Harmonious Colourings of Temporal Matchings

Duncan Adamson
Leverhulme Centre for Functional Material Design,
University of Liverpool, Liverpool, United Kingdom

Abstract

Graph colouring is a fundamental problem in computer science, with a large body of research dedicated to both the general colouring problem and restricted cases. *Harmonious colourings* are one such restriction, where each edge must contain a globally unique pair of colours, i.e. if an edge connects a vertex coloured x with a vertex coloured y, then no other pair of connected vertices can be coloured x and y. Finding such a colouring in the traditional graph setting is known to be NP-hard, even in trees. This paper considers the generalisation of harmonious colourings to *Temporal Graphs*, specifically (k, t)-*Temporal matchings*, a class of temporal graphs where the underlying graph is a matching (a collection of disconnected components containing pairs of vertices), each edge can appear in at most t timesteps, and each timestep can contain at most k other edges. We provide a complete overview of the complexity landscape of finding *temporal harmonious colourings* for (k, t)-matchings. We show that finding a *Temporal Harmonious Colouring*, a colouring that is harmonious in each timestep, is NP-hard for *(k,t)-Temporal Matchings* when $k \geq 4, t \geq 2$, or when $k \geq 2$ and $t \geq 3$. We further show that this problem is inapproximable for $t \geq 2$ and an unbounded value of k, and that the problem of determining the temporal harmonious chromatic number of a $(2, 3)$-temporal matching can be determined in linear time. Finally, we strengthen this result by a set of upper and lower bounds of the temporal harmonious chromatic number both for individual temporal matchings and for the class of (k, t)-temporal matchings.

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1 Introduction

In real-world settings, networks are changing structures with connections between vertices changing at each time step. Temporal graphs provide a natural means of modelling such a network. Formally, a *temporal graph* G is defined by a static collection of vertices V and sequence of edge sets $E_1, E_2, \dots, E_T$, where T is the *lifetime* of the graph. The underlying graph $\mathcal{G}$ of a temporal graph G is the static graph formed by taking the vertex set V and the union of edge sets $E_1 \cup E_2 \cup \dots \cup E_T$. Temporal graphs have recently become a well-studied object, with a particular focus on reachability [3, 5, 11] and exploration [6, 7, 13].

Graph colouring, despite being a fundamental problem in computer science, has remained relatively unstudied within temporal graphs. The main reason for this is that, for the general problems of vertex and edge colouring, finding such a static colouring on a temporal graph is equivalent to finding a static colouring on the underlying graph. Recent work on graph colouring in the temporal setting can be split into two broad directions. First is the work of Yu, Bar, Basu, and Ramanathan [14] and Ghosal and Ghosh [8], who focused on finding a sequence of colourings for each node, with the twin goals of minimising the total number of colours and the number of changes of the colour of each vertex. Second is the work by Mertzios, Molter, and Zamaraev [12] on *sliding window* colourings, where the goal is to provide a colouring such that each (active) edge is coloured properly at least once within

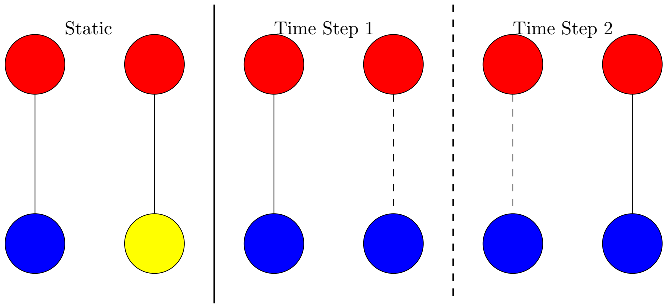

Figure 1 An example of two harmonious colourings. Left is the harmonious colouring for the static underlying graph, while the right provides a simple example of a harmonious temporal colouring. Observe that the static colouring needs three colours in order to avoid having two edges with the same pair of colours. In the temporal example, note that neither edge is active at the same time-step. Therefore, both can have the same colours.

each window of a given size. In [12], the authors prove a series of results on the hardness of this problem, as well as a number of exact and approximation algorithms for several special cases of the problem. The computational aspects of this problem have been further studied by Marino and Silva [10] who, in particular, considered the problem on temporal graphs where each edge is either active for at least t times steps in a row or at least t snapshots over the lifetime of the graph.

In this paper, we are interested in finding a *harmonious colouring* of a temporal graph. Harmonious colourings of static graphs are colourings where the colour pair on each edge is globally unique. In the static setting, this problem is known to be very challenging, with hardness results for a wide variety of otherwise simple graph classes such as interval and permutation graphs [1], bipartite permutation and quasi-threshold graphs [2], and trees [4]. Despite the challenge, the temporal setting offers a slight relaxation of the problem, namely a requirement that the pair of colours on each edge is unique only within the same snapshot. This means that any pair of edges that are not active at the same time may share a colouring. Such a colouring is called a *temporal harmonious colouring*. Figure 1 illustrates that such a colouring can be found using fewer colours than in the underlying graph.

Noting that this problem is trivially hard when any snapshot includes a graph that is known to be hard to harmoniously colour, this paper focuses on the class of *temporal matchings*. In a temporal path, the underlying graph is a matching graph. Beyond the general class, we look at (k, t)-temporal matchings, where at each snapshot the graph contains at most t active edges, and each edge can be active for at most k time steps within the lifetime of the graph. Restricting the graphs in this way allows a more precise understanding of the complexity landscape. Further, showing that this problem remains hard for perhaps the simplest non-trivial graph class highlights the difference between the complexity of problems on static graphs and temporal ones.

Our Contribution

This paper provides two main results. In Section 3, we show that the problem of finding a temporal harmonious colouring is NP-complete for $(2, 4)$-temporal matchings and $(3, 2)$-temporal matchings. This shows the problem to be highly challenging even for a relatively simple class of temporal graphs. In addition to the hardness result, we show that the problem of determining the temporal harmonious chromatic number (the minimum number of colours needed for a temporal harmonious colouring) can not be approximated within polynomial time within a factor of $n^{(1-\epsilon)/2}$ for any positive value ϵ for any $(k, 2)$-temporal paths when the value of k is unbounded. In Section 4, we show that the temporal chromatic number (the minimum number of colours needed for a temporal harmonious colouring) of $(2, 3)$-matchings can be determined in linear time. Finally, in Section 5, we provide a series of bounds on the temporal harmonious chromatic number of (k, t)-temporal paths, including a $\sqrt{4\min(k, t) - 2}$ lower bound and a $t(k - 1) + 2$ upper bound.

2 Notation and Definitions

We use the notation $[n]$ to denote the set $\{1, 2, \ldots, n\}$. For a graph G, we denote by $V(G)$ and $E(G)$ its vertex set and edge set respectively. A *temporal graph* $\mathcal{G}$ is an ordered sequence $(G_1, G_2, \ldots, G_T)$ of static graphs over the common set V of vertices. The static graphs G_i, $i \in [T]$ are called *snapshots* of $\mathcal{G}$, and T is called *the lifetime* of the temporal graph. We say that the edges in $E(G_i)$ are *active* at time step i. The *underlying graph* of the temporal graph $\mathcal{G}$ is the graph formed by taking the union of its snapshots, i.e. $(V, \bigcup_{i\in[T]} E(G_i))$.

▶ **Definition 1.** *A temporal graph $\mathcal{G}$ of lifetime T is a* (k, t)-temporal graph *if every edge of its underlying graph is active in at most t time steps, and every snapshot has at most k edges.*

If the underlying graph of $\mathcal{G}$ is a path, we say that $\mathcal{G}$ is a (k, t)-temporal path*; similarly, if the underlying graph is a matching (i.e. graph of maximum degree at most 1), we say that $\mathcal{G}$ is a* (k, t)-temporal matching*.*

For a natural number c, a *c-colouring* of a graph G is a mapping $\psi : V(G) \to [c]$ such that for any two adjacent vertices u, v we have $\psi(u) \neq \psi(v)$. If G admits a c-colouring we say that G is c-colourable. The *chromatic number* of G is the smallest c such that G is c-colourable. A c-colouring ψ of G is *harmonious* if, for every pair of edges $\{v_1, u_1\}, \{v_2, u_2\} \in E$, $\{\psi(v_1), \psi(u_1)\} \neq \{\psi(v_2), \psi(u_2)\}$. The *harmonious chromatic number* of G is the smallest value c such that G admits a harmonious c-colouring. Given a graph G and a natural number c the Harmonious Colouring problem asks whether G admits a harmonious c-colouring or not. The optimisation variant of this problem asks to find the harmonious chromatic number of G.

▶ **Definition 2.** *For a temporal graph $\mathcal{G} = (G_1, G_2, \ldots, G_T)$ over a vertex set V, a c-colouring $\phi : V \to [c]$ of its underlying graph is a* c-temporal harmonious colouring *if ψ is a harmonious colouring of every snapshot G_i, $i \in [T]$. The smallest c such that $\mathcal{G}$ admits a c-temporal harmonious colouring is the* temporal harmonious chromatic number*.*

TEMPORAL HARMONIOUS COLOURING (THC)
Input: A temporal graph $\mathcal{G}$, and an integer c.
Output: YES, if there exists a temporal harmonious c-colouring of $\mathcal{G}$; NO otherwise.

As in the static case, the Temporal Harmonious Colouring can be phrased as an optimisation problem, asking for the temporal harmonious chromatic number of the input graph $\mathcal{G}$.

Since temporal graphs generalise (static) graphs, TEMPORAL HARMONIOUS COLOURING is at least as hard as the Harmonious Colouring problem. In particular, the NP-hardness of the latter problem on trees [4] implies NP-hardness of the former on temporal graphs where snapshots are restricted to trees. On the other hand, the Harmonious Colouring problem is rather simple in matchings. Indeed, if G is a matching with m edges, then its harmonious chromatic number is the smallest c such that $\binom{c}{2} = \frac{c(c-1)}{2} \geq m$, i.e. the smallest number of colours providing at least m unique pairs of different colours.

It turns out that in the temporal setting, the problem becomes much harder, even in the simplest case of temporal graphs whose underlying graph is a matching. In particular, we will show that Temporal Harmonious Colouring is NP-complete on (k, t)-temporal matchings when $k \geq 4$ and $t \geq 2$ or when $k \geq 2$ and $t \geq 3$. On the other hand, we show that the temporal harmonious chromatic number can be determined in linear time for $(2, 3)$-temporal matchings. Thus, our results provide computational complexity dichotomy for Temporal Harmonious Colouring on (k, t)-temporal matchings. We further provide a set of bounds on the chromatic number of (k, t)-temporal paths.

3 Hardness of Harmonious Colourings on (k,t)-Temporal Matching

In this section, we show that the problem of finding a temporal harmonious colouring is NP-hard even for $(2, 4)$-temporal matchings and $(3, 2)$-temporal matching. We start by providing a tool for constructing temporal matchings from a static graph. Informally, the goal is to construct a matching with a temporal harmonious chromatic number that can be used to determine the chromatic number of the static graph for low-degree graphs. We strengthen this reduction by showing that each snapshot contains at most 2 edges.

▶ **Lemma 3.** *Let $G = (V, E)$ be a static graph with a chromatic number χ and maximum degree $\Delta > 1$, then there exists a $(2, \Delta)$-temporal matching $G' = (V', E')$ such that the temporal harmonious chromatic number of G' is χ' such that χ' is the smallest value for which $\chi \leq \frac{\chi'(\chi'-1)}{2}$. Further, G' has a lifetime of $|E|$.*

Proof. For each vertex $v \in V$, a pair of vertices v_1, v_2 are added to V'. A snapshot is constructed for each edge $(v, u) \in E$, with the edge set $E_{v,u}$ is constructed containing the edges (v_1, v_2) and (u_1, u_2). Note that as each vertex v has degree at most Δ, the edge (v_1, v_2) appears in at most Δ timesteps. Under the current construction, the graph is a matching rather than a path.

Let ψ be a temporal harmonious colouring of G'. Observe that at each snapshot, there exists exactly 2 edges, with each edge corresponding to a vertex in G and the snapshot corresponding to an edge in G. Therefore, for any colouring to be harmonious, given any edge $(v, u) \in E$, the pairs $(\psi(v_1), \psi(v_2))$ and $(\psi(u_1), \psi(u_2))$ must be distinct. Therefore, there must be a mapping from the set of distinct pairs of colours from $[\psi']$ to some set of colours of size at most $\gamma = \frac{\chi'(\chi'-1)}{2}$. Let ϕ be a γ-colouring of G such that each vertex of G is coloured using the pair of colours given by the χ' colouring of G'.

Assume, for the sake of contradiction, that ϕ is not a valid colouring of G. Then, there must exist some edge $(v, u) \in E$ such that $\phi(v) = \phi(u)$. In this case, in the colouring of G', $(\psi(v_1), \psi(v_2)) = (\psi(u_1), \psi(u_2))$. However, as there exists some snapshot of G' containing the edges (v_1, v_2) and (u_1, u_2), this contradicts the assumption that G' has a valid temporal harmonious colouring. Therefore, ϕ must be a valid colouring of G. Further, if $\gamma < \chi$, then there must exist a colouring of G using fewer than χ colours, contradicting the assumption that χ is the chromatic number of G.

In the other direction, let ψ be a χ-colouring of G. A colouring ψ' of G' is constructed via a bijective mapping $\lambda : [\chi] \mapsto \{(x, y) | x, y \in [\chi'], x > y\}$. Assume, for the sake of contradiction, that ϕ' is not a valid colouring of G'. Then, there must exist some snapshot at time step i such that for the pair of edges $(v_1, v_2), (u_1, u_2) \in E_i$, $(\phi(v_1), \phi(v_2)) = (\phi(u_1), \phi(u_2))$.

Following the above construction, such a snapshot must correspond to an edge $(v, u) \in E$. As ψ is a valid χ-colouring of G $\psi(v) \neq \psi(u)$. Therefore, the $\lambda(\psi(v)) \neq \lambda((\psi(u)))$, and hence $(\psi'(v_1), \psi'(v_2)) \neq (\psi'(u_1), \psi'(u_2))$, contradicting the assumption that ψ' is not a valid colouring. Further, χ' must be the smallest value such that $\chi \leq \frac{\chi'(\chi'-1)}{2}$. ◀

▶ **Theorem 4.** *The problem of determining if a given (k, t)-temporal matching $\mathcal{G}$ has a temporal harmonious chromatic number of c is NP-complete for any $k \geq 2$ and $t \geq 4$.*

Proof. Let $G = (V, E)$ be a 3-regular graph. As established by Leven and Galil [9], determining if G has an edge colouring of size 3 is an NP-complete problem. In order to reduce the problem of finding an edge colouring of G to finding a temporal harmonious colouring on a $(2, 4)$-temporal graph, let H be the edge adjacency graph of G. Note that H has a maximum degree of 4. Using Lemma 3, H can be transformed into a $(2, 4)$-temporal matching G'.

If H has a chromatic number of 3, then the temporal harmonious chromatic number of G' is exactly 3. Otherwise, if the chromatic number of H is either 4 or 5, the temporal harmonious chromatic number of G' is 4. Therefore, any algorithm to determine the temporal harmonious chromatic number of a $(2, 4)$-temporal matching can also determine if a 3-regular graph has a 3-edge colouring. Hence finding the temporal harmonious chromatic number of a $(2, 4)$-temporal matching is NP-hard.

To show that the problem is NP-Complete, note that any colouring can be verified as a temporal harmonious colouring in polynomial time. ◀

▶ **Corollary 5.** *The problem of finding the temporal harmonious chromatic number of a (k, t)-temporal path is NP-hard for any $t \geq 2, k \geq 4$.*

▶ **Corollary 6.** *The problem of finding the temporal harmonious chromatic number of a (k, t)-temporal cycle is NP-hard for any $t \geq 2, k \geq 4$.*

Building on the above results, we now show that determining the harmonious chromatic number of a $(3, 2)$-temporal path is NP-complete.

▶ **Theorem 7.** *The problem of determining if a given (k, t)-temporal matching $\mathcal{G}$ has a temporal harmonious chromatic number of c is NP-complete for any $k \geq 3$ and $t \geq 2$.*

Proof. This proof follows a similar outline to the proof of Theorem 4. As before, we take a cubic graph $G = (V, E)$ and construct a $(3, 2)$-temporal matching G', such that the edge-chromatic number of G is equal to the harmonious temporal harmonious chromatic number of G'. The temporal matching G' is constructed as follows. For each edge $e \in E$, a pair of vertices e_1, e_2 are constructed and connected in the underlying graph of G'. For each vertex $v \in V$, a snapshot is constructed containing the pair of vertices (e_1, e_2) for every edge e incident to v. As in Lemma 3, this matching may be transformed into a path by adding dummy vertices between the pairs.

To show that this problem is NP-hard, first assume that G has an edge chromatic number of 3. For the sake of contradiction, assume further that the temporal harmonious chromatic number of G' is greater than 3. Let ψ be an edge colouring of G using 3 colours, and let $f(c) = \begin{cases} (1,2) & c = 1 \\ (1,3) & c = 2 \\ (2,3) & c = 3 \end{cases}$ be a function mapping the colours given by ψ to pairs of colours. For each edge $e \in E$, let the colours of e_1 and e_2 be equal to the colours given by $f(\psi(e))$. As the mapping f does not allow any pair of incident vertices to share a colour, this colouring must be valid. Further, for each snapshot in G', note that the only active edges are those corresponding to edges incident to a given vertex in V. Therefore, any given snapshot is not harmonious if and only if there is a pair of edges in G sharing a colour, contradicting the assumption that ψ is a valid colouring of G. Hence, G' has a temporal harmonious chromatic number greater than 3 if and only if G does not have an edge colouring of size 3.

In the other direction, assume for the sake of contradiction that the temporal harmonious chromatic number of G' is 3, while the edge chromatic number of G is greater than 3. Let ϕ be a vertex colouring of G' and $f'(c) = \begin{cases} 1 & (1,2) \text{ or } (2,1) \\ 2 & (1,3) \text{ or } (3,1) \\ 3 & (2,3) \text{ or } (3,2) \end{cases}$ be a function mapping pairs of vertex colours to a set of 3 colours. For each edge $e \in E$, let e be coloured $f'(\phi(e_1), \phi(e_2))$. For the edge chromatic number of G to be greater than 3, this colouring must not be feasible. Observe that at each snapshot of G', the colouring is harmonious. Therefore, given any pair of edges $e, h \in E$ such that e and h are incident to the vertex v, the pairs $(\psi(e_1), \psi(e_2))$ and $(\psi(h_1), \psi(h_2))$ must be distinct. Hence, $f(\psi(e_1), \psi(e_2)) \neq f(\psi(h_1), \psi(h_2))$ and by extension the edges coloured using this mapping must be distinct. Therefore using the mapping given by f, either G has a proper edge colouring with 3 colours, or G' does not have a temporal harmonious chromatic number of 3, contradicting the original assumption.

Hence, the temporal harmonious chromatic number of G' is 3 if and only if the edge chromatic number of G is 3. By extension the problem of computing the temporal harmonious chromatic number of a $(3, 2)$-temporal matching is NP-hard. Further, any colouring can be verified as a temporal harmonious colouring in polynomial time. Therefore the problem of computing the temporal harmonious chromatic number of a $(3, 2)$-temporal matching is NP-complete. ◀

▶ **Corollary 8.** *The problem of finding the temporal harmonious chromatic number of a (k, t)-temporal path is NP-hard for any $t \geq 3, k \geq 2$.*

▶ **Corollary 9.** *The problem of finding the temporal harmonious chromatic number of a (k, t)-temporal cycle is NP-hard for any $t \geq 3, k \geq 2$.*

3.1 Hardness of Approximation

Building on Theorem 4, this section shows that the temporal harmonious colouring problem is hard to approximate even on (k, t)-temporal matchings. This bound utilises the tools of Lemmas 3 as a basis for converting existing results on the inapproximability of colouring problems to the temporal harmonious setting. In this section, we consider the more general class of $(\infty, 2)$-temporal matchings, where there is no bound on the number of times each edge appears in the graph. By focusing on the restricted case of $(\infty, 2)$-temporal graphs, we show that the general case is at least as hard, and indeed likely to be much harder.

▶ **Theorem 10.** *It is NP-hard to approximate the temporal harmonious number of a $(\infty, 2)$-temporal matching within a factor of $n^{(1-\epsilon)/2}$ for any $\epsilon > 0$, where n is the number of vertices in the graph.*

Proof. From Zuckerman [15], it is known that it is NP-hard to approximate the harmonious number of a graph G within a factor of $n^{1-\epsilon}$, for any $\epsilon > 0$. Following the construction given in Lemma 3, given any graph G with maximum degree Δ, and chromatic number χ, a $(2, \Delta)$-temporal matching can be constructed with a temporal harmonious number γ such that γ is the smallest value satisfying $\chi \leq \frac{\gamma(\gamma-1)}{2} \leq \gamma^2$.

Let α be a polynomial time approximation of χ, and β be a polynomial time approximation of γ. Following [15], $\alpha \geq n$. Similarly, note that any approximation of γ provides an upper bound of χ using the inequality $\gamma^2 \geq \alpha$. Hence β is the smallest value such that $\beta^2 \geq \alpha$ and by extension $\beta \geq \sqrt{\alpha}$. Therefore, any $\sqrt{n^{1-\epsilon}}$ approximation of γ provides a $n^{1-\epsilon}$ approximation of χ. Therefore, γ can not be approximated in polynomial time within $n^{(1-\epsilon)/2}$ for any positive value ϵ for the class $(\infty, 2)$-temporal matchings unless $P = NP$. ◀

4 Temporal Harmonious Chromatic Number of (2,3)-Temporal Matchings

In this section, we strengthen the hardness result from Section 3 by showing that the temporal harmonious chromatic number of $(2, 3)$-temporal matchings can be determined in linear time. Note that any $(2, 2)$-temporal matching is also a $(2, 3)$-temporal matching, and thus this linear bound also holds. This shows that the hardness bound from Section 3 is "tight", in the sense that the case of $(2, 4)$ -temporal matchings and $(3, 2)$-temporal matchings are the smallest values of k and t for which the problem is hard. Further, this highlights a large gulf in the complexity space, moving from a problem that is solvable in linear time to an NP-complete problem with a relatively small change in the parameters.

We start with the simple case of finding a temporal harmonious colouring of a $(1, k)$-temporal matching.

▶ **Lemma 11.** *A temporal harmonious colouring of any $(1, k)$-temporal matching with n vertices and 2 colours can be found in $O(n)$ time.*

Proof. Observe that in a $(1, k)$-temporal matching, each snapshot contains at most 1 edge. Therefore, any valid colouring is also a temporal harmonious colouring. As the underlying graph is a matching, a 2-colouring can be found by a greedy algorithm, iterating over the set of edges and colouring one end node colour 1, and the other node colour 2. ◀

We now provide the main result of this section, namely a proof that the temporal harmonious chromatic number of $(2, 3)$-temporal matchings can be determined in linear time. The high-level idea behind this proof is to provide a construction of the *temporal edge adjacency graph* of the temporal matching $\mathcal{G}$. Informally, such a graph represents edges with vertices and connects them if and only if the corresponding edges are active in the same snapshot. By finding a colouring of this graph, a mapping can be used to connect the colours of the edges to the colours of vertex pairs. As the temporal edge adjacency graph has a maximum degree of 3, the chromatic number of the graph can be determined in linear time using Brooks' Theorem.

▶ **Theorem 12.** *The temporal harmonious chromatic number of a $(2, 3)$-temporal matching $\mathcal{G}$ can be determined in linear time.*

Proof. At a high level, this is done by reversing the construction from Lemma 3. We assume, without loss of generality, that $\mathcal{G}$ does not contain any vertices of degree 0. Note that any such vertices may be coloured arbitrarily without conflicting with the temporal harmonious colouring condition. Let $G' = \{V', E'\}$ be the *temporal edge adjacency graph* of $\mathcal{G}$. Formally, G' is constructed as follows. For every edge $e \in \mathcal{G}$, a vertex is constructed in V' and labelled with the edge e. Given a pair of edges $e_1, e_2 \in \mathcal{G}$, an edge is constructed between v_{e_1} and v_{e_2} if and only if there exists some snapshot of $\mathcal{G}$ in which both e_1 and e_2 are active. Note that for a $(2, 3)$-temporal matching, each edge appears in at most 3 snapshots, and each time step contains at most 2 edges. Therefore, G' has a degree of at most 3.

Let ψ be a colouring of G' using c-colours. Let c' be the smallest value such that $c \leq \frac{c'(c'-1)}{2}$. A temporal harmonious c'-colouring ϕ of $\mathcal{G}$ is constructed from ψ by constructing a mapping f from c to the set $\{\{x,y\}|x,y \in [c], x \neq y\}$. Using this mapping, each edge $e \in \mathcal{G}$ is coloured using $f(v_e)$. We first show that ϕ is a valid c'-colouring. Observe that each pair in $\{\{x,y\}|x,y \in [c], x \neq y\}$ contains two unique colours from $[c']$, and further each vertex belongs to only a single edge. Therefore, by assigning the colours from the pair $f(v_e)$ to the two vertices, ϕ produces a valid c'-colouring of $\mathcal{G}$. Further, as each pair of edges $e_1, e_2 \in \mathcal{G}$ that are active in the same snapshot are assigned different colours by ψ, the pair of colours on the vertices incident to e_1 and e_2 are distinct. Therefore, ψ is a temporal harmonious c'-colouring. Therefore, given a c-colouring of the edge temporal graph G, a c' colouring of $\mathcal{G}$ can be determined in polynomial time.

As G' is a cubic graph, G' has a chromatic number of 4 if and only if G' contains the complete graph K_4. As the clique K_4 can be detected in linear time, it is possible to determine if G' has a chromatic number of 4 in polynomial time. By extension, if G' has a chromatic number of 4, then the smallest value c' such that $\frac{c'(c'-1)}{2} \geq 4$ is 4. Hence the temporal harmonious chromatic number of $\mathcal{G}$ is 4 if and only if G' contains the graph K_4 as a subgraph. On the other hand, G' has a chromatic number of 2 if and only if it is bipartite, and further, it is possible to determine this in linear time. If G' is bipartite, then the temporal harmonious chromatic number c' of $\mathcal{G}$ is 3 as 3 is the smallest value such that $\frac{c'(c'-1)}{2} \geq 2$. Further, G' has a chromatic number of 1 if and only if G' contains only disconnected vertices. In this case, no pair of edges in $\mathcal{G}$ are active at the time step. Therefore any proper colouring of $\mathcal{G}$ is also a temporal harmonious colouring. Finally, if G' does not have a chromatic number of $1, 2$ or 4, then the chromatic number of G' must be 3, and by extension, the temporal harmonious chromatic number of $\mathcal{G}$ is 3. Therefore, the temporal harmonious chromatic number of $\mathcal{G}$ can be determined in linear time for any $(2, 3)$-temporal matching. ◀

5 Further Bounds

This section strengthens the results of Section 3 by providing stronger bounds on the temporal harmonious chromatic number of (k,t)-*temporal paths*. Note that any bounds on (k,t)-temporal paths also apply to (k,t)-matchings. This is done in two ways. First, we provide an upper bound by constructing a linear time greedy algorithm for finding a temporal harmonious colouring using at most $t(k-1)+2$ colours. Secondly, we provide a series of lower bounds to strengthen the upper bound.

▶ **Lemma 13.** *Algorithm 1 finds a $(t(k-1)+2)$-colouring of any (k,t)-temporal path $G = (V, E_1, E_2, \ldots, E_T)$ in $O(n \cdot k \cdot t)$ time, where n is the number of vertices in G.*

Proof. We assume, without loss of generality, that each node in G is labelled from 1 to n such that vertex 1 is a terminal vertex on the path $\mathcal{G}$ and vertex i is incident to $i+1$ for every $i \in [n-1]$. Note that the first vertex can be arbitrarily coloured in the first step without violating the colouring constraint. Similarly, the second vertex can be coloured any colour other than the colour of vertex 1. The remaining vertices are coloured in order from 3 to n. At each step, we treat the vertex as though were the terminal vertex. In doing so, it becomes only necessary to check that the edge $(i-1, i)$ satisfies the harmonious condition and that the colour of i is distinct from $i-1$ to satisfy the colouring condition. Therefore, by an exhaustive search of each previous edge, a list of colours that can be allowed at position i can be determined.

As there are at most t time steps in which the edge $(i-1, i)$ is active, and at most k edges at each snapshot, $(i-1, i)$ can conflict with at most $t(k-1)$ edges. Further, for each edge $(j-1, j)$ appearing at the same timestep as $(i-1, i)$, the colour $col(j-1)$ is removed

Algorithm 1 Greedy Algorithm.

```
procedure Colour(G = (V, E_1, E_2, ..., E_T), c)
    col(1) ← 1
    col(2) ← 2.
    for i ∈ V \ {1,2} do
        Colours ← {1,2,...c} \ {col(i − 1)}
        for k ∈ ActiveTimesteps((i − 1, i)) do
            for (j − 1, j) ∈ E_k do
                if ((j, j + 1), (i − 1, i)) ∈ E_k then
                    if col(j) = col(i − 1) then
                        Colours ← Colours \ {col(j + 1)}
                    end if
                    if col(j + 1) = col(i − 1) then
                        Colours ← Colours \ {col(j)}
                    end if
                end if
            end for
        end for
        col(i) ← min(Colours)
    end for
end procedure
```

from the set of potential colours of i if and only if $col(j) = col(i-1)$. Similarly, the colour $col(j)$ is removed from the set of candidate colours of i if and only if $col(j-1) = col(i-1)$. As $col(j) \neq col(j-1)$, at most 1 colour can be removed for each edge that appears in the same timestep as $(i-1, i)$. Hence at most $t(k-1)+1$ colours are forbidden for i, therefore as long as i has a palette of size at least $t(k-1)+2$, there must always be at least one colour that i can choose. Therefore, G must have a $(t(k-1)+2)$ colouring.

To get the time complexity, we assume that each edge is labelled with the time step at which it appears. Note that such a list can be computed by checking the set of active edges for each snapshot in the graph. As there are at most $n-1$ edges, each of which are active for at most t time steps, this will take at most $O(t \cdot n)$ time. For each snapshot, at most k edges need to be checked for each vertex. Therefore, as there are at most k time steps at which each edge is active, the total complexity of this algorithm is $O(n \cdot k \cdot t)$. ◀

Finally, we provide a lower bound on the temporal harmonious chromatic number of the class of (k,t)-temporal matchings of $\sqrt{8(\min(k,t)-1)}$. In doing so, we provide a clear gap between the upper and lower bounds and leave open the question of the optimal bound for $(2,2)$-temporal matchings and $(2,3)$ temporal matchings. Further, we provide a lower bound of $\sqrt{t}$ on the temporal harmonious chromatic number for any (k,t)-temporal matching.

▶ **Lemma 14.** *For any $k, t \in \mathbb{N}$, there exists a (k,t)-temporal matching that has a temporal harmonious chromatic number of $\sqrt{4\min(k,t)-2}$.*

Proof. This lemma is proven by constructing a $(\min(k,t), \min(k,t))$-temporal matching $G = (V, E_1, E_2, \dots, E_T)$ with a temporal harmonious chromatic number of $\sqrt{4\min(k,t)-2}$. We assume without loss of generality that $k = t$. Note that any $(\min(k,t), \min(k,t))$-temporal matching is also a (k,t)-temporal matching. The set V is constructed by forming two sets $\mathcal{T}$ and $\mathcal{K}$. The set $\mathcal{T}$ contains t vertices labelled $v_1, v_2, \dots, v_t$. The set $\mathcal{K}$ contains k vertices labelled $u_1, u_2, \ldots, u_t$. For every $i \in [t-1]$, an edge is constructed between v_i and v_{i+1}. Similarly, for every $j \in [k-1]$, an edge is constructed between u_j and u_{j+1}. Finally, an edge is constructed between v_t and u_1.

The first k snapshots are constructed by having every edge of the form (v_i, v_{i+1}) active, as well as exactly one edge of the form (u_j, u_{j+1}). Formally, for $l \in [k]$, the time step l contains has the active edges $\{(v_1, v_2), (v_2, v_3), \ldots, (v_{t-1}, v_t), (u_l, u_{l+1})\}$. Note that each such step contains exactly t members. The final snapshot contains every edge between members of $\mathcal{K}$ and the edge (v_t, u_1).

To determine the chromatic number of G, observe that by construction, every edge in G requires a different pair of colours. As there are $k + t - 1 = 2k - 1$ edges in G, the temporal harmonious chromatic number of G, χ must be satisfy $2k - 1 \leq \frac{\chi(\chi-1)}{2} \leq \frac{\chi^2}{2}$. Hence $4k - 2 \leq \chi^2$ and by extension $\chi \geq \sqrt{4k-2}$- In the general case, when $k \neq t$, this can be rewritten as $\sqrt{4\min(k,t) - 2}$ ◀

▶ **Lemma 15.** *For any (k,t)-temporal matching G that is not also a $(k-1,t)$-temporal matching, the temporal harmonious chromatic number of G is at least $\sqrt{2 \cdot k}$.*

Proof. Note that if G is not a $(k-1,t)$-temporal matching, then there must be at least one snapshot containing k edges. Let χ be the temporal harmonious chromatic number of G. Therefore, to colour every edge in this timestep with a unique pair of colours, χ must satisfy $\frac{\chi(\chi-1)}{2} \geq k$. Hence $\chi^2 \geq 2k$ and by extension $\chi \geq \sqrt{2k}$. ◀

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