Poset Positional Games

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Abstract

We propose a generalization of positional games, supplementing them with a restriction on the order in which the elements of the board are allowed to be claimed. We introduce poset positional games, which are positional games with an additional structure – a poset on the elements of the board. Throughout the game play, based on this poset and the set of the board elements that are claimed up to that point, we reduce the set of available moves for the player whose turn it is – an element of the board can only be claimed if all the smaller elements in the poset are already claimed.

We proceed to analyze these games in more detail, with a prime focus on the most studied convention, the Maker-Breaker games. First we build a general framework around poset positional games. Then, we perform a comprehensive study of the complexity of determining the game outcome, conditioned on the structure of the family of winning sets on the one side and the structure of the poset on the other.

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1 Introduction

1.1 General motivation

Positional games. Positional games are a class of combinatorial games that have been extensively studied in recent literature – see books [6] and [12] for an overview of the field. They include popular recreational games like Tic-Tac-Toe, Hex and Sim. Structurally, a positional game is a pair $(X, \mathcal{F})$, where X is a (finite) set that we call *the board*, and $\mathcal{F} \subseteq 2^X$ is a collection of sets that we call *the winning sets*. The pair $(X, \mathcal{F})$ is referred to as the *hypergraph of the game*. The game is played in the following way: two players alternately claim unclaimed elements of the board, until all the elements are claimed. There are several standard conventions, defining the way the winner is determined: Maker-Maker games, Maker-Breaker games, Avoider-Enforcer games, etc.

In *Maker-Maker games*, also known as strong making games, players compete to fill up one of the winning sets i.e. claim all its elements, and whoever does it first wins. If there is no winner by the time all the board elements are claimed, the game is declared a draw. Tic-Tac-Toe (“3-in-a-line”) is a notable representative of this type of games and every child knows the game ends in a draw, provided that both players play optimally. It can be generalized to the $n \times n$ board, going under the name “n-in-a-line”, and it is also known to be a drawn game [6] for all $n \geq 3$. More generally, Tic-Tac-Toe can be played on the hypercube $[n]^d$, where the winning sets are all the geometric lines of cardinality n. The Hales-Jewett Theorem [11] states that, for every n, there exists a positive number d, also called the Hales-Jewett number $HJ(n)$, such that for all $k \geq d$, the game $[n]^k$ is a first player win.

Generally speaking, the strong making games are natural to introduce and study, and many recreational games are of this type. Hence, it is no wonder that numerous questions about them have been asked in the literature. Yet, it may come as a surprise that the majority of those questions remain unanswered. The thing is, in a typical strong making game, each player’s goal can be seen as two-fold: they are simultaneously trying to claim a winning set and to block their opponent’s attempts at claiming a winning set. This makes most such games notoriously hard to analyze, and hardly any general tools are known for this purpose. As a consequence, there are very few published results about this convention, when compared to Maker-Breaker games.

In *Maker-Breaker games*, we call the players Maker and Breaker, and they have different goals – Maker wins if she fills up a winning set, while Breaker wins otherwise, i.e. if he claims at least one element in each winning set. Note that no draw is possible in this convention. Maker-Breaker games are the most researched convention of positional games, ever since Erdős and Selfridge [9] first introduced them in 1973. Looking at Tic-Tac-Toe on the standard 3×3 board in the Maker-Breaker convention, it is straightforward to convince oneself that Makers wins when playing first.

The main problem on positional games consists in determining the *outcome* i.e. the identity of the player who has a winning strategy (possibly depending on who starts), if there is one, assuming that both players are playing optimally. The study of the complexity of computing the outcome of a given positional game can be traced back to Schaefer [14], who was first to prove that Maker-Breaker games are PSPACE-complete, even when the winning sets are of size at most 11. This was later improved by Rahman and Watson [13], requiring only winning sets of size 6. On the other hand, Galliot et al. [10] proved that the outcome of any Maker-Breaker game with winning sets of size at most 3 can be determined in polynomial time. Maker-Maker games are also known to be PSPACE-complete, as shown by Byskov in [7].

Poset positional games. Let us now take a closer look at a popular recreational game, Connect-4, which has a lot in common with the Tic-Tac-Toe family of positional games. In Connect-4, two players play on a board that is 7-wide and 6-high. They move alternately by placing a token of theirs in a column of their choice. Each placed token drops down with "gravity", landing on top of the last previously placed token in that column, or heading all the way to the bottom of the column if it is empty. The first player with tokens on four consecutive positions in a line (vertical, horizontal or diagonal) wins, and, if neither of the players manages to do that by the time all the columns are full, then a draw is declared. Although the game is similar to Tic-Tac-Toe at first glance, there is one crucial difference – the players cannot choose freely from all the empty positions of the board, as at any point a column offers at most one available position (the lowest token-free position).

The outcome of the game is known as a first player win, as shown by Allis in [3], and independently by Allen [1] who later wrote a whole book [2] about this game. Connect-4 was solved in [17] for some nonstandard board sizes, and for some others in [16].

Recently, Avadhanam and Jena [4] studied *Connect-Tac-Toe*, where they introduce a restriction on which unclaimed elements can be claimed, combining the nature of Connect-4 with the positional game of Tic-Tac-Toe. They also look at the generalized $[n]^d$ Connect-Tac-Toe, played on a hypercube $[n]^d$, relating it to the restricted version of the $[n]^d$ Tic-Tac-Toe. Additionally, they give a lower bound on an analogue of the Hales-Jewett number, $HJ(n)$, in this setting.

In the present paper, we propose a new framework which, in full generality, enables us to combine the move restrictions with positional games (like it is done in Connect-4). Namely, we introduce *poset positional games*, which are positional games with an additional structure – a poset on the elements of the board. Throughout the game play, based on this poset and the set of elements that are claimed up to that point, we reduce the set of available moves for the player whose turn it is – an element of the board can only be claimed if all the smaller elements (in the poset) are already claimed.

We proceed to analyze these games in more detail, with a prime focus on the most studied convention, the Maker-Breaker games. After setting out a formal introduction of poset positional games, we go on to build a general framework around them. Then, we perform a comprehensive study of the complexity of determining the game outcome, conditioned on the structure of the family of winning sets on the one side and the structure of the poset on the other.

1.2 Framework of poset positional games

A *poset* P on a set X is defined by a partial order relation $\leq$. On top of that, we use $<$ to denote the same relation acting on distinct elements. In this paper, posets will be depicted by directed graphs in the usual way, where the elements are vertices, and two elements x and y satisfy $x \leq y$ if and only if there exists a directed path from x to y. Two elements are deemed *incomparable* if there exists no directed path between them.

In addition, we will use the following standard definitions about posets. A *chain* is a set of elements that are pairwise comparable. An *antichain* is a set of elements that are pairwise incomparable. The *height* of a poset is the cardinality of its longest chain. The *width* of a poset is the cardinality of its largest antichain. Given an element x, we say that y is a *predecessor* (resp. a *successor*) of x if $y < x$ (resp. $x < y$) and there is no other element between x and y. An element x is said to be *maximal* (resp. *minimal*) in the poset if it has no successor (resp. no predecessor).

We are now able to formally define a poset positional game as follows. A *poset positional game* is a triple $(X, \mathcal{F}, P)$ where X is a finite set of elements (also called *vertices*), $\mathcal{F}$ is a collection of subsets of X corresponding to the *winning sets*, and P is a poset on X.

The game is played by two players that alternately claim an unclaimed vertex x of X such that all vertices smaller than x have already been claimed. The rest stays the same as before. In the Maker-Maker convention, the first player that fills up a winning set $S \in \mathcal{F}$ wins. If no player manages to fill up a winning set, the game ends in a draw. In the Maker-Breaker convention, Maker wins if she fills up a winning set at some point during the game, otherwise Breaker wins. During a game, when a player claims a single vertex, we call that a *move*, whereas a *round* corresponds to a pair of moves made successively by both players.

Note that a standard positional game is a poset positional game where all the vertices are pairwise incomparable. Furthermore, in poset positional games, from any given position, the set of moves that are available to the next player forms an antichain of the poset. In particular, there are at most w available moves at any point of a game, where w is the width of the poset.

It is a well-known result that, for any standard positional game played in the Maker-Maker convention, the second player cannot have a winning strategy, so that the only possible outcomes are a first player win $\mathcal{FP}$ or a draw $\mathcal{D}$. Similarly, for any standard positional game played in the Maker-Breaker convention, there are only three possible outcomes (if we do not specify who starts):

- $\mathcal{M}$ if Maker has a winning strategy no matter who starts the game,
- $\mathcal{B}$ if Breaker has a winning strategy no matter who starts the game,
- $\mathcal{N}$ if the next player (i.e. the player whose turn it is) has a winning strategy.

When switching to the framework of poset positional games, the property asserting that the second player never wins is not true any more. As a consequence, there are game positions for which the next player may have interest in skipping their turn. The corresponding outcome will be denoted by $\mathcal{P}$ (standing for "Previous player wins", or equivalently meaning that the second player has a winning strategy). Such phenomena are generally called *zugzwangs* in the literature of combinatorial games [8].

From now on, unless explicitly stated, we will only consider the Maker-Breaker convention of poset positional games. Though the computation of the outcome is generally the main issue when investigating such games, this study can, under certain circumstances, be reduced to the case of a particular player starting the game. Indeed, starting with a game $\mathcal{G} = (X, \mathcal{F}, P)$ where Maker (resp. Breaker) is the first player and claims a vertex x, we get a resulting game $\mathcal{G}' = (X', \mathcal{F}', P')$ where Breaker (resp. Maker) is the first player, defined by $X' = X - x$, $P' = P - x$ and $\mathcal{F}' = \{W \setminus \{x\} | \ W \in \mathcal{F}\}$ (resp. $\mathcal{F}' = \{W | \ W \in \mathcal{F}, x \notin W\}$). Therefore, when studying a class of games which is stable under Maker's (resp. Breaker's) moves in terms of that update, we may freely assume that Breaker (resp. Maker) is the first player, up to considering all possibilities for their opponent's first move otherwise. In this paper, all studied classes will be stable under Breaker's moves, so we will always assume that Maker is the first player. Therefore, the decision problem that will be mainly investigated is the following:

MB Poset Positional Game
Input: A poset positional game $\mathcal{G} = (X, \mathcal{F}, P)$.
Output: The player having a winning strategy (i.e. Maker or Breaker) when Maker starts.

Since standard positional games are included in poset positional games, the problem MB Poset Positional Game is PSPACE-complete from a result of Schaefer [14].

1.3 Exposition of the results

The main objective of this paper is to consider the complexity of MB Poset Positional Game related to some parameters of the instance. More precisely, we have chosen to focus on the properties of the poset, as it is the main distinctiveness of the current contribution, comparing the results we obtain to the previous results about standard positional games. In Section 2, we will firstly examine the problem depending on the height of the poset. As it is already known that, even for height 1 (i.e. all the elements are pairwise incomparable) and winning sets of size 6, the problem is PSPACE-complete [13], we will also refine our classification according to the number and/or the size of the winning sets. The main contribution of this section addresses the case where there is only one winning set of size 1. By adding such a condition, the problem admits a complexity jump between instances of height 2, proved to be polynomial, and height 3, proved to be NP-hard.

Section 3 deals with the width of the poset. As the case of width 1 is straightforward (P is made of only one chain, so the order of the moves is completely predetermined), we start by considering instances with width 2. We show that MB Poset Positional Game is PSPACE-hard in this case, even if all the winning sets are of size 3. This illustrates a major difference with standard positional games, which are known to be tractable in this setting. We then give a polynomial-time algorithm that solves the general case where the width of the poset and the number of winning sets are fixed.

Section 4 is devoted to the case where the poset is a union of disjoint chains. This case is a direct generalization of the game Connect-4. Our first result is a full characterization of the outcome when all the winning sets are of size 1. When the winning sets are of size at most 2, things are more tricky, but we do provide a polynomial-time algorithm when the width of the poset (i.e. the number of chains in the union) is fixed. Finally, by adding the restriction that the height of each chain is at most 2, we get a polynomial-time algorithm regardless of the number of chains.

Table 1 summarizes all the results about the complexity of MB Poset Positional Game, referring either to the literature of standard positional games, or to results proved in the current paper. In the table, recall that the complexity class XP defines the class of problems parameterized by a parameter k and that can be solved in time $O(|X|^{f(k)})$, where $|X|$ is the size of the instance and f is a computable function.

In order to satisfy the page limit, some of our statements are given without a proof. All the proofs are available in the full version of our paper [5].

2 Posets of small height

2.1 Posets of height 2

We first look at what happens when all the winning sets are of size 1. This case is rarely straightforward, and often deserves to be studied depending on some parameters of the poset. In addition, it is closely correlated to the case where there is a unique winning set (of any size). The following remark explains the link between both situations.

▶ Remark 1. Up to switching the roles of the players, a poset positional game with a single winning set of size k is equivalent to a poset positional game with k winning sets of size 1.

In the above remark, the equivalence means that the two games have the same outcome. Indeed, Maker wins when there is a single winning set of size k if and only if she manages to claim all the elements of this set. In the second game, a win of Breaker consists in claiming all the winning sets (of size 1), thus corresponding to the equivalence. Note that Breaker is starting in one of the two games, but this is not a pitfall since both these classes are stable under Breaker's moves.

Table 1 Complexity of MB POSET POSITIONAL GAME depending on some parameters of the poset and of the collection of winning sets.

winning sets \ poset	general	height h	width w	disjoint chains
general	PSPACE-c [13]	$h = 1$: PSPACE-c [13]	$w = 1$: P $w = 2$: PSPACE-c (Th. 7)	$h = 1$: PSPACE-c [13]
size s	$s = 3$: PSPACE-c (Th. 7)	$h = 1$, $s = 6$: PSPACE-c [13] $h = 2$, $s = 3$: NP-h (Th. 5)	$w = 2$, $s = 3$: PSPACE-c (Th. 7)	$s = 1$: P (Th. 11) $s = 2$: XP by w (Th. 14) $h = 2$, $s = 2$: P (Th. 15)
number m	NP-hard (Th. 6)	$m = 1$, $h = 3$: NP-hard (Th. 6)	XP (Th. 8)	$m = 1$: P (Th. 11 and Remark 1)
number m and size s	$m = 1$, $s = 1$: NP-hard (Th. 6)	$h = 2$, $m = 1$, $s = 1$: P (Th. 3) $h = 3$, $m = 1$, $s = 1$: NP-hard (Th. 6)	XP (Th. 8)	XP (Th. 8)

We continue with a minor result which is useful when dealing with winning sets of size 1. Given a poset P and a vertex x, we denote by $p(x)$ the number of vertices that are not greater or equal to x i.e. $p(x) := |X| - |\{y|\ y \geq x\}|$. This quantity is used in the following lemma, which yields a necessary and sufficient condition for Maker to win when there is a unique winning set of size 1.

▶ **Lemma 2.** *Let $\mathcal{G} = (X, \{\{x\}\}, P)$ be a poset positional game. Suppose that, after some rounds of play in $\mathcal{G}$, all predecessors of x have been claimed apart from one, which we denote by y. Then, in this position, Maker wins if and only if $p(y)$ is odd.*

▶ **Theorem 3.** MB POSET POSITIONAL GAME *can be solved in polynomial time when: the poset is of height 2, there is only one winning set, and this winning set is of size 1.*

Sketch of the proof. Let $\{x\}$ be the winning set. We partition the predecessors of x into two sets M and B, where M (resp. B) contains the predecessors y of x such that $p(y)$ is even (resp. odd). Then Maker wins if and only if $|M| \leq |B|$. ◀

The situation is already much more complicated with two winning sets of size 1 – we leave this as an open problem. However, for winning sets of size 1 and a poset of height 2 whose nonminimal elements are all winning sets, one can compute the outcome in polynomial time, as we now show. Note that the same situation is PSPACE-complete in the Maker-Maker convention (see the full version [5]).

▶ **Theorem 4.** MB POSET POSITIONAL GAME *can be solved in polynomial time when: the poset is of height 2, all the winning sets are of size 1, and all nonminimal elements are winning sets.*

Sketch of the proof. Let P be a poset of height 2 on a set X. We say the minimal elements of P are the "bottom" vertices, and the others are the "top" vertices. We say that x is a *private predecessor* of y if y is the only successor of x.

The poset positional game we consider is $\mathcal{G} = (X, \{\{y\}, y \text{ top vertex}\}, P)$. We prove that Maker wins if and only if (1) $|X|$ is odd or (2) there exists a top vertex y which does not have more private predecessors than non-private ones.

If (1) holds then, since Maker starts, she also plays last, which means she can win by waiting for Breaker to claim the last predecessor of some top vertex. If (2) holds, then Maker can ensure that all the private predecessors of y are claimed before all the non-private ones are, after which she waits for Breaker to claim the last predecessor of y. She might have to claim it herself, but that will make at least two top vertices available at once, meaning she will win with her next move.

If none of the conditions hold, then Breaker can maintain the property that "all the top vertices have strictly more private predecessors that non-private ones" during the game. This allows Breaker to claim all the top vertices, meaning he wins. ◀

For posets of height $h = 2$, as the size s of the winning sets increases, the value $s = 3$ is the smallest one for which we have identified an algorithmic complexity jump.

▶ **Theorem 5.** MB Poset Positional Game *is* NP*-hard, even when restricted to instances where the poset has height* 2 *and all the winning sets are of size 3.*

Sketch of the proof. The proof is a reduction from 3-SAT. Let ϕ be a 3-SAT formula. We build a poset positional game $\mathcal{G} = (X, \mathcal{F}, P)$ with P of height 2 and winning sets of size 3 as follows:

- For any variable x_i of ϕ, we add four vertices $u_i, v_i, \overline{v_i}$ and $\widetilde{v_i}$ in X.
- For any clause $C_j = l_{j_1} \vee l_{j_2} \vee l_{j_3}$ in ϕ, we add the winning set S_j with, for $1 \leq k \leq 3$, $v_{j_k} \in S_j$ if $l_{j_k} = x_{j_k}$, and $\overline{v_{j_k}} \in S_j$ if $l_{j_k} = \neg x_{j_k}$.
- For any variable x_i of ϕ, we add the relations $u_i < v_i$, $u_i < \overline{v_i}$ and $u_i < \widetilde{v_i}$ in P.

We prove that ϕ is satisfiable if and only if Breaker has a winning strategy in $\mathcal{G}$. ◀

2.2 Posets of height 3

We now consider posets of height 3. Using Lemma 2, we prove that MB Poset Positional Game is NP-hard even if there is only one winning set and that winning set is of size 1. The reduction is done from the problem Set Cover.

▶ **Theorem 6.** MB Poset Positional Game *is* NP*-hard even when restricted to instances where: the poset has height 3, there is only one winning set, and that winning set is of size 1.*

3 Posets of bounded width

In this section, we consider posets of bounded width. For width 1, all moves are forced, so the outcome is obviously computed in polynomial time. However, for width 2, we prove that MB Poset Positional Game is already PSPACE-hard even if the winning sets are of size 3.

▶ **Theorem 7.** MB Poset Positional Game *is* PSPACE*-complete even when restricted to instances where the poset is of width 2 and the winning sets are of size 3.*

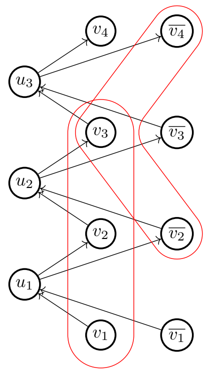

Figure 1 The poset positional game obtained by reduction of $\forall x_1 \exists x_2 \forall x_3 \exists x_4 (x_1 \vee x_2 \vee x_3) \wedge (\neg x_2 \vee x_3 \vee \neg x_4)$.

Sketch of the proof. The reduction is done from 3-QBF, which has been proved PSPACE-complete by Stockmeyer and Meyer [15]. In 3-QBF, the input is a formula of the form $\psi = \forall x_1 \exists x_2 \dots \forall x_{2n-1} \exists x_{2n} \ \ \phi(x_1, \dots, x_{2n})$ where ϕ is a 3-CNF formula. The ouput is the truth value of ψ. We build a poset positional game $\mathcal{G} = (X, \mathcal{F}, P)$ as follows:

- For any $1 \leq i \leq 2n$, we add two vertices v_i and $\overline{v_i}$ in X.
- For any $1 \leq i \leq 2n-1$, we also add a vertex u_i.
- For any clause $C_j = l_{i_1} \vee l_{i_2} \vee l_{i_3}$ in ϕ, we add a winning set $S_j \in \mathcal{F}$. For $1 \leq k \leq 3$, we have $v_{i_k} \in S_j$ if $l_{i_k} = x_{i_k}$ and $\overline{v_{i_k}} \in S_j$ if $l_{i_k} = \neg x_{i_k}$.
- For any $1 \leq i \leq 2n-1$, we add in P the relations $v_i < u_i$, $\overline{v_i} < u_i$, $u_i < v_{i+1}$ and $u_i < \overline{v_{i+1}}$.

See Figure 1 for an example. It can be proved that ψ is True if and only if Breaker wins in $\mathcal{G}$. ◀

If the number of winning sets is also bounded (by m), we give an algorithm running in time $O(|X|^w 2^m w^4)$.

▶ **Theorem 8.** MB Poset Positional Game *can be decided in time $O(|X|^w 2^m w^4)$ for instances where the poset is of width at most w and there are at most m winning sets.*

Sketch of the proof. The proof uses a dynamic approach. We characterize the games that can be reached from a game played on $\mathcal{G} = (X, \mathcal{F}, P)$ using an antichain Y of X, that represents the current available moves, and a boolean vector $B = (b_1, ..., b_m) \in \{0, 1\}^m$ that represents, for each winning set, if Maker has claimed all its elements below Y (and thus can still hope to claim it entirely). ◀

4 Posets made of pairwise disjoint chains

In this section, motivated by the game Connect-4, we consider posets made of pairwise disjoint chains. In the rest of this section, we give positive results for poset positional games on pairwise disjoint chains with very small winning sets (but an unbounded number of them). We will often use parity arguments when discussing games on posets made of pairwise disjoint chains. Therefore, we introduce the following framework, which will be useful in this section. Consider a poset positional game $\mathcal{G} = (X, \mathcal{F}, P)$ where P is made of pairwise disjoint chains $C_1, ..., C_w$. The elements of the chain C_i are denoted as $x_{i,1} > x_{i,2} > ... > x_{i,\ell_i}$. This numbering of the vertices from top to bottom is best adapted to define the following coloring: a vertex $x_{i,j}$ is colored white if j has same parity as $|X|$, otherwise it is colored black.

4.1 Winning sets of size 1

The next two lemmas give sufficient conditions for Maker to win under particular conditions that are based on the above coloring. They can be proved with simple parity arguments. Combined together in the next theorem, they allow to have a complete characterization of the general case where all the winning sets are of size 1.

▶ **Lemma 9.** *If $\{u\} \in \mathcal{F}$ for some white vertex u, then Maker wins.*

▶ **Lemma 10.** *If $|X|$ is odd and $\{u\}, \{v\} \in \mathcal{F}$ for some black vertices u and v sitting on different chains, then Maker wins.*

▶ **Theorem 11.** *Let $\mathcal{G} = (X, \mathcal{F}, P)$ be a poset positional game where P is made of pairwise disjoint chains and all elements of $\mathcal{F}$ are of size 1. Maker wins $\mathcal{G}$ if and only if at least one of the following conditions hold:*
- *there is a winning set that is a minimal element; or*
- *there is a white winning set; or*
- *$|X|$ is odd and there are two black winning sets on different chains.*

In particular, MB Poset Positional Game *can be solved in linear time for such games.*

Sketch of the proof. The "only if" direction is the one that remains to be proved. For even $|X|$, we show that Breaker can claim all the nonminimal black vertices. For odd $|X|$, we show that Breaker can claim all the nonminimal black vertices of any given chain. ◀

Using this result, we can solve a basic case of $w \times h$ *Connect-k* in Maker-Breaker convention, which is the poset positional game played on w chains of height h where the winning sets are the alignments of k consecutive vertices (horizontal, vertical or diagonal). Note that the game is easily solved if $k \leq 2$, or if $k > w$, or if $k > h$ and wh is even.

▶ **Corollary 12.** *Let k, w, h be integers with $3 \leq k \leq w$ and $h > 1$. If w and h are both odd, then Maker wins the $w \times h$ Connect-k game in Maker-Breaker convention.*

4.2 Winning sets of size at most 2

Things become more complicated when the winning sets are not all of size 1. In the presence of a black winning set of size 1, we can simplify the game. Using the same notations as before, assume that there exists a winning set $\{x_{i,j}\}$ for some nonminimal black vertex $x_{i,j}$ with $j \geq 3$. We define a reduced game $\mathcal{G}' = (X', \mathcal{F}', P')$ as follows.

If $|X|$ is even, then define $Y = \{x_{i,1}, ..., x_{i,j-1}\}$, otherwise define $Y = \{x_{i,1}, ..., x_{i,j-2}\}$. In both cases, let $X' = X \setminus Y$ and let P' be the poset induced by P on X'. Note that, since $x_{i,j}$ is black, we have removed an even number of vertices in both cases. In particular, the coloring of the vertices is the same for (X', P') as for (X, P). We now define the winning sets of $\mathcal{G}'$ as follows. Let $S \in \mathcal{F}$. If $S \subseteq X'$, then $S \in \mathcal{F}'$. If S only intersects Y on white vertices, then $S \cap X' \in \mathcal{F}'$. Otherwise, i.e. if S contains a black vertex greater than $x_{i,j}$, then we ignore S. See Figure 2 for an illustration in the case where $|X|$ is odd.

▶ **Lemma 13.** *Assume that $\mathcal{G}$ has no winning set containing only white vertices that are greater than $x_{i,j}$. Then the two games $\mathcal{G}$ and $\mathcal{G}'$ have the same outcome when Maker starts.*

Sketch of the proof. We prove that, if a player has a winning strategy in $\mathcal{G}'$, then that player has also a winning strategy in $\mathcal{G}$. The idea is the same for both players: they follow their strategy in $\mathcal{G}'$, as long as possible. When $x_{i,j}$ is claimed (by Breaker, otherwise Maker wins immediately), Maker can ensure to claim all the white vertices above $x_{i,j}$, thus completing a

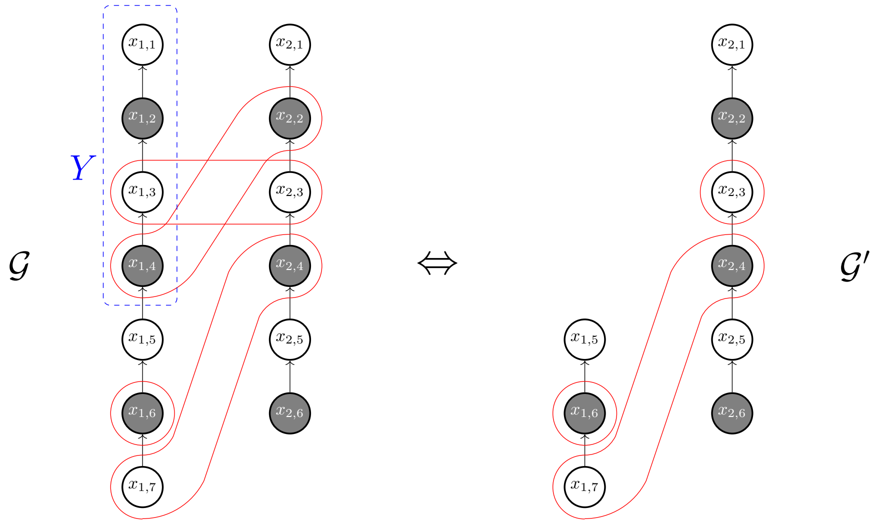

Figure 2 Reduction of a game $\mathcal{G}$ containing a black winning set of size 1 to a game $\mathcal{G}'$. The winning set $\{x_{1,4}, x_{2,2}\}$ disappears since it contains a black vertex in Y. The winning set $\{x_{1,3}, x_{2,3}\}$ becomes the winning set $\{x_{2,3}\}$ in $\mathcal{G}'$ since $x_{1,3}$ is white. By Lemma 13, the two games have the same outcome. In this case, since $\mathcal{G}'$ has a white winning set of size 1, both games are winning for Maker.

winning set she would have filled up in $\mathcal{G}'$. If she cannot fill up a winning set of $\mathcal{G}'$ (i.e. if Breaker wins $\mathcal{G}'$), then Breaker can claim all the black vertices above $x_{i,j}$ and thus claim a vertex in all the winning sets of $\mathcal{G}$. ◀

Now equipped with this reduction, we can provide an algorithm when all the winning sets have size at most 2. Our algorithm is based on a dynamic programming approach. Thanks to Lemmas 9 and 13, we can reduce the number of sub-positions to consider. Indeed, we can assume that each chain contains at most one (black) winning set of size 1. Our algorithm dynamically computes the outcome of all these "useful" positions.

▶ **Theorem 14.** MB Poset Positional Game *can be solved in time $O(h^{2(w+1)})$ for instances where the poset consists of w pairwise disjoint chains of height at most h and all winning sets have size at most 2.*

4.3 Winning sets of size at most 2, chains of height at most 2

The algorithm from Theorem 14 is not efficient for posets of unbounded width. However, when restricting the problem to posets of height at most 2, we do get a polynomial-time algorithm in this case also. Since the chains are of height at most 2, we will simply use the "top/bottom" terminology already adopted in the proof of Theorem 4, rather than the coloring previously used in this section.

▶ **Theorem 15.** MB Poset Positional Game *can be solved in time $O(|X|^4)$ for instances where the poset on X consists of pairwise disjoint chains of height at most 2 and all winning sets are of size at most 2.*

Sketch of the proof. Let $x_1, \ldots, x_t, y_1, \ldots, y_t$ be the vertices of the chains of height 2, with $x_i < y_i$ for all $1 \le i \le t$. Let $x_{t+1}, \ldots, x_w$ be the other vertices if there are any. We say the x_i are the "bottom" vertices and the y_j are the "top" vertices. The proof distinguishes two cases depending on the parity of $|X|$. When $|X|$ is odd, there are several possibilities, but each of them is a straightforward win for one of the two players. When $|X|$ is even, the situation is more tricky. In that case, the key argument is based on two reductions R1 and R2 that preserve the outcome:

R1: Assume there exists a winning set of the form $\{x_i, x_j\}$ that is disjoint from all the other winning sets. Then remove x_i and x_j.

R2: Assume there exists a nonempty set of indices $I \subseteq \{1, \ldots, t\}$ such that all the winning sets that contain some x_i with $i \in I$ also contain some y_j with $j \in I$. Then remove all the vertices with indices in I. ◀

5 Conclusion and future work

The current work introduces a new framework that opens the door to a variety of perspectives. Our analysis with respect to the parameters of the poset led to a first classification of the complexity of MB Poset Positional Game, which is a step towards a better understanding of the boundary between tractability and hardness. The results presented here directly induce a list of open problems that arise naturally in order to refine this boundary. Among them, we have identified the following two questions that seem the most relevant for us:

- When there is exactly one winning set of size 1, the height of the poset makes a difference between tractability ($h = 2$) and NP-hardness ($h = 3$). Therefore, when $h = 2$, it is worthwhile to examine how the conditions on the number m of winning sets and their size s impact the algorithmic complexity. A next study would be to investigate the case $m = 2$ and $s = 1$.
- In the case of disjoint chains, an analysis of the complexity according to their width (i.e. the number of chains) is a natural perspective. In particular, one may focus on the case $w = 2$, even when restricted to $s \leq 3$ (since we have a polynomial-time algorithm for $s = 2$). It would also be interesting to obtain a hardness results for disjoint chains.

Going back to our initial motivation of giving a general framework for Connect-k games, one could also examine the case of boards of even size for this game. For example, the famous case $(k, w, h) = (4, 7, 6)$, is known to be a Maker win in the Maker-Breaker convention since it a first player win in the Maker-Maker convention. However, a direct proof for the Maker-Breaker convention could be considered, that could also be extended to other sizes.

Finally, while we mostly studied the poset positional games in the Maker-Breaker convention, our definition can be transposed to all other conventions of positional games. It would therefore be interesting to perform a similar analysis for Maker-Maker or Avoider-Enforcer poset positional games.

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