On the Number of Digons in Arrangements of Pairwise Intersecting Circles

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Abstract

A long-standing open conjecture of Branko Grünbaum from 1972 states that any arrangement of n pairwise intersecting pseudocircles in the plane can have at most $2n-2$ digons. Agarwal et al. proved this conjecture for arrangements in which there is a common point surrounded by all pseudocircles. Recently, Felsner, Roch and Scheucher showed that Grünbaum's conjecture is true for arrangements of pseudocircles in which there are three pseudocircles every pair of which creates a digon. In this paper we prove this over 50-year-old conjecture of Grünbaum for any arrangement of pairwise intersecting circles in the plane.

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1 Introduction

A family of *pseudocircles* is a set of closed Jordan curves such that every two of them are either disjoint, intersect at exactly one point in which they touch or intersect at exactly two points in which they properly cross each other. The bounded regions whose boundaries are the pseudocircles are called *pseudodiscs*. An *arrangement* of pseudocircles is the cell complex into which the plane is decomposed by the pseudocircles and consists of *vertices*, *edges* and *faces*. If there are two points that lie on every pseudocircle, then the arrangement is *trivial*. If there is no point that lies on three pseudocircles, then the arrangement is *simple*.

A *digon* is a face whose boundary consists of two edges. It is easy to see that there are $2n$ digons in a trivial arrangement of n pseudocircles, for $n > 1$. More than 50 years ago Grünbaum conjectured that non-trivial arrangements of *pairwise intersecting* pseudocircles have fewer digons.

▶ **Conjecture 1** (Grünbaum's digon conjecture [8, Conjecture 3.6])**.** *Every non-trivial arrangement of n pairwise intersecting pseudocircles has at most $2n - 2$ digons.*

Grünbaum's conjecture is still open, however, some special cases were settled. Agarwal et al. [1] proved the conjecture for *cylindrical* arrangements, that is, for arrangements in which there is a region which is contained in each pseudocircle. Recently, Felsner, Roch and Scheucher [7] showed that the conjecture also holds for simple arrangements in which there are three pseudocircles such that every two of them form a digon. Here we prove that the conjecture holds for simple arrangements of circles.

▶ **Theorem 2.** *Every non-trivial simple arrangement of n pairwise intersecting circles has at most $2n - 2$ digons.*

The bound in Theorem 2 is tight, as can be seen from the construction in Figure 1 (taken from [8]). For better readability this construction is presented using pseudocircles, however, it can be easily implemented using circles.

1.1 Lenses and lunes

Consider an arrangement of $n > 2$ pseudocircles. If a pseudocircle C contains an edge which is part of the boundary of a digon, then we say that C (resp., the pseudodisc that is bounded by C) *supports* that digon. Thus, every digon is supported by two pseudocircles (resp., pseudodiscs). By taking a suitable inversion of the plane that is centered in a non-digon face of the arrangement, we might assume that the unbounded face of the arrangement is not a digon. Note that an inversion maps digons to digons, hence the number of digons did not change.

Therefore each digon is contained in at least one circle. We distinguish two different types of digons. A *lens* is a digon that is equal to the intersection of the two pseudodiscs supporting it, whereas a *lune* is a digon that is equal to the difference of the two pseudodiscs supporting it. See Figure 2 for an example where the blue digons are lunes and the red digons are lenses.

1.2 Digons and touching points

It is easy to see that in a simple arrangement of pseudocircles one can turn every digon into a touching point between the two pseudocircles that support this digon and vice versa. Note that this does not hold for non-simple arrangements. Indeed, one can easily construct n (pseudo)circles that are all pairwise touching at a single common point, whereas the number of digons is always sub-quadratic (see below). Still, we believe that the following strengthening of Grünbaum's conjecture holds.

▶ **Conjecture 3.** *Every arrangement of n pairwise intersecting pseudocircles contains at most $2n - 2$ touching points.*

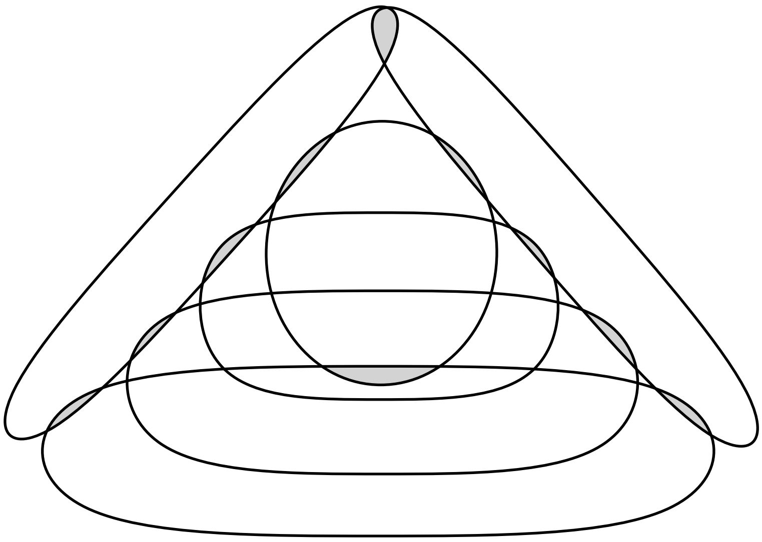


Figure 1 An arrangement of n (pseudo)circles which has $2n-2$ digons.

Note that for simple arrangements the number of touching points and the number of touching pairs are the same. As for arrangements of circles, one has to be a little more careful. Suppose that $\mathcal{A}$ is a simple arrangement of circles that contains some touching pairs of circles. Let C be the smallest circle in $\mathcal{A}$ that touches some other circle. Thus, C cannot encircle any of the circles that it touches. Therefore, by slightly "inflating" C, every touching point between C and another circle becomes a digon, no other touching points are destroyed, no new touching points are introduced and the arrangement remains simple. Continuing in this manner, we obtain a simple arrangement $\mathcal{A}'$ without touching points such that the number of touching points in $\mathcal{A}$ is at most the number of digons in $\mathcal{A}'$. Hence, by Theorem 2 we have:

▶ **Corollary 4.** *Every simple arrangement of n pairwise intersecting circles has at most $2n-2$ touching pairs of circles.*

1.3 Related work

Alon et al. [2] proved that every arrangement of n pairwise intersecting circles contains $O(n)$ digons. Specifically, they showed that such an arrangement contains at most $2n-2$ lunes and at most $18n$ lenses. Hence, Theorem 2 improves their $20n-2$ upper bound. A worse yet still linear upper bound was proved in [1] for pairwise intersecting pseudocircles. For arrangements of pairwise intersecting *unit circles*, Pinchasi [14] proved that there are at most n lenses and at most 3 lunes, hence at most $n+3$ digons.

Concerning arrangements of (pseudo)circles that are not necessarily pairwise intersecting, it is well known that one can construct such arrangements of n circles with $\Omega(n^{4/3})$ digons based on a famous construction of Erdős of n lines and n points admitting that many point-line incidences, by replacing points with small circles and lines with very large circles (see [11]). The best known upper bound for pseudocircles is $O(n^{3/2}\log n)$ by Marcus and Tardos [10]. A slightly better upper bound of $O(n^{3/2})$ for the number of touching points among n circles follows from a result of Ellenberg, Solymosi and Zahl [5]. For unit circles counting the number of tangencies is equivalent to the famous unit distance problem of Erdős, as among the center points of the unit circles exactly those are at distance 2 whose corresponding circles touch. For this problem the best known lower and upper bounds are $\Omega(n^{1+c/\log\log n})$ [6] and $O(n^{4/3})$ [13, 16, 17], respectively.

2 More Terminology and Tools

We prove Theorem 2 by constructing a graph G whose vertices and edges correspond to circles and pairs of circles forming digons, respectively, and then bounding the number of edges in G. After establishing several properties of G, we may conclude the proof in two different ways. The first and shorter way is by showing that G has a *parity embedding* in the real projective plane, therefore it can be drawn as a *generalized thrackle* which implies the required upper bound on its size. The second way uses a more basic tool, namely, the *Hanani-Tutte theorem*. This way is somewhat longer, however, yields a more general result that might be of independent interest (see Theorem 9 below). Next we present the definitions and results that will be used later for proving Theorem 2.

A *topological graph* is a graph drawn in the plane such that its vertices are drawn as distinct points and its edges as Jordan arcs connecting the corresponding points. Apart from its endpoints an edge of a topological graph may not contain any drawn vertex. Furthermore, every two edges in a topological graph intersect at a finite number of points, each of which is either a common endpoint or a crossing point.

The *rotation* of a vertex v in a topological graph is the clockwise cyclic order of the edges incident with v. The *rotation system* of a topological graph is the set of rotations of all its vertices.

▶ **Theorem 5** (Strong Hanani-Tutte theorem [18]). *A graph is planar if and only if it can be drawn as a topological graph in which every two non-adjacent edges cross an even number of times.*

▶ **Theorem 6** (Weak Hanani-Tutte theorem [3, Lemma 3]). *A topological graph in which every two edges cross an even number of times can be embedded (that is, drawn crossing-free) in the plane keeping the same rotation system at its vertices.*

A topological graph is a *generalized thrackle* if every pair of its edges intersects an odd number of times. We use the following result of Cairns and Nikolayevsky [3] and also adapt its proof for our purposes.

▶ **Theorem 7** ([3]). *A generalized thrackle with n vertices has at most $2n-2$ edges.*

Following [4], we say that a closed curve on a surface is *two-sided* (resp., *one-sided*) if the local orientation of the surface is preserved (resp., reversed) when we complete a circuit of the curve. A *parity embedding* of a graph into a surface maps simple cycles of even length to two-sided curves and simple cycles of odd length to one-sided curves. Cairns and Nikolayevsky [4] proved the following.

▶ **Theorem 8** ([4]). *A graph can be drawn in the plane as a generalized thrackle if and only if it has a parity embedding on the real projective plane.*

As the Hanani-Tutte theorem concerns topological graphs in which every pair of edges crosses an even number of times and in generalized thrackles every pair of edges intersects an odd number of times, it is no surprise that the two are closely related. In fact, Theorem 6 was used in the proof Theorem 7. En-route of proving Theorem 2 we prove the following result about topological graphs in which some pairs of edges cross evenly and some cross oddly.

▶ **Theorem 9.** *Let H be an n-vertex topological graph such that its vertex set can be partitioned into two subsets X and Y and its edge set can be partitioned into two subsets of red edges and blue edges such that the following properties hold:*

(1) *Every blue edge connects a vertex in X and a vertex in Y;*
(2) *every red edge connects two vertices in Y;*
(3) *each blue edge crosses every other edge an even number of times; and*
(4) *every two red edges cross an odd number of times.*

Then H has at most $2n-2$ edges. If H is bipartite, then it has at most $2n-4$ edges.

Note that if H has no blue edges, then it is a generalized thrackle and has at most $2n-2$ edges by Theorem 7. If, on the other hand, H has no red edges, then it is a bipartite planar graph by Theorem 5 and therefore has at most $2n-4$ edges. Conversely, every generalized thrackle and bipartite planar graph has the properties required by Theorem 9. Since both of these bounds are tight, so are the bounds in Theorem 9.

3 Proof of Theorem 2

For $n \leqslant 2$ the statement is trivial. Let $\mathcal{C}$ be a family of $n > 2$ pairwise intersecting circles such that there is no point that lies on three of them. Without loss of generality we assume that every circle in $\mathcal{C}$ supports some digon. Since $n > 2$ there is a non-digon face of the arrangement. By applying a generic inversion centered in a non-digon face we might assume and that the unbounded face of the arrangement is not a digon and furthermore that no three of the circles have collinear centers. Consider the geometric graph G whose vertices are the centers of the circles in $\mathcal{C}$. We connect two centers by a straight line segment if the corresponding two circles create a digon. We will assume without loss of generality that G is a connected graph.

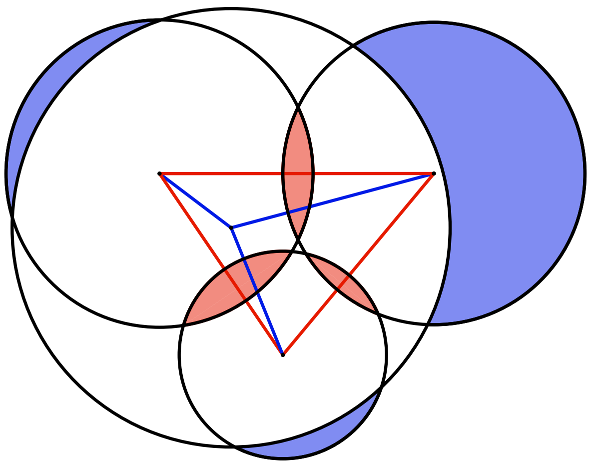


Figure 2 An arrangement of circles and the corresponding geometric graph. Red edges correspond to lenses whereas blue edges correspond to lunes.

We say that a circle in $\mathcal{C}$ is *exterior* and we call the corresponding vertex in G an *exterior vertex*, if it supports a digon (necessarily a lune) that it does not surround. We say that a circle in $\mathcal{C}$ is *interior*, and we call the corresponding vertex in G an *interior vertex*, if it supports a digon that it surrounds.

We notice the following two simple observations.

▷ Claim 10. Every circle in $\mathcal{C}$ is either interior or exterior.

Proof. Suppose that there is a circle C that supports one digon which it does not surround and another digon which it surrounds. Let C' be the other circle that supports the former digon and let C'' be the other circle that supports the latter digon. Then it is easy to see that C' and C'' do not intersect. ◁

We color blue the edges in G that correspond to lunes, and we color red the edges in G that correspond to lenses. Note that each blue edge connects an interior vertex to an exterior vertex, and the red edges run between interior vertices.

▷ Claim 11. Every odd cycle in G must contain an odd number of red edges.

Proof. Consider an odd cycle C in G. Because the set of blue edges in G forms a bipartite graph, C must contain a red edge. Consider a maximal path of blue edges in C. The path alternates between interior and exterior vertices and the two extreme vertices must be interior vertices since they are endpoints of red edges. This implies that the number of blue edges in every maximal blue path is even. Consequently, the number of blue edges in C is even and the number of red edges in C must be odd. ◁

We will rely on the following two results about the red and blue edges in G. The first result is from a paper by Alon, Last, Sharir, and Pinchasi [2]. The second result appears in a recent paper by Pinchasi [15].

▶ **Lemma 12** ([2, Lemma 3.3, Theorem 3.1]). *No two blue edges cross each other. Consequently, G has at most $2n-4$ blue edges as they form a planar bipartite graph.*

Two edges in a geometric graph are called *avoiding* if they are two opposite edges of a convex quadrilateral. Or in other words, if no line containing one edge crosses the other edge.

▶ **Lemma 13** ([15]). *No two red edges in G are avoiding.*

We notice that as a consequence of Lemma 13, G has at most $2n-2$ red edges, by a result of Katchalsky, Last, and Valtr [9, 19].

We will also need the following lemma whose proof is postponed to Section 5.

▶ **Lemma 14.** *There cannot be a red edge and a blue edge in G that are disjoint such that the line containing the red edge crosses the blue edge.*

Figure 3 shows the local restrictions implied by Lemmas 12, 13 and 14.

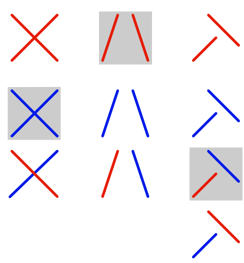


Figure 3 The configurations with gray background are forbidden in G.

We may now conclude the theorem in two different ways. First, suppose that G is drawn in the real projective plane and replace every red edge with its complement. That is, if $\{u,v\}$ is a red edge which is drawn as the straight line segment uv and ℓ is the line containing uv, then we replace uv with $\ell \setminus uv$. Denote by G' the new drawing and note that it follows from Lemmas 12, 13 and 14 that G' is an embedding of G, that is, it contains no crossing edges (consult Figure 3). Moreover, it follows from Claim 11 that G' is a parity embedding of G and therefore G can be drawn as a generalized thrackle by Theorem 8. Thus, G has at most $2n-2$ edges by Theorem 7. This completes the proof of Theorem 2.

For an alternative way of finishing the proof of Theorem 2, we use a projection of the plane on a sphere S touching the plane in a point not part of the drawing of G, through the center of S. We get a drawing of G on the lower hemisphere of S in which every edge is represented by a great arc. Those arcs inherit the color of the original edges in G. We now replace every red arc on S by its complement on the great circle in S containing it. Finally, we project the resulting drawing on the plane through the north pole of S (note that it cannot be part of any arcs due to our choice of the sphere).

The resulting topological graph that we denote by H has the following properties: (1) Every blue edge connects an interior vertex and an exterior vertex; (2) every red edge connects two interior vertices; (3) no two blue edges cross; (4) a red edge cannot cross a blue edge (because of Lemma 14); and (5) every two red edges, including two red edges with a common vertex, cross precisely once (because of Lemma 13). Theorem 2 now follows from Theorem 9 since H satisfies the properties mentioned in the statement of the latter.

4 Proof of Theorem 9

Let H be an n-vertex topological graph whose edges are colored by red and blue and whose vertices are partitioned into two subsets X and Y such that the properties in the statement of the theorem hold. That is, every blue edge connects a vertex in X and a vertex in Y; every red edge connects two vertices in Y; each blue edge crosses every other edge an even number of times; and every pair of red edges crosses an odd number of times.

We consider first the case that H is bipartite and show that H is planar and therefore has at most $2n-4$ edges. Let $A \cup B$ be the partitioning of the vertices of H, as a bipartite graph. Let $A_Y = A \cap Y$, $A_X = A \cap X$, $B_Y = B \cap Y$ and $B_X = B \cap X$. Since red edges connect vertices in Y and each blue edge connects a vertex in Y and a vertex in X, the structure of the graph H is as follows: all the red edges are between vertices in A_Y and B_Y whereas the blue edges are between A_Y and B_X or between A_X and B_Y.

Next, we use a standard redrawing procedure for turning odd crossings into even crossings (see, e.g., [12]). Namely, we distort H in the plane without creating or removing crossings in such a way that in the resulting drawing of H the x-axis separates $A_X \cup B_Y$ and $A_Y \cup B_X$. Note that every red edge crosses the x-axis an odd number of times whereas every blue edge crosses the x-axis an even number of times. Then we further change the drawing of H by flipping horizontally (about the y-axis) the half-plane bounded below the x-axis without cutting the edges in H that cross the x-axis. We claim that in the resulting distortion of H every two edges cross an even number of times, therefore, by the strong Hanani-Tutte theorem (Theorem 5) H is a planar graph.

Indeed, let e_1 and e_2 be two edges and let k_i denote the number of times the edge e_i crosses the x-axis. If we ignore the other edges, then the flipping introduces $\binom{k_1+k_2}{2}$ new crossings from which $\binom{k_1}{2}$ are self-crossings of e_1 and $\binom{k_2}{2}$ are self-crossings of e_2.[1] Therefore,

[1] We usually do not allow self-crossing edges in a topological graph, however, such crossings can be easily eliminated by redrawing the edges near every self-crossing point.

there are $\binom{k_1+k_2}{2} - \binom{k_1}{2} - \binom{k_2}{2} = k_1k_2$ new crossings between e_1 and e_2. If both e_1 and e_2 are red, then before the flipping they cross an odd number of times and the flipping introduces an odd number of new crossings, since k_1 and k_2 are odd. Hence e_1 and e_2 cross an even number of times after the flipping. If at least one of e_1 and e_2 is blue, then before the flipping they cross an even number of times and the flipping introduces an even number of new crossings, since at least one of k_1 and k_2 is even. Therefore, e_1 and e_2 cross an even number of times after the flipping in this case as well. This completes the proof of the second part of Theorem 9.

Suppose now that H is non-bipartite. As in the proof of Claim 11, we can conclude:

▶ **Observation 15.** *Every odd cycle in H must contain an odd number of red edges.*

Furthermore, we have:

▷ Claim 16. Every two odd cycles in H must have a common vertex.

Proof. Let C_1 and C_2 be two odd cycles in H with no common vertex. Then they must properly cross an even number of times, as every two closed curves do. Recall that only pairs of red edges cross an odd number of times. Since by Observation 15 each of C_1 and C_2 contains an odd number of red edges, C_1 and C_2 cross an odd number of times, which is impossible. ◁

Since H is non-bipartite it contains at least one odd cycle. Fix a shortest odd cycle C and denote by $V(C)$ its vertices. Observe that it follows from Claim 16 that $H \setminus C$ is a bipartite graph. In the rest of the proof we apply on C the doubling technique of Cairns and Nikolayevski [3] adapted to our case.

The curve that corresponds to the cycle C in H divides the plane into two parts one of which is *Jordan-surrounded* by C and the other is not. (These parts are not necessarily connected.) A point x is called *Jordan-surrounded* by C if any curve starting at x and going to infinity crosses C an odd number of times. Let $\mathcal{I}$ (resp., $\mathcal{O}$) denote the region that is (resp., not) Jordan-surrounded by C.

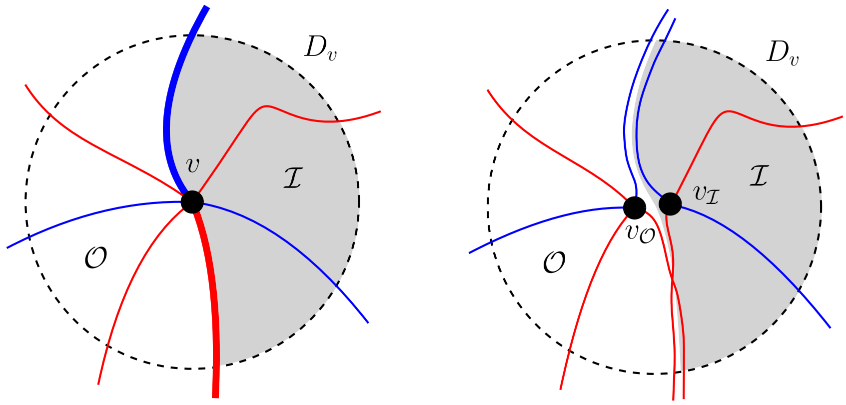
Denote by $\mathcal{I}_Y$ the vertices in $Y \setminus V(C)$ that are Jordan-surrounded by C. Let $\mathcal{O}_Y$ be the set of vertices in $Y \setminus V(C)$ that are not Jordan-surrounded by C. Similarly, $\mathcal{I}_X$ and $\mathcal{O}_X$ denote the subsets of vertices in $X \setminus V(C)$ that are Jordan-surrounded by C and are not Jordan-surrounded by C, respectively.

Notice that there cannot be a blue edge between a vertex in $\mathcal{I}_Y$ and a vertex in $\mathcal{O}_X$, nor between a vertex in $\mathcal{I}_X$ and a vertex in $\mathcal{O}_Y$. This is because every blue edge not in C crosses every edge in C and hence also C an even number of times. Therefore, each blue edge connects either a vertex in $\mathcal{I}_Y$ and a vertex in $\mathcal{I}_X$, or a vertex in $\mathcal{O}_Y$ and a vertex in $\mathcal{O}_X$. Additionally, there can be blue edges connecting a vertex in C and a vertex not in C.

Similarly, we notice that there can be red edges in $H \setminus C$ only between vertices in $\mathcal{I}_Y$ and vertices in $\mathcal{O}_Y$. This is because every red edge in $H \setminus C$ crosses every red edge in C an odd number of times and every blue edge in C an even number of times. Hence, every red edge in $H \setminus C$ crosses C an odd number of times, by Observation 15. Additionally, there can be red edges with one endpoint in $Y \cap V(C)$ and one in $Y \setminus V(C)$.

Doubling the cycle C

Let v be a vertex in $V(C)$ and let D_v be a small enough disc centered at v such that it does not intersect any edge which is not incident to v and does not contain any crossing point of H. Denote by $\{v, u\}$ and $\{v, w\}$ the edges of C that are incident to v. Then the intersection of



(a) before **(b)** after

Figure 4 Doubling of C at the neighborhood of v. The gray region is Jordan-surrounded by C.

these edges with D_v partitions D_v into two regions, one which is Jordan-surrounded by C and one which is not. The edges of H which are incident to v and different from $\{v, u\}$ and $\{v, w\}$ are partitioned into ones whose intersection with $D_v \setminus \{v\}$ is in $\mathcal{I}$ and ones whose intersection with $D_v \setminus \{v\}$ is in $\mathcal{O}$. Denote by $E_{v,\mathcal{I}}$ the first set and by $E_{v,\mathcal{O}}$ the second set and observe that the edges in each of these sets are consecutive around v. We replace v with two new vertices $v_{\mathcal{I}}$ and $v_{\mathcal{O}}$ such that $v_{\mathcal{I}}$ is very close to v in $\mathcal{I}$ and $v_{\mathcal{O}}$ is very close to v in $\mathcal{O}$. The edges of $E_{v,\mathcal{I}}$ (resp., $E_{v,\mathcal{O}}$) are redrawn in D_v such that $v_{\mathcal{I}}$ (resp., $v_{\mathcal{O}}$) replaces v and their endpoint, without introducing any new crossings (see Figure 4).

Next, we replace every edge of C by two new edges as follows. We slightly inflate every such edge so we can refer to each such edge having two "sides". Suppose that $\{x, y\}$ is a blue edge of C. Then we pick arbitrarily one of its endpoints, say x, and redraw $\{x, y\}$ starting from $x_{\mathcal{I}}$ and following the side of $\{x, y\}$ which is inside $\mathcal{I}$ within D_x. We keep following this side of $\{x, y\}$ until reaching D_y. Along the way the edge may switch between $\mathcal{I}$ and $\mathcal{O}$, however, since $\{x, y\}$ crosses every other edge an even number of times, the side of $\{x, y\}$ that we follow lies in $\mathcal{I}$ when reaching D_y. We keep following this side of $\{x, y\}$ within D_y and end up at $y_{\mathcal{I}}$. In a similar and symmetric way we draw an edge connecting $x_{\mathcal{O}}$ and $y_{\mathcal{O}}$ by following the other side of $\{x, y\}$. Note that if we add just these two edges to H, then whenever a part of $\{x_{\mathcal{I}}, y_{\mathcal{I}}\}$ lies in $\mathcal{I}$ then its counterpart at $\{x_{\mathcal{O}}, y_{\mathcal{O}}\}$ lies in $\mathcal{O}$ and vice versa. Therefore $\{x_{\mathcal{I}}, y_{\mathcal{I}}\}$ and $\{x_{\mathcal{O}}, y_{\mathcal{O}}\}$ do not intersect.

Suppose now that $\{x, y\}$ is a red edge of C. Then, as in the case of a blue edge, we pick arbitrarily one of its endpoints, say x, and redraw $\{x, y\}$ starting from $x_{\mathcal{I}}$ and following the side of $\{x, y\}$ which is inside $\mathcal{I}$ within D_x. However, midway within D_x we switch to the other side of $\{x, y\}$ that lies in $\mathcal{O}$ within D_x and keep following this side until reaching D_y. Recall that $\{x, y\}$ crosses every blue edge an even number of times and every red edge an odd number of times. Furthermore, there is an even number of red edges in C different from $\{x, y\}$, since the number of red edges in C is odd. Therefore, $\{x, y\}$ crosses the other edges of C an even number of times. This implies that the side of $\{x, y\}$ that we follow lies in $\mathcal{O}$ within D_y. We keep following this side of $\{x, y\}$ within D_y and end up at $y_{\mathcal{O}}$. In a similar and symmetric way we draw an edge connecting $x_{\mathcal{O}}$ and $y_{\mathcal{I}}$ and observe that it can be done such that $\{x_{\mathcal{I}}, y_{\mathcal{O}}\}$ and $\{x_{\mathcal{O}}, y_{\mathcal{I}}\}$ intersect at exactly one point in which they cross in D_x. See Figure 4 for an illustration of the doubling of C at the neighborhood of a vertex.

We denote by H' the topological graph which we get after doubling C as above and observe that the edges in H' inherit their colors from their corresponding edges in H. Next, we establish several properties of H'.

▷ Claim 17. Let e' and f' be two distinct edges in H' and let e and f be their corresponding edges in H, respectively. If $e \neq f$ then e' and f' cross if and only if e and f cross. If $e = f$ then e' and f' cross if and only if e is a red edge.

Proof. We have observed above that if $e = f$ is a red edge, then e' and f' cross once and that if $e = f$ is a blue edge then e' and f' do not cross. For the case $e \neq f$ observe that the claim follows from the drawing procedure by which we follow the original edges without introducing new crossings. ◁

The following is implied by Claim 17, the properties of H and the doubling procedure.

▶ **Observation 18.** *Every blue edge in H' crosses every other edge in H' an even number of times and both of its endpoints are in $\mathcal{I}$ or both of them are in $\mathcal{O}$. Every red edge in H' crosses every other red edge in H' an odd number of times and has one endpoint in $\mathcal{I}$ and the other in $\mathcal{O}$.*

▶ **Lemma 19.** *H' is bipartite.*

Proof. We have already seen that $H \setminus C$ is a bipartite graph and the only edges in this subgraph are between a vertex from $\mathcal{I}_Y \cup \mathcal{O}_X$ and a vertex from $\mathcal{I}_X \cup \mathcal{O}_Y$.

To see that H' is indeed a bipartite graph we consider the following partition of the vertices of the doubling of C: $C_{Y,\mathcal{I}} = \{v_{\mathcal{I}} \mid v \in V(C) \cap Y\}$, $C_{Y,\mathcal{O}} = \{v_{\mathcal{O}} \mid v \in V(C) \cap Y\}$, $C_{X,\mathcal{I}} = \{v_{\mathcal{I}} \mid v \in V(C) \cap X\}$ and $C_{X,\mathcal{O}} = \{v_{\mathcal{O}} \mid v \in V(C) \cap X\}$. We will show that H' is bipartite with the bipartition $(\mathcal{I}_Y \cup \mathcal{O}_X \cup C_{Y,\mathcal{I}} \cup C_{X,\mathcal{O}}, \mathcal{I}_X \cup \mathcal{O}_Y \cup C_{Y,\mathcal{O}} \cup C_{X,\mathcal{I}})$.

▷ Claim 20. A vertex in $C_{Y,\mathcal{I}} \cup C_{X,\mathcal{O}}$ cannot be connected by an edge e' in H' to a vertex in $\mathcal{I}_Y \cup \mathcal{O}_X \cup C_{Y,\mathcal{I}} \cup C_{X,\mathcal{O}}$.

Proof. Suppose that such an edge e' exists and let e be the edge in H that corresponds to e'. We consider the possible cases.

Case 1: e connects two vertices in C. In this case e must be an edge of C because of our assumption that C is the shortest odd cycle in H. If e is a red edge, then both of its vertices must be in Y. Then under the contrary assumption, the only possibility is that e' is an edge between a vertex from $C_{Y,\mathcal{I}}$ and another vertex of $C_{Y,\mathcal{I}}$. But this is impossible by Observation 18.

If, on the other hand, e is a blue edge, then it must connect a vertex in Y to a vertex in X. Under the contrary assumption and keeping in mind that both vertices of e are in C, it must be that e' connects a vertex from $C_{Y,\mathcal{I}}$ to a vertex in $C_{X,\mathcal{O}}$. However, this is again impossible by Observation 18.

Case 2: e' is a red edge connecting a vertex in $C_{Y,\mathcal{I}} \cup C_{X,\mathcal{O}}$ and a vertex in $\mathcal{I}_Y \cup \mathcal{O}_X$. Since red edges connect vertices in Y, it follows that e' connects a vertex in $C_{Y,\mathcal{I}}$ and a vertex in $\mathcal{I}_Y$. This is impossible by Observation 18.

Case 3: e' is a blue edge connecting a vertex in $C_{Y,\mathcal{I}} \cup C_{X,\mathcal{O}}$ and a vertex in $\mathcal{I}_Y \cup \mathcal{O}_X$. It follows from Observation 18 that e' cannot connect a vertex in $C_{Y,\mathcal{I}}$ to a vertex in $\mathcal{O}_X$ nor can it connect a vertex in $C_{X,\mathcal{O}}$ to a vertex in $\mathcal{I}_Y$. This completes the proof of Claim 20. ◁

The following claim is similar to Claim 20 and its proof goes almost verbatim and is symmetric to the proof of Claim 20. Therefore, we skip the proof and leave it to the reader.

▷ Claim 21. A vertex in $C_{Y,\mathcal{O}} \cup C_{X,\mathcal{I}}$ cannot be connected by an edge in H' to a vertex in $\mathcal{I}_X \cup \mathcal{O}_Y \cup C_{Y,\mathcal{O}} \cup C_{X,\mathcal{I}}$.

Claims 20 and 21 along with the fact that the only edges in $H \setminus C$ are between a vertex in $\mathcal{I}_X \cup \mathcal{O}_Y$ and vertex in $\mathcal{I}_Y \cup \mathcal{O}_X$ imply that H' is bipartite with the bipartition $(\mathcal{I}_Y \cup \mathcal{O}_X \cup C_{Y,\mathcal{I}} \cup C_{X,\mathcal{O}}, \mathcal{I}_X \cup \mathcal{O}_Y \cup C_{Y,\mathcal{O}} \cup C_{X,\mathcal{I}})$. This concludes the proof of Lemma 19. ◀

The graph H' satisfies the same properties satisfied by H with X replaced by $(X \setminus V(C)) \cup C_{X,\mathcal{I}} \cup C_{X,\mathcal{O}}$ and Y replaced by $(Y \setminus V(C)) \cup C_{Y,\mathcal{I}} \cup C_{Y,\mathcal{O}}$. Therefore, it follows from the proof of the second part of the theorem (which appears above) that H' is planar.

Denoting by m the number of edges in H, we may conclude from the fact that H' is planar and bipartite with $n + |C|$ vertices such that $m + |C| \leqslant 2(n + |C|) - 4$. This gives the upper bound $m \leqslant 2n - 4 + |C|$ which is almost tight when $|C|$ is small but is far off $2n - 2$ if $|C|$ is larger. We can improve on this bound and get rid of the dependency in $|C|$ by arguing that H' can be embedded in the plane such that one of its faces has size $2|C|$. In fact we will show that the doubling of C becomes a face of size $2|C|$ in a planar embedding of H'. This will imply an upper bound of $2(n + |C|) - 4 - (|C| - 2) = 2n - 2 + |C|$ on the number of edges in H', that is, on $m + |C|$. Thus, we will conclude that $m \leqslant 2n - 2$ as desired.

▶ **Lemma 22.** *H' can be embedded in the plane such that the doubling of C is the boundary of a face.*

Proof. We will use Theorem 6 by which a topological graph in which every two edges cross an even number of times can be embedded in the plane keeping the cyclic order at which the edges sharing a common vertex are drawn in a small neighborhood of that vertex.

We redraw H' in the plane, without adding or removing crossing, by “pushing” above the x-axis all the vertices that are in $\mathcal{I}$, that is, $\mathcal{I}_Y \cup \mathcal{I}_X \cup C_{Y,\mathcal{I}} \cup C_{X,\mathcal{I}}$. Similarly, we “push” below the x-axis all the vertices that are in $\mathcal{O}$, that is, $\mathcal{O}_Y \cup \mathcal{O}_X \cup C_{Y,\mathcal{O}} \cup C_{X,\mathcal{O}}$. Note that it follows from Observation 18 that every blue edge has both of its endpoints in the same half-plane bounded by the x-axis, while every red edge in H' has its two endpoints separated by the x-axis.

As in the proof of the second part of Theorem 9, we now flip horizontally (that is, about the y-axis) the half-plane below the x-axis without breaking the arcs crossing it to obtain a topological graph in which every pair of edges crosses an even number of times. Therefore, by Theorem 6, we can embed this graph, and hence H', keeping the same rotation system at its vertices. Let H'' be such an embedding.

We now concentrate on the doubling of the cycle C in H'' and show that it becomes the boundary of a face of size $2|C|$. This will complete the proof of Theorem 2.

We first notice that the doubling of C results indeed in a cycle of length $2|C|$ and not in two cycles of length $|C|$. The reason is that if we had two cycles of length $|C|$, each having an odd number of red edges, then they would have crossed each other an odd number of times which is impossible. We denote this cycle of length $2|C|$ by C'.

Note that in a planar drawing a cycle forms a face if and only if we can traverse the cycle such that each time we arrive through an edge e to a vertex v, then we leave through the edge that comes clockwise next after e among the edges incident to v.

Recall that for every vertex $v \in V(C)$ the following holds. All the edges that are incident to $v_{\mathcal{I}}$ are consecutive around $v_{\mathcal{I}}$ and “start” in $\mathcal{I}$. Similarly all the edges that are incident to $v_{\mathcal{O}}$ are consecutive around $v_{\mathcal{O}}$ and “start” in $\mathcal{O}$.

This implies that before the flipping the half-plane below the x-axis we can traverse C' along the drawing in some appropriate direction such that at each vertex $v \in \mathcal{I}$ we arrive via an edge and leave through the clockwise next edge incident at v, and for each vertex $v \in \mathcal{O}$ we leave through the counterclockwise next edge incident at v, see Figure 4b.

After flipping the half-plane below the x-axis, the cyclic order of edges at each vertex in $C_{Y,\mathcal{O}} \cup C_{X,\mathcal{O}}$ is reversed. Hence we can traverse C' along the drawing in some appropriate direction after flipping H' such that at each vertex v we leave through the clockwise next edge incident to v. Since we have the same rotation system in H'' as in the drawing of the flipped H' it follows that C' bounds a face. ◀

5 Proof of Lemma 14

Proof of Lemma 14. We start with a simple observation that we will apply multiple times.

▶ **Observation 23.** *Suppose two circles K_1 and K_2 with centers e_1 and e_2 intersect. Then the e_1e_2 ray intersects K_1 either on K_2 or inside K_2.*

Let U_1 and U_2 be two circles forming a lune edge and let E_1 and E_2 be two circles forming a lens edge. We may assume that U_1 supports the inner arc of the lune. Let u_1, u_2, e_1, e_2 be the centers of these circles. We rotate the circles such that u_1 and u_2 share x-coordinates and u_1 lies above u_2. Suppose on the contrary that the e_2e_1 ray intersects the u_1u_2 segment.

We can assume without loss of generality that e_1 and e_2 has a larger x-coordinate than u_1 and u_2, and that if we look around at u_2 starting from u_1, going in a clockwise direction, then we see e_1 sooner than e_2. See Figure 5.

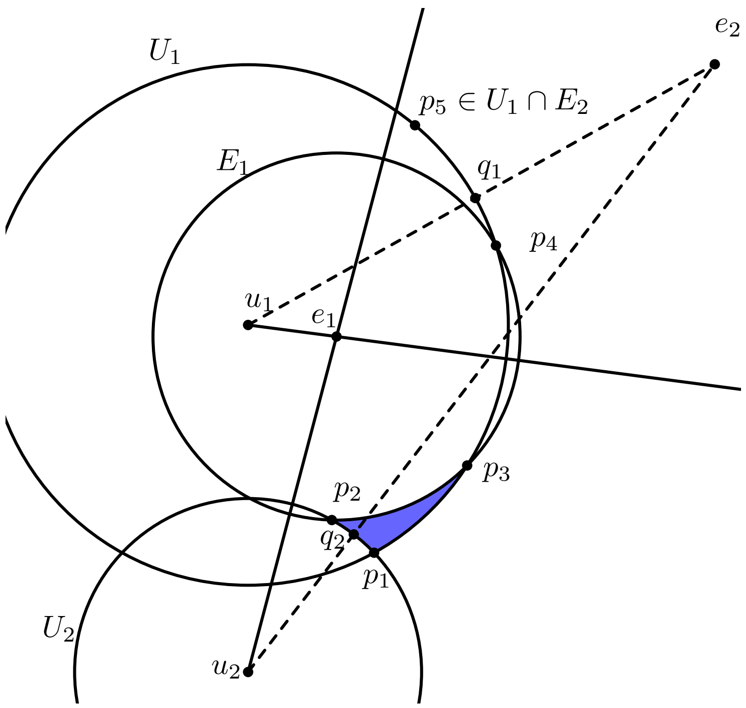


Figure 5 Region A is indicated by blue.

Since the e_2e_1 ray hits the u_1u_2 segment and e_1 comes before e_2 in a clockwise order when we look around at u_2 starting from u_1, e_2 must lie in the cone that has apex e_1 and is bounded by the ray u_1e_1 minus the segment u_1e_1 and the ray u_2e_1 minus the segment u_2e_1. This means that if we look around at u_1 starting from u_2 in counterclockwise direction, then we see e_1 sooner than e_2.

Let p_1 be the intersection point of U_1 and U_2 to the right of u_2u_1.

Let p_2 be the intersection point of E_1 and U_2 to the right of u_2e_1. Since U_1 and U_2 form a lune, p_2 must lie within U_1.

Let p_3 and p_4 be the intersections of E_1 and U_1 so that p_3 is to the right of u_1e_1. (see Figure 5). Since U_1 and U_2 form a lune, p_3 appears counterclockwise later than p_1 when we look around at u_1 starting from u_2. All these together imply that p_1, p_2, p_3 are corners of a region bounded by three circular arcs, let us call this region A.

Let q_1 be the intersection of the u_1e_2 ray and U_1. By Observation 23 q_1 is inside the disk of E_2. Since E_1 and E_2 form a lens, q_1 cannot be in the disk of E_1. This implies that there is some $p_5 \in U_1 \cap E_2$ that is counterclockwise later than p_4 when we look around at u_1 starting from u_2 and p_5 is not inside E_1.

We will show that E_2 intersects the p_1p_2 arc of U_2 twice. To see this, it suffices to show that there is a point of the p_1p_2 arc inside E_2 and that p_1 and p_2 are not in E_2. Let q_2 be the intersection of the u_2e_2 ray and U_2. By Observation 23 q_2 is also in the disk of E_2. We claim that q_2 is on the arc p_1p_2. Indeed, it is to the right of the ray u_2e_1 yet it cannot be on the arc of U_2 which is part of the empty lune of U_2 and U_1 and it also cannot be in the disk of E_1 (as it would then be in the empty lens defined by E_1 and E_2). Since E_1 and E_2 form an empty lens, p_2 cannot be inside E_2. Since p_1 is part of the lune of U_1 and U_2 it cannot be in E_2. Hence the endpoints of the arc p_1p_2 are not in the disk of E_2 but a point q_2 on it is in the disk of E_2, therefore E_2 intersects the p_1p_2 arc of U_2 twice.

Consider the part of E_2 which lies outside U_2. It is an arc which starts and ends on the p_1p_2 arc of U_2. It must also contain p_5 which is outside A. This implies that there must be at least two other points x_1, x_2 where it leaves the region A. Since E_2 intersects the U_1 circle at p_5 which is counterclockwise later than p_3 when we look around at u_1 starting from u_2, there can be at most one of x_1, x_2 on the p_1p_3 arc of A. Hence, E_2 intersects the p_2p_3 arc of A. But p_2 and p_3 cannot be part of the lens of E_1 and E_2, so E_2 must intersect the p_2p_3 arc of A twice. Hence, E_2 intersects the p_1p_3 arc of A zero or two times. As we have seen, it cannot be two times, so there is no intersection. Therefore the part of E_2 that lies outside U_2 lies entirely in the union of A and the disk of E_1 and therefore it cannot contain p_5, a contradiction. ◀

References

1. Pankaj K. Agarwal, Eran Nevo, János Pach, Rom Pinchasi, Micha Sharir, and Shakhar Smorodinsky. Lenses in arrangements of pseudo-circles and their applications. *J. ACM*, 51(2):139–186, 2004. doi:10.1145/972639.972641.
2. Noga Alon, Hagit Last, Rom Pinchasi, and Micha Sharir. On the complexity of arrangements of circles in the plane. *Discret. Comput. Geom.*, 26(4):465–492, 2001. doi:10.1007/s00454-001-0043-x.
3. Grant Cairns and Yury Nikolayevsky. Bounds for generalized thrackles. *Discret. Comput. Geom.*, 23(2):191–206, 2000. doi:10.1007/PL00009495.
4. Grant Cairns and Yury Nikolayevsky. Generalized thrackle drawings of non-bipartite graphs. *Discret. Comput. Geom.*, 41(1):119–134, 2009. doi:10.1007/s00454-008-9095-5.
5. Jordan S. Ellenberg, József Solymosi, and Joshua Zahl. New bounds on curve tangencies and orthogonalities. *Discrete Anal.*, November 2016. doi:10.19086/da.990.
6. Paul Erdős. On sets of distances of n points. *Am. Math. Mon.*, 53(5):248–250, 1946. doi:10.2307/2305092.
7. Stefan Felsner, Sandro Roch, and Manfred Scheucher. Arrangements of pseudocircles: on digons and triangles. In *Graph drawing and network visualization*, volume 13764 of *Lecture Notes in Computer Science*, pages 441–455. Springer, Cham, 2023. doi:10.1007/978-3-031-22203-0_32.

8 Branko Grünbaum. *Arrangements and spreads.* Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 10. American Mathematical Society, Providence, R.I., 1972.

9 Meir Katchalski and Hagit Last. On geometric graphs with no two edges in convex position. *Discret. Comput. Geom.*, 19:399–404, 1998. `doi:10.1007/PL00009357`.

10 Adam Marcus and Gábor Tardos. Intersection reverse sequences and geometric applications. *J. Comb. Theory, Ser. A*, 113(4):675–691, 2006. `doi:10.1016/j.jcta.2005.07.002`.

11 János Pach and Pankaj K. Agarwal. *Combinatorial Geometry*, chapter 11, pages 177–178. John Wiley and Sons Ltd, 1995. `doi:10.1002/9781118033203.ch11`.

12 János Pach and Géza Tóth. Disjoint edges in topological graphs. In *Combinatorial Geometry and Graph Theory, Indonesia-Japan Joint Conference, IJCCGGT 2003, Revised Selected Papers*, volume 3330 of *Lecture Notes in Computer Science*, pages 133–140. Springer, 2003. `doi:10.1007/978-3-540-30540-8_15`.

13 János Pach and Gábor Tardos. Forbidden paths and cycles in ordered graphs and matrices. *Isr. J. Math.*, 155:359–380, 2006. `doi:10.1007/BF02773960`.

14 Rom Pinchasi. Gallai-Sylvester Theorem for Pairwise Intersecting Unit Circles. *Discret. Comput. Geom.*, 28(4):607–624, 2002. `doi:10.1007/s00454-002-2892-3`.

15 Rom Pinchasi. A note on lenses in arrangements of pairwise intersecting circles in the plane, 2024. `arXiv:2403.05270`.

16 Joel Spencer, Endre Szemerédi, and William T. Trotter. *Unit distances in the Euclidean plane*, pages 294–304. Academic Press, United Kingdom, 1984.

17 László A. Székely. Crossing Numbers and Hard Erdős Problems in Discrete Geometry. *Comb. Probab. Comput.*, 6(3):353–358, 1997. `doi:10.1017/S0963548397002976`.

18 William T. Tutte. Toward a theory of crossing numbers. *J. Comb. Theory*, 8(1):45–53, 1970. `doi:10.1016/S0021-9800(70)80007-2`.

19 Pavel Valtr. On geometric graphs with no k pairwise parallel edges. *Discret. Comput. Geom.*, 19:461–469, 1998. `doi:10.1007/PL00009364`.