An Analysis of the Recurrence/Transience of Random Walks on Growing Trees and Hypercubes

Shuma Kumamoto
Graduate School of Mathematical Science, Kyushu University, Fukuoka, Japan

Shuji Kijima
Faculty of Data Science, Shiga University, Hikone, Japan

Tomoyuki Shirai
Institute of Mathematics for Industry, Kyushu University, Fukuoka, Japan

Abstract

It is a celebrated fact that a simple random walk on an *infinite* k-ary tree for $k \geq 2$ returns to the initial vertex at most finitely many times during infinitely many transitions; it is called *transient*. This work points out the fact that a simple random walk on an infinitely *growing* k-ary tree can return to the initial vertex infinitely many times, it is called *recurrent*, depending on the growing speed of the tree. Precisely, this paper is concerned with a simple specific model of a *random walk on a growing graph* (*RWoGG*), and shows a phase transition between the recurrence and transience of the random walk regarding the growing speed of the graph. To prove the phase transition, we develop a coupling argument, introducing the notion of *less homesick as graph growing* (*LHaGG*). We also show some other examples, including a random walk on $\{0,1\}^n$ with infinitely growing n, of the phase transition between the recurrence and transience. We remark that some graphs concerned in this paper have infinitely growing degrees.

2012 ACM Subject Classification Theory of computation → Random walks and Markov chains

Keywords and phrases Random walk, dynamic graph, recurrent, transient

Digital Object Identifier 10.4230/LIPIcs.SAND.2024.17

Related Version *Full Version*: https://arxiv.org/abs/2405.09102

Funding *Shuma Kumamoto*: This work is supported by JST SPRING, Grant Number JPMJSP2136.
Shuji Kijima: This work is partly supported by JSPS KAKENHI Grant Number JP21H03396.
Tomoyuki Shirai: This work is partly supported by JSPS KAKENHI Grant Number JP20K20884, JP22H05105 and JP23H01077.

1 Introduction

The recurrence or transience is a classical and fundamental topic of random walks on *infinite* graphs, see e.g., [18]: let $X_0, X_1, X_2, \ldots$ be a random walk (or a Markov chain)[1] on an infinite state space V, e.g., $V = \mathbb{Z}$, with $X_0 = v$ for $v \in V$. For convenience, let

$$R(t) = \Pr[X_t = v] \qquad (= \Pr[X_t = v \mid X_0 = v])$$

denote the probability that a random walk returns to the initial state at time step t ($t = 1, 2, \ldots$), and then the initial point v is *recurrent* by the random walk if

$$\sum_{t=1}^{\infty} R(t) = \infty \tag{1}$$

[1] This paper is concerned with discrete time and space processes. We will be mainly concerned with time-*inhomogeneous* Markov chains, but here you may assume a time-homogeneous chain, i.e., the transition probability $\Pr[X_{t+1} = v \mid X_t = u]$ is independent of the time t, but depends on u, v.

3rd Symposium on Algorithmic Foundations of Dynamic Networks (SAND 2024).
Editors: Arnaud Casteigts and Fabian Kuhn; Article No. 17; pp. 17:1–17:17
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

holds, otherwise it is *transient*. Intuitively, (1) means that the random walk is "expected" to return to the initial state infinitely many times. It is well known that a simple random walk on $\mathbb{Z}^d$ is recurrent for $d = 1, 2$, while it is transient for $d \geq 3$, cf. [18]. Another celebrated fact is that a simple random walk on an infinite k-ary tree is transient [28, 29].

Analysis of random walks on *dynamic graphs* has been developed in several contexts. In probability theory, random walks in random environments are a major topic, where self-interacting random walks including reinforced random walks and excited random walks have been intensively investigated as a relatively tractable non-Markovian process, see e.g., [11, 6, 17, 31, 32, 24]. The recurrence or transience of a random walk in a random environment is a major topic there, particularly random walks on growing subgraphs of $\mathbb{Z}^d$ or on infinitely growing trees are the major targets [13, 14, 21, 1]. In distributed computing, analysis of algorithms, including random walk, on dynamic graph attracts increasing attention due to the fact that real networks are often dynamic [8, 25, 3, 30]. Searching or covering networks, related to hitting or cover times of random walks, are major topics there [9, 4, 16, 5, 26, 7, 23].

This work is concerned with the recurrence/transience of a random walk on a growing graph. We show the fact that a simple random walk on an infinitely growing complete k-ary tree can be recurrent *depending on the growing speed* of the tree, while a simple random walk on an infinite k-ary tree is transient as we mentioned above. More precisely, this paper follows the model of the random walk on growing graph (RWoGG) [23], where the network gradually grows such that the growing network keeps its shape $G(n)$ for $\mathfrak{d}(n)$ steps, and then changes the shape to $G(n+1)$ by adding some vertices to $G(n)$ (see Section 2.1 for detail). Then, we show a phase transition between the recurrence and transience of a random walk on a growing k-ary tree, regarding the growing speed of the graph. For a proof, we develop the notion of *less-homesick as graph growing* (LHaGG), which is a quite natural property of RWoGG, and gives a simple proof by a *coupling* argument, that is an elementary technique of random walks or Markov chains based on a comparison method. We also show some other examples of the phase transition, including as a random walk on $\{0, 1\}^n$ with infinitely growing n.

1.1 Existing works and contribution of the paper

The recurrence/transience of a random walk on a dynamic graph has been mainly developed in the context of random walks in random environment including reinforced random walks and excited walks. Here we briefly review some existing works concerning the recurrence of a random walk on $\mathbb{Z}^d$ and infinite (or infinitely growing) trees, directly related to this paper.

Random walks on (asymptotically) $\mathbb{Z}^d$. It is a celebrated fact that the initial point, say origin $\mathbf{0}$, in the infinite integer grid $\mathbb{Z}^d$ is recurrent when $d = 1$ and 2 by a simple random walk, and it is transient for $d \geq 3$, see e.g., [18].

Dembo et al. [14] is concerned with a random walk on an infinitely growing subgraph of $\mathbb{Z}^d$, and gave a phase transition, that is roughly speaking a random walk is recurrent if and only if $\sum_{t=1}^{\infty} \pi_t(\mathbf{0}) = \infty$ holds under a certain condition, where π_t denotes the stationary distribution of the transition matrix at time t. Huang [21] extended the argument of [14] and gave a similar or essentially the same phase transition for more general graphs. The proofs are based on the edge conductance and a central limit theorem, on the assumptions that every vertex of the dynamic graph has a degree at most *constant* to time (or the size of the graph), and the random walk is "lazy" such that it has at least a *constant* probability of self-loops at every vertex. Those arguments are sophisticated and enhanced using the argument of evolving set and the heat kernel by recent works [12, 15].

Random walks on infinitely growing trees. Lyons [28] studied sufficient conditions for a random walk being recurrent/transient, see also [29]. Roughly speaking, the initial point, say the root r, is recurrent if and only if the random walk is enough *homesick*, meaning that a random walk probabilistically tends to choose the direction to the root.

Amir et al. [1] introduced a random walk in changing environment model, and investigated the recurrence and transience of random walks in the model. They gave a conjecture about the conditions for the recurrence and transience regarding the limit of a graph sequence, and proved it for trees. Huang's work [21], which we mentioned above, implies that a simple random walk starting from a vertex v on growing k-ary tree is recurrent if and only if $\sum \pi_t(v) = \infty$, that is similar to or essentially the same as a main result of this paper under a certain condition. We remark that a k-ary tree with height n is not an (edge induced) subgraph of $\mathbb{Z}^d$ for a *constant* d.

There is a lot of work on the recurrence or transience of a random walk on a growing tree, related to self-interacting random walks including reinforced random walks and excited random walks, e.g., [22, 19]. They are non-Markovian processes, and in a bit different line from [14, 1, 21] and this paper.

Contribution of this work. This paper is concerned with a specific model of dynamic graphs with an increasing number of vertices, which we will describe in Section 2.1, and gives a phase transition by the growing speed regarding a random walk being recurrent/transient. The phase transition is very similar to or essentially the same as [14, 21], while this paper contains mainly three contributions. One is the proof technique: we employ a coupling argument while the existing works are based on the conductance and a central limit theorem. The coupling arguments is a classical and elementary comparison technique of random walks, and we introduce the notion of LHaGG to use the comparison technique. Since the coupling technique is relatively simple, we can drop two assumptions in the existing works, namely a random walk being *lazy* and a growing graph having *uniformly bounded degree*, which are naturally required in the conductance argument to make the arguments simple. This paper is mainly concerned with reversible random walks of *period 2*, which contains simple random walks on undirected bipartite graphs; this is the second contribution. We also show an example of random walk on $\{0,1\}^n$ with increasing n, where the (maximum) degree of the dynamic graph, that is n, infinitely grows; this is the third contribution.

While the coupling technique is relatively easy, it often selects the applicable target. In fact, the results by [14, 21] are widely applied to general setting as far as it satisfies appropriate assumptions, while our result is limited to specific targets. Such an argument about conductance and coupling seems known as an implicit knowledge in the literature of mixing time analysis, cf. [2, 20]. However, we emphasize that the coupling technique often gives an easy proof of an interesting phenomena, as this paper shows.

1.2 Organization

As a preliminary, we describe the model of random walk on growing graph (RWoGG) in Section 2. Section 3 introduces the notion of less homesickness as graph growing (LHaGG), and presents some general theorems for sufficient conditions of a RWoGG being recurrent/transient. Section 4 shows a phase transition between the recurrence and transience of a random walk on growing k-ary tree. Section 5 shows a phase transition for a random walk on $\{0,1\}^n$ with increasing n.

2 Preliminaries

2.1 Model

A growing graph is a sequence of (static) graphs $\mathcal{\boldsymbol{G}} = \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \dots$ where $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$ for $t = 0, 1, 2, \dots$ denotes a graph[2] with a finite vertex set $\mathcal{V}_t$ and an edge set $\mathcal{E}_t \subseteq \binom{\mathcal{V}_t}{2}$. For simplicity, this paper assumes[3] $\mathcal{V}_t \subseteq \mathcal{V}_{t+1}$ and $\mathcal{E}_t \subseteq \mathcal{E}_{t+1}$. In this paper, we assume $|\mathcal{V}_\infty| = \infty$, otherwise the subject (recurrence) is trivial. A random walk on a growing graph is a Markovian series $X_t \in \mathcal{V}_t$ $(t = 0, 1, 2, \dots)$.

In particular, this paper is concerned with a specific model, described as follows, cf. [23]. A *random walk on a growing graph* (*RWoGG*), in this paper, is formally characterized by a 3-tuple of functions $\mathcal{D} = (\mathfrak{d}, G, P)$. The function $\mathfrak{d} \colon \mathbb{Z}_{>0} \to \mathbb{Z}_{\geq 0}$ denotes the duration. For convenience, let $T_n = \sum_{i=1}^{n} \mathfrak{d}(i)$ for $n = 1, 2, \dots$[4] and $T_0 = 0$. We call the time interval $[T_{n-1}, T_n]$ *phase* n for $n = 1, 2, \dots$; thus $T_{n-1} = \sum_{i=1}^{n-1} \mathfrak{d}(i)$ is the beginning of the n-th phase, but we also say that T_{n-1} is the end of the $(n-1)$-st phase, for convenience. The function $G \colon \mathbb{Z}_{>0} \to \mathfrak{G}$ represents the graph $G(n) = (V(n), E(n))$ for the phase n, where $\mathfrak{G}$ denotes the set of all (static) graphs, i.e., our growing graph $\mathcal{\boldsymbol{G}}$ satisfies $\mathcal{G}_t = G(n)$ for $t \in [T_{n-1}, T_n)$. Similarly, the function $P \colon \mathbb{Z}_{>0} \to \mathfrak{M}$ is a function that represents the "transition probability" of a random walk on graph $G(n)$ where $\mathfrak{M}$ denotes the set of all stochastic matrices.

A RWoGG X_t $(t = 0, 1, 2, \dots)$ characterized by $\mathcal{D} = (\mathfrak{d}, G, P)$ is temporally a time-homogeneous finite Markov chain according to $P(n)$ with the state space $V(n)$ during the time interval $[T_{n-1}, T_n]$; precisely, a transition from X_t to X_{t+1} follows $P(n)$ for any $t \in [T_{n-1}, T_n)$. We specially remark for $t = T_n$ that $X_t \in V(n) \subseteq V(n+1)$, meaning that X_t is a state of $V(n+1)$ but actually X_t must be in $V(n)$ by the definition of the transition. Suppose $X_0 = v$ for $v \in V(1)$. We define the return probability at v by

$$R(t) = \Pr[X_t = v] \ \ (= \Pr[X_t = v \mid X_0 = v]) \tag{2}$$

at each time $t = 0, 1, 2, \dots$. We say v is *recurrent* by RWoGG $\mathcal{D} = (\mathfrak{d}, G, P)$ if

$$\sum_{t=1}^{\infty} R(t) = \infty \tag{3}$$

holds, otherwise, i.e., $\sum_{t=1}^{\infty} R(t)$ is finite, v is *transient* by $\mathcal{D}$.

2.2 Terminology on time-homogeneous Markov chains

We here briefly introduce some terminology for random walks on static graphs, or time-homogeneous Markov chains, according to [27].

2.2.1 Ergodic random walks

Suppose that $X_0, X_1, X_2, \dots$ is a random walk on a static graph $G = (V, E)$ characterized by a time-homogeneous transition matrix $P = (P(u, v)) \in \mathbb{R}_{\geq 0}^{V \times V}$ where $P(u, v) = \Pr[X_{t+1} = v \mid X_t = u]$. A random walk is *reversible* if there exists a positive function $\mu \colon V \to \mathbb{R}_{>0}$ such that $\mu(u)P(u, v) = \mu(v)P(v, u)$ hold for all $u, v \in V$. A transition matrix P is *irreducible* if

[2] Every static graph is simple and undirected in this paper, for simplicity of the arguments.
[3] Thus, the current position does not disappear in the next step.
[4] We do not exclude $T_{n-1} = T_n$; if $\mathfrak{d}(n) = 0$ then $T_{n-1} = T_n$.

$\forall u,v \in V$, $\exists t > 0$, $(P^t)(u,v) > 0$. The period of P is given by $\mathrm{period}(P) = \min_{v\in V} \gcd\{t > 0 : (P^t)(v,v) > 0\}$. It is well known that $\gcd\{t > 0 : (P^t)(v,v) > 0\}$ is common for any $v \in V$ if P is irreducible.

If $\mathrm{period}(P) = 1$ then P is said to be *aperiodic*. A transition matrix P is *ergodic* if it is irreducible and aperiodic. We say a random walk is (γ-)*lazy* if $P(v,v) \geq \gamma$ holds for any $v \in V$ for a constant γ $(0 < \gamma < 1)$. A lazy random walk is clearly aperiodic. A probability distribution π over V is a *stationary distribution* if it satisfies $\pi P = \pi$. It is well known that an ergodic P has a unique stationary distribution [27]. The *mixing time* of P is given by

$$\tau(\epsilon) \overset{\text{def.}}{=} \min\left\{ t \;\middle|\; t \in \mathbb{Z}_{>0},\ \frac{1}{2}\max_{u\in V}\sum_{v\in V}\left|P^t(u,v) - \pi(v)\right| \leq \epsilon \right\} \tag{4}$$

for $\epsilon \in (0,1)$.

2.2.2 Random walk with period 2

A *simple* random walk (or "busy" simple random walk) on an undirected graph $G = (V,E)$ is given by $P(u,v) = 1/\deg(u)$ for $\{u,v\} \in E$ where $\deg(u)$ denotes the degree of $u \in V$ on G. This paper is mainly concerned with bipartite graphs, such as trees, integer grids, and 0-1 hypercubes, and then the most targeted random walks are irreducible and reversible, but *not aperiodic*.

▶ **Observation 1.** *If P is reversible then its period is at most 2.*

Suppose P is irreducible and reversible, and it has period 2. Then, the underlying graph is a connected bipartite $(U,\overline{U};E)$, where $U = \{u \in V \mid \exists t',\ P^{2t'}(v,u) \neq 0\}$ for any $v \in U$, $\overline{U} = \{u \mid \forall t,\ P^{2t}(v,u) = 0\}$, i.e., $\overline{U} = V \setminus \overline{U}$, and $E = \{\{u,v\} \in V^2 \mid P(u,v) > 0\}$. Notice that E does not contain any self-loop, otherwise, P is aperiodic.

Here, we introduce some unfamiliar terminology for periodic Markov chains. We say $\mathring{x} \in \mathbb{R}^V_{\geq 0}$ is *even-time distribution* if it satisfies $\sum_{v\in V} \mathring{x}(v) = 1$ and $\mathring{x}(u) = 0$ for any $u \in \overline{U}$. We say $\mathring{\pi} \in \mathbb{R}^V_{\geq 0}$ is *even-time stationary distribution* if it is an even-time distribution and satisfies $\mathring{\pi}P^2 = \mathring{\pi}$.

▶ **Proposition 2** (limit distribution). *Suppose P is irreducible and reversible, and it has period 2. Then, P has a unique even-time stationary distribution $\mathring{\pi}$, and $\lim_{t\to\infty} \mathring{x}P^{2t} = \mathring{\pi}$ for any even-time distribution $\mathring{x}$.*

We define the *even mixing-time* of P by

$$\mathring{\tau}(\epsilon) = \min\left\{ 2t' \;\middle|\; t' \in \mathbb{Z}_{>0},\ \frac{1}{2}\max_{u\in U}\sum_{v\in U}\left|P^{2t'}(u,v) - \mathring{\pi}(v)\right| \leq \epsilon \right\} \tag{5}$$

for $\epsilon \in (0,1)$. We remark that the even mixing-time of P is equal to the twice of the mixing time of $P^2[U]$, where $P^2[U]$ denotes the submatrix of P induced by U. Thus, we can use some standard arguments, e.g., coupling technique, about the even mixing-time of P. Finally, we remark on a proposition, that plays a key role in our analysis.

▶ **Proposition 3** (Proposition 10.25 in [27]). *If P is reversible then $\mathring{\pi}(v) \leq P^{2t+2}(v,v) \leq P^{2t}(v,v)$ for any $t = 0,1,2,\ldots$.*

3 Analytical Framework: LHaGG

This section introduces the notion of less-homesickness as graph growing (LHaGG), and presents general theorems (Lemmas 5 and 6) describing some sufficient conditions of a RWoGG being recurrent or transient. See the following sections for specific RWoGGs, namely, RW on growing k-ary tree in Section 4, RW on $\{0,1\}^n$ hypercube skeleton with increasing n in Section 5, etc.

3.1 Less-homesick as graph growing

Let $\mathcal{D} = (f, G, P)$ and $\mathcal{D}' = (f', G', P')$ be RWoGG, and let $R(t)$ and $R'(t)$ respectively denote their return probabilities to respective initial vertices at time $t = 1, 2, \ldots$. We say $\mathcal{D}$ is *less-homesick* than $\mathcal{D}' = (f', G', P')$ at time t if $R(t) \leq R'(t)$ holds.

In particular, this paper is mainly concerned with the less-homesick relationship between $\mathcal{D} = (f, G, P)$ and $\mathcal{D}' = (g, G, P)$ with the same P, G and the initial vertex v. We say $\mathcal{D}$ is *less-homesick as graph growing* (*LHaGG*)[5] if $\mathcal{D} = (f, G, P)$ is less-homesick than for any $\mathcal{D}' = (g, G, P)$ satisfying that

$$\sum_{i=1}^{n} f(i) \leq \sum_{i=1}^{n} g(i) \tag{6}$$

for any $n \in \mathbb{Z}_{>0}$. The condition (6) intuitively implies that the graph in $\mathcal{D}$ grows faster than $\mathcal{D}'$. For instance, we will prove that the simple random walk on growing k-regular tree is LHaGG, in Section 4.

▶ **Lemma 4.** *Suppose RWoGG $\mathcal{D} = (f, G, P)$ is LHaGG. Let X_t $(t = 0, 1, 2, \ldots)$ be a RWoGG according to $\mathcal{D}$ with $X_0 = v \in V(1)$. Let Y_t $(t = 0, 1, 2, \ldots)$ be a random walk on (a static graph) $G(n)$ according to $P(n)$ with $Y_0 = v$, where G, P and v are common with $\mathcal{D}$. Then, Y_t is less-homesick than X_t at any time $t \in [T_n, T_{n+1}]$, i.e., $R(t) \geq R'(t)$ holds for $t \in [T_n, T_{n+1}]$, where $R(t) = \Pr[X_t = v]$ and $R'(t) = \Pr[Y_t = v]$.*

Proof. Let

$$g(i) = \begin{cases} 0 & (i < n), \\ \sum_{j=1}^{n} f(j) & (i = n), \\ f(i) & (i > n). \end{cases}$$

Then, the static random walk Y_t on $G(n)$ also follows $\mathcal{D}' = (g, G, P)$ for $t \leq T_{n+1}$. Clearly, $\sum_{i=1}^{n} f(i) \geq \sum_{i=1}^{n} g(i)$ for any n. Since $\mathcal{D}$ is LHaGG by the hypothesis, $R(t) \geq R'(t)$. ◀

We remark that if all P_n takes period 2 then $R(t) = R'(t) = 0$ for any odd t.

3.2 Recurrent

We prove the following lemma, presenting a sufficient condition for a RWoGG to be recurrent.

---

[5] Strictly speaking, LHaGG should be a property of the sequence of transition matrices $P(1), P(2), P(3), \ldots$. For the convenience of the notation, we say $\mathcal{D} = (f, G, P)$ is LHaGG, in this paper.

▶ **Lemma 5.** *Suppose that RWoGG $\mathcal{D} = (\mathfrak{d}, G, P)$ is LHaGG, and that every $P(n) = P_n$ ($n = 1, 2, \ldots$) is irreducible, reversible and* $\mathrm{period}(P_n) = 2$. *Let* $p(n) = \mathring{\pi}_n(v)$ *where* $\mathring{\pi}_n$ *denote the even-time stationary distribution of* P_n. *If* $\mathfrak{d}$ *satisfies*

$$\sum_{n=1}^{\infty} (\mathfrak{d}(n) - 1) p(n) = \infty \tag{7}$$

then v is recurrent by $\mathcal{D}$.

Proof. Let $f(n) = 2\lfloor \frac{\mathfrak{d}(n)}{2} \rfloor$, i.e., $f(n) = \mathfrak{d}(n)$ if $\mathfrak{d}(n)$ is even, otherwise $f(n) = \mathfrak{d}(n) - 1$. For convenience, let $T'_n = \sum_{k=1}^{n} f(k)$ for $n = 1, 2, \ldots$, and let $T'_0 = 0$. Let X_t (resp. X'_t) for $t = 0, 1, 2, \ldots$ be a RWoGG according to $\mathcal{D} = (\mathfrak{d}, G, P)$ (resp. $\mathcal{D}' = (f, G, P)$), and let $R(t)$ (resp. R') denote the return probability of X_t (resp. X'_t). The hypothesis LHaGG implies $R(t) \geq R'(t)$. Let Y^n_t ($t = 0, 1, \ldots, T'_n$) be a time-homogeneous random walk according to $P(n)$, and let $R''_n(t)$ ($t = 1, \ldots, T'_n$) denote the return probability of Y^n_t. The hypothesis LHaGG and Lemma 4 implies

$$R'(t) \geq R''_n(t) \tag{8}$$

for $t \in (T'_{n-1}, T'_n]$. Then, we can see

$$\begin{aligned}
\sum_{t=1}^{\infty} R(t) \geq \sum_{t=1}^{\infty} R'(t) & \qquad \text{(by LHaGG)} \\
= \sum_{n=1}^{\infty} \sum_{t=T'_{n-1}+1}^{T'_n} R'(t) & \\
\geq \sum_{n=1}^{\infty} \sum_{t=T'_{n-1}+1}^{T'_n} R''_n(t) & \qquad \text{(by (8))} \\
= \sum_{n=1}^{\infty} \sum_{i=1}^{f(n)} R''_n(T'_{n-1} + i) & \qquad \text{(recall } T'_n = T'_{n-1} + f(n)) \\
= \sum_{n=1}^{\infty} \sum_{i'=1}^{\frac{f(n)}{2}} R''_n(T'_{n-1} + 2i') & \qquad \text{(notice that } R''_n(T'_{n-1} + 2i' - 1) = 0) \\
\geq \sum_{n=1}^{\infty} \sum_{i'=1}^{\frac{f(n)}{2}} p(n) & \qquad \text{(by Proposition 3)} \\
= \frac{1}{2} \sum_{n=1}^{\infty} f(n) p(n) & \\
\geq \frac{1}{2} \sum_{n=1}^{\infty} (\mathfrak{d}(n) - 1) p(n) &
\end{aligned} \tag{9}$$

hold. If (7) holds then (9) is ∞, meaning that v is recurrent by $\mathcal{D}$. ◀

It is not difficult to see that a similar proposition holds for *lazy* random walks.

3.3 Transient

This section establishes the following lemma, which suggests Lemma 5 is nearly optimal. In fact, we will provide an example of a random walk on a growing k-ary tree in Section 4, that shows a tight example of Lemma 5.

▶ **Lemma 6.** *Suppose that a RWoGG $\mathcal{D} = (\mathfrak{d}, G, P)$ is LHaGG, and that every $P(n) = P_n$ $(n = 1, 2, \ldots)$ is irreducible and reversible with* $\mathrm{period}(P_n) = 2$. *Let $p(n) = \mathring{\pi}_n(v)$ where $\mathring{\pi}_n$ denote the even-time stationary distribution of P_n. Let $\mathring{\tau}_n(\epsilon)$ denote the even mixing-time of $P(n)$, and let*

$$\mathring{\mathfrak{t}}(n) = \mathring{\tau}_n(p(n))$$

for $n = 2, 3, \ldots$. If

$$\max\left\{\mathfrak{d}(1), \mathring{\mathfrak{t}}(1)\right\} + \sum_{n=2}^{\infty} \max\left\{\mathfrak{d}(n), \mathring{\mathfrak{t}}(n)\right\} p(n-1) < \infty \tag{10}$$

holds then v is transient by $\mathcal{D}$.

Proof. Let

$$f(n) = \max\left\{\mathfrak{d}(n), \mathring{\mathfrak{t}}(n)\right\}$$

for $n = 1, 2, 3, \ldots$. Let $R(t)$ and $R'(t)$ respectively denote the return probabilities of $\mathcal{D} = (\mathfrak{d}, G, P)$ and $\mathcal{D}' = (f, G, P)$. Clearly, $f(n) \geq \mathfrak{d}(n)$ for any n, LHaGG implies

$$R(t) \leq R'(t) \tag{11}$$

for any $t = 0, 1, 2, \ldots$. For convenience, let

$$T'_n = \sum_{k=1}^{n} f(k) \tag{12}$$

for $n = 1, 2, \ldots$.

We carry a tricky argument in the following: roughly speaking we compare $\mathcal{D}'$ with P_{n-1} in the n-th round, i.e., $[T_{n-1}, T_n]$, for $n = 2, 3, \ldots$. Let

$$g_{n-1}(k) = \begin{cases} f(k) & (k \leq n-2) \\ \infty & (k = n-1) \end{cases}$$

for $n = 2, 3, \ldots$. Let $Z_t^{(n-1)}$ $(t = 0, 1, 2, \ldots)$ denote a RWoGG (g_{n-1}, G, P), where $Z_0^{(n-1)} = v$. Let $R''_{n-1}(t)$ denote the return probability of $Z_t^{(n-1)}$, Clearly, $\sum_{i=1}^{j} f(i) \leq \sum_{i=1}^{j} g_{n-1}(i)$ holds for any j, hence the LHaGG assumption implies

$$R'(t) \leq R''_{n-1}(t) \tag{13}$$

for any $t = 0, 1, 2 \ldots$ for any $n = 2, 3, \ldots$.

Notice that $Z_t^{(n-1)}$ for $t \in [T_{n-2}, T_n]$ is nothing but a time-homogeneous random walk according to P_{n-1} with the "initial state" $Z_{T_{n-2}} = v$ for $n = 2, 3, \ldots$. Since

$$T'_{n-1} = T'_{n-2} + f(n-1) \geq T'_{n-2} + \mathring{\mathfrak{t}}(n-1) = T'_{n-2} + \mathring{\tau}_{n-1}(p(n-1))$$

$Z_t^{(n-1)}$ mixes well for $t > T'_{n-1}$, meaning that $|\Pr[Z_t^{(n-1)} = v] - \mathring{\pi}_{n-1}(v)| \leq p(n-1)$ for any even $t \in (T'_{n-1}, T_n]$. This implies

$$R''_{n-1}(t) = \Pr[Z_t^{(n-1)} = v] \leq \mathring{\pi}_{n-1}(v) + p(n-1) = 2p(n-1) \tag{14}$$

holds[6] for $t \in (T'_{n-1}, T'_n]$, where we remark that $R''_{n-1}(t) = 0$ for any odd t. Then,

$$\begin{aligned}
\sum_{t=1}^{\infty} R(t) &\leq \sum_{t=1}^{\infty} R'(t) && \text{(by (11))}\\
&= \sum_{n=1}^{\infty} \sum_{t=T'_{n-1}+1}^{T'_n} R'(t)\\
&\leq f(1) + \sum_{n=2}^{\infty} \sum_{t=T'_{n-1}+1}^{T'_n} R'(t)\\
&\leq f(1) + \sum_{n=2}^{\infty} \sum_{t=T'_{n-1}+1}^{T'_n} R''_{n-1}(t) && \text{(by (13))}\\
&\leq f(1) + \sum_{n=2}^{\infty} \sum_{t=T'_{n-1}+1}^{T'_n} 2p(n-1) && \text{(by (14))}\\
&= f(1) + 2\sum_{n=2}^{\infty} f(n)p(n-1)
\end{aligned}$$

holds. Now it is easy to see that (10) implies $\sum_{t=1}^{\infty} R(t) < \infty$, meaning that v is transient by $\mathcal{D}$. ◀

It is not difficult to see that a similar proposition holds for *lazy* random walks.

4 Random Walk on a Growing Complete k-ary Tree

Lyons gave sufficient conditions that a random walk on an infinite tree gets recurrent or transient at the root (initial point), cf. [28, 29], as a consequence, it is a celebrated fact that a simple random walk on an infinite k-ary tree is transient. This section shows that a simple random walk on a *moderately* growing complete k-ary tree is recurrent at the root.

4.1 Result summary

Let k be an integer greater than one, and let $G_n = (V_n, E_n)$ denote a *complete k-ary tree* with height n for $n = 1, 2, \ldots$, i.e., $|V_n| = \sum_{i=0}^{n} k^i = \frac{k^{n+1}-1}{k-1}$, every internal node (including the root) has exactly k children, and every leaf places the same height n. Let $r \in V_n$ denote the root, that is the unique vertex of height 0. For convenience, let $h(v)$ denote the height of vertex $v \in V_n$, i.e., $h(r) = 0$, and $h(v) = n$ if and only if v is a leaf of G_n. Let

$$U_n = \{v \in V_n \mid h(v) \equiv 0 \pmod 2\} \tag{15}$$

denote the vertices of even heights, and thus $\overline{U}_n = V_n \setminus U_n$ is the vertices of odd heights. Clearly, $G_n = (U_n, \overline{U}_n; E_n)$ is a bipartite graph. See [10] for a standard terminology about a complete k-ary tree, e.g., parent, child, root, internal node, leaf, height.

[6] We remark this argument requires only *point-wise additive error* bound, instead of total variation. Clearly, point-wise additive error is upper bounded by total variation. We here use the mixing time for total variation just because it has been better analyzed than the other.

Next, we define a transition probability of a random walk over G_n according to [28, 29]. Let λ be a fixed positive real[7], and we define a transition probability on the k-ary tree G_n with height n by

$$P_n(u,v) = \begin{cases} \frac{1}{k} & \text{if } u = r \text{ and } v \text{ is a child of } u, \\ \frac{1}{\lambda+k} & \text{if } u \neq r \text{ and } v \text{ is a child of } u, \\ \frac{\lambda}{\lambda+k} & \text{if } u \text{ is an internal node and } v \text{ is the parent of } u, \\ 1 & \text{if } u \text{ is a leaf and } v \text{ is the parent of } u, \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

for $u, v \in V_n$. Notice that (16) denotes a *simple random walk* over T_n when $\lambda = 1$. We also remark that λ and k are constants to n. As a consequence of [28], we know the following fact about a random walk on an infinite k-ary tree T_∞.

▶ **Proposition 7** ([28, 29])**.** *If $\lambda \geq k$ (resp. $\lambda < k$) then the root r is recurrent (resp. transient) by P_∞.*

Then, we are concerned with a RWaGG $\mathcal{D}_\mathrm{T} = (\mathfrak{d}, G, P)$ starting from the root r where $G(n) = G_n$ and $P(n) = P_n$. Our goal of the section is to establish the following theorem.

▶ **Theorem 8.** *Let $k \geq 2$ and $\lambda > 0$ be constants to n. Then, the root r is recurrent by $\mathcal{D}_\mathrm{T}$ if*

$$\sum_{n=1}^{\infty} \mathfrak{d}(n) \left(\frac{\lambda}{k}\right)^n = \infty \tag{17}$$

holds, otherwise, transient.

For instance, Theorem 8 implies the following corollary, about a simple random walk on an infinitely growing k-ary tree.

▶ **Corollary 9.** *Let $\lambda = 1$, i.e., every P_n denotes a* simple *random walk on the complete k-ary tree T_n. If $\mathfrak{d}(n) = \Omega(k^n/(n\log n))$ then r is recurrent by $\mathcal{D}_\mathrm{T}$. If $\mathfrak{d}(1) < \infty$ and $\mathfrak{d}(n) = \mathrm{O}(k^n/(n(\log n)^{1+\epsilon}))$ for $n \geq 2$ with a constant $\epsilon > 0$ then r is transient by $\mathcal{D}_\mathrm{T}$.*

Proof. Suppose $\mathfrak{d}(n) \geq ck^n/(n\log n)$ for some constant $c > 0$. Then, $\sum_{n=1}^{\infty} \mathfrak{d}(n)(\frac{1}{k})^n \geq \sum_{n=1}^{\infty} c\frac{k^n}{n\log n}(\frac{1}{k})^n = c\sum_{n=1}^{\infty}\frac{1}{n\log n} \geq c\int_2^\infty \frac{1}{n\log n} = c[\log\log n]_2^\infty = \infty$, and Theorem 8 implies that r is recurrent.

Suppose $\mathfrak{d}(n) \leq c'k^n/(n(\log n)^{1+\epsilon})$for some constant $c' > 0$. Then, $\sum_{n=1}^{\infty} \mathfrak{d}(n)(\frac{1}{k})^n \leq \mathfrak{d}(1) + \sum_{n=2}^{\infty} c'\frac{k^n}{n(\log n)^{1+\epsilon}}(\frac{1}{k})^n \leq \mathfrak{d}(1) + c'\frac{1}{2(\log 2)^{1+\epsilon}} + c'\int_2^\infty \frac{1}{x(\log x)^{1+\epsilon}}\mathrm{d}x = \mathfrak{d}(1) + c'\frac{1}{2(\log 2)^{1+\epsilon}} + c'k\left[-\frac{1}{\epsilon(\log x)^\epsilon}\right]_2^\infty < \infty$, and Theorem 8 implies that r is transient. ◀

4.2 Proof of Theorem 8

We prove Theorem 8. As a preliminary step, we remark on the following two facts.

▶ **Lemma 10.** *(i) Every P_n ($n = 1, 2, \ldots$) is reversible: precisely, let*

$$\phi(v) = \begin{cases} \frac{k}{\lambda+k} & \text{if } h(v) = 0 \text{ (i.e., } v = r), \\ \lambda^{-h(v)} & \text{if } 0 < h(v) < n, \\ \frac{\lambda}{\lambda+k}\lambda^{-n} & \text{if } h(v) = n \text{ (i.e., } v \text{ is a leaf).} \end{cases} \tag{18}$$

[7] For simplicity of notation, Lyons [28] and Lyons and Peres [29] assume $\lambda > 1$, but many arguments are naturally extended to $\lambda > 0$ by modifications with some bothering notations.

Then, the detailed balance equation

$$\phi(u)P_n(u,v) = \phi(v)P_n(v,u)$$

holds for any $u, v \in V_n$*. (ii) Every* P_n *is irreducible and* $\mathrm{period}(P_n) = 2$*. Thus the even-time stationary distribution of* P_n *is*

$$\mathring{\pi}_n(v) = \frac{\phi(v)}{\sum_{u \in U_n} \phi(u)} \tag{19}$$

for any $v \in U_n$*.*

Let $p(n) = \mathring{\pi}_n(r)$, then

$$p(n) = \begin{cases} \frac{\frac{k}{\lambda+k}}{\frac{k}{\lambda+k} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{k}{\lambda}\right)^{2i}} & \text{if } n \text{ is odd,} \\ \frac{\frac{k}{\lambda+k}}{\frac{k}{\lambda+k} + \sum_{i=1}^{\frac{n}{2}-1} \left(\frac{k}{\lambda}\right)^{2i} + \frac{\lambda}{\lambda+k}\left(\frac{k}{\lambda}\right)^{n}} & \text{if } n \text{ is even} \end{cases} \tag{20}$$

by (18) and (19) considering the fact $|\{v \in V_n \mid h(v) = i\}| = k^i$ for $i = 0, 1, \ldots, n$.

▶ **Lemma 11.** *If* $\lambda < k$*, then*

$$\tfrac{k-\lambda}{k} \left(\frac{\lambda}{k}\right)^{n+1} \leq p(n) \leq \left(\frac{\lambda}{k}\right)^{n-1}. \tag{21}$$

Proof. Firstly, we prove the upper bound of (21). When n is odd,

$$p(n) = \frac{\frac{k}{\lambda+k}}{\frac{k}{\lambda+k} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{k}{\lambda}\right)^{2i}} \leq \frac{1}{\frac{k}{\lambda+k} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{k}{\lambda}\right)^{2i}} \leq \frac{1}{\left(\frac{k}{\lambda}\right)^{2\lfloor \frac{n}{2} \rfloor}} = \left(\frac{\lambda}{k}\right)^{2\lfloor \frac{n}{2} \rfloor} = \left(\frac{\lambda}{k}\right)^{n-1}$$

and we obtain the upper bound in the case. When n is even, similarly,

$$\begin{aligned} p(n) &= \frac{\frac{k}{\lambda+k}}{\frac{k}{\lambda+k} + \sum_{i=1}^{\frac{n}{2}-1} \left(\frac{k}{\lambda}\right)^{2i} + \frac{\lambda}{\lambda+k}\left(\frac{k}{\lambda}\right)^{n}} = \frac{1}{1 + \frac{\lambda+k}{k} \sum_{i=1}^{\frac{n}{2}-1} \left(\frac{k}{\lambda}\right)^{2i} + \left(\frac{k}{\lambda}\right)^{n}} \\ &\leq \frac{1}{\left(\frac{k}{\lambda}\right)^{n}} \leq \left(\frac{\lambda}{k}\right)^{n-1} \end{aligned}$$

and we obtain the upper bound. Then, we prove the lower bound of (21). When n is odd,

$$p(n) = \frac{\frac{k}{\lambda+k}}{\frac{k}{\lambda+k} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{k}{\lambda}\right)^{2i}} \geq \frac{\frac{k}{\lambda+k}}{1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{k}{\lambda}\right)^{2i}} = \frac{\frac{k}{\lambda+k}}{\frac{\left(\left(\frac{k}{\lambda}\right)^2\right)^{\lfloor \frac{n}{2} \rfloor + 1} - 1}{\left(\frac{k}{\lambda}\right)^2 - 1}} = \frac{\frac{k}{\lambda+k}}{\frac{\left(\frac{k}{\lambda}\right)^{n+1} - 1}{\left(\frac{k}{\lambda}\right)^2 - 1}}$$

and we obtain the lower bound in the case. When n is even,

$$p(n) = \frac{\frac{k}{\lambda+k}}{\frac{k}{\lambda+k} + \sum_{i=1}^{\frac{n}{2}-1} \left(\frac{k}{\lambda}\right)^{2i} + \frac{\lambda}{\lambda+k}\left(\frac{k}{\lambda}\right)^{n}} \geq \frac{\frac{k}{\lambda+k}}{1 + \sum_{i=1}^{\frac{n}{2}-1} \left(\frac{k}{\lambda}\right)^{2i} + \left(\frac{k}{\lambda}\right)^{n}} = \frac{\frac{k}{\lambda+k}}{\frac{\left(\frac{k}{\lambda}\right)^{n+2} - 1}{\left(\frac{k}{\lambda}\right)^2 - 1}}$$

holds. In both cases,

$$\begin{aligned} p(n) &\geq \frac{\frac{k}{\lambda+k}}{\frac{\left(\frac{k}{\lambda}\right)^{n+2} - 1}{\left(\frac{k}{\lambda}\right)^2 - 1}} = \tfrac{k}{\lambda+k} \left(\left(\tfrac{k}{\lambda}\right)^2 - 1\right) \frac{1}{\left(\frac{k}{\lambda}\right)^{n+2} - 1} \\ &\geq \tfrac{k}{\lambda+k} \left(\left(\tfrac{k}{\lambda}\right)^2 - 1\right) \frac{1}{\left(\frac{k}{\lambda}\right)^{n+2}} = \tfrac{k-\lambda}{k} \left(\frac{\lambda}{k}\right)^{n+1} \end{aligned}$$

holds and we obtain the lower bound. ◀

The following lemma is a key of the proof of Theorem, 8.

▶ **Lemma 12.** *If $\lambda < k$ then $\mathcal{D}_{\mathrm{T}}$ is LHaGG.*

Proof. Let f and g satisfy $\sum_{i=1}^{n} f(i) \leq \sum_{i=1}^{n} g(i)$ for any $n = 1, 2, \ldots$, and let X_t and Y_t ($t = 0, 1, 2, \ldots$) respectively follow (f, G, P) and (g, G, P), i.e., the tree of (f, G, P) grows faster than (g, G, P). Let $X_0 = Y_0 = r$, and we prove $\Pr[X_t = r] \leq \Pr[Y_t = r]$ for any $t = 1, 2, \ldots$ (recall Section 3.1 for LHaGG).

We construct a coupling of $\boldsymbol{X} = \{X_t\}_{t\geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t\geq 0}$ such that $h(X_t) \geq h(Y_t)$ holds for any $t = 1, 2, \ldots$. The proof is an induction concerning t. Clearly, $h(X_0) = h(Y_0) = 0$. Inductively assuming $h(X_t) \geq h(Y_t)$, we prove $h(X_{t+1}) \geq h(Y_{t+1})$. If $h(X_t) > h(Y_t)$ then $h(X_t) \geq h(Y_t) - 2$ since every P_n is period$(P_n) = 2$ for $n = 1, 2, \ldots$. It is easy to see that $h(X_{t+1}) \geq h(X_t) - 1 \geq h(Y_t) + 1 \geq h(Y_{t+1})$, and we obtain $h(X_{t+1}) \geq h(Y_{t+1})$ in the case.

Suppose $h(X_t) = h(Y_t)$. We consider four cases: (i) $X_t = Y_t = r$, (ii) both X_t and Y_t are internal nodes, (iii) both X_t and Y_t are leaves, i.e., both trees of (f, G, P) and(g, G, P) take the same height at time t, (iv) X_t is not a leaf but Y_t is a leaf, i.e., the tree of (f, G, P) is higher than that of (g, G, P) at time t. In the case (i),

$$\Pr[h(X_{t+1}) = h(X_t) + 1] = \Pr[h(Y_{t+1}) = h(Y_t) + 1] = 1$$

hold, and hence we can couple them to satisfy $h(X_{t+1}) = h(Y_{t+1})$. In the case (ii), since both X_t and Y_t are internal nodes,

$$\Pr[h(X_{t+1}) = h(X_t) - 1] = \Pr[h(Y_{t+1}) = h(Y_t) - 1] = \frac{\lambda}{k + \lambda}$$

and

$$\Pr[h(X_{t+1}) = h(X_t) + 1] = \Pr[h(Y_{t+1}) = h(Y_t) + 1] = \frac{k}{k + \lambda}$$

hold, and hence we can couple them to satisfy $h(X_{t+1}) = h(Y_{t+1})$. In the case (iii), since both X_t and Y_t are leaves,

$$\Pr[h(X_{t+1}) = h(X_t) - 1] = \Pr[h(Y_{t+1}) = h(Y_t) - 1] = 1$$

holds, and hence we can couple them to satisfy $h(X_{t+1}) = h(Y_{t+1})$. In the case (iv), since X_t is not a leaf but Y_t is a leaf,

$$\Pr[h(X_{t+1}) = h(X_t) - 1] = \frac{\lambda}{k + \lambda} \leq \Pr[h(Y_{t+1}) = h(Y_t) - 1] = 1$$

holds, and hence we can couple them to satisfy $h(X_{t+1}) \geq h(Y_{t+1})$.

Now we obtain a coupling of $\boldsymbol{X} = \{X_t\}_{t\geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t\geq 0}$ such that $h(X_t) \geq h(Y_t)$ hold for any $t = 1, 2, \ldots$, which implies that $h(Y_t) = 0$ as long as $h(X_t) = 0$. This means that $\Pr[X_t = r] \leq \Pr[Y_t = r]$ for any $t = 1, 2, \ldots$. We obtain the claim. ◀

By Lemma 5 with Lemma 12, we get a sufficient condition for recurrence in Theorem 8. On the other hand, we cannot directly apply Lemma 6 to the sufficient condition for transient in Theorem 8, because the "mixing time" of P_n is proportional to k^n, see e.g., [27]. Then, we estimate $R(t)$ by another random walk.

Let $Z_t = h(X_t)$, where X_t is a random walk on a growing k-ary tree $\mathcal{D}_{\mathrm{T}} = (\mathfrak{d}, G, P)$. Then Z_t is a RWoGG $\mathcal{D}_{\mathrm{L}} = (\mathfrak{d}, L, Q)$ where $L(n) = (\{0, 1, \ldots, n\}, \{\{i, i+1\} \mid i = 0, 1, 2, \ldots, n-1\})$ is a path graph of length n, and the transition probability matrix $Q(n) = Q_n$ is given by

$$\begin{cases} Q_n(0,1) = 1, & \\ Q_n(i, i+1) = \frac{k}{\lambda+k} & \text{for } i = 1, 2, \ldots, n-1, \\ Q_n(i, i-1) = \frac{\lambda}{\lambda+k} & \text{for } i = 1, 2, \ldots, n-1, \\ Q_n(n, n-1) = 1. & \end{cases}$$

The following Lemmas 13 and 14 are easy to observe.

▶ **Lemma 13.** *Let X_t (resp. Z_t) follow $\mathcal{D}_{\mathrm{T}} = (f, G, P)$ (resp. $\mathcal{D}_{\mathrm{L}} = (f, L, Q)$). Let $R(t) = \Pr[X_t = r]$ (resp. $R'(t) = \Pr[Z_t = r]$), and let $\mathring{\pi}_n$ (resp. $\mathring{\pi}'_n$) denote the even-time stationary distribution of P_n (resp. Q_n). Then, $R(t) = R'(t)$ for any $t = 1, 2, \ldots,$ as well as $\mathring{\pi}_n(r) = \mathring{\pi}'_n(r)$.*

▶ **Lemma 14.** *If $\lambda < k$ then $\mathcal{D}_{\mathrm{L}}$ is LHaGG.*

The following lemma about the mixing time of Q_n is easily obtained by a standard coupling argument for the mixing time, and we here omit the proof.

▶ **Lemma 15.** *Let $\mathring{\tau}'_n(\epsilon)$ denote the even mixing-time of Q_n then $\mathring{\tau}'_n(\epsilon) \le n^2 \log \epsilon^{-1}$.*

Then, we can prove the condition for $\mathcal{D}_{\mathrm{L}}$ being transient from Lemma 6.

▶ **Lemma 16.** *If $\lambda < k$ and $\sum_{n=1}^{\infty} \mathfrak{d}(n)(\frac{\lambda}{k})^n < \infty$ then 0 is transient by $\mathcal{D}_{\mathrm{L}}$.*

Proof. Let $p(n) = \mathring{\pi}_n(r)$ and $p'(n) = \mathring{\pi}'_n(r)$, then $p(n) = p'(n)$ by Lemma 13. By Lemma 15, $\mathring{\mathfrak{t}}'(n) = \mathring{\tau}'{}_n(p(n-1)) \le n^2 \log(p(n-1)) \le n^2 \log\big((\frac{\lambda}{k})^n\big) \le c'n^3$, and hence $\sum_{n=1}^{\infty} \mathring{\mathfrak{t}}'(n)p(n-1) \le \sum_{n=1}^{\infty} n^3 c'(\frac{\lambda}{k})^{n-1} < \infty$. If $\sum_{n=1}^{\infty} \mathfrak{d}(n)(\frac{\lambda}{k})^n < \infty$, then $\sum_{n=1}^{\infty} \max\{\mathfrak{d}(n), \mathring{\mathfrak{t}}'(n)\}p(n-1) \le \sum_{n=1}^{\infty}(\mathfrak{d}(n) + \mathring{\mathfrak{t}}'(n))(\frac{\lambda}{k})^{n-1} < \infty$, which implies $\sum_{t=1}^{\infty} R'(t) < \infty$ by Lemma 6 with Lemma 14. ◀

Now, we are ready to prove Theorem 8.

Proof of Theorem 8. First, we consider the (interesting) case $\lambda < k$.

(Recurrent) Assuming $\sum_{n=1}^{\infty} \mathfrak{d}(n)(\frac{\lambda}{k})^n = \infty$, we prove $\sum_{n=1}^{\infty}(\mathfrak{d}(n) - 1)p(n) = \infty$. Notice that $\sum_{n=1}^{\infty} p(n) \le c\sum_{n=1}^{\infty}(\frac{\lambda}{k})^n = \frac{1}{1-\frac{\lambda}{k}} < \infty$. Let $C = \sum_{n=1}^{\infty} p(n)$, then $\sum_{n=1}^{\infty}(\mathfrak{d}(n) - 1)p(n) = \sum_{n=1}^{\infty} \mathfrak{d}(n)p(n) - C \ge \sum_{n=1}^{\infty} \mathfrak{d}(n)c(\frac{\lambda}{k})^n - C$, which is ∞ from the assumption. Thus, r is recurrent by Lemma 5.

(Transient) By Lemma 13, $\sum_{t=1}^{\infty} R(t) = \sum_{t=1}^{\infty} R'(t)$. If $\sum_{n=1}^{\infty} \mathfrak{d}(n)(\frac{\lambda}{k})^n < \infty$ then $\sum_{t=1}^{\infty} R'(t) < \infty$ by Lemma 16, meaning that r is transient.

In the case of $\lambda \ge k$, it is always recurrent. The proof follows that of Lemma 5, but here we omit the proof. ◀

5 Random Walk on $\{0, 1\}^n$ with Increasing n

5.1 Main result

This section shows an interesting example. Let $C_n = (V_n, E_n)$ where

$$V_n = \{0, 1\}^n \tag{22}$$

$$E_n = \left\{ \{\boldsymbol{u}, \boldsymbol{v}\} \in \binom{V_n}{2} \mid \|\boldsymbol{u} - \boldsymbol{v}\|_1 = 1 \right\} \tag{23}$$

for $n = 1, 2, \ldots$. Let $\mathbf{0} \in V_n$ denote the (common) origin vertex $(0, \ldots, 0)$ for each n. Let

$$P_n(\boldsymbol{u}, \boldsymbol{v}) = \begin{cases} \frac{1}{n} & \text{if } \|\boldsymbol{u} - \boldsymbol{v}\|_1 = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{24}$$

for $\boldsymbol{u}, \boldsymbol{v} \in V_n$. Then, we are concerned with $\mathcal{D}_{\mathrm{C}} = (\mathfrak{d}, G, P)$ starting from $\mathbf{0}$ where $G(n) = C_n$ and $P(n) = P_n$.

▶ **Theorem 17.** *If $\mathcal{D}_{\mathrm{C}}$ satisfies*

$$\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty \tag{25}$$

then $\mathbf{0}$ is recurrent, otherwise $\mathbf{0}$ is transient.

The following lemma is not very difficult, but nontrivial.

▶ **Lemma 18.** *$\mathcal{D}_{\mathrm{C}}$ is LHaGG.*

Proof. Let f and g satisfy $\sum_{i=1}^n f(i) \leq \sum_{i=1}^n g(i)$ for any $n = 1, 2, \ldots$, and let X_t and Y_t ($t = 0, 1, 2, \ldots$) respectively follow (f, G, P) and (g, G, P), i.e., the box of (f, G, P) grows faster than (g, G, P). Let n_t (resp. n'_t) denote the dimension of (f, G, P) (resp. (g, G, P)) at time t, and then notice that $n_t \geq n'_t$ hold for any $t = 0, 1, \ldots$ by the assumption that (f, G, P) grows faster. Let $X_0 = Y_0 = \mathbf{0}$, and we prove $\Pr[X_t = \mathbf{0}] \leq \Pr[Y_t = \mathbf{0}]$ for any $t = 1, 2, \ldots$.

Let $h(\boldsymbol{u}) = |\{i \in \{1, \ldots, n\} \mid u_i = 1\}|$ for $\boldsymbol{u} = (u_1, \ldots, u_n) \in V_n$. We construct a coupling of $\boldsymbol{X} = \{X_t\}_{t \geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t \geq 0}$ such that $h(X_t) \geq h(Y_t)$ holds for any $t = 1, 2, \ldots$. The proof is an induction concerning t. Clearly, $h(X_0) = h(Y_0) = 0$. Inductively assuming $h(X_t) \geq h(Y_t)$, we prove $h(X_{t+1}) \geq h(Y_{t+1})$. If $h(X_t) > h(Y_t)$ then $h(X_t) \geq h(Y_t) - 2$ since every P_n is $\mathrm{period}(P_n) = 2$ for $n = 1, 2, \ldots$. It is easy to see that $h(X_{t+1}) \geq h(X_t) - 1 \geq h(Y_t) + 1 \geq h(Y_{t+1})$, and we obtain $h(X_{t+1}) \geq h(Y_{t+1})$ in the case. Suppose $h(X_t) = h(Y_t)$. Then,

$$\Pr[h(X_{t+1}) = h(X_t) - 1] = \frac{h(X_t)}{n_t} \leq \frac{h(Y_t)}{n'_t} = \Pr[h(Y_{t+1}) = h(Y_t) - 1], \qquad \text{and}$$

$$\Pr[h(X_{t+1}) = h(X_t) + 1] = 1 - \frac{h(X_t)}{n_t} \geq 1 - \frac{h(Y_t)}{n'_t} = \Pr[h(Y_{t+1}) = h(Y_t) + 1]$$

hold, which implies that a coupling exists such that $h(X_{t+1}) \geq h(Y_{t+1})$.

Now we obtain a coupling of $\boldsymbol{X} = \{X_t\}_{t \geq 0}$ and $\boldsymbol{Y} = \{Y_t\}_{t \geq 0}$ satisfying $h(X_t) \geq h(Y_t)$ for any $t = 1, 2, \ldots$, which implies that $h(Y_t) = 0$ as long as $h(X_t) = 0$. This means that $\Pr[X_t = \mathbf{0}] \leq \Pr[Y_t = \mathbf{0}]$ for any $t = 1, 2, \ldots$. We obtain the claim. ◀

The following two lemmas are well known.

▶ **Lemma 19.** *Let $\mathring{\tau}_n(\epsilon)$ denote the mixing time of P_n. Then, $\mathring{\tau}_n(\epsilon) = \mathrm{O}(n \log(n/\epsilon))$.*

▶ **Lemma 20.** $p(n) = \frac{1}{\frac{2^n}{2}} = 2^{-n+1}$.

Proof of Theorem 17. (Recurrence) By Lemma 18, $\mathcal{D}_{\mathrm{C}}$ is LHaGG. Since $p(n) = 2^{-n+1}$ by Lemma 20, Lemma 5 implies that if $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} = \infty$ then $\mathbf{0}$ is recurrent.

(Transience) By Lemma 19, $\mathring{\mathfrak{t}}(n) = \mathring{\tau}_n(p(n-1)) \leq n \log \frac{n}{p(n-1)} \leq n \log(n 2^n) \leq c' n^2 \log n$, and hence $\sum_{n=1}^{\infty} \mathring{\mathfrak{t}}(n) p(n-1) \leq \sum_{n=1}^{\infty} c' n^2 \log n \frac{1}{2^{n-2}} < \infty$. If $\sum_{n=1}^{\infty} \frac{\mathfrak{d}(n)}{2^n} < \infty$, then $\sum_{n=1}^{\infty} \max\{\mathfrak{d}(n), \mathring{\mathfrak{t}}(n)\} p(n-1) \leq \sum_{n=1}^{\infty} (\mathfrak{d}(n) + \mathring{\mathfrak{t}}(n)) \frac{1}{2^n} < \infty$, which implies $\sum_{t=1}^{\infty} R(t) < \infty$ by Lemma 6 with Lemma 18. ◀

5.2 An Interesting fact: every finite point becomes recurrent

We can easily observe the following fact from Theorem 17.

▶ **Corollary 21.** *If $\mathfrak{d}(n) = \Omega(2^n/n)$ then $\mathbf{0}$ is recurrent. If $\mathfrak{d}(n) = \mathrm{O}(2^n/n^{1+\epsilon})$ then $\mathbf{0}$ is transient.*

Notice that the maximum degree of $G(n)$ is unbounded asymptotic to n, clearly. Nevertheless, we can see the following interesting fact.

▶ **Proposition 22.** *If $\mathfrak{d}(n) = \Omega(n2^n)$ then $\mathcal{D}_{\mathrm{C}}$ starting from $\mathbf{0}$ visits $\boldsymbol{v} \in V_m$ infinitely many times for any $m < \infty$.*

Proof. Notice that $\mathring{\tau}_n(2^{-n-1}) = \mathrm{O}(n \log(n2^{n+1})) = \mathrm{O}(n^2 \log n)$ by Lemma 19. Thus, in the n-th phase, i.e, $[T_{n-1}, T_n]$, the random walk X visits $\boldsymbol{v} \in V_n$ with probability at least 2^{-n-1} in every $\mathrm{O}(n^2 \log n)$ steps (even if $\boldsymbol{v} \in \overline{U}_n$, here we omit the proof). Thus the probability that X never visit $\boldsymbol{v}$ during the n-th phase is at most $(1-2^{-n-1})^{n2^{n+1}/n^2 \log n} \leq \exp(-\frac{1}{n \log n})$. This implies that the probability that X never visits $\boldsymbol{v} \in V_m$ forever is at most $\prod_m^\infty \exp(-\frac{1}{n \log n}) = \exp(-\sum_{n=m}^\infty \frac{1}{n \log n}) \leq \exp(-\int_m^\infty \frac{1}{x \log x} \mathrm{d}x) = \exp(-[\log \log x]_m^\infty) = \exp(-\infty) = 0$. This means that the RWoGG X visits $\boldsymbol{v} \in V_m$ at least once in finite steps with probability 1.

Once we know that X visits $\boldsymbol{v}$ in a finite steps, the claim is trivial thanks to the vertex transitivity of the hypercube skeleton. ◀

We think that the hypothesis of Proposition 22 can be relaxed from $\Omega(n2^n)$ to $\Omega(2^n/n)$, but we are not sure.

6 Concluding Remark

In this paper, we have developed a coupling method to prove the recurrence and transience of a RWoGG, by introducing the notion of LHaGG. Then, we showed the phase transition between the recurrence and transience of random walks on a growing k-ary tree (Theorem 8) and on a growing hypercube (Theorem 17). We also have other examples of LHaGG, such as growing integer grids and growing level trees (see a full paper version). It is a future work to develop an extended technique to prove the phase transitions for more general growing trees and integer grids.

References

1 Gideon Amir, Itai Benjamini, Ori Gurel-Gurevich, and Gady Kozma. Random walk in changing environment. *Stochastic Processes and their Applications*, 130(12):7463–7482, 2020. doi:10.1016/j.spa.2020.08.003.

2 V.S. Anil Kumar and H. Ramesh. Coupling vs. conductance for the Jerrum–Sinclair chain. *Random Structures & Algorithms*, 18(1):1–17, 2001. doi:10.1002/1098-2418(200101)18:1<1::AID-RSA1>3.0.CO;2-7.

3 John Augustine, Gopal Pandurangan, and Peter Robinson. Distributed algorithmic foundations of dynamic networks. *SIGACT News*, 47(1):69–98, March 2016. doi:10.1145/2902945.2902959.

4 Chen Avin, Michal Koucký, and Zvi Lotker. How to explore a fast-changing world (cover time of a simple random walk on evolving graphs). In Luca Aceto, Ivan Damgård, Leslie Ann Goldberg, Magnús M. Halldórsson, Anna Ingólfsdóttir, and Igor Walukiewicz, editors, *Automata, Languages and Programming*, ICALP 2008, pages 121–132, 2008.

5 Chen Avin, Michal Koucký, and Zvi Lotker. Cover time and mixing time of random walks on dynamic graphs. *Random Structures & Algorithms*, 52(4):576–596, 2018. doi:10.1002/rsa.20752.

6 Itai Benjamini and David Wilson. Excited Random Walk. *Electronic Communications in Probability*, 8:86–92, 2003. doi:10.1214/ECP.v8-1072.

7 Leran Cai, Thomas Sauerwald, and Luca Zanetti. Random walks on randomly evolving graphs. In Andrea Werneck Richa and Christian Scheideler, editors, *Structural Information and Communication Complexity*, SIROCCO 2020, pages 111–128, 2020.

8 Colin Cooper. Random walks, interacting particles, dynamic networks: Randomness can be helpful. In Adrian Kosowski and Masafumi Yamashita, editors, *Structural Information and Communication Complexity*, SIROCCO 2011, pages 1–14, 2011.

9 Colin Cooper and Alan Frieze. Crawling on simple models of web graphs. *Internet Mathematics*, 1(1):57–90, 2003.

10 Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms*. The MIT Press, Second edition, 2001.

11 Burgess Davis. Reinforced random walk. *Probability Theory and Related Fields*, 84(2):203–229, 1990. doi:10.1007/BF01197845.

12 Amir Dembo, Ruojun Huang, Ben Morris, and Yuval Peres. Transience in growing subgraphs via evolving sets. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 53(3):1164–1180, 2017. doi:10.1214/16-AIHP751.

13 Amir Dembo, Ruojun Huang, and Vladas Sidoravicius. Monotone interaction of walk and graph: recurrence versus transience. *Electronic Communications in Probability*, 19:1–12, 2014. doi:10.1214/ECP.v19-3607.

14 Amir Dembo, Ruojun Huang, and Vladas Sidoravicius. Walking within growing domains: recurrence versus transience. *Electronic Journal of Probability*, 19:1–20, 2014. doi:10.1214/EJP.v19-3272.

15 Amir Dembo, Ruojun Huang, and Tianyi Zheng. Random walks among time increasing conductances: heat kernel estimates. *Probability Theory and Related Fields*, 175(1):397–445, October 2019. doi:10.1007/s00440-018-0894-1.

16 Oksana Denysyuk and Luís Rodrigues. Random walks on evolving graphs with recurring topologies. In Fabian Kuhn, editor, *Distributed Computing*, DISC 2014, pages 333–345, 2014.

17 Dmitry Dolgopyat, Gerhard Keller, and Carlangelo Liverani. Random walk in Markovian environment. *The Annals of Probability*, 36(5):1676–1710, 2008. doi:10.1214/07-AOP369.

18 Richard Durrett. *Probability: theory and examples*. Cambridge University Press, Fifth edition, 2019.

19 Daniel Figueiredo, Giulio Iacobelli, Roberto Oliveira, Bruce Reed, and Rodrigo Ribeiro. On a random walk that grows its own tree. *Electronic Journal of Probability*, 26:1–40, 2021. doi:10.1214/20-EJP574.

20 Venkatesan Guruswami. Rapidly mixing markov chains: A comparison of techniques (a survey), 2016. arXiv:1603.01512.

21 Ruojun Huang. On random walk on growing graphs. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 55(2):1149–1162, 2019. doi:10.1214/18-AIHP913.

22 Giulio Iacobelli, Daniel R. Figueiredo, and Giovanni Neglia. Transient and slim versus recurrent and fat: Random walks and the trees they grow. *Journal of Applied Probability*, 56(3):769–786, 2019. doi:10.1017/jpr.2019.43.

23 Shuji Kijima, Nobutaka Shimizu, and Takeharu Shiraga. How many vertices does a random walk miss in a network with moderately increasing the number of vertices? In *Proceedings of the Thirty-Second Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA 2021, pages 106–122, 2021. doi:10.1137/1.9781611976465.8.

24 Elena Kosygina and Martin PW Zerner. Excited random walks: results, methods, open problems. *Bulletin of the Institute of Mathematics Academia Sinica (New Series)*, 8(1), 2013.

25 Fabian Kuhn and Rotem Oshman. Dynamic networks: models and algorithms. *SIGACT News*, 42(1):82–96, March 2011. doi:10.1145/1959045.1959064.

26 Ioannis Lamprou, Russell Martin, and Paul Spirakis. Cover time in edge-uniform stochastically-evolving graphs. *Algorithms*, 11(10), 2018. doi:10.3390/a11100149.

27 David A. Levin and Yuval Peres. *Markov chains and mixing times*. American Mathematical Society, Second edition, 2017.

28 Russell Lyons. Random walks and percolation on trees. *The Annals of Probability*, 18(3):931–958, 1990.

29 Russell Lyons and Yuval Peres. *Probability on Trees and Networks*. Cambridge University Press, USA, 2017.

30 Othon Michail and Paul G. Spirakis. Elements of the theory of dynamic networks. *Communications of the ACM*, 61(2):72, January 2018. doi:10.1145/3156693.

31 Laurent Saloff-Coste and Jessica Zúñiga. Merging for time inhomogeneous finite Markov chains, Part I: Singular values and stability. *Electronic Journal of Probability*, 14:1456–1494, 2009. doi:10.1214/EJP.v14-656.

32 Laurent Saloff-Coste and Jessica Zúñiga. Merging for time inhomogeneous finite Markov chains, Part II: Nash and log-Sobolev inequalities. *The Annals of Probability*, 39(3):1161–1203, 2011.