/j.ipl.2014.05.010.

13 Miika Hannula. Conditional independence on semiring relations. *CoRR*, abs/2310.01910, 2023. doi:10.48550/arXiv.2310.01910.

14 Christian Herrmann. On the undecidability of implications between embedded multivalued database dependencies. *Information and Computation*, 122(2):221–235, 1995. doi:10.1006/inco.1995.1148.

15 Douglas Stott Parker Jr. and Kamran Parsaye-Ghomi. Inferences involving embedded multivalued dependencies and transitive dependencies. In *SIGMOD Conference*, pages 52–57. ACM Press, 1980. doi:10.1145/582250.582259.

16 Batya Kenig and Dan Suciu. Integrity constraints revisited: From exact to approximate implication. *Log. Methods Comput. Sci.*, 18(1), 2022. doi:10.46298/LMCS-18(1:5)2022.

17 Mahmoud Abo Khamis, Phokion G. Kolaitis, Hung Q. Ngo, and Dan Suciu. Decision problems in information theory. In *ICALP*, volume 168 of *LIPIcs*, pages 106:1–106:20, 2020. doi:10.4230/LIPIcs.ICALP.2020.106.

18 Lukas Kühne and Geva Yashfe. On entropic and almost multilinear representability of matroids. *CoRR*, abs/2206.03465, 2022. doi:10.48550/arXiv.2206.03465.

19 Tony T. Lee. An information-theoretic analysis of relational databases - part I: data dependencies and information metric. *IEEE Trans. Software Eng.*, 13(10):1049–1061, 1987. doi:10.1109/TSE.1987.232847.

20 Cheuk Ting Li. Undecidability of network coding, conditional information inequalities, and conditional independence implication. *IEEE Trans. Inf. Theory*, 69(6):3493–3510, 2023. doi:10.1109/TIT.2023.3247570.

21 J. Pearl. *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. Morgan Kaufmann, San Mateo, CA, 1988.

22 Yehoshua Sagiv, Claude Delobel, Douglas Stott Parker Jr., and Ronald Fagin. An equivalence between relational database dependencies and a fragment of propositional logic. *J. ACM*, 28(3):435–453, 1981. doi:10.1145/322261.322263.

23 Glenn Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, 1976.

24 Wolfgang Spohn. Ordinal conditional functions. a dynamic theory of epistemic states. In W. L. Harper and B. Skyrms, editors, *Causation in Decision, Belief Change, and Statistics, vol. II*. Kluwer Academic Publishers, 1988.

25 Milan Studený. Multiinformation and the problem of characterization of conditional independence relations. *Problems of Control and Information Theory*, 18(1):3–16, 1989.

26 Milan Studený. Conditional independence relations have no finite complete characterization. *Transactions of the 11th Prague Conference on Information Theory*, pages 377–396, 1992.

27 Milan Studený. Formal properties of conditional independence in different calculi of AI. In *ECSQARU*, volume 747 of *Lecture Notes in Computer Science*, pages 341–348. Springer, 1993. doi:10.1007/BFb0028219.

28 Milan Studený. Conditional independence and natural conditional functions. *Int. J. Approx. Reason.*, 12(1):43–68, 1995. doi:10.1016/0888-613X(94)00014-T.

29 L.A Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1(1):3–28, 1978.

A Embeddings

Let τ be a first-order vocabulary consisting of function and relation symbols (constant symbols can be viewed as 0-ary function symbols). We write write $\mathrm{ar}(\alpha)$ for the *arity* of a symbol $\alpha \in \tau$. Given a τ-structure $\mathcal{M}$ and an element α from τ, we write $\alpha^{\mathcal{M}}$ for the interpretation of α in $\mathcal{M}$. Let $\mathcal{A}$ and $\mathcal{B}$ be two τ-structures with domains A and B. We call $\mathcal{A}$ a *submodel* of $\mathcal{B}$, written $\mathcal{A} \subseteq \mathcal{B}$, if $A \subseteq B$, and the interpretation of every function symbol and relation symbol in τ is inherited from $\mathcal{B}$; i.e., for each $\alpha \in \tau$, $\alpha^{\mathcal{A}}$ is the restriction $\alpha^{\mathcal{B}} \upharpoonright A^k$ of $\alpha^{\mathcal{B}}$ to A^k. We say that $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic*, written $\mathcal{A} \cong \mathcal{B}$, if there exists a bijection (called an *isomorphism* between $\mathcal{A}$ and $\mathcal{B}$) $\pi : A \to B$ such that

- $\pi(f^{\mathcal{A}}(a_1, \dots, a_{\mathrm{ar}(f)})) = f^{\mathcal{B}}(\pi(a_1), \dots, \pi(a_{\mathrm{ar}(f)}))$, for all function symbols $f \in \tau$ and elements $a_1, \dots, a_k \in A$, and
- $(a_1, \dots, a_{\mathrm{ar}(R)}) \in R^{\mathcal{A}} \iff (\pi(a_1), \dots, \pi(a_{\mathrm{ar}(R)})) \in R^{\mathcal{B}}$, for all relation symbols $R \in \tau$ and elements $a_1, \dots, a_k \in A$.

We say that $\mathcal{A}$ *embeds in* $\mathcal{B}$, written $\mathcal{A} \preccurlyeq \mathcal{B}$, if $\mathcal{A}$ and some submodel of $\mathcal{B}$ are isomorphic.

B Graphoid axioms

We will use the following helping lemma in the proof of Theorem 12.

▶ **Lemma 22.** *Let $R(\boldsymbol{X})$ be a K-relation, where K is $\oplus$-positive. Let t be a tuple of R, and let $\boldsymbol{Y}, \boldsymbol{Z}$ be variable sets such that $\boldsymbol{Z} \subseteq \boldsymbol{Y} \subseteq \boldsymbol{X}$. Then, $t[\boldsymbol{Y}] \in R[\boldsymbol{Y}]'$ implies $t[\boldsymbol{Z}] \in R[\boldsymbol{Z}]'$.*

Proof. By Lemma 4, $R[\boldsymbol{Y}][\boldsymbol{Z}] = R[\boldsymbol{Z}]$. Using Eq. (2) we have

$$R(t[\boldsymbol{Z}]) = \bigoplus_{\substack{u\in\mathrm{Tup}(\boldsymbol{Y})\\ u[\boldsymbol{Z}]=t[\boldsymbol{Z}]}} R(u) = R(t[\boldsymbol{Y}]) \oplus \bigoplus_{\substack{u\in\mathrm{Tup}(\boldsymbol{Y})\\ u[\boldsymbol{Z}]=t[\boldsymbol{Z}]\\ u[\boldsymbol{Y}]\neq t[\boldsymbol{Y}]}} R(u).$$

Since by assumption $R(t[\boldsymbol{Y}]) \neq 0$, we obtain by $\oplus$-positivity of K that $R(t[\boldsymbol{Z}]) \neq 0$, i.e., $t[\boldsymbol{Z}] \in R[\boldsymbol{Z}]'$. ◀

▶ **Theorem 12.** *Triviality, Symmetry, and Decomposition are sound for K-relations. Weak union and Contraction are sound for K-relations where K is positive and multiplicatively cancellative. Interaction is sound for total K-relations where K is positive and multiplicatively cancellative.*

Proof. Triviality and Symmetry are clearly sound for all K-teams. We thus consider only Decomposition, Weak union, and Contraction. Fix a K-relation $R(\boldsymbol{V})$ over some variable set $\boldsymbol{V}$ that contains $\boldsymbol{XYZW}$.

- Decomposition: Suppose R satisfies $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{ZW} \mid \boldsymbol{X}$. Then, for all tuples $t \in \mathrm{Tup}(\boldsymbol{V})$ it holds that

$$R(t[\boldsymbol{XY}])R(t[\boldsymbol{XZW}]) = R(t[\boldsymbol{X}])R(t[\boldsymbol{XYZW}]) \tag{10}$$

This implies

$$\begin{aligned} R(t[\boldsymbol{XY}])R(t[\boldsymbol{XZ}]) &= R(t[\boldsymbol{XY}]) \bigoplus_{\substack{t'\in\mathrm{Tup}(\boldsymbol{XZW})\\ t'[\boldsymbol{XZ}]=t[\boldsymbol{XZ}]}} R(t'[\boldsymbol{XZW}]) \\ &= \bigoplus_{\substack{t'\in\mathrm{Tup}(\boldsymbol{XYZW})\\ t'[\boldsymbol{XYZ}]=t[\boldsymbol{XYZ}]}} R(t'[\boldsymbol{XY}])R(t'[\boldsymbol{XZW}]) = \bigoplus_{\substack{t'\in\mathrm{Tup}(\boldsymbol{XYZW})\\ t'[\boldsymbol{XYZ}]=t[\boldsymbol{XYZ}]}} R(t'[\boldsymbol{X}])R(t'[\boldsymbol{XYZW}]) \\ &= R(t[\boldsymbol{X}]) \bigoplus_{\substack{t'\in\mathrm{Tup}(\boldsymbol{XYZW})\\ t'[\boldsymbol{XYZ}]=t[\boldsymbol{XYZ}]}} R(t'[\boldsymbol{XYZW}]) = R(t[\boldsymbol{X}])R(t[\boldsymbol{XYZ}]). \end{aligned}$$

Having showed

$$R(t[\boldsymbol{XY}])R(t[\boldsymbol{XZ}]) = R(t[\boldsymbol{X}])R(t[\boldsymbol{XYZ}]), \tag{11}$$

we conclude that R satisfies $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$.

- Weak union: Suppose again R satisfies $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{ZW} \mid \boldsymbol{X}$, in which case Eq. (10) holds for all tuples $t \in \mathrm{Tup}(\boldsymbol{V})$. Multiplying both sides by $R[t(\boldsymbol{XZ})]R[t(\boldsymbol{XYZ})]$ yields

$$\begin{aligned} &R(t[\boldsymbol{XY}])R(t[\boldsymbol{XZW}])R[t(\boldsymbol{XZ})]R[t(\boldsymbol{XYZ})] \\ =\, &R(t[\boldsymbol{X}])R(t[\boldsymbol{XYZW}])R[t(\boldsymbol{XZ})]R[t(\boldsymbol{XYZ})]. \end{aligned}$$

If $R(t[\boldsymbol{XY}])R(t[\boldsymbol{XZ}]) \neq 0$, we may apply Eq. (11), which is implied by Eq. (10), and multiplicative cancellativity to obtain

$$R(t[\boldsymbol{XZW}])R(t[\boldsymbol{XYZ}]) = R[t(\boldsymbol{XZ})]R[t(\boldsymbol{XYZW})]. \tag{12}$$

Suppose then $R(t[\boldsymbol{XY}])R(t[\boldsymbol{XZ}]) = 0$. Since K lacks zero divisors, either $R(t[\boldsymbol{XY}]) = 0$ or $R(t[\boldsymbol{XZ}]) = 0$. By positivity and and Lemma 22, it follows that $R(t[\boldsymbol{XYZ}]) = 0$ and $R(t[\boldsymbol{XYZW}]) = 0$. In particular, both sides of Eq. (12) vanish. Hence we may conclude that R satisfies $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{W} \mid \boldsymbol{XZ}$.

- Contraction: Suppose R satisfies $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$ and $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{W} \mid \boldsymbol{XZ}$, in which case we have Eq. (12) as well as

$$R(t[\boldsymbol{XY}])R(t[\boldsymbol{XZ}]) = R(t[\boldsymbol{X}])R(t[\boldsymbol{XYZ}]), \tag{13}$$

for all $t \in \mathrm{Tup}(\boldsymbol{V})$. Multiplying both left-hand sides and right-hand sides of Eqs. (12) and (13) with one another yields

$$\begin{aligned} &R(t[\boldsymbol{XZW}])R(t[\boldsymbol{XYZ}])R(t[\boldsymbol{XY}])R(t[\boldsymbol{XZ}]) \\ =&R(t[\boldsymbol{XZ}])R(t[\boldsymbol{XYZW}])R(t[\boldsymbol{X}])R(t[\boldsymbol{XYZ}]). \end{aligned}$$

If $R(t[\boldsymbol{XZ}])R(t[\boldsymbol{XYZ}]) \neq 0$, we obtain Eq. (10) by multiplicative cancellativity. On the other hand, assuming $R(t[\boldsymbol{XZ}])R(t[\boldsymbol{XYZ}]) = 0$ we have three cases:

1. $R(t[\boldsymbol{XZ}]) = R(t[\boldsymbol{XYZ}]) = 0$. Then $R(t[\boldsymbol{XZW}]) = R(t[\boldsymbol{XYZW}]) = 0$ by positivity of R and Lemma 22, wherefore both sides of Eq. (10) vanish.
2. $R(t[\boldsymbol{XZ}]) = 0$ and $R(t[\boldsymbol{XYZ}]) \neq 0$. Then $R(t[\boldsymbol{X}]) = 0$ by positivity of K and Eq. (13). Again, both sides of Eq. (10) vanish.
3. $R(t[\boldsymbol{XZ}]) \neq 0$ and $R(t[\boldsymbol{XYZ}]) = 0$. This time $R(t[\boldsymbol{XY}]) = 0$ by positivity of K and Eq. (13), and once more we obtain Eq. (10).

Since Eq. (10) always holds, we conclude that R satisfies $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{WZ} \mid \boldsymbol{X}$.

- Interaction: Suppose R satisfies $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{XW}$ and $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{W} \mid \boldsymbol{XZ}$. Then, we have

$$R(t[\boldsymbol{XZW}])R(t[\boldsymbol{XYW}]) = R(t[\boldsymbol{XW}])R(t[\boldsymbol{XYZW}]), \tag{14}$$
$$R(t[\boldsymbol{XYZ}])R(t[\boldsymbol{XZW}]) = R(t[\boldsymbol{XZ}])R(t[\boldsymbol{XYZW}]), \tag{15}$$

for all tuples $t \in \mathrm{Tup}(\boldsymbol{V})$. By Proposition 1, K embeds in some semifield F. Given $a \in K \setminus \{0\}$, let us write a^{-1} for its multiplicative inverse in F. By positivity of K, totality of R, and Lemma 22, we observe $R(t[\boldsymbol{U}]) \neq 0$ for all $\boldsymbol{U} \subseteq \boldsymbol{V}$ and $t \in \mathrm{Tup}(\boldsymbol{V})$. Thus, the expression $R(t[\boldsymbol{U}'] \mid t([\boldsymbol{U}]) \coloneqq R(t[\boldsymbol{UU}'])R(t[\boldsymbol{U}])^{-1}$ is well defined for all $\boldsymbol{U}, \boldsymbol{U}' \subseteq \boldsymbol{V}$ and $t \in \mathrm{Tup}(\boldsymbol{V})$. We may now rewrite Eqs. (14) and (15) as

$$R(t[\boldsymbol{XYW}] \mid t[\boldsymbol{XW}]) = R(t[\boldsymbol{XYZW}] \mid t[\boldsymbol{XZW}]) = R(t[\boldsymbol{XYZ}] \mid t[\boldsymbol{XZ}])$$

from which we obtain

$$R(t[\boldsymbol{XYW}])R(t[\boldsymbol{XZ}]) = R(t[\boldsymbol{XYZ}])R(t[\boldsymbol{XW}])$$

for all $t \in \mathrm{Tup}(\boldsymbol{V})$. Thus, for arbitrary $t \in \mathrm{Tup}(\boldsymbol{V})$,

$$\begin{aligned} R(t[\boldsymbol{XY}])R(t[\boldsymbol{XZ}]) =& R(t[\boldsymbol{XZ}]) \bigoplus_{\substack{t' \in \mathrm{Tup}(\boldsymbol{XYW}) \\ t'[\boldsymbol{XY}] = t[\boldsymbol{XY}]}} R(t'[\boldsymbol{XYW}]) \\ =& \bigoplus_{\substack{t' \in \mathrm{Tup}(\boldsymbol{XYZW}) \\ t'[\boldsymbol{XYZ}] = t[\boldsymbol{XYZ}]}} R(t'[\boldsymbol{XZ}])R(t'[\boldsymbol{XYW}]) \\ =& \bigoplus_{\substack{t' \in \mathrm{Tup}(\boldsymbol{XYZW}) \\ t'[\boldsymbol{XYZ}] = t[\boldsymbol{XYZ}]}} R(t'[\boldsymbol{XYZ}])R(t'[\boldsymbol{XW}]) \\ =& R(t[\boldsymbol{XYZ}]) \bigoplus_{\substack{t' \in \mathrm{Tup}(\boldsymbol{XW}) \\ t'[\boldsymbol{X}] = t[\boldsymbol{X}]}} R(t'[\boldsymbol{XW}]) \\ =& R(t[\boldsymbol{XYZ}])R(t[\boldsymbol{X}]), \end{aligned}$$

by which we conclude that R satisfies $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{Z} \mid \boldsymbol{X}$. Hence, R also satisfies $\boldsymbol{Y} \perp\!\!\!\perp \boldsymbol{ZW} \mid \boldsymbol{X}$ by soundness of Contraction. ◀