Saturation Results Around the Erdős–Szekeres Problem

Gábor Damásdi
HUN-REN Alfréd Rényi Institute of Mathematics, Budapest, Hungary
ELTE Eötvös Loránd University, Budapest, Hungary

Zichao Dong
HUN-REN Alfréd Rényi Institute of Mathematics, Budapest, Hungary
Extremal Combinatorics and Probability Group (ECOPRO), Institute for Basic Science (IBS), Daejeon, South Korea

Manfred Scheucher
Institut für Mathematik, Technische Universität Berlin, Germany

Ji Zeng
University of California San Diego, La Jolla, CA, USA
HUN-REN Alfréd Rényi Institute of Mathematics, Budapest, Hungary

Abstract

In this paper, we consider saturation problems related to the celebrated Erdős–Szekeres convex polygon problem. For each $n \geq 7$, we construct a planar point set of size $(7/8) \cdot 2^{n-2}$ which is saturated for convex n-gons. That is, the set contains no n points in convex position while the addition of any new point creates such a configuration. This demonstrates that the saturation number is smaller than the Ramsey number for the Erdős–Szekeres problem. The proof also shows that the original Erdős–Szekeres construction is indeed saturated.

Our construction is based on a similar improvement for the saturation version of the cups-versus-caps theorem. Moreover, we consider the generalization of the cups-versus-caps theorem to monotone paths in ordered hypergraphs. In contrast to the geometric setting, we show that this abstract saturation number is always equal to the corresponding Ramsey number.

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1 Introduction

Two types of problems are of central significance in extremal combinatorics. Turán-type problems originate from the work of Turán [19] (and earlier Mantel [14]) that determines the maximum number of edges in a graph without a k-clique. Ramsey-type problems begin with the work of Ramsey [16] which states that any large enough graph must contain either a k-clique or a k-independent set.

In 1964, Erdős, Hajnal, and Moon [4] investigated a variation of Turán's theorem, called *the saturation problem*, where one aims to minimize the number of edges in a graph that does not contain a k-clique, but the addition of any edge to this graph yields a k-clique. More generally, saturation problems can be considered in various settings, see our incomplete list [9, 13, 10, 7, 2]. Typically, an object that is maximal with respect to certain property is said to be *saturated*. While the classical extremal problems such as Turán-type problems or Ramsey-type problems ask for certain maximum quantity possibly achieved, their saturation versions aim at the corresponding minimum quantity possibly achieved by saturated objects.

The Erdős–Szekeres problem is a classical extremal problem in combinatorial geometry proposed in the seminal paper [5] back to 1935. It asks for the maximum size of a planar point set that does not contain k points in convex position. In this paper, we consider the saturation version of the Erdős–Szekeres problem, as well as the saturation versions of the related Ramsey-type results in [5] and some later graph-theoretic generalizations.

1.1 The Erdős–Szekeres lemma on monotone sequences

Let $\mathrm{ram}_{\mathsf{s}}(k,\ell)$ be the maximum length of a sequence of distinct real numbers that is (k,ℓ)-*seq.-free* (i.e. containing no increasing subsequence of length k or decreasing subsequence of length ℓ). The first result in [5], later known as the Erdős-Szekeres lemma, states that

$$\mathrm{ram}_{\mathsf{s}}(k,\ell) = (k-1)(\ell-1). \tag{1}$$

Like most problems considered in this paper, this extremal quantity can be considered as Ramsey-type where the edge coloring is whether each pair is increasing or decreasing.

In [2], the authors studied the saturation version of (1). In their setting, a sequence is called (k,ℓ)-*seq.-saturated* if it is (k,ℓ)-seq.-free and the insertion of any distinct real number anywhere into this sequence yields such a monotone subsequence. The saturation number $\mathrm{sat}_{\mathsf{s}}(k,\ell)$ is the minimum length of a sequence that is (k,ℓ)-seq.-saturated. It is obvious that $\mathrm{sat}_{\mathsf{s}}(k,\ell) \le \mathrm{ram}_{\mathsf{s}}(k,\ell)$. Interestingly, the saturation number and the Ramsey number are equal in this setting.

▶ **Theorem 1** (Damásdi et al. [2]). *For any integers $k,\ell \ge 1$, we have* $\mathrm{sat}_{\mathsf{s}}(k,\ell) = \mathrm{ram}_{\mathsf{s}}(k,\ell)$.

For a new proof of Theorem 1 see the full version of the paper [1]. Our proof leads to a hypergraph variation of this result, see Theorem 5.

1.2 The Erdős–Szekeres theorem on cups-versus-caps

The main objects we consider in this paper are planar point sets. To simplify our discussion, a planar point set is said to be *generic* if it is *in general position* (meaning without three collinear points throughout this paper) and its members have distinct x-coordinates. A point p is said to be *generic* with respect to a set P if $P \cup \{p\}$ is a generic point set.

For a sequence of generic points $p_1,\dots,p_n$ ordered increasingly with respect to x-coordinates, we say that $p_1,\dots,p_n$ form a *cup* (resp. *cap*) if the slopes of the lines p_ip_{i+1} (for $1 \le i < k$) form an increasing (resp. decreasing) sequence. Moreover, a k*-cup* refers to a cup of size k and an ℓ*-cap* refers to a cap of size ℓ. Let $\mathrm{ram}_{\mathsf{c}}(k,\ell)$ denote the maximum size of a generic planar point set that is (k,ℓ)*-cup-cap-free* (i.e. containing neither a k-cup nor an ℓ-cap). The second result in [5], later known as the Erdős–Szekeres theorem, states that

$$\mathrm{ram}_{\mathsf{c}}(k,\ell) = \binom{k+\ell-4}{k-2}. \tag{2}$$

In this paper, we study the saturation version of (2). A generic planar point set is said to be (k,ℓ)*-cup-cap-saturated* if it is (k,ℓ)-cup-cap-free and the addition of any generic point with respect to the set yields a k-cup or an ℓ-cap. The saturation number $\mathrm{sat}_{\mathsf{c}}(k,\ell)$ is defined as the minimum size of a point set that is (k,ℓ)-cup-cap-saturated. In contrast to Theorem 1, we will show that $\mathrm{sat}_{\mathsf{c}}(k,\ell)$ is in general significantly smaller than $\mathrm{ram}_{\mathsf{c}}(k,\ell)$.

▶ **Theorem 2.** *For all integers $k, \ell \geq 4$, we have*

$$2k + 2\ell - 14 \leq \mathrm{sat}_{\mathsf{c}}(k, \ell) \leq \binom{k+\ell-4}{k-2} - 2\binom{k+\ell-8}{k-4}.$$

Here, we use the conventions $\binom{-1}{0} = 0$ and $\binom{0}{0} = 1$. In Section 3 where this theorem is proven, we also determine $\mathrm{sat}_{\mathsf{c}}(k, \ell)$ for small values of the pair (k, ℓ). For example, we establish that $\mathrm{sat}_{\mathsf{c}}(4, 5) = 8 < 10 = \mathrm{ram}_{\mathsf{c}}(4, 5)$, which is the lexicographically smallest case where "$\mathrm{sat}_{\mathsf{c}}(k, \ell) < \mathrm{ram}_{\mathsf{c}}(k, \ell)$" happens.

1.3 The Erdős–Szekeres problem on convex polygons

Another landmark in the paper [5] of Erdős and Szekeres is the following famous conjecture.

▶ **Conjecture 3.** *Every set of $2^{n-2} + 1$ points in the plane that are in general position contains n points in convex position, and this bound is tight in the worst case.*

Denote by $\mathrm{ram}_{\mathsf{g}}(n)$ the maximum size of a planar point set in general position that is *n-gon-free* (i.e. containing no n points in convex position). The Erdős–Szekeres conjecture can be phrased as

Is it true that $\mathrm{ram}_{\mathsf{g}}(n) = 2^{n-2}$ for all $n \geq 2$?

In a subsequent paper [6] from 1961, Erdős and Szekeres constructed a generic point set of size 2^{n-2} that is n-gon-free for all integers $n \geq 2$. Thus, for any integer $n \geq 2$, we have

$$\mathrm{ram}_{\mathsf{g}}(n) \geq 2^{n-2}. \tag{3}$$

Notice that $\mathrm{ram}_{\mathsf{g}}(n) \leq \mathrm{ram}_{\mathsf{c}}(n, n)$, since any n-cup or n-cap is an n-gon. Hence, (2) implies that $\mathrm{ram}_{\mathsf{g}}(n) \leq \binom{2n-4}{n-2} = 4^{n-o(n)}$. There have been only small improvements of this upper bound, until Suk [17] made a breakthrough in 2017 by proving that $\mathrm{ram}_{\mathsf{g}}(n) \leq 2^{n+o(n)}$ (see also [11]). The Erdős–Szekeres conjecture has been verified for $n \leq 6$ in [18].

We consider the saturation version of the Erdős–Szekeres problem. A planar point set is *n-gon-saturated* if it is n-gon-free while any $q \notin P$ with $P \cup \{q\}$ being in general position is part of an n-gon in $P \cup \{q\}$. Denote by $\mathrm{sat}_{\mathsf{g}}(n)$ the smallest size of an n-gon-saturated set. The main result of our paper is that $\mathrm{sat}_{\mathsf{g}}(n)$ is significantly smaller than $\mathrm{ram}_{\mathsf{g}}(n)$ in general.

▶ **Theorem 4.** *For any integer $n \geq 7$, we have $\mathrm{sat}_{\mathsf{g}}(n) \leq \frac{7}{8} \cdot 2^{n-2} \leq \frac{7}{8} \cdot \mathrm{ram}_{\mathsf{g}}(n)$.*

We prove this theorem by modifying the original construction of [6], replacing the substructures in their point set with smaller ones we found in Theorem 2. The proof also shows that the original Erdős–Szekeres construction in [6] is indeed saturated. As far as we know, this is widely believed (for otherwise the Erdős–Szekeres conjecture would be disproved immediately) but never been verified in the literature since its existence from 1961.

1.4 Graph-theoretic generalizations

The increasing or decreasing sequences and cups or caps can be generalized to monotone paths in a graph-theoretic setting. Inside any r-uniform complete hypergraph H with a linear-ordered vertex set $V(H)$, a *monotone path of length k* (or a *monotone k-path*) is a subgraph of H with vertices being $v_1 < v_2 < \cdots < v_{k+r-1}$ and (hyper)edges being $\{v_i, v_{i+1}, \ldots, v_{i+r-1}\}$ for $i = 1, \ldots, k$. Here we define the length of a monotone path as the number of its edges rather than its vertices.

Let $\mathrm{ram}_{\mathsf{p}}^{(r)}(k,\ell)$ be the maximum size of an r-uniform vertex-ordered complete hypergraph H that is (k,ℓ)*-path-free* (i.e. each edge is colored by one of red and blue such that H contains neither a red monotone k-path nor a blue monotone ℓ-path). The determination of $\mathrm{ram}_{\mathsf{p}}^{(r)}(k,\ell)$ is a Ramsey-type problem that connects to the Erdős–Szekeres results in the following way: Given a generic planar point set P, we can create a 3-uniform complete hypergraph H_P whose vertices are P ordered by their x-coordinates. Color each triple of H_P red or blue based on whether it is a cup or a cap. Then a monochromatic monotone ℓ-path in H_P corresponds to an $(\ell+2)$-cup or cap. So, $\mathrm{ram}_{\mathsf{p}}^{(3)}(k,\ell) \geq \mathrm{ram}_{\mathsf{c}}(k+2,\ell+2)$. Similarly, $\mathrm{ram}_{\mathsf{p}}^{(2)}(k,\ell) \geq \mathrm{ram}_{\mathsf{s}}(k+1,\ell+1)$. The Ramsey numbers $\mathrm{ram}_{\mathsf{p}}^{(r)}(k,\ell)$ are extensively studied, and their bounds are considered as abstract generalizations of (1) and (2). See [3, 8, 15] for more details.

Another major question investigated in this paper is the saturation problem for monotone paths. We say that a 2-colored vertex-ordered complete hypergraph H is (k,ℓ)*-path-saturated* if H contains neither red monotone k-path nor blue monotone ℓ-path, and any H^+ properly containing H contains either a red monotone k-path or a blue monotone ℓ-path. Here, H^+ is again a 2-colored vertex-ordered complete hypergraph such that the containment preserves both the ordering and the coloring. The saturation number $\mathrm{sat}_{\mathsf{p}}^{(r)}(k,\ell)$ is defined to be the minimum size of an r-uniform 2-colored ordered complete hypergraph that is (k,ℓ)-path-saturated. In contrast to the geometric setting of Theorems 2 and 4, we show that this abstract saturation number is always equal to the Ramsey number.

▶ **Theorem 5.** *For any integers $r \geq 2$ and $k,\ell \geq 1$, we have $\mathrm{sat}_{\mathsf{p}}^{(r)}(k,\ell) = \mathrm{ram}_{\mathsf{p}}^{(r)}(k,\ell)$.*

The rest of this paper is organized as follows: In Section 2, we use a labeling technique of Moshkovitz and Shapira [15] to prove Theorem 5 completely. Section 3 is devoted to the saturation problem for cups-versus-caps. In particular, we prove Theorem 2 there. Section 4 is devoted to the proof of Theorem 4. Finally, we include remarks and open problems in Section 5. In the arxiv version of this paper [1] we also show a new proof of Theorem 1 using the labeling technique.

2 Monotone paths in ordered hypergraphs

In this section, we prove Theorem 5. Our proof is based on an enumerative result of Moshkovitz and Shapira [15]. For ease of notation, we only establish the result for monotone paths of the same lengths in 2-colored hypergraphs (i.e. $k=\ell=n$), our proof is easily generalizable to the cases when the desired monotone paths have different lengths for different (possibly even more than two) colors.

Set $\mathcal{P}_2(n) \stackrel{\text{def}}{=} [n]^2$. With an abuse of notations, for any $x=(x_1,x_2)$, $y=(y_1,y_2) \in \mathcal{P}_2(n)$, write $x \subseteq y$ if $x_1 \leq y_1$ and $x_2 \leq y_2$. Inductively, we define $\mathcal{P}_k(n)$ for $k=3,4,\dots$ as follows:

- A subset $\mathcal{F} \subseteq \mathcal{P}_{k-1}(n)$ is in $\mathcal{P}_k(n)$ if $S \in \mathcal{F}$ implies $S' \in \mathcal{F}$ for any $S' \subseteq S$.

In other words, $\mathcal{P}_k(n)$ contains those subsets of $\mathcal{P}_{k-1}(n)$ that are closed under taking subsets. If we consider the $\mathcal{P}_{k-1}(n)$-s as a poset with respect to $\subseteq$, then $\mathcal{P}_k(n)$ is the family of down-sets of $\mathcal{P}_{k-1}(n)$. We refer to this defining condition of $\mathcal{P}_k(n)$ as the *hereditary property*.

Given any red-blue colored r-uniform complete ordered hypergraph H on the vertex set $[N]$ containing no monochromatic monotone path of length n, we assign labels to each k-tuple with $1 \leq k \leq r-1$ in $V(H)$ as follows:

- For every $(r-1)$-tuple of vertices $v_1 < \cdots < v_{r-1}$, set $L(v_1,\dots,v_{r-1}) \stackrel{\text{def}}{=} (1+\ell_{\mathsf{r}}, 1+\ell_{\mathsf{b}})$, where ℓ_{r} (ℓ_{b}) is the length of the longest red (blue) monotone path ending at $v_1,\dots,v_{r-1}$.

- For $k = r-2, r-3, \dots, 1$ and every k-tuple of vertices $v_1 < \cdots < v_k$, recursively define

$$L(v_1, \dots, v_k) \overset{\text{def}}{=} \{S \in \mathcal{P}_{r-k}(n) : S \subseteq L(v_0, v_1, \dots, v_k) \text{ for some } v_0 < v_1\}.$$

Obviously, the labels of k-tuples are elements in $\mathcal{P}_{r+1-k}(n)$ for $1 \le k \le r-1$. The following result, albeit not specifically stated, was proved by Moshkovitz and Shapira (Lemma 3.2 in [15]). Its proof essentially relies on the hereditary property of $\mathcal{P}$.

▶ **Lemma 6.** *For each $1 \le k \le r-1$ and every $v_1 < v_2 < \cdots < v_{k+1}$ in $V(H)$, we have*

$$L(v_1, \dots, v_k) \not\supseteq L(v_2, \dots, v_{k+1}).$$

In particular, $u \neq v$ implies $L(u) \neq L(v)$, and so $\mathrm{ram}_{\mathsf{p}}^{(r)}(n) \le |\mathcal{P}_r(n)|$.

We use abbreviated notations $\mathrm{ram}_{\mathsf{p}}^{(r)}(n) \overset{\text{def}}{=} \mathrm{ram}_{\mathsf{p}}^{(r)}(n,n)$ and $\mathrm{sat}_{\mathsf{p}}^{(r)}(n) \overset{\text{def}}{=} \mathrm{sat}_{\mathsf{p}}^{(r)}(n,n)$. Moshkovitz and Shapira also provided constructions (Lemma 3.4 in [15]) to prove that

$$\mathrm{ram}_{\mathsf{p}}^{(r)}(n) = |\mathcal{P}_r(n)|. \tag{4}$$

Now we are ready to state the following result. Together with (4) it implies Theorem 5.

▶ **Theorem 7.** *For any integers $r \ge 2$ and $n \ge 1$, we have $\mathrm{sat}_{\mathsf{p}}^{(r)}(n) = |\mathcal{P}_r(n)|$.*

If we have less than $|\mathcal{P}_r(n)|$ vertices, then after creating the vertex labeling, one of the possible labels is missing. We will extend the hypergraph and its coloring so that the new vertex receives one of the missing labels while the labels of other vertices do not change.

Proof of Theorem 7. Let H be a red-blue colored r-uniform complete ordered hypergraph H on the vertex set $V(H) = [N]$ containing no monochromatic monotone path of length n. Suppose $N < |\mathcal{P}_r(n)|$. Our goal is to show that H is not saturated. Let L be the labeling for H as described earlier in this section. Since $N < |\mathcal{P}_r(n)|$, there exists a missing label $M \in \mathcal{P}_r(n)$ with $M \neq L(v)$ for each vertex $v \in V(H)$.

Picking the position. Let w be the largest vertex in $V(H)$ such that $L(w) \subseteq M$. Such a w exists because $L(1)$ takes the minimum value of $\mathcal{P}_r(n)$ with respect to "$\subseteq$". We construct a new r-uniform complete ordered hypergraph H^+ by adding a new vertex v^+ into $V(H)$ right after w and keeping the colors of the edges originally in H. If suffices to show that we can color the additional edges of H^+ without creating monochromatic monotone paths of length n.

Coloring the edges. We begin with assigning potential labels for the hypergraph H^+. For each $1 \le k \le r-1$ and k-tuple $v_1 < \cdots < v_k$ in $V(H^+)$, assign a potential label $\widetilde{L}(v_1, \dots, v_k) \in \mathcal{P}_{r+1-k}(n)$ with

(i) $\widetilde{L}(v_1, \dots, v_k) \not\supseteq \widetilde{L}(v_2, \dots, v_{k+1})$ for all $v_1 < \cdots < v_{k+1}$ in $V(H^+)$, and

(ii) $\widetilde{L}(v_1, \dots, v_k) = L(v_1, \dots, v_k)$ for all $v_1 < \cdots < v_k$ from the original $V(H)$.

Define $\widetilde{L}(v^+) \overset{\text{def}}{=} M$, the missing label, and $\widetilde{L}(v) \overset{\text{def}}{=} L(v)$ for all $v \in V(H)$. By Lemma 6 and our construction of H^+, the conditions (i) and (ii) are satisfied for $k = 1$.

Inductively, suppose the potential labels for all $(k-1)$-tuples have been assigned, and we are in the position to define $\widetilde{L}(v_1, \dots, v_k)$ for every $v_1 < \cdots < v_k$ in $V(H^+)$. Due to (ii), we have to set $\widetilde{L}(v_1, \dots, v_k) \overset{\text{def}}{=} L(v_1, \dots, v_k)$ if this k-tuple comes from $V(H)$. For other k-tuples containing v^+, we define $\widetilde{L}(v_1,\dots,v_k)$ as an arbitrary fixed element from $\widetilde{L}(v_2,\dots,v_k)\setminus\widetilde{L}(v_1,\dots,v_{k-1})$. Since condition (i) is satisfied for $k-1$, such an element always exists. In fact, for any $v_1<\cdots<v_k$ in $V(H)$, we also have

$$\widetilde{L}(v_1,\dots,v_k)\in\widetilde{L}(v_2,\dots,v_k)\setminus\widetilde{L}(v_1,\dots,v_{k-1}).$$

Indeed, due to $\widetilde{L}=L$ on $V(H)$, the only thing we need to check is that $L(v_1,\dots,v_k)\notin L(v_1,\dots,v_{k-1})$. And this follows from the definition of L and Lemma 6.

We need to check that condition (i) is satisfied for k. Suppose for the sake of contradiction that $\widetilde{L}(v_2,\dots,v_{k+1})\subseteq\widetilde{L}(v_1,\dots,v_k)$ for $v_1<\cdots<v_{k+1}$. Then by hereditary property,

$$\widetilde{L}(v_1,\dots,v_k)\in\widetilde{L}(v_2,\dots,v_k)\implies\widetilde{L}(v_2,\dots,v_{k+1})\in\widetilde{L}(v_2,\dots,v_k),$$

which contradicts the fact that $\widetilde{L}(v_2,\dots,v_{k+1})\in\widetilde{L}(v_3,\dots,v_{k+1})\setminus\widetilde{L}(v_2,\dots,v_k)$. So, (i) holds. We conclude that the potential labels $\widetilde{L}$ can be recursively assigned.

Now, we can color the new edges using $\widetilde{L}$, the potential labels. For any edge $v_1\cdots v_r\in E(H^+)$ with $v^+\in\{v_1,\dots,v_r\}$, the condition (i) implies that $\widetilde{L}(v_1,\dots,v_{r-1})\not\supseteq\widetilde{L}(v_2,\dots,v_r)$ as elements in $\mathcal{P}_2(n)=[n]^2$. So, at least one coordinate of $\widetilde{L}(v_2,\dots,v_r)$ is larger than that of $\widetilde{L}(v_1,\dots,v_{r-1})$. Color $v_1\cdots v_r$ red if the first coordinate is larger, and blue if the second coordinate is larger. For edges that are both red and blue, we arbitrarily assign a color.

Finishing the proof. We show that H^+ contains no monochromatic monotone path of length n. For every $(r-1)$-tuple $v_1<\cdots<v_{r-1}$ in $V(H^+)$, set $L^+(v_1,\dots,v_{r-1})\overset{\text{def}}{=}(1+\ell_{\mathsf{r}},1+\ell_{\mathsf{b}})$ where ℓ_{r} (resp. ℓ_{b}) is the length of the longest red (resp. blue) monotone path in H^+ ending at $v_1,\dots,v_{r-1}$. We shall prove that

$$L^+(v_1,\dots,v_{r-1})\subseteq\widetilde{L}(v_1,\dots,v_{r-1})\text{ for all }v_1<\cdots<v_{r-1}\text{ in }V(H^+).\tag{5}$$

Since $\widetilde{L}(v_1,\dots,v_{r-1})$ takes its value in $\mathcal{P}_2(n)=[n]^2$, Theorem 7 follows from (5).

For a contradiction, suppose (5) is violated by some $(r-1)$-tuple in $V(H^+)$. Let $v_1<\cdots<v_{r-1}$ be the smallest such tuple under the lexicographic order. According to the definition of L^+, this violation is witnessed by a monochromatic (say red) monotone path P ending at $v_1,\dots,v_{r-1}$. Let $e\overset{\text{def}}{=}v_0v_1\cdots v_{r-1}$ be the last edge and ℓ be the length of this red path. Then $1+\ell$ is larger than the first coordinate of $\widetilde{L}(v_1,\dots,v_{r-1})$. The minimum assumption on $v_1,\dots,v_{r-1}$ implies that

$$L^+(v_0,\dots,v_{r-2})\subseteq\widetilde{L}(v_0,\dots,v_{r-2}).\tag{6}$$

We then separate our indirect proof into two cases:

- If $v^+\in e$, then $\widetilde{L}(v_1,\dots,v_{r-1})$ has a larger first coordinate than $\widetilde{L}(v_0,\dots,v_{r-2})$ since e is red, and so ℓ is larger than the first coordinate of $\widetilde{L}(v_0,\dots,v_{r-2})$. On the other hand, notice that $P\setminus\{e\}$ is a red monotone path of length $\ell-1$ ending at $v_0,v_1,\dots,v_{r-2}$. This means the first coordinate of $L^+(v_0,\dots,v_{r-2})$ is at least ℓ, a contradiction to (6).
- If $v^+\notin e$, then the vertices $v_0,v_1\dots,v_{r-1}$ are all in $V(H)$. From condition (ii) of the potential labeling and (6) we obtain $L(v_0,\dots,v_{r-2})=\widetilde{L}(v_0,\dots,v_{r-2})\supseteq L^+(v_0,\dots,v_{r-2})$. Again, the path $P\setminus\{e\}$ implies that the first coordinate of $L^+(v_0,\dots,v_{r-2})$ is at least ℓ, and so there is a red monotone path in H ending at $v_0,\dots,v_{r-2}$ of length at least $\ell-1$. Together with e, we have a red monotone path in H ending at $v_1,\dots,v_{r-1}$ of length at least ℓ. It follows that the first coordinate of $L(v_1,\dots,v_{r-1})$ is at least $1+\ell$, which is larger than the first coordinate of $\widetilde{L}(v_1,\dots,v_{r-1})$. This contradicts condition (ii) of the potential labeling. ◀

3 Saturation for cups-versus-caps

In this section, we study the saturation problem for cups and caps. Recall that the definitions directly imply $\mathrm{sat}_{\mathsf{c}}(k,\ell) \le \mathrm{ram}_{\mathsf{c}}(k,\ell)$ for any integers k,ℓ. By reflecting over any horizontal line, it is easily seen that $\mathrm{sat}_{\mathsf{c}}(k,\ell) = \mathrm{sat}_{\mathsf{c}}(\ell,k)$.

The study begins with a basic property of (k,ℓ)-cup-cap-saturated sets.

▶ **Proposition 8.** *Let $k,\ell \ge 2$ be integers. If P is a (k,ℓ)-cup-cap-saturated set, then*

- *there exist $k-1$ points of P that form a $(k-1)$-cup, and*
- *there exist $\ell-1$ points of P that form an $(\ell-1)$-cap.*

Proof. Due to the symmetry, it suffices to prove the first statement. Assume, for the sake of contradiction, that $P \subset \mathbb{R}^2$ is a (k,ℓ)-cup-cap-saturated set without any $(k-1)$-cup subset. Choose an arbitrary point q such that

- the x-coordinate of q is bigger than every point from P, and
- the y-coordinate of q is big enough so that q is above every line spanned by P.

Then $P \cup \{q\}$ is generic and p_1, p_2, q form a 3-cup for any choice of $p_1, p_2 \in P$. This implies that $P \cup \{q\}$ is (k,ℓ)-cup-cap-free, which contradicts the saturation property of P. ◀

We work out some values of $\mathrm{sat}_{\mathsf{c}}(k,\ell)$ where at least one of k and ℓ is small.

▶ **Proposition 9.** *Let ℓ be a positive integer.*

- $\mathrm{sat}_{\mathsf{c}}(1,\ell) = 0 = \mathrm{ram}_{\mathsf{c}}(1,\ell)$ *for any* $\ell \ge 1$.
- $\mathrm{sat}_{\mathsf{c}}(2,\ell) = 1 = \mathrm{ram}_{\mathsf{c}}(2,\ell)$ *for any* $\ell \ge 2$.
- $\mathrm{sat}_{\mathsf{c}}(3,\ell) = \ell - 1 = \mathrm{ram}_{\mathsf{c}}(3,\ell)$ *for any* $\ell \ge 3$.
- $\mathrm{sat}_{\mathsf{c}}(4,4) = 6 = \mathrm{ram}_{\mathsf{c}}(4,4)$.

Proof. The facts $\mathrm{sat}_{\mathsf{c}}(1,\ell) = 0$ and $\mathrm{sat}_{\mathsf{c}}(2,\ell) = 1$ are strightforward corollaries of the definitions. From Proposition 8 we deduce that $\mathrm{ram}_{\mathsf{c}}(3,\ell) = \ell - 1 \le \mathrm{sat}_{\mathsf{c}}(3,\ell)$, and so $\mathrm{sat}_{\mathsf{c}}(3,\ell) = \ell - 1$.

Obviously, $\mathrm{sat}_{\mathsf{c}}(4,4) \ge 3$. Let $P = \{(x_1,y_1),\dots,(x_m,y_m)\}$ be a generic $(4,4)$-cup-cap-free set with $x_1 < \dots < x_m$ and $3 \le m \le 5$. Consider $q \overset{\text{def}}{=} (\frac{x_2+x_3}{2}, y)$ such that $P \cup \{q\}$ is generic. If q is very high up (i.e. above any line formed by two points of P) and $P \cup \{q\}$ is not $(4,4)$-cup-cap-free, then the only case to make this happen is that $(x_3,y_3),(x_4,y_4),(x_5,y_5)$ form a 3-cup. Indeed, q has to be part of some 4-cup or 4-cap in $P \cup \{q\}$, yet it cannot make a 4-cap since the x-coordinate of q is between x_2 and x_3. It follows that $P \cup \{q\}$ is $(4,4)$-cup-cap-free if y is chosen to make q very low below (i.e. below any line formed by two points of P). We conclude that $\mathrm{sat}_{\mathsf{c}}(4,4) \ge 6 = \mathrm{ram}_{\mathsf{c}}(4,4)$, and so $\mathrm{sat}_{\mathsf{c}}(4,4) = 6$. ◀

The following result gives the first example with $\mathrm{sat}_{\mathsf{c}}(k,\ell) < \mathrm{ram}_{\mathsf{c}}(k,\ell)$, which is crucial for our proof of the upper bound in Theorem 2.

▶ **Theorem 10.** *We have* $\mathrm{sat}_{\mathsf{c}}(4,5) = 8 < 10 = \mathrm{ram}_{\mathsf{c}}(4,5)$.

Before proving Theorem 10, we need some preparation. For any point $p = (a,b)$ in the plane, set $x(p) \overset{\text{def}}{=} a$. When denote by $\overline{p_1 \cdots p_m}$ an m-cup or cap, we implicitly assume $x(p_1) < \dots < x(p_m)$. For any m-cup or cap $\overline{p_1 \cdots p_m}$, we define its *shadow* as the open interval $\big(x(p_1), x(p_m)\big)$ in $\mathbb{R}$.

▶ **Lemma 11.** *Let $P \subset \mathbb{R}^2$ be a (k,ℓ)-cup-cap-saturated set with $k \ge 4$ and $\ell \ge 4$. Then P contains two $(k-1)$-cups with disjoint shadows and two $(\ell-1)$-caps with disjoint shadows.*

Proof. We show the existence of two $(k-1)$-cups with disjoint shadows, and there exist two such $(\ell-1)$-caps for similar reasons. For a contradiction, suppose there are no $(k-1)$-cups with disjoint shadows. Then the intersection of all shadows from $(k-1)$-cups in P is nonempty, and so there exists some $x_* \in \mathbb{R}$ in this intersection with $x(p) \neq x^*$ for all $p \in P$. Consider a generic point $q \overset{\text{def}}{=} (x^*, y)$, where y is some sufficiently large number such that, together with q,

- every $p_i, p_j \in P$ with $x(p_i) < x(p_j) < x(q)$ or $x(q) < x(p_i) < x(p_j)$ form a 3-cup, and
- every $p_i, p_j \in P$ with $x(p_i) < x(q) < x(p_j)$ form a 3-cap.

We argue that $P \cup \{q\}$ is (k,ℓ)-cup-cap-free, a contradiction to the hypothesis that P is saturated. Indeed, the construction shows that q cannot be part of any 4-cap in $P \cup \{q\}$, and so $P \cup \{q\}$ contains no ℓ-cap. If q is part of some k-cup $\overline{a_1 \cdots a_s q b_1 \cdots b_t}$ in $P \cup \{q\}$, then $s \geq 1$ and $t \geq 1$, for otherwise x^* would lie outside the shadow of the $(k-1)$-cup $\overline{a_1 \cdots a_s b_1 \cdots b_t}$. However, this contradicts the fact that $\overline{a_1 q b_1}$ is a 3-cap, and so $P \cup \{q\}$ contains no k-cup. ◀

Proof of Theorem 10. First, we show that $\mathrm{sat_c}(4,5) \leq 8$. It suffices to construct a generic 8-point set that is $(4,5)$-cup-cap-saturated. Consider the following 8 points as P:

$$(-60,40),\ (-40,20),\ (-20,16),\ (0,10),\ (5,-50),\ (15,-40),\ (25,-40),\ (125,-230).$$

It is straightforward to verify that P is $(4,5)$-cup-cap-free. We use a computer program to check the saturation property. This program first computes all the lines generated by pairs in P and the vertical lines through every point of P and works out all the regions of $\mathbb{R}^2$ enclosed by these lines. Then the program picks a point from each region, adds it to the point set P, and verifies that the new set contains a 4-cup or a 5-cap. See the full version [1] for our supplementary code.

Next, we prove that $\mathrm{sat_c}(4,5) \geq 8$. Let P be a $(4,5)$-cup-cap-saturated point set, we argue that $|P| \geq 8$. By Lemma 11, we can find two 4-caps in P, $\overline{p_1p_2p_3p_4}$ and $\overline{p_5p_6p_7p_8}$ with $x(p_4) \leq x(p_5)$, whose shadows are disjoint. If $p_4 \neq p_5$, then $|P| \geq 8$ and the proof done. Assume $p_4 = p_5 = p$. Then the shadow of any 3-cup within $P_- \overset{\text{def}}{=} \{p_1, p_2, p_3, p, p_6, p_7, p_8\}$ must contain $x(p)$. By Lemma 11, there are two 3-cups with disjoint shadows in P. So, at least one point from P does not belong to P_-, and hence $|P| \geq 8$. ◀

We shall use Theorem 10 to upper bound $\mathrm{sat_c}(k,\ell)$. It follows from the next lemma.

▶ **Lemma 12.** *For any $k, \ell \geq 3$, we have* $\mathrm{sat}_c(k,\ell) \leq \mathrm{sat}_c(k-1,\ell) + \mathrm{sat}_c(k,\ell-1)$.

Proof. Suppose $P_{k-1,\ell} \subset \mathbb{R}^2$ and $P_{k,\ell-1} \subset \mathbb{R}^2$ are $(k-1,\ell)$- and $(k,\ell-1)$-cup-cap-saturated, respectively. We construct P with $|P| = |P_{k-1,\ell}| + |P_{k,\ell-1}|$ that is (k,ℓ)-cup-cap-saturated.

We begin with a fixed vertical line h. Put a translated copy A of $P_{k-1,\ell}$ to the left of h, and a translated copy B of $P_{k,\ell-1}$ to the right of h. Vertically shift B to somewhere very high so that every point in A is below all lines spanned by B, and every point in B is above all lines spanned by A. This implies that

- any one point from A and two points from B form a 3-cap, and
- any two points from A and one point from B form a 3-cup.

These conditions guarantee that $P \overset{\text{def}}{=} A \cup B$ is (k,ℓ)-cup-cap-free.

We claim that P is (k,ℓ)-saturated. It suffices to disprove that some point $q \notin P$ makes a generic $P \cup \{q\}$ that is (k,ℓ)-cup-cap-free. Without loss of generality, assume that q lies to the left of h (possibly on h). Since A is $(k-1,\ell)$-cup-cap-saturated and $P \cup \{q\}$ is (k,ℓ)-cup-cap-free, the point q has to be part of a $(k-1)$-cup $C_1 = \overline{a_1 \ldots a_{k-1}}$ in $A \cup \{q\}$, where a_{k-1} (possibly be q) lies to the left of h (possibly on h). Since B is $(k,\ell-1)$-cup-cap-saturated and $P \cup \{q\}$ is (k,ℓ)-cup-cap-free, the point a_{k-1} has to be part of an $(\ell-1)$-cap $C_2 = \overline{b_1 \ldots b_{\ell-1}}$

where $b_1 = a_{k-1}$. Now the cup C_1 and the cap C_2 share $a_{k-1} = b_1$, and it follows that either C_1 can be extended by b_2 or C_2 can be extended by a_{k-2}, which contradicts the fact that $P \cup \{q\}$ is (k,ℓ)-cup-cap-free. This completes the proof. ◀

Proof of Theorem 2. The upper bound follows from solving (for $k, \ell \geq 1$) the recursion

$$\mathrm{sat}_{\mathsf{c}}(k,\ell) \leq \mathrm{sat}_{c}(k-1,\ell) + \mathrm{sat}_{c}(k,\ell-1)$$

from Lemma 12, with initial values given by Proposition 9 and Theorem 10.

Notice that any two cups (resp. caps) with disjoint shadows share no more than one point. Since any cup and any cap share no more than two points, from Lemma 11 we deduce that

$$\mathrm{sat}_{\mathsf{c}}(k,\ell) \geq 2(k-1) + 2(\ell-1) - 1 - 1 - 2 - 2 - 2 - 2 = 2k + 2\ell - 14. \quad ◀$$

4 Saturation for convex polygons

This section is devoted to the proof of Theorem 4. Recall that Erdős and Szekeres proved $\mathrm{ram}_{\mathsf{g}}(n) \geq 2^{n-2}$ by taking a union of $(i+1, n+1-i)$-cup-cap-free sets appropriately along a convex curve. Roughly speaking, we shall establish the lower bound on $\mathrm{sat}_{\mathsf{g}}(n)$ via replacing each $(i+1, n+1-i)$-cup-cap-free set before by an $(i+1, n+1-i)$-cup-cap-saturated set.

For a technical issue in Lemma 14, we need the following notion: a planar point set is called *very generic* if it is in general position, and its members together with the intersection points of the lines spanned by it all have distinct x-coordinates. The main result of this section is the following proposition.

▶ **Proposition 13.** *For any positive integer $n \geq 3$, suppose that P_i is a very generic planar point set that is $(i+1, n+1-i)$-cup-cap-saturated for each $1 \leq i < n$, then there exists an n-gon-saturated set P such that $|P| = \sum_{i=1}^{n-1} |P_i|$.*

We point out that our upper bounds in previous sections, Proposition 9, Theorems 2 and 10, all give us (k,ℓ)-cup-cap-saturated sets that have the claimed size and are very generic, although the very generic property is not stated. Together with Proposition 13 and (3), these upper bounds imply that for every $n \geq 7$,

$$\mathrm{sat}_{\mathsf{g}}(n) \leq \sum_{i=1}^{n-1} \binom{n-2}{i-1} - 2\sum_{i=1}^{n-1} \binom{n-6}{i-3} = 2^{n-2} - 2^{n-5} = \frac{7}{8} \cdot 2^{n-2} \leq \frac{7}{8}\,\mathrm{ram}_{\mathsf{g}}(n).$$

This concludes the proof of Theorem 4.

In Proposition 13, the $n = 3$ and $n = 4$ cases are easy. Indeed, every 2-point set is 3-gon-saturated, and every 4-point set not in convex position is 4-gon saturated. For the rest of this section, we implicitly assume $n \geq 5$. To prove Proposition 13, our strategy is to place an appropriate copy of a $(i+1, n+1-i)$-cup-cap-saturated set around the point (i, i^2) for $i = 1, \ldots, n$. This is motivated by the original Erdős–Szekeres construction [6] showing $\mathrm{ram}_{\mathsf{g}}(n) \geq 2^{n-2}$. We remark that a (k,ℓ)-cup-cap-free point set of size $\mathrm{ram}_{\mathsf{c}}(k,\ell)$ is always (k,ℓ)-cup-cap-saturated. Furthermore, a small rotation can be applied such that it becomes very generic. In this way, our Proposition 13 implies that the construction of Erdős and Szekeres is also n-gon-saturated.

We begin with some preparation. For two points $p, q \in \mathbb{R}^2$ with $x(p) \neq x(q)$, denote by

- $\mathsf{line}(p)$ the vertical line through p,
- $\mathsf{line}(p,q)$ the unique line through p and q,
- $\mathsf{ray}(p,q)$ the ray emanating from p through q,
- $\mathsf{slope}(p,q)$ the slope of $\mathsf{line}(p,q)$.

We say that a point set P is $(k, \ell; \varphi)$-*cup-cap-saturated* if P will be (k, ℓ)-cup-cap-saturated after a rotation of any angle $\theta \in (-\varphi, \varphi)$ with arbitrary center. The next result captures the idea that, given φ, if we flatten a (k, ℓ)-cup-cap-saturated set enough, then it becomes $(k, \ell; \varphi)$-cup-cap-saturated. The proof of this lemma can be fund in the full version [1].

▶ **Lemma 14.** *For a very generic (k, ℓ)-cup-cap-saturated set P_0 and a positive real number $\varphi < \pi/2$, there exists some sufficiently small $\delta > 0$ such that the "flattening" map $\sigma : (x, y) \mapsto (x, \delta y)$ produces a set $P \stackrel{\text{def}}{=} \sigma(P_0)$ which is $(k, \ell; \varphi)$-cup-cap-saturated.*

We shall deduce the existence of a large convex polygon by combining a lower cup and a higher cap in many situations. The following combination lemmas will be quite useful (see Figure 1). We omit their easy proofs. Recall that when we denote by $\overline{p_1 \cdots p_m}$ an m-cup or cap, we implicitly assume $x(p_1) < \cdots < x(p_m)$.

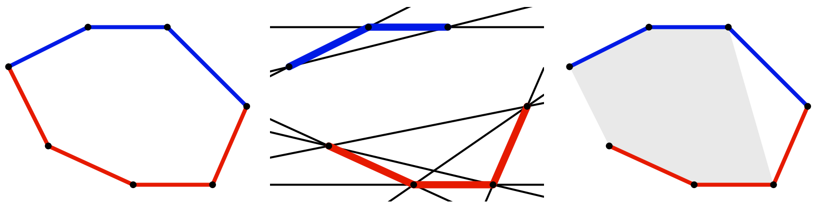

Figure 1 Three different ways of combining cups and caps.

▶ **Observation 15.** *Let $C_- = \overline{sp_1 \cdots p_k t}$ be a cup and $C_+ = \overline{sq_1 \cdots q_\ell t}$ be a cap with $k, \ell \geq 0$. Then $s, p_1, \ldots, p_k, t, q_l, \ldots, q_1$ form a convex $(k + \ell + 2)$-gon with vertices in this order.*

▶ **Observation 16.** *Let $C_- = \overline{p_1 \cdots p_k}$ be a cup and $C_+ = \overline{q_1 \cdots q_\ell}$ be a cap with $k, \ell \geq 2$. If*
- $\mathsf{line}(p_i, p_j)$ *is below C_+ for every $1 \leq i < j \leq k$, and*
- $\mathsf{line}(q_i, q_j)$ *is above C_- for every $1 \leq i < j \leq \ell$,*

then $p_1, \ldots, p_k, q_\ell, \ldots, q_1$ form a convex $(k + \ell)$-gon with vertices in this order.

▶ **Observation 17.** *Suppose $p_1, \ldots, p_k, q_\ell, \ldots, q_1$ form a convex $(k + \ell)$-gon with vertices in this order, where $k \geq 2$, $\ell \geq 2$. If t is a point such that $C_- = \overline{p_1 \cdots p_k t}$ is a $(k + 1)$-cup and $C_+ = \overline{q_1 \cdots q_\ell t}$ is a $(\ell + 1)$-cap, then $p_1, \ldots, p_k, t, q_\ell, \ldots, q_1$ form a convex $(k + \ell + 1)$-gon with vertices in this order.*

We are ready to construct an n-gon-saturated set of small size. Let $p_i \stackrel{\text{def}}{=} (i, i^2)$ and $\varepsilon \gg \delta > 0$ be some sufficiently small constants depending on n that will be fixed later. For each i we will place a "small and flat" $(i + 1, n + 1 - i)$-cup-cap-saturated set in an δ-neighborhood of p_i. To be precise, for any fixed ε we can pick δ and P_i such that

I) P_i is a $(i + 1, n + 1 - i; \pi/3)$-cup-cap-saturated set;
II) $P_i \subset \mathbf{B}(p_i, \delta)$, where $\mathbf{B}(x, r)$ denotes the open disk of radius r around x;
III) $|\mathsf{slope}(x, y)| < \varepsilon$ holds for any distinct $x, y \in P_i$;
IV) $\mathsf{line}(x, y)$ is below P_j and $\mathsf{line}(z, w)$ is above P_i for any distinct $x, y \in P_i$ and any distinct $z, w \in P_j$ with $i < j$;
V) $\mathsf{line}(x, y)$ and $\mathsf{line}(z, w)$ intersect in $\mathbf{B}(p_i, \varepsilon)$ for any $(x, y, z, w) \in P_i \times P_{i+1} \times P_{i+1} \times P_{i+2}$.

For any ε and δ properties I), II), III), IV) can be achieved by sufficient scaling and flattening of an arbitrary very generic $(i + 1, n + 1 - i)$-cup-cap-saturated set. In particular, I) follows from the very generic property and Lemma 14. Finally, by picking δ small enough with respect to ε, V) follows from II). We will refer to the property IV) as the *height hierarchy* of P. We want one more property which needs some explanation. Suppose we have a cup (cap) C of size at least 3 in some P_i. Consider where a point x in the plane might lie such that x extends C to a bigger cup (cap) but x is not the first or the last point in the extended cup (cap). Since the size of the cup (cap) is at least 3, the possible positions of x are bounded regions in the plane. Therefore, by further flattening the sets $P_1, \ldots, P_{n-1}$ we may assume that this bounded region is in $\mathbf{B}(p_i, \delta)$.

VI) For any P_i and cup (cap) C with $|C| \geq 3$ within P_i, all the points of the plane that extend C not at the ends lie in $\mathbf{B}(p_i, \delta)$.

According to Proposition 9, P_1 and P_{n-1} contains a single point. Hence, we pick $P_1 = \{p_1\}$ and $P_{n-1} = \{p_{n-1}\}$ for convenience. Note that the angle between any vertical line and any $\mathsf{line}(p_i, p_{i+1})$ is less than $\pi/3$. Set $P \overset{\text{def}}{=} \bigcup_{i=1}^{n-1} P_i$. Our aim is to show that P is n-gon-saturated.

For this purpose, we will need three different arguments based on the position of the point q. The first case is when q is close to one of the $P_2, \ldots, P_{n-2}$. Denote by $\mathbf{D}_i \overset{\text{def}}{=} \mathbf{B}(p_i, \varepsilon)$ for $i = 2, \ldots, n-2$. We will refer to them as *disks*, and we have $n-3$ disks in total. Recall that a point q is generic (with respect to P) if $P \cup \{q\}$ is generic, and we call q *good* if q is part of an n-gon in $P \cup \{q\}$.

▶ **Proposition 18.** *For $i = 2, 3, \ldots, n-2$, every generic point $q \in \mathbf{D}_i$ is good.*

Proof. Let $a \in P_{i-1}$ be the counterclockwise next vertex on the boundary of $\operatorname{conv}(P_{i-1} \cup \{q\})$ after q, and $b \in P_{i+1}$ be the clockwise next vertex on the boundary of $\operatorname{conv}(P_{i+1} \cup \{q\})$ after q. Then $\ell_- \overset{\text{def}}{=} \mathsf{line}(a, q)$ and $\ell_+ \overset{\text{def}}{=} \mathsf{line}(b, q)$ partition the disk $\mathbf{D}_i$ into four open regions, denote them as $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$, and $\mathcal{R}_4$ according to Figure 2. If some point $p \in P_i$ is in $\mathcal{R}_1 \cup \mathcal{R}_3$, then any sequence of points $q_i \in P_i$ for $i \in \{1, 2, \ldots, i-2, i+2, i+3, \ldots, n-1\}$ together with a, p, q, b form a convex n-gon. To see this note that if ε is small enough, so q is very close to p and therefore we only need that a and b lie on the same side of $\mathsf{line}(p, q)$. This holds exactly when p is in $\mathcal{R}_1 \cup \mathcal{R}_3$.

Assume that P_i lies entirely in $\mathcal{R}_2 \cup \mathcal{R}_4$ then. In this case, we are going to find an n-gon from $P_{i-1} \cup P_i \cup P_{i+1} \cup \{q\}$ using the saturation properties of these sets. The idea is simple, q is part of either a large cup or a large cap in P_i, and we try to combine that with either a cup from P_{i-1} or a cap from P_{i+1}. To make the idea work, we start with a careful rotation of the point set P.

Let τ denote a rotation around q with angle θ. For any $S \subset \mathbb{R}^2$, denote by S' the image of S under τ. We will show that θ can be chosen such that the following properties hold:

(a) $|\theta| < \frac{\pi}{3}$.
(b) $x(q) = x(\tau(q)) > x(\tau(p))$ holds for all $p \in P_{i-1} \cup P_{i+1}$.
(c) P'_{i-1}, P'_i, P'_{i+1} obey the same height hierarchy as P_{i-1}, P_i, P_{i+1}.
(d) $P' \cup \{q\}$ is generic.

Since ε is sufficiently small, our definition of P_i implies that θ can be chosen to meet (a) and (b). (c) is ensured by III), (a), and ε being sufficiently small, as no line from those sets passes through a vertical state during the rotation. Finally, (d) is easy as it only forbids finitely many values for θ. From (b) it follows that for any point $p \in \mathcal{R}_2 \cup \mathcal{R}_4$, in particular for any $p \in P'_i$, $\mathsf{line}(q, p)$ is above P'_{i-1} and below P'_{i+1}. This is the crucial advantage of applying the rotation τ (see the second part of Figure 2).

From (a) and I) we deduce that P'_i is $(i+1, n+1-i)$-cup-cap-saturated, hence (d) implies that the point $q = q'$ is part of some $(i+1)$-cup or $(n+1-i)$-cap in $P'_i \cup \{q\}$. Assume without loss of generality that C_+ is such a cap. Since P'_{i-1} is $(i, n+2-i)$-cup-cap-saturated by I), from Proposition 8 it follows that there exists an $(i-1)$-cup C_- in P'_{i-1}. We claim that C_+ and C_- together form a convex n-gon. From Observation 16 we know that it is enough to check that all lines spanned by C_+ run above C_- and all lines spanned by C_- run below C_+. This follows from (c) for almost all the cases, except for the lines $\mathsf{line}(q,p)$ with $p \in C_+$. We have seen that $\mathsf{line}(q,p)$ is above P'_{i-1} for all $p \in P'_i$, hence these cases are also satisfied. We conclude that $C_+ \cup C_-$ forms a convex n-gon, and the proof is complete. ◀

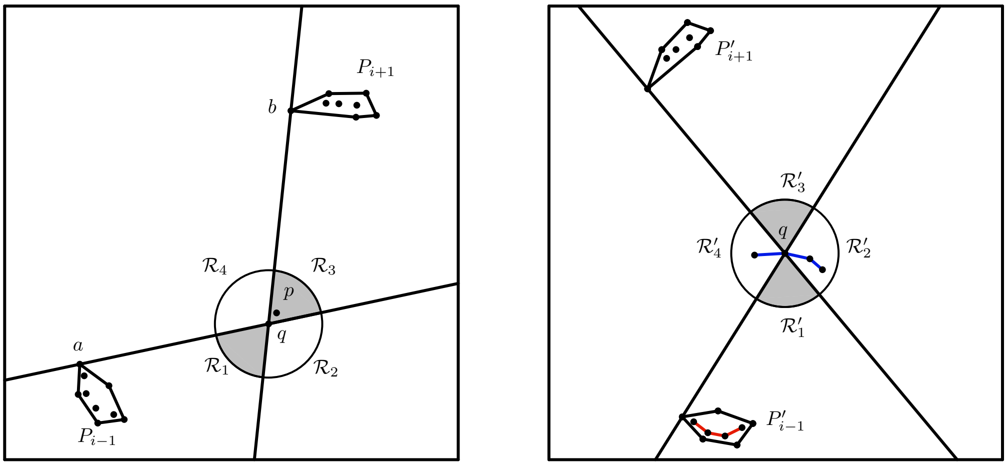

Figure 2 P_{i-1}, P_i and P_{i+1} before and after the rotation.

The next case is when the new point q lies roughly between P_i and P_{i+1}. Define

$$\mathbf{T}_i \stackrel{\text{def}}{=} \operatorname{conv}(P_i \cup P_{i+1}) \setminus (\mathbf{D}_i \cup \mathbf{D}_{i+1})$$

for $i = 1, \ldots, n-2$. These regions are called *tubes*. However, we need two more unbounded tube regions for technical reasons. From Proposition 9 we know that the set P_2 itself is an $(n-2)$-cap, and the set P_{n-2} itself is an $(n-2)$-cup. Suppose P_2 is $\overline{p_2^1 \cdots p_2^{n-2}}$, P_{n-2} is $\underline{p_{n-2}^1 \cdots p_{n-2}^{n-2}}$, and let

- $\mathbf{T}_0$ denote the region enclosed by $\mathsf{ray}(p_2^1, p_1)$, $\mathsf{ray}(p_2^{n-2}, p_1)$ with apex p_1,
- $\mathbf{T}_{n-1}$ denote the region enclosed by $\mathsf{ray}(p_{n-2}^1, p_{n-1}), \mathsf{ray}(p_{n-2}^{n-2}, p_{n-1})$ with apex p_{n-1}.

Note that we have defined n *tubes* $\mathbf{T}_0, \ldots, \mathbf{T}_{n-1}$ in total.

▶ **Proposition 19.** *For $i = 0, 1, \ldots, n-1$, every generic point $q \in \mathbf{T}_i$ is good.*

The proof of this proposition is can be found in the full version [1].

Our third case deals with any new point that is not in disks or tubes. Recall that P_2 is an $(n-2)$-cap $\overline{p_2^1 \cdots p_2^{n-2}}$ and P_{n-2} is an $(n-2)$-cup $\underline{p_{n-2}^1 \cdots p_{n-2}^{n-2}}$. For indices $i = 2$ and $n-2$, we define $\overleftrightarrow{C_i}$ as the piecewise linear curve that consists of $\mathsf{ray}(p_i^2, p_i^1)$, the line segment $p_i^2 p_i^{n-3}$, and $\mathsf{ray}(p_i^{n-3}, p_i^{n-2})$. Denote by $\mathbf{O}$ the points of the plane that lie outside all disks $\mathbf{D} \stackrel{\text{def}}{=} \bigcup_{i=2}^{n-2} \mathbf{D}_i$ and all tubes $\bigcup_{i=0}^{n-1} \mathbf{T}_i$. Then $\overleftrightarrow{C_2}$ and $\overleftrightarrow{C_{n-2}}$ partition $\mathbf{O}$ into six connected regions $\mathbf{O}_{\mathrm{UL}}$, $\mathbf{O}_{\mathrm{UR}}$, $\mathbf{O}_{\mathrm{DL}}$, $\mathbf{O}_{\mathrm{DR}}$, $\mathbf{O}_{\mathrm{L}}$, $\mathbf{O}_{\mathrm{R}}$, where the indices suggest their relative positions (such as upper-left, lower-right, and so on). Figure 3 illustrates the $n = 5$ case.

▶ **Proposition 20.** *Every generic point of $\mathbf{O}$ is good.*

The proof of this proposition can be found in the full version [1].

Finally, we are ready to finish the proof and hence conclude this section.

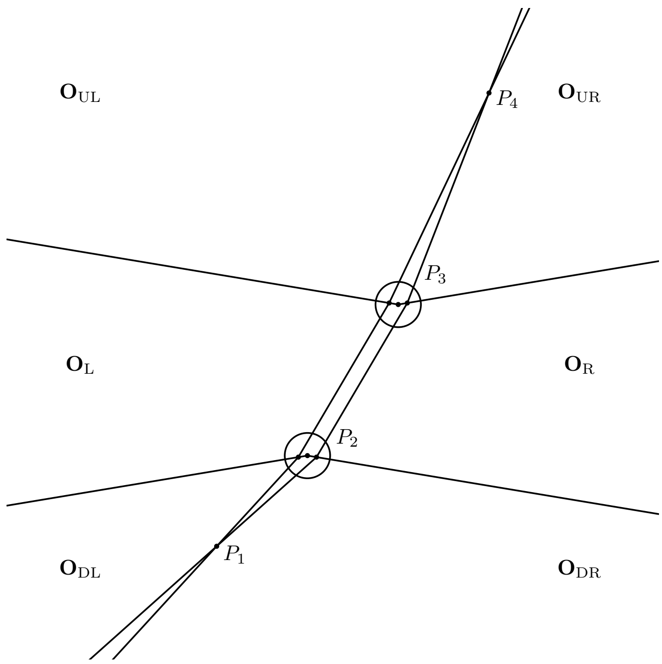

Figure 3 Regions around the construction.

Proof of Proposition 13. Let P be as described in this section. By its construction, we clearly have $|P| = \sum_{i=1}^{n-1} |P_i|$. It then suffices to argue that P is n-gon-saturated.

First, we show that P is n-gon-free. This is parallel to the original Erdős–Szekeres proof of their construction. Suppose that $G \subset P$ is in convex position. G cannot be contained in a single P_i because any n-gon contains either a $(i+1)$-cup or an $(n+1-i)$-cap. So, we can assume that G intersects at least two of $P_1, \ldots, P_{n-1}$. It follows from the height hierarchy that any four points $q_1 \in P_i$, $q_2, q_3 \in P_j$, $q_4 \in P_k$ with $i < j < k$ are not in convex position. Thus, there are at most two of $P_1, \ldots, P_{n-1}$ containing more than one point of G, and the other points of G are between these two groups. So, there exist $1 \leq i_1 < i_2 \leq n-1$ such that $|P_j \cap G| \leq 1$ for $j = i_1+1, \ldots, i_2-1$ and $|P_j \cap G| = 0$ for $j < i_1$ and $j > i_2$. Since G is not contained in a single group, $P_{i_1} \cap G$ must be a cup and $P_{i_2} \cap G$ must be a cap. From the saturation properties of P_{i_1}, P_{i_2} we obtain

$$|G| \leq i_1 + (i_2 - i_1 - 1) \cdot 1 + (n - i_2) = n - 1.$$

We then argue that any $q \notin P$ is part of an n-gon as long as $P \cup \{q\}$ is in general position. Consider all lines spanned by P. These lines cut the plane into polygonal cells and since $P \cup \{q\}$ is in general position q lies in the interior of a cell. If any other point in its cell is part of a convex n-gon, then q is part of a convex n-gon with the same $n-1$ other vertices from P. Hence, if needed, we can move q within its cell such that $P \cup \{q\}$ becomes generic. Then, by putting Propositions 18–20 together, the proof is complete. ◀

5 Final remarks

Our current upper bounds on $\mathrm{sat}_{\mathtt{c}}(k, \ell)$ and $\mathrm{sat}_{\mathtt{g}}(n)$ are exponential, while the lower bounds are linear ($\mathrm{sat}_{\mathtt{g}}(n) \geq n-1$ is trivial). The obvious problem is to obtain better bounds.

▶ **Problem 21.** *What is the correct asymptotics for* $\mathrm{sat}_{\mathtt{c}}(k, \ell)$ *and* $\mathrm{sat}_{\mathtt{g}}(n)$*?*

It is known that the Ramsey number equals to $\binom{k+\ell-4}{k-2}$ for monotone paths in 3-uniform complete ordered hypergraphs with transitive 2-colorings, see e.g. [15].

▶ **Problem 22.** *Is the saturation number equal to* $\binom{k+\ell-4}{k-2}$ *again for monotone paths in* 3*-uniform complete ordered hypergraphs with transitive* 2*-colorings?*

Inside a generic planar point set P, a subset S is called an *n-hole* if S forms an n-gon whose convex hull contains no points of P in its interior. A construction due to Horton [12] shows that there are arbitrarily large point sets without n-holes for every $n \geq 7$.

▶ **Problem 23.** *For $n \geq 7$, is the saturation number for n-holes bounded (i.e. finite)?*

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