No Tiling of the 70×70 Square with Consecutive Squares

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Abstract

The total area of the 24 squares of sizes $1, 2, \ldots, 24$ is equal to the area of the 70×70 square. Can this equation be demonstrated by a tiling of the 70×70 square with the 24 squares of sizes $1, 2, \ldots, 24$? The answer is "NO", no such tiling exists. This has been demonstrated by computer search. However, until now, no proof without use of computer was given. We fill this gap and give a complete combinatorial proof.

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1 Introduction

The total area of the 24 squares of sizes $1, 2, \ldots, 24$ is equal to the area of the 70×70 square. In fact, this is the only nontrivial solution of the Diophantine equation $1^2 + 2^2 + \cdots + n^2 = m^2$, see [7]. Can this equation be demonstrated by a tiling of the 70×70 square with the 24 squares of sizes $1, 2, \ldots, 24$?

This natural question was popularized by Martin Gardner [3], who attributes the problem to R. B. Britton (unpublished). The answer is "NO", no such tiling exists. This has been demonstrated by computer search, as claimed first in [2] without giving details and later in [4, 5]. However, until now, no proof without use of computer was given. We fill this gap and give a complete combinatorial proof.

Some initial steps towards a combinatorial proof were given in [6]. Using an extensive case analysis, it is shown there that squares of size up to 5 cannot be placed on the edges of the 70×70 board; in addition some further combinations of squares on edges, typically involving 6, 7, and/or 8, are also excluded.

This problem belongs to a wide class of Mrs. Perkins's Quilt problems, that in general ask for "squaring the square", i.e., tiling of a given square with smaller squares. See [1] for an overview of this rich area. Our problem is one special case. In addition to exact tiling, there are numerous results on packings that do not cover the whole square or coverings where some small squares overlap. One can then state various optimization problems, like minimizing the bounding box that allows packing a given set of small squares, or minimizing the overlap. One can also allow to omit some small squares and find a packing of a subset of squares in the given square that minimizes the empty space. In our case, the 70×70 square can accommodate all smaller squares except for the 7×7 square and the computer search in [5] also shows that 49 is indeed the smallest possible empty space.

1.1 New techniques and proof overview

The key to a shorter proof avoiding a long case analysis is a new insight into the global structure of a tiling. We assume that we have some tiling of the 70×70 board with the 24 squares and seek an eventual contradiction.

First observe that no three squares can together intersect all the rows as $22 + 23 + 24 = 69 < 70$. This allows us to select four **important rows** such that each square intersects at most one of them. Namely, select as two important rows the bottom row and the row just above the largest square intersecting the bottom row. Similarly, select the top row and the row just below the largest square intersecting the top row. Actually, there is some flexibility in this selection, and we may choose the important rows so that the distance among any two of them is at least 22 by moving the middle two important rows towards the center, if necessary. This simplifies some arguments later. Analogously we define four **important columns**. See Section 2 for precise definitions.

Now we classify all squares in the solution. A square is called

Major if it intersects both an important row and an important column.

Minor if it intersects an important row or an important column but not both.

Orphan if it intersects neither an important row nor an important column.

See Fig. 1.1 for an illustration of the square types.

We continue by a counting argument that shows that there are at most two orphans which in addition have total size at most 4, see Theorem 2.4.

The limit of two orphans, additionally of small size, is a serious restriction. E.g., when trying to sketch a possible tiling, one realizes that a corner square often has an orphan adjacent to it, similarly to the top left corner in Figure 1.1, unless the corner square is very large. One can actually prove that a smallest square on an edge must have an orphan adjacent to it whenever it has size at most 11.

With this classification and restrictions in mind, we examine the tiling and exclude possible cases step by step.

We start by excluding small squares on the edges of the 70×70 board, somewhat similar to [6]. The bounds on orphans allow us to exclude squares of size up to 9 fairly easily, see Sections 3 and 4.

If all the squares on an edge have size at least 10, we show that there are only four squares on each edge, total of twelve squares along all edges, including the four corner squares touching two edges, see Lemma 3.6. With this restriction, it is not hard to exclude also a square of size 10 on an edge, see the end of Section 4.

More importantly, this restriction to only four squares on each edge allows us to use another counting technique introduced at the beginning of Section 5. We illustrate it on the case where all the squares on the edges have size at least 13. This means that the twelve

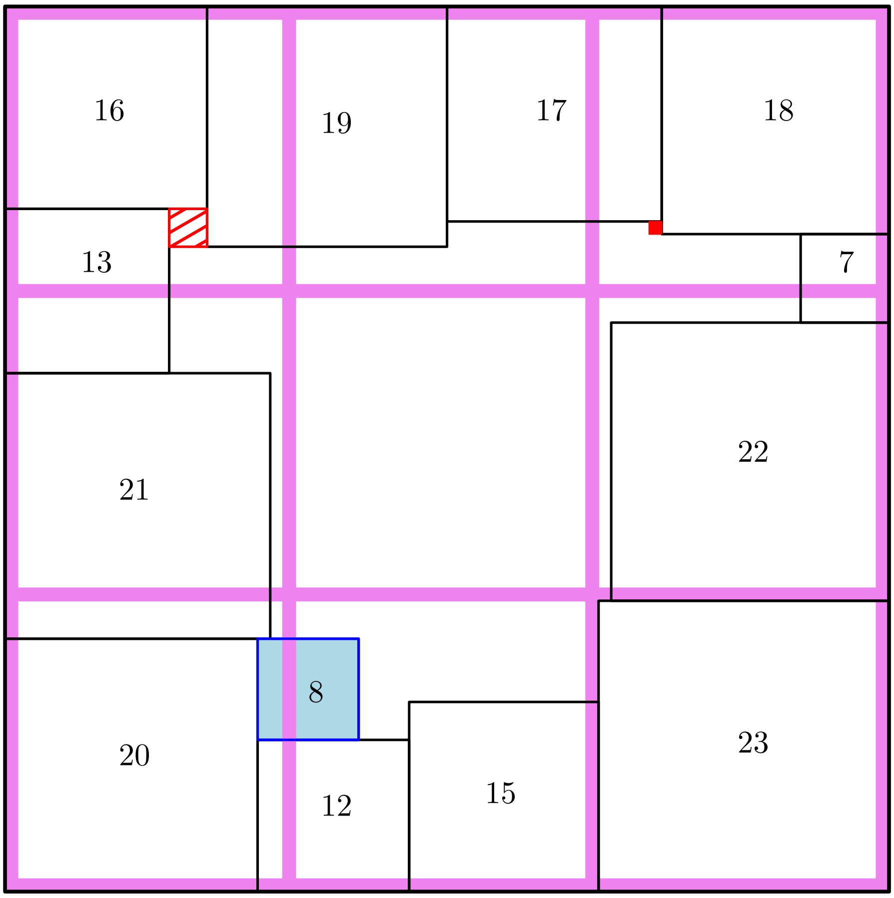


Figure 1.1 Important rows and columns; square types.
As a convention, in all our figures we use colors and shading as follows:
- the important rows and columns are drawn as violet thick lines;
- major squares are drawn in black with no fill;
- minor squares are drawn in blue with a light fill;
- orphans are drawn in red, filled if size is 1, hatched if size is larger.

squares on edges are the twelve largest squares and have total size equal to $13+\cdots+24 = 222$. Furthermore, every corner square is adjacent to a smaller square on one of the edges and these squares are unique. It follows that the sizes of corner squares are at least 14, 16, 18, 20 and thus their total size is at least 68. However, this would imply that the circumference of the board is at least $222 + 68 = 290$, contradicting the obvious fact that the circumference is equal to $4 \cdot 70 = 280$.

The remaining case where a square of size 11 or 12 is on an edge is the most delicate part of the proof and needs a careful combination of all the techniques above, this is covered in Section 5. First we exclude the option that both 11 and 12 touch an edge. This gives a good lower bound on the total size of the twelve squares on the edges. Then, using some case analysis, we give a lower bound on the size of the corner squares. This leads to the final contradiction.

2 Important lines and classification of squares

As already used above, the board refers to the 70×70 square which should be tiled. The board consists of 70 rows and 70 columns of 1×1 squares that we call cells.

From now on, by a square we always mean one of the square tiles as positioned in the solution. Edges always mean edges of the board, while sides are the sides of a square. A square that is adjacent to an edge of the board is called an edge square; a square adjacent to two edges of the board is called a corner square.

We assume that we have some solution, i.e., a tiling of the board with the 24 squares. It is obvious that every square in the solution is aligned with the cells, as the solution covers the board perfectly. From now on, we consider only such positions of squares, i.e., only positions that have an integral offset from a corner of the board.

▶ **Definition 2.1.** *We define four important rows of cells:*

Bottom row: *The row along the bottom edge.*

Low row: *Let $\mathbb{A}$ be the largest square touching the bottom edge, let $x = \max(22, \mathbb{A})$. The low row is the $(x+1)$-st row from bottom.*

High row: *Let $\mathbb{B}$ be the largest square touching the top edge, let $y = \max(22, \mathbb{B})$. The high row is the $(y+1)$-st row from top.*

Top row: *The row along the top edge.*

Similarly, we define the first, left, right, and last columns and together call them important columns.

See Fig. 1.1 for an illustration of important rows and columns on a partial tiling. Note that the second important row and column do not touch any square on the top or left edge, as all these squares are smaller than 22 and thus the important row and column are moved towards the center, to maintain the desired spacing.

▶ **Observation 2.2.**

(i) *No square intersects two of the important rows.*

(ii) *Any 22 consecutive rows contain at most one important row.*

(iii) *No square intersects two of the important columns.*

(iv) *Any 22 consecutive columns contain at most one important column.*

Proof. We prove the first two items for rows. The claims for columns follow by symmetry.

By definition, no square intersects both bottom and low rows, and also no square intersects both top and high rows.

Let x be the number of rows below the low row and y the number of rows above the high row (as in the definition). For contradiction, assume that a square of size z intersects both low and high rows. Note that this together covers all rows, implying $x + y + z \geq 70$.

At most one of x, y, z can be equal to 24, as these options imply that the square 24 is either at the top edge, at the bottom edge, or none of the two, respectively. Similarly, at most one of x, y, z can be equal to 23. Thus $x + y + z \leq 24 + 23 + 22 = 69$, a contradiction.

For the second claim, we need to observe that there are at least 21 rows between two important rows. This follows by definition for bottom and low rows, and also for top and high rows. As $x + y \leq 23 + 24 = 47$, there are at least $70 - 47 = 23$ rows between low and high rows including these two important rows, and the claim follows as well. ◀

For completeness, we repeat the classification of the squares in the solution.

▶ **Definition 2.3.** *A square is called*
Major *if it intersects both an important row and an important column.*
Minor *if it intersects an important row or an important column but not both.*
Orphan *if it intersects neither an important row nor an important column.*

There are exactly 16 major squares, as there are exactly 16 cells in the intersections of important rows and important columns, each of these cells is covered by a square and no square contains two of these cells due to Observation 2.2.

In the next theorem we show that the total size of orphans is at most 4. This implies that there are at most two orphans and if there are two orphans, the smaller one has size equal to 1. In addition, we give all combinations of sizes of minor and major squares for all orphan sizes.

▶ **Theorem 2.4.** *The total size of orphans is at most* 4*. There are at most two orphans and at most one orphan larger than* 1*. All possible combinations of sizes of orphans, minor and major squares are given in Table 2.1.*

Table 2.1 Possible combinations of minor and major squares. The dots in the first two rows in the column of minor squares denote all the smaller squares that are not among the orphans.

orphans	minor squares	major squares
1,3 4	...,5,6,7,8	9,10,11,12,13,14,...,24
1,2 3	...,4,5,6,7,9	8,10,11,12,13,14,...,24
2	1,3,4,5,6,8,9 1,3,4,5,6,7,10	7,10,11,12,13,14,...,24 8,9,11,12,13,14,...,24
1	2,3,4,5,7,8,9 2,3,4,5,6,8,10 2,3,4,5,6,7,11	6,10,11,12,13,14,...,24 7,9,11,12,13,14,...,24 8,9,10,12,13,14,...,24
none	1,2,3,4,6,7,8,9 1,2,3,4,5,7,8,10 1,2,3,4,5,6,9,10 1,2,3,4,5,6,8,11 1,2,3,4,5,6,7,12	5,10,11,12,13,14,...,24 6,9,11,12,13,14,...,24 7,8,11,12,13,14,...,24 7,9,10,12,13,14,...,24 8,9,10,11,13,14,...,24

Proof. Let MAJOR be the sum of sizes of all major squares, MINOR the sum of sizes of all minor squares, and ORPHAN the sum of sizes of orphans.

The expression $2 \cdot \text{MAJOR} + \text{MINOR}$ is equal to the sum of sizes of squares on all important rows and columns. Thus we have $2 \cdot \text{MAJOR} + \text{MINOR} = 8 \cdot 70 = 560$. The sum of all 24 square sizes is $1 + \cdots + 24 = 12 \cdot 25 = 300$, so $\text{MAJOR} + \text{MINOR} = 300 - \text{ORPHAN}$. Together this implies $\text{MAJOR} = 260 + \text{ORPHAN}$.

Furthermore, MAJOR is at most the sum of the 16 largest squares, which gives $\text{MAJOR} \leq 9 + 10 + \cdots + 24 = 8 \cdot 33 = 264$. Thus $\text{ORPHAN} \leq 4$. As the three smallest square sizes sum to 6, there are at most two orphans. Also, there cannot be two orphans larger than 1, as their total size would be at least 5.

It remains to examine all possible combinations of squares that sum up to ORPHAN and the corresponding value of MAJOR. The results are given in Table 2.1. ◀

3 Preliminary observations and small $\mathbb{MIN}$ cases

As in previous literature on the problem, it is convenient to consider various cases where we have some partial tiling of the 70×70 board and we try to extend it. Our goal is to achieve a contradiction, eventually covering all possibilities. To make our presentation complete, we cover all cases, including those already excluded in the previous literature.

We denote the squares by blackboard-bold letters $\mathbb{A}$, $\mathbb{B}$, etc. We overload this notation and use it for the size of squares, too. Similarly, square 1 denotes the unique square of size 1, square 2 denotes the unique square of size 2, etc.

The smallest edge square is denoted $\mathbb{MIN}$. (Recall that an edge square is a square touching an edge of the board.) We will gradually restrict the possible sizes of $\mathbb{MIN}$. Typically we will assume that $\mathbb{MIN}$ touches the bottom edge. Some of our claims speak about an edge square that is a local minimum, which means that on both sides it has either a larger edge square or an edge (i.e., it is a corner square). Obviously, $\mathbb{MIN}$ is always a local minimum.

The following observation shows that $\mathbb{MIN}$ is never a corner square, but is useful in other situations, too.

▶ **Observation 3.1.** *A corner square is always adjacent to a smaller edge square.*

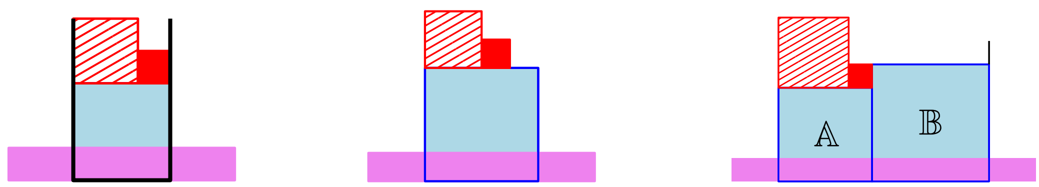
Proof. If a corner square is adjacent to two larger edge squares, then these two squares intersect, which is impossible. As there are no two equal squares, it follows that one of the adjacent edge squares is smaller than the corner square. ◀

Some configurations lead to an easy contradiction based only on local considerations. One of them is so-called corridor.

Given a partial tiling, a corridor is a rectangular space bounded by already placed squares from three sides and open on the last side. Its width is the length of the middle bounding segment and its height is the minimum of the two remaining bounding segments. Narrow corridors are quite restrictive. The only way to fill a corridor of width 1 or 2 is to use square 1 or 2, respectively, so if such a corridor has height of 3 or more, it cannot be filled at all. Similarly, a corridor of width 3 and height 5 cannot be filled, as the only way to cover the width is to use square 3 or squares 1 and 2 together. See Fig. 3.1(a).

Any square $\mathbb{A}$ on the bottom edge that is a local minimum creates a corridor of width $\mathbb{A}$ on top of it. This corridor must be filled with at least two squares, as size $\mathbb{A}$ is already used. This immediately excludes the case $\mathbb{MIN} \leq 2$, as the corridor on top of $\mathbb{MIN}$ cannot be filled.

Observation 3.1 excludes the situation where square 1 is placed in a corner, as it cannot be adjacent to two larger edge squares. This applies also to a corner-like configuration similar to Figure 3.1(b) where instead of a corner of the board we have two squares.



(a) A corridor of width 3 and height 5 that cannot be filled.

(b) A corner-like situation where square 1 is excluded.

(c) The situation of $\mathbb{A} = \mathbb{MIN} \in \{3, 4\}$ in Lemma 3.2 and the text below.

Figure 3.1 Some configurations that are easy to exclude.

The following observation will be useful at multiple places. See Figures 3.1(c) and 3.3 for an illustration.

▶ **Lemma 3.2.** *Suppose a square $\mathbb{A}$ on the bottom edge is a local minimum, i.e., any adjacent edge square on the bottom edge is larger, and it has square 1 on top. Then square 1 is the leftmost or rightmost square on top of $\mathbb{A}$ and the adjacent edge square exists and has size $\mathbb{A}+1$.*

Proof. If square 1 is in the middle of the top side of $\mathbb{A}$, it creates a corridor of width 1 that cannot be filled, as 1 is already used. Similarly, if it is next to a vertical edge or a square larger than $\mathbb{A}+1$, it creates a corridor of width 1 that cannot be filled. ◀

Now we are ready to exclude the case of $\mathbb{MIN} \leq 4$, which is known but the proof is significantly easier with our classification of squares. Suppose $\mathbb{MIN} \in \{3,4\}$ is at the bottom edge. By Theorem 2.4, $\mathbb{MIN}$ is a minor square, i.e., it cannot intersect an important column. The only possible way to fill the corridor above it is to use squares 1 and $\mathbb{MIN}-1$, and both of these squares are orphans. See Fig. 3.1(c) for reference. Lemma 3.2 now implies that the neighboring edge square $\mathbb{B}$ has size $\mathbb{MIN}+1$. This is necessarily also a minor square, as we now have orphans. Also, the neighbor of $\mathbb{B}$ on the other side than $\mathbb{MIN}$ is larger than $\mathbb{B}$ or $\mathbb{B}$ is a corner square. In both cases, there is a corridor above 1 and $\mathbb{B}$, thus any square on top of $\mathbb{B}$ is another orphan, using also the fact that such squares cannot reach the low row. This is a contradiction as we have three orphans.

Similar ideas lead to following observations that limit the possibilities on top of local minimum squares and solve some more cases.

▶ **Lemma 3.3.** *Suppose that $\mathbb{A}$ is a local minimum, w.l.o.g. on the bottom edge, and $5 \leq \mathbb{A} \leq 11$. Then $\mathbb{A}$ is a major square, one square on top of it is minor and the other square(s) on top of it are orphans.*

Proof. Since $\mathbb{A} \leq 11$, all squares on top of it have size at most 10. Thus none of these squares intersects the low row. It follows that these squares are orphans or minor. They cannot be all orphans, as $\mathbb{A} \geq 5$. Thus one, and only one, of them is intersected by an important column. Consequently, $\mathbb{A}$ is a major square. ◀

We immediately see that $\mathbb{MIN} = 5$ is excluded, as Theorem 2.4 implies that a major square of size 5 cannot exist if there is an orphan.

At this point, it is good to summarize possible configurations on top of a local minimum $\mathbb{A} \leq 11$, see Fig. 3.2. In addition to the minor square, there are one or two orphans, i.e., at most three squares. If there are two orphans, one of them must have size 1 and it is not the middle square.

The following lemma shows that for $\mathbb{A} = \mathbb{MIN} \leq 10$, only the configuration in Fig. 3.2(a) is possible.

▶ **Lemma 3.4.** *Suppose that $6 \leq \mathbb{MIN} \leq 10$. Then $\mathbb{MIN}$ cannot have square 1 on top.*

Proof. Suppose for a contradiction that square 1 is on top of $\mathbb{MIN}$. Lemma 3.3 implies that it is an orphan as otherwise there are orphans of total size $\mathbb{MIN}-1 \geq 5$. Then by Lemma 3.2, it is next to an adjacent edge square $\mathbb{B}$ of size $\mathbb{MIN}+1$. See Fig. 3.3. The squares $\mathbb{B}$ and 1 together create a corridor of width $\mathbb{MIN}+2$. This corridor is not intersected by an important column, as together with $\mathbb{MIN}$ they intersect at most $\mathbb{MIN}+\mathbb{MIN}+1 \leq 21$ consecutive columns.

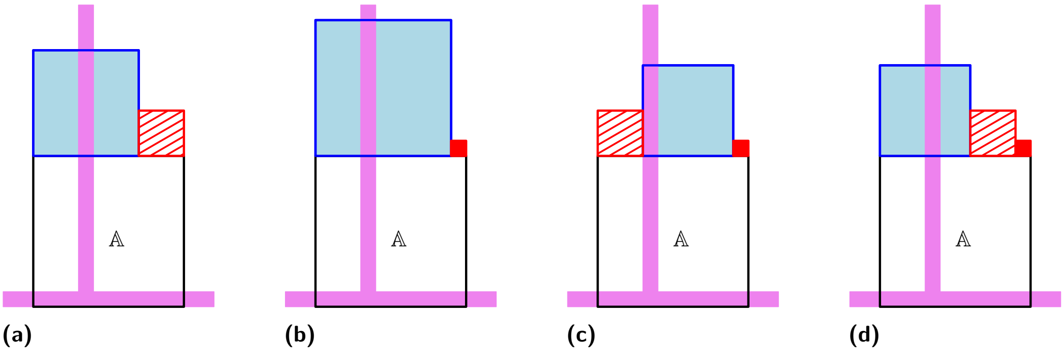


Figure 3.2 Possible configurations in the corridor above a local minimum $\mathbb{A}$.

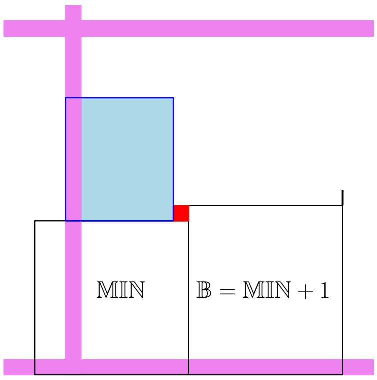


Figure 3.3 An illustration of the proof of Lemma 3.4. On the right, a big square or the edge of the board creates a corridor.

Thus any square filling this corridor must be an orphan, unless it intersects the low row. The only configuration where the low row may be intersected is when $\mathbb{MIN} = 10$ and the whole corridor of width 12 is filled by a single square of size 12; as it sits on square $\mathbb{MIN}+1 = 11$, it reaches row 23. However, then the square 12 is minor, as it is not intersected by an important column. By Theorem 2.4 this is impossible, as there is an orphan.

In all the other cases the corridor is filled by orphans of total size $\mathbb{MIN} + 2 \geq 8$, which is also a contradiction. ◀

The last lemma excludes the case $\mathbb{MIN} = 6$, as a major square of size 6 is not compatible with orphans larger than 1.

The following proposition summarizes the cases excluded in this section.

▶ **Proposition 3.5.** *We have $\mathbb{MIN} \geq 7$. Furthermore, if $\mathbb{MIN} \leq 10$, then $\mathbb{MIN}$ is major and there is a single orphan on top of $\mathbb{MIN}$ and it has size at least 2.*

We note that already our Proposition 3.5 is as strong as the results of approximately 40 pages of [6].

The next lemma limits the number of edge squares for large $\mathbb{MIN}$. We use it in the later parts of the proof.

▶ **Lemma 3.6.** *If $\mathbb{MIN} \geq 10$, then each edge touches exactly four squares. There are exactly twelve edge squares, all of them are major squares.*

Proof. For a contradiction, assume that some edge touches more than four squares. Then there must be a minor edge square.

If $\mathbb{MIN} = 10$ then Lemmata 3.3 and 3.4 imply that $\mathbb{MIN}$ is a major square and there is an orphan of size at least 2. This in turn, using Theorem 2.4, excludes the existence of a minor square of size 11 or more, thus we have a contradiction.

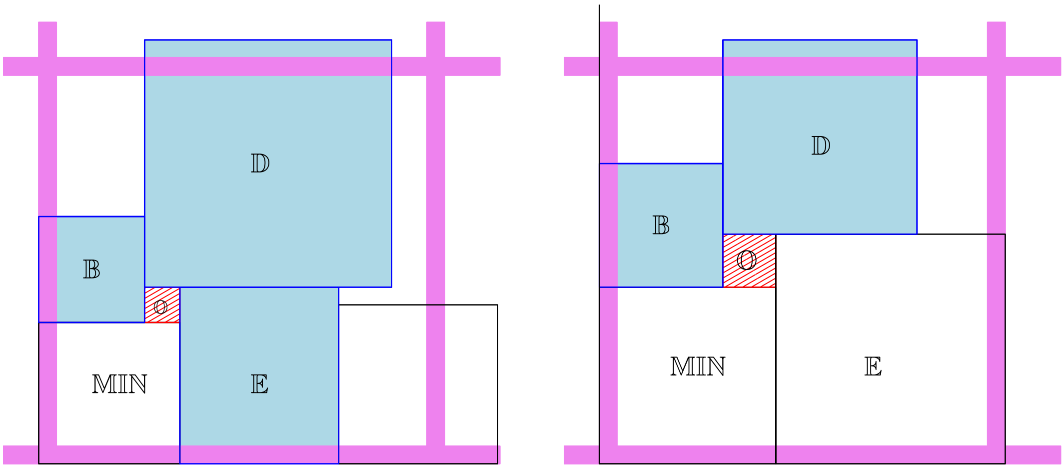
If $\mathbb{MIN} = 11$ then by Lemma 3.3 it is a major square and there is an orphan on top of it. This excludes a minor square of size 12 or more, thus we have a contradiction.

Otherwise the minor edge square must have size 12 and $\mathbb{MIN} = 12$. Then at least one of the squares on top of it does not intersect the low row, and thus it is an orphan. We have a contradiction again.

We have shown that each edge touches four squares. Thus we have four corner squares plus eight remaining edge squares, two per edge, a total of twelve edge squares. ◀

4 Medium $\mathbb{MIN}$ cases

In this section we exclude the remaining cases with $7 \leq \mathbb{MIN} \leq 10$. Proposition 3.5 and Theorem 2.4 imply that $\mathbb{MIN}$ is a major square and there are two squares on top of it, a minor square $\mathbb{B}$ and an orphan $\mathbb{O} \geq 2$; w.l.o.g. we assume that $\mathbb{B}$ is on the left of $\mathbb{O}$. Let $\mathbb{D}$ be the square touching both $\mathbb{B}$ and $\mathbb{O}$. See Fig. 4.1.



(a) The case of $\mathbb{MIN} \leq 8$, including the square to the right of $\mathbb{E}$ in the proof of Proposition 4.2.

(b) The case of $\mathbb{MIN} = 10$. Note the boundary of the board or the big square depicted on the left.

Figure 4.1 An illustration of the medium $\mathbb{MIN}$ case.

▶ **Lemma 4.1.** *Suppose that* $7 \leq \mathbb{MIN} \leq 10$ *and* $\mathbb{B}$, $\mathbb{O}$, *and* $\mathbb{D}$ *are as above. Then* $\mathbb{D} \geq 9$ *and if* $\mathbb{MIN} \leq 8$ *then* $\mathbb{D} \geq 17$. *Furthermore, the edge square touching* $\mathbb{MIN}$ *on the right exists and has size exactly* $\mathbb{MIN} + \mathbb{O}$.

Proof. We examine the possible types and sizes of $\mathbb{D}$.

If $\mathbb{D}$ is an orphan, then $\mathbb{D} = 1$ and $\mathbb{O} \leq 3$. This in turn implies that $\mathbb{B} = \mathbb{MIN} - \mathbb{O} \geq 4$. However, the combination of a major square $\mathbb{MIN} = 7$ and an orphan $\mathbb{O} = 3$ is excluded by Theorem 2.4, so $\mathbb{B} \geq 5 \geq \mathbb{O} + 2$. It follows that $\mathbb{O}$ is in a corner-like position similar to Fig. 3.1(a), which leads to a contradiction.

If $\mathbb{D}$ is not an orphan, it has to intersect an important row or column or both. If $\mathbb{D}$ intersects the low row, then $\mathbb{MIN}+\mathbb{O}+\mathbb{D} \geq 23$, thus $\mathbb{D} \geq 23-\mathbb{O}-\mathbb{MIN} \geq 19-\mathbb{MIN} \geq 9$. If $\mathbb{D}$ intersects an important column, then it is not the same important column as the one intersecting $\mathbb{MIN}$ and $\mathbb{B}$, thus $\mathbb{B}$ and $\mathbb{D}$ span at least 23 columns; we obtain $23 \leq \mathbb{B}+\mathbb{D} = \mathbb{D}+\mathbb{MIN}-\mathbb{O}$ and thus $\mathbb{D} \geq 23+\mathbb{O}-\mathbb{MIN} \geq 23+2-10 = 15$. The bound $\mathbb{D} \geq 9$ follows.

Furthermore, if $\mathbb{MIN} \leq 8$ then we get a stronger bound $\mathbb{D} \geq 19-\mathbb{MIN} \geq 11$. However, as we have an orphan $\mathbb{O} \geq 2$, Theorem 2.4 implies that $\mathbb{D} \geq 11$ has to be a major square. Thus it intersects both an important row and column and the same calculation as in the previous paragraph yields $\mathbb{D} \geq 23+\mathbb{O}-\mathbb{MIN} \geq 23+2-8 = 17$.

As $\mathbb{D} \geq 9$, it extends at least $\mathbb{D}-\mathbb{O} \geq 5$ squares to the right of $\mathbb{MIN}$ and $\mathbb{O}$. We claim that this implies that the edge square touching $\mathbb{MIN}$ on the right exists an has size at most $\mathbb{MIN}+\mathbb{O}$. Suppose for a contradiction that it has a strictly smaller size. The size needs to be strictly larger than $\mathbb{MIN}$. Then above this edge square and below $\mathbb{D}$ we have a corridor of width 1, 2, or 3 and length at least 5, which cannot be filled, see Fig. 3.1(b), a contradiction. Thus the size of the edge square is equal to $\mathbb{MIN}+\mathbb{O}$ and the last claim of the lemma follows. ◀

▶ **Proposition 4.2.** *We have* $\mathbb{MIN} \geq 11$.

Proof. Let $\mathbb{B}$, $\mathbb{O}$, and $\mathbb{D}$ be as above and let $\mathbb{E}$ be the edge square touching $\mathbb{MIN}$ on the right.

If $\mathbb{MIN} \leq 8$ then $\mathbb{O} \leq 3$ and $\mathbb{E} = \mathbb{MIN}+\mathbb{O} \leq 11$. See Fig. 4.1(a) for an illustration. Lemma 4.1 implies that $\mathbb{D}$ extends at least 3 cells to the right of $\mathbb{E}$. Thus there exists an edge square to the right of $\mathbb{E}$ and it has size strictly between $\mathbb{MIN}$ and $\mathbb{E} \leq \mathbb{MIN}+3$. Thus above this edge square and below $\mathbb{D}$ we have a corridor of width 1 or 2 and length at least 3, which cannot be filled, a contradiction.

If $\mathbb{MIN} = 9$ then $11 \leq \mathbb{E} \leq 13$. Thus $\mathbb{MIN}+\mathbb{E} \leq 22$ and by Observation 2.2 only one important column intersects $\mathbb{MIN}$ and $\mathbb{E}$; we know that it intersects the major square $\mathbb{MIN}$. Thus $\mathbb{E}$ is a minor square. However, Theorem 2.4 excludes a combination of an orphan $\mathbb{O} \geq 2$ with a minor square $\mathbb{E} \geq 11$, a contradiction.

If $\mathbb{MIN} = 10$ then Lemma 3.6 implies that there are only four edge squares on the edge of $\mathbb{MIN}$. See Fig. 4.1(b) for an illustration. As $\mathbb{E} = \mathbb{MIN}+\mathbb{O} \leq 14$, it follows that the remaining two edge squares are either 24 and 23 (and $\mathbb{E} = 13$) or 24 and 22 (and $\mathbb{E} = 14$). Thus $\mathbb{O} \in \{3,4\}$ and $\mathbb{B} = 10-\mathbb{O} \in \{6,7\}$; also note that $\mathbb{MIN}$ and $\mathbb{B}$ span at most 17 rows. Observe that on the left $\mathbb{B}$ touches the edge or a square of size at least 22 and on the right it touches $\mathbb{D} \geq 9$ which extends above $\mathbb{B} \leq 7$ by at least $\mathbb{D}+\mathbb{O}-\mathbb{B} \geq 5$ cells. Thus there is a corridor on top of $\mathbb{B}$ of length at least 5. It follows that there are at least two squares on top of $\mathbb{B}$. They cannot have size 1, as the cell above 1 cannot be covered, so they have size at least 2 and at most 5. None of them intersect the low row as they reach at most $17+5 = 22$ rows from bottom, and only one of them intersects an important column. Thus we have another orphan of size at least 2, a contradiction. ◀

5 Large $\mathbb{MIN}$ cases

At this point we know that $\mathbb{MIN} \geq 11$. To exclude the cases with $\mathbb{MIN} \geq 11$, we use a technique based on counting the sizes of edge squares.

Recall that by Lemma 3.6 there are exactly four squares along each edge, a total of twelve squares, all of them major. For the rest of the section, let E denote the sum of sizes of the edge squares and C the sum of sizes of the four corner squares.

5.1 Edge-square counting technique and $\mathbb{MIN} = 13$

We introduce and demonstrate the new technique on the case $\mathbb{MIN} \geq 13$, which is relatively easy.

▶ **Observation 5.1.** *We have* $E + C = 280$.

Proof. The sum of all the squares touching a given edge is 70. Summing over all edges we get $4 \cdot 70 = 280$. Considering that all corners touch two edges and the remaining edge squares only one edge, the sum is equal to $E + C$. ◀

▶ **Proposition 5.2.** *We have* $\mathbb{MIN} \leq 12$.

Proof. Suppose for a contradiction that $\mathbb{MIN} \geq 13$.

Then the sizes of the twelve edge squares are $13, \ldots, 24$ (note that $\mathbb{MIN} > 13$ is impossible). Thus $E = 13 + \cdots + 24 = 6 \cdot 37 = 222$.

Every corner square is adjacent to a smaller edge square and these are unique. It follows that the sizes of corner squares are at least $14, 16, 18, 20$ and thus $C \geq 14 + 16 + 18 + 20 = 68$.

We have $E + C \geq 222 + 68 = 290$, a contradiction. ◀

5.2 Small corners and edge squares

To use the edge counting technique for $\mathbb{MIN} = 11, 12$, we use the following ingredients, to get the necessary bounds on E and C.

(1) We prove that only one of 11 and 12 is used as an edge square. This gives us a bound $E \geq 209$ for $\mathbb{MIN} = 11$ and $E \geq 210$ for $\mathbb{MIN} = 12$.
(2) We prove that $\mathbb{MIN}$ has an orphan of size at least 2 adjacent to it.
(3) We prove that any corner square has size at least 14.
(4) We carefully analyze certain corner squares of small size, to show that each of them enforces an orphan. This typically allows us to conclude that there is only one corner of size at most 17, giving us a bound $C \geq 71$.
(5) In case $\mathbb{MIN} = 11$, we need to work a bit harder to obtain the bound $C \geq 72$.

The first three items are covered in this subsection.

▶ **Observation 5.3.** *Suppose that a corner square $\mathbb{C}$ neighbors with two edge squares such that $\mathbb{A}$ has size $\mathbb{C} - \delta$ for $\delta \in \{1, 2\}$ and $\mathbb{B}$ is larger than $\mathbb{C}$. Then $\mathbb{B}$ has size at most $\mathbb{C} + \delta$.*

Proof. Suppose w.l.o.g. that $\mathbb{C}$ is the bottom right corner and $\mathbb{A}$ is on the bottom edge. Then we have a corridor of width δ on top of $\mathbb{A}$ below $\mathbb{B}$. This corridor cannot be longer than δ, as it can be filled only by square δ. Thus $\mathbb{C}$ cannot extend more than δ to the left of $\mathbb{C}$ and the claim follows. ◀

▶ **Lemma 5.4.** *There is no corner square of size* 12.

Proof. Suppose for a contradiction that 12 is a corner square.

The corner 12 must have a smaller neighbor, which is necessarily $\mathbb{MIN} = 11$. Observation 5.3 implies that the other neighbor of 12 has size 13. Assume w.l.o.g. that 12 is the bottom right corner and $\mathbb{MIN}$ neighbors it on the bottom edge.

It follows that the other two squares along the bottom edge have size 23 and 24. The remaining two squares along the right edge then have sizes at most 21 and 22. However, then the total size along the right edge is at most $12 + 13 + 21 + 22 = 68 < 70$, a contradiction. ◀

▶ **Lemma 5.5.** *We cannot have two edge squares of sizes* 11 *and* 12.

Proof. For a contradiction, assume that we have edge squares $\mathbb{MIN} = 11$ and $\mathbb{B} = 12$. We distinguish two cases.

Case 1. $\mathbb{MIN} = 11$ and $\mathbb{B} = 12$ are on the same edge, w.l.o.g. the bottom one. As none of them can be a corner by Lemma 5.4, they are the two middle squares. The two bottom corners have sizes 23 and 24.

Now consider the top row. All its squares have size at most 22, as 23 and 24 are the two bottom corners. Thus the two top corners have size at least $70 - 22 - 21 = 37$ and overall we get $C \geq 37 + 23 + 24 = 84$. Using again the fact that 23 and 24 are on the bottom edge, we can bound $E \geq (11 + 12 + \cdots + 20) + 23 + 24 = 5 \cdot 31 + 47 = 202$. Thus we have $E + C \geq 202 + 84 = 284 > 280$, a contradiction.

Case 2. $\mathbb{MIN} = 11$ and $\mathbb{B} = 12$ are on different edges. W.l.o.g. assume that $\mathbb{B}$ is on the bottom edge and note that it is a local minimum.

Consider the squares on top of $\mathbb{B}$. As 11 is used elsewhere, they have size at most 10 and do not intersect the low row. It follows that one of them is an orphan of size at least 2.

As 12 is not adjacent to $\mathbb{MIN} = 11$, it follows by Lemma 3.2 and 3.3, that there is an orphan of size at least 2 adjacent to $\mathbb{MIN}$. This is a contradiction as there cannot be two orphans of size at least 2. ◀

▶ **Corollary 5.6.** *If* $\mathbb{MIN} = 11$ *then* $E \geq 209$*. If* $\mathbb{MIN} = 12$ *then* $E \geq 210$.

Proof. We have $E \geq \mathbb{MIN} + 13 + 14 + \cdots + 23 = \mathbb{MIN} + 198$. The claim follows. ◀

▶ **Corollary 5.7.** *If* $\mathbb{MIN} = 11$*, then* $\mathbb{MIN}$ *is adjacent to an orphan of size at least 2.*

Proof. By Lemma 3.3 there is an orphan on top of $\mathbb{MIN} = 11$. If the orphan has size one, Lemma 3.2 implies that $\mathbb{MIN}$ is adjacent to 12, which is excluded by Lemma 5.5 ◀

▶ **Lemma 5.8.** *There is no corner square of size* 13.

Proof. Suppose for a contradiction that 13 is a corner square.

There is a smaller edge square adjacent to the corner 13, and Lemma 5.5 implies that this is $\mathbb{MIN} \in \{11, 12\}$, as no other edge square smaller than 13 is available. Assume w.l.o.g. that 13 is the bottom right corner and $\mathbb{MIN}$ neighbors it on the bottom edge. Observation 5.3 implies that the other neighbor of 13 has size 14 or 15.

Now we examine the total size of the seven squares along the left and top edges. The total, counting the top left corner twice, is equal to 140. The seven squares can contain one of 14 and 15 (the one not adjacent to the corner 13) and then squares of size at least 16. Thus the total size of the seven squares is at least $14 + 16 + 17 + \cdots + 21 = 14 + 3 \cdot 37 = 125$. Furthermore, the size of the top left corner is at least 16, as it has a smaller adjacent edge square. Thus the total along the left and top edges is at least $125 + 16 = 141 > 140$, a contradiction. ◀

We conclude this subsection by showing that $\mathbb{MIN}$ is adjacent to an orphan of size at least 2. We already know this for $\mathbb{MIN} = 11$. The proof for $\mathbb{MIN} = 12$ follows the line of proof of Lemma 5.5.

▶ **Lemma 5.9.** *If* $\mathbb{MIN} = 12$*, then we cannot have two adjacent edge squares of sizes* 12 *and* 13.

Proof. For a contradiction, assume that we have adjacent edge squares $\mathbb{MIN} = 12$ and 13; w.l.o.g. assume they are at the bottom edge. By Lemmata 5.4 and 5.8, these are not corners. It follows that the two bottom corners have total size $70 - 12 - 13 = 45$. The top corners have sizes at least 15 and 17, as they are adjacent to smaller edge squares. Thus $C \geq 45 + 15 + 17 = 76$. Together with $E \geq 210$ we have $E + C \geq 210 + 76 = 286 > 280$, a contradiction. ◀

▶ **Corollary 5.10.** *If $\mathbb{MIN} = 12$, then $\mathbb{MIN}$ is adjacent to an orphan of size at least 2.*

Proof. If there is 1 on top of $\mathbb{MIN} = 12$ then Lemma 3.2 implies that $\mathbb{MIN}$ is adjacent to 13, which is excluded by Lemma 5.9.

Thus all squares on top of $\mathbb{MIN}$ have size at least 2. As there are at least two squares on top of $\mathbb{MIN}$, they are all of size at most 10. Thus they do not intersect the low row. This in turn implies that one of them is an orphan, as at most one of them is intersected by an important column, and we have obtained an orphan of size at least 2. ◀

5.3 Medium corners and the final analysis

The core of the remaining argument is to examine the corners of medium size more carefully. Essentially, we need to extend Observation 3.1 to see that on one side of the corner square, the adjacent squares do not extend beyond the corner square. The neighboring edge square on that side is then called the key neighbor.

▶ **Definition 5.11.** *Suppose $\mathbb{C}$ is a corner square and $\mathbb{A}$ is an adjacent edge square. W.l.o.g. assume that $\mathbb{C}$ is the bottom right corner and $\mathbb{A}$ is on the bottom edge.*

We say that $\mathbb{A}$ is a key neighbor of $\mathbb{C}$ if $\mathbb{A} < \mathbb{C}$ and no square adjacent to the left side of $\mathbb{C}$ extends beyond the top of $\mathbb{C}$.

▶ **Observation 5.12.** *Any corner square has at least one key neighbor.*

Proof. Suppose that $\mathbb{C}$ is the bottom right corner, $\mathbb{A} < \mathbb{C}$ is an edge square adjacent to $\mathbb{C}$ on the bottom edge and $\mathbb{B}$ is an edge square adjacent to $\mathbb{C}$ on the right edge.

If $\mathbb{B} > \mathbb{C}$ then no square adjacent to the left side of $\mathbb{C}$ may extend above the top of $\mathbb{C}$, as it would intersect $\mathbb{B}$. So $\mathbb{A}$ is a key neighbor of $\mathbb{C}$.

Suppose now that $\mathbb{B} < \mathbb{C}$ and it is not a key neighbor of $\mathbb{C}$. Then there exists a square $\mathbb{D}$ adjacent to the top side of $\mathbb{C}$ and extending to the left of $\mathbb{C}$. Now again no square adjacent to the left side of $\mathbb{C}$ may extend above the top of $\mathbb{C}$, as it would intersect $\mathbb{D}$. So $\mathbb{A}$ is a key neighbor of $\mathbb{C}$. ◀

Now we are ready to formulate the needed extension of Observation 5.3 for small corners.

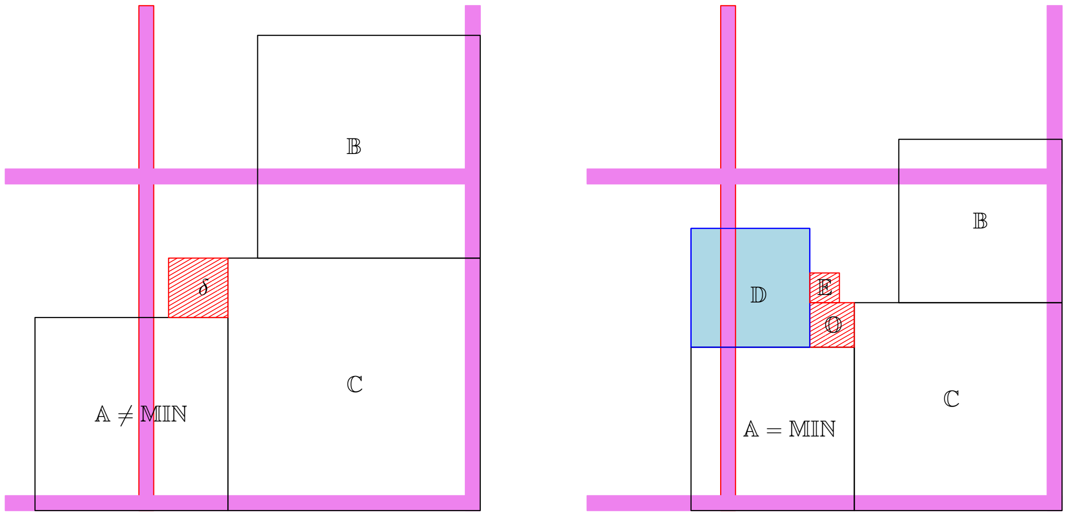
▶ **Lemma 5.13.** *Suppose $\mathbb{C}$ is a corner square and $\mathbb{A}$ is its key neighbor and $\mathbb{B}$ the other edge square adjacent to $\mathbb{C}$; furthermore let $\delta = \mathbb{C} - \mathbb{A}$. W.l.o.g. assume that $\mathbb{C}$ is the bottom right corner and $\mathbb{A}$ is on the bottom edge, thus $\mathbb{B}$ is on the right edge.*

Then the following holds:

(1) *If $\mathbb{A} \neq \mathbb{MIN}$ and $\mathbb{C} + \delta \leq 22$ then $\delta = 1$ and $\mathbb{B} \leq \mathbb{C} + \delta$.*

(2) *If $\mathbb{A} = \mathbb{MIN}$ and $\mathbb{C} \leq 17$ then $\delta \in \{2, 3, 4\}$ and either $\mathbb{B} = \mathbb{C} + \delta$ or $\mathbb{B} = 13$, $\mathbb{C} = 16$, $\mathbb{MIN} = 12$, and $\mathbb{O} = 4$.*

Furthermore, in both cases there is an orphan of size δ adjacent to $\mathbb{C}$.



(a) The case of $\mathbb{A} \neq \mathbb{MIN}$. **(b)** The case of $\mathbb{A} = \mathbb{MIN}$.

Figure 5.1 An illustration of the proof of Lemma 5.14.

Proof. Suppose first that $\mathbb{A} \neq \mathbb{MIN}$. See Fig. 5.1(a) for an illustration. Consider the δ cells adjacent to the left side of $\mathbb{C}$ and above $\mathbb{A}$. These cells are covered by squares that do not extend above $\mathbb{C}$. Thus all these squares have size at most δ and do not intersect any important column due to the condition $\mathbb{C} + \delta \leq 22$; they also do not intersect the low row, as the low row is above $\mathbb{C}$. It follows that all these squares are orphans. However, by Corollaries 5.7 and 5.10 we already have a different orphan of size at least 2 adjacent to $\mathbb{MIN}$ (it is different from the squares above $\mathbb{A}$ as $\mathbb{MIN} \neq \mathbb{A}$); this implies that all these squares are in fact only a single orphan of size 1. Thus $\delta = 1$, we have an orphan of size 1 adjacent to $\mathbb{C}$ and the lemma follows.

Suppose now that $\mathbb{A} = \mathbb{MIN}$. See Fig. 5.1(b) for an illustration. We know that there are exactly two squares on top of $\mathbb{MIN}$, namely one orphan $\mathbb{O} \in \{2, 3, 4\}$ and one minor square $\mathbb{D}$ of size $\mathbb{MIN} - \mathbb{O} \geq 11 - 4 = 7$. Since $\mathbb{C} \leq 17$, we have $\delta \leq 17 - \mathbb{MIN} \leq 6$. It follows that $\mathbb{D}$ cannot be adjacent to $\mathbb{C}$. Thus $\mathbb{O}$ is adjacent to $\mathbb{C}$ and $\mathbb{D}$ is to the left of $\mathbb{O}$. We claim first that $\delta = \mathbb{O}$ and thus $\mathbb{C} = \mathbb{MIN} + \mathbb{O}$. Indeed, if $\delta > \mathbb{O}$ then the cells just above $\mathbb{O}$ must be covered by at least two orphans, which leads to a contradiction.

Finally, we claim that $\mathbb{B} = \mathbb{C} + \mathbb{O}$, except for one special case described in the lemma. Indeed, if $\mathbb{B} < \mathbb{C} + \mathbb{O}$, there is a corridor above $\mathbb{O}$ between $\mathbb{D}$ and $\mathbb{B} \geq 13$ of nonzero width $\mathbb{C} + \mathbb{O} - \mathbb{B} = \mathbb{MIN} + 2 \cdot \mathbb{O} - \mathbb{B} \leq 12 + 2 \cdot 4 - 13 = 7$ (the corridor may be wider than $\mathbb{O}$ if $\mathbb{B} < \mathbb{C}$). If the width of the corridor is 7, the previous calculation has to be tight, which leads to $\mathbb{MIN} = 12$, $\mathbb{O} = 4$, $\mathbb{C} = \mathbb{MIN} + \mathbb{O} = 16$, and $\mathbb{B} = 13$, i.e., exactly the special case in the lemma. In the remaining case the corridor has width at most 6; this leads to a contradiction as follows: Consider the square $\mathbb{E}$ adjacent to $\mathbb{O}$ and $\mathbb{D}$. It reaches at most the row $\mathbb{A} + \mathbb{O} + 6 \leq 22$, so it does not reach the low row. It also does not intersect any important column, so $\mathbb{E}$ is an orphan. It cannot have size 1, as it is in a corner-like position which we excluded, see Fig. 3.1(2). It also cannot have size at least 2, as we would have two orphans of size at least 2. So we have a contradiction. ◀

▶ **Lemma 5.14.** *If* $\mathbb{MIN} = 12$ *then* $C \geq 71$. *If* $\mathbb{MIN} = 11$ *then* $C \geq 72$.

Proof. We start with several observations.

First, suppose we prove that there is a single corner of size at most 17. Then this corner has size at least 14 and altogether we have $C \geq 14 + 18 + 19 + 20 = 71$. Thus we are almost done, except that we need to improve the bound by 1 if $\mathbb{MIN} = 11$.

Second, if we have a corner $\mathbb{C} \leq 17$ with a key neighbor $\mathbb{A} \neq \mathbb{MIN}$, then in Lemma 5.13 we have $\delta \leq 17 - 13 = 4$ and $\mathbb{C} + \delta \leq 21$; thus the case (1) of Lemma 5.13 applies.

Third, we can have only one corner where the case (1) of Lemma 5.13 applies, as there is only one orphan of size 1.

With these observations in mind, we proceed to examine some cases.

Suppose first that we have a corner $\mathbb{C}$ of size at most 17 with a key neighbor $\mathbb{MIN}$. We examine all possible cases which we list in the form of triples listing $\mathbb{MIN}$, the size of the corner, and the size of the other neighbor. We have $\mathbb{C} \geq 14$, furthermore Lemma 5.13 implies that $\delta = \mathbb{C} - \mathbb{MIN} \leq 4$ and the size of the other neighbor equals $\mathbb{C} + \delta$, except for one special case. Thus for the key neighbor $\mathbb{MIN} = 11$, the only possibilities are $(11, 14, 17)$ and $(11, 15, 19)$, while for $\mathbb{MIN} = 12$, the possibilities are $(12, 14, 16)$, and $(12, 15, 18)$, $(12, 16, 20)$, and $(12, 16, 13)$ (the special case in Lemma 5.13).

$(11, 15, 19)$, $(12, 16, 20)$, or $(12, 16, 13)$: Then we have an orphan of size $\delta = 4$, so no other corner of size at most 17 exists. Thus $C \geq 15 + 18 + 19 + 20 = 72$ and the lemma holds.

$(12, 14, 16)$: Then there is no other corner of size at most 17, as there is no consecutive pair of sizes for the corner and its key neighbor. Thus $C \geq 14 + 18 + 19 + 20 = 71$ and the lemma holds. (Note that $\mathbb{MIN} = 12$ in this case.)

$(12, 15, 18)$: If there is no other corner of size at most 17, we are done. The only remaining possibility is a corner 17 with a key neighbor 16. Then $C \geq 15 + 17 + 19 + 20 = 71$ and the lemma holds as well. (Note that $\mathbb{MIN} = 12$ in this case, too.)

$(11, 14, 17)$: We have two somewhat subtle subcases. If there is another corner of size at most 17, the only possibility is a corner 16 with a key neighbor 15 and the other neighbor 13. This exhausts all edge squares up to size 17. Now Lemma 5.13 implies that the other corners have size at least 21. Indeed, for a corner of size at most 20 with a key neighbor of size at least 18 we have $\delta \leq 2$ and thus the case (1) of Lemma 5.13 would apply; this would imply another orphan of size 1, which is impossible. It follows that $C \geq 14 + 16 + 21 + 22 = 73$ and the lemma holds.

In the second subcase we have no other corner of size at most 17. We already know that $C \geq 14 + 18 + 19 + 20 = 71$ and we need to improve the bound by 1. To do this, consider the edge of $\mathbb{MIN}$. It has squares 11 and 14, thus its other corner has size at least $70 - 11 - 14 - 24 = 21$. Thus $C \geq 14 + 18 + 19 + 21 = 72$ and the lemma holds.

In the remaining case, we have no corner of size at most 17 with a key neighbor $\mathbb{MIN}$. From the observations at the beginning of the proof, it follows that there is at most a single corner $\mathbb{C} \leq 17$. Thus we already know that $C \geq 14 + 18 + 19 + 20 = 71$ and we need to improve the bound by 1 if $\mathbb{MIN} = 11$. This means that it is sufficient to exclude the case where the corner sizes are exactly 14, 18, 19, and 20.

For a contradiction, assume that $\mathbb{MIN} = 11$ and the corners are $\mathbb{C} = 14$, 18, 19, and 20. Then $\mathbb{C} = 14$ has a key neighbor 13 and another neighbor of size at most 15; this other neighbor is either 15 or $\mathbb{MIN} = 11$, as no other small edge square is available. Now consider the key neighbors of the two corners 18 and 19. One of them may have size at most 15, namely it may be the one of 15 and $\mathbb{MIN} = 11$ not adjacent to $\mathbb{C}$. The other key neighbor of

18 or 19 has size at least 16. Thus for this corner $\delta \leq 19 - 16 = 3$ and case (1) of Lemma 5.13 applies. This implies the existence of a second orphan of size 1, in addition to the one adjacent to $\mathbb{C}$. This is the final contradiction. ◀

▶ **Theorem 5.15.** *There exists no tiling of the* 70×70 *board with squares of sizes* $1, 2, \ldots, 24$.

Proof. Suppose we have a perfect tiling.

By Propositions 4.2 and 5.2 we have $\mathbb{MIN} \in \{11, 12\}$.

If $\mathbb{MIN} = 11$ then Corollary 5.6 and Lemma 5.14 imply that $E + C \geq 209 + 72 = 281$.

If $\mathbb{MIN} = 12$ then Corollary 5.6 and Lemma 5.14 imply that $E + C \geq 210 + 71 = 281$.

In both cases we get a contradiction with $E + C = 280$ from Observation 5.1. ◀

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