Colorful Intersections and Tverberg Partitions

Michael Gene Dobbins
Department of Mathematics and Statistics, Binghamton University, NY, USA

Andreas F. Holmsen
Department of Mathematical Sciences, KAIST, Daejeon, South Korea
Discrete Mathematics Group, Institute for Basic Science (IBS), Daejeon, South Korea

Dohyeon Lee
Department of Mathematical Sciences, KAIST, Daejeon, South Korea
Discrete Mathematics Group, Institute for Basic Science (IBS), Daejeon, South Korea

Abstract

The colorful Helly theorem and Tverberg's theorem are fundamental results in discrete geometry. We prove a theorem which interpolates between the two. In particular, we show the following for any integers $d \geq m \geq 1$ and k a prime power. Suppose $F_1, F_2, \ldots, F_m$ are families of convex sets in $\mathbb{R}^d$, each of size $n > (\frac{d}{m}+1)(k-1)$, such that for any choice $C_i \in F_i$ we have $\bigcap_{i=1}^{m} C_i \neq \emptyset$. Then, one of the families F_i admits a Tverberg k-partition. That is, one of the F_i can be partitioned into k nonempty parts such that the convex hulls of the parts have nonempty intersection. As a corollary, we also obtain a result concerning r-dimensional transversals to families of convex sets in $\mathbb{R}^d$ that satisfy the colorful Helly hypothesis, which extends the work of Karasev and Montejano.

2012 ACM Subject Classification Theory of computation → Computational geometry; Mathematics of computing → Combinatorics; Mathematics of computing → Topology

Keywords and phrases Tverberg's theorem, geometric transversals, topological combinatorics, configuration space/test map, discrete Morse theory

Digital Object Identifier 10.4230/LIPIcs.SoCG.2024.52

Funding *Michael Gene Dobbins*: Support from KAIST Advanced Institute for Science-X (KAI-X).
Andreas F. Holmsen: Supported by the Institute for Basic Science (IBS-R029-C1).
Dohyeon Lee: Supported by the Institute for Basic Science (IBS-R029-C1).

1 Introduction

1.1 Background

A *k-partition* of a finite set X is an *ordered* partition of X into k-nonempty parts, that is, a k-tuple

$$(X_1, X_2, \ldots, X_k),$$

such that $X = X_1 \cup X_2 \cup \cdots \cup X_k$ and $X_i \neq \emptyset$ for all i. If the elements of X are points (or convex sets) in $\mathbb{R}^d$, a k-partition of X, $(X_1, X_2, \ldots, X_k)$, is called a *Tverberg k-partition* provided that

$$(\text{conv } X_1) \cap (\text{conv } X_2) \cap \cdots \cap (\text{conv } X_k) \neq \emptyset.$$

A fundamental result of discrete geometry is the celebrated theorem of Tverberg [17], which asserts that if $|X| > (d+1)(k-1)$, then X admits a Tverberg k-partition.

40th International Symposium on Computational Geometry (SoCG 2024).
Editors: Wolfgang Mulzer and Jeff M. Phillips; Article No. 52; pp. 52:1–52:13
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

Let $F_1, F_2, \dots, F_m$ be families of sets. We say that $F_1, F_2, \dots, F_m$ satisfy *the colorful intersection property* if

$$C_1 \cap C_2 \cap \dots \cap C_m \neq \emptyset$$

for every choice $C_1 \in F_1, C_2 \in F_2, \dots, C_m \in F_m$.

Another fundamental result of discrete geometry is the colorful Helly theorem [1]. It asserts that if $F_1, F_2, \dots, F_{d+1}$ are finite families of convex sets in $\mathbb{R}^d$ that satisfy the colorful intersection property, then there is a point in common to every member of one of the families. Observe that it is no loss in generality to assume that $|F_i| = n$ for all i, in which case the conclusion asserts (in our terminology) that one of the F_i admits a Tverberg n-partition.

Tverberg's theorem and the colorful Helly theorem have both played important roles in the development of discrete geometry, and there are a number of generalizations and extensions. For further information on the subject, we suggest the reader consult [2, 3, 5, 6, 8, 12] and the references therein.

In the last decade, a particular intriguing question related to the colorful Helly theorem has been under investigation: *What conclusions (if any) can be drawn if we are given fewer than $d + 1$ families of convex sets in $\mathbb{R}^d$ which satisfy the colorful intersection property?* See [15, 11, 14] for some answers to this question, as well as [11, Conjecture 1] and [3, Problems 8.1 and 8.2].

1.2 Main results

Our main result can be viewed as an interpolation between the colorful Helly theorem and Tverberg's theorem.

▶ **Theorem 1.** *Given integers $d \geq m \geq 1$ and k a prime power. Suppose $F_1, F_2, \dots, F_m$ are families of convex sets in $\mathbb{R}^d$, with $|F_i| = n > (\frac{d}{m} + 1)(k - 1)$, that satisfy the colorful intersection property. Then one of the F_i admits a Tverberg k-partition.*

▶ Remark 2. Note that for $m = d$, our theorem follows immediately from the colorful Helly theorem. First project the sets into a hyperplane and apply the colorful Helly theorem. There is a point in common to all the members of one of the projected families, and the preimage of this point is a line that intersects all the members of one of the F_i. Now, just apply Tverberg's theorem within this line.

For smaller m this argument becomes generally less effective. Projecting into an $(m-1)$-flat and applying the colorful Helly theorem would give a $(d - m + 1)$-flat transversal to one of the families F_i, but to obtain a Tverberg k-partition within this $(d - m + 1)$-flat would require $|F_i| > (d - m + 2)(k - 1)$, which is significantly worse than the size of F_i that we require in Theorem 1.

▶ Remark 3. Whenever d is a multiple of m, our theorem is optimal with respect to the size of the families. Indeed, consider the following construction in $\mathbb{R}^d = \mathbb{R}^t \times \mathbb{R}^t \times \dots \times \mathbb{R}^t$ with m copies of $\mathbb{R}^t$, where $t = \frac{d}{m}$. Let $X \subset \mathbb{R}^t$ be a set of $(t + 1)(k - 1)$ points which does not admit a Tverberg k-partition, and for $1 \leq i \leq m$ define the family

$$F_i := \big\{\mathbb{R}^t \times \dots \times \mathbb{R}^t \times \underbrace{\{x\}}_{i\text{th factor}} \times \mathbb{R}^t \times \dots \times \mathbb{R}^t : x \in X\big\}.$$

Observe that these families satisfy the colorful intersection property, but none of them has a Tverberg k-partition.

▶ Remark 4. The prime power assumption in our Theorem 1 appears to be just an artifact of our proof method, and we conjecture that the theorem also holds when k is any positive integer.

Another interesting instance that our proof method falls short of would be to generalize Theorem 1 to the case where the F_i may have distinct sizes and we ask for one of the F_i to admit a Tverberg k_i-partition. We leave it to the reader's imagination to formulate (and prove) such a conjecture.

1.3 An application to geometric transversals

Some of the earliest results dealing with the colorful intersection property for less than $d + 1$ families of convex sets in $\mathbb{R}^d$ are the transversal theorems of Karasev and Montejano [15, 14]. The simplest case asserts: *Given three red convex and three blue convex sets in* $\mathbb{R}^3$*, where each red set intersects each blue set, then there is a line that intersects every red set or a line that intersects every blue set.* More generally, we have the following

▶ **Corollary 5.** *Let $F_1, F_2, \ldots, F_m$ be families of convex sets in $\mathbb{R}^d$, each of size $k+r$, satisfying the colorful intersection property, where k is a prime power. If $d < \frac{(r+1)m}{k-1}$, then one of the F_i have an r-dimensional affine flat transversal.*

Proof. Theorem 1 implies that one of the F_i admits a Tverberg k-partition. That is, for $F_i = \{C_1, C_2, \ldots, C_{k+r}\}$, there is a k-partition $(X_1, X_2, \ldots, X_k)$ of $[k+r]$ and a point $x \in \mathbb{R}^d$ such that $x \in \operatorname{conv}\left(\bigcup_{i \in X_j} C_i\right)$ for every $1 \leq j \leq k$. This means that we can choose a point $x_i \in C_i$, for every $1 \leq i \leq k + r$, such that $x \in \operatorname{conv}\{x_i\}_{i \in X_j}$ for every $1 \leq j \leq k$. Let A_j denote the affine hull of $\{x_i\}_{i \in X_j}$, and let A be the affine hull of $A_1 \cup A_2 \cup \cdots \cup A_k$. Then $x \in A_1 \cap A_2 \cap \cdots \cap A_k$, and so

$$\dim A \leq \sum_{j=1}^{k} \dim A_j = \sum_{j=1}^{k} (|X_j| - 1) = r.$$

Since $x_i \in A$ for every $1 \leq i \leq k + r$, it follows that A is an affine flat of dimension at most r that intersects every C_i. ◀

▶ Remark 6. We note that the work of Karasev and Montejano deals with the cases $k = 2$ [15, Theorem 8] and $k = m$ [15, Corollary 7], but without any primality condition on m. For $k = 2$, their theorem requires $d < r + m + 1$, while Corollary 5 allows for $d < (r + 1)m$. However, for the case $k = m$, their theorem requires $d < (\frac{r}{m-1} + 1)m$ which is slightly better than our bound $d < (\frac{r+1}{m-1})m$ given by Corollary 5.

▶ Remark 7. The work of Karasev and Montejano makes use of Schubert calculus and the Lusternik–Schnirelmann category of the Grassmannian. In contrast, our proof uses a combination of the configuration space / test map scheme from topological combinatorics (see e.g. [13]) and Sarkaria's tensor method from discrete geometry (see e.g. [12, section 8.3]). The latter appear more frequently in discrete and computational geometry literature, so we expect these methods to attract broader interest from the community.

2 Proof of Theorem 1

We will show that a hypothetical counter-example to Theorem 1 would contradict the following

▶ **Theorem** (Volovikov [18]). *Let $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ be the product of finitely many copies, with p prime. Let X and Y be fixed point free G-spaces, where X is n-connected and Y is finite dimensional and homotopy equivalent to S^n. Then there is no G-equivariant map $X \to Y$.*

To reach a contradiction, we use Sarkaria's tensor method to get geometric criteria for the existence of a Tverberg k-partition, which we then use to construct such an equivariant map. We then use discrete Morse theory to show that the domain of this map is sufficiently connected to violate Volovikov's theorem.

2.1 The configuration space $K_{n,k}^{*m}$

Given integers $n > k \geq 1$, we let $V_{n,k}$ denote the set of *surjective* maps

$$\varphi : [n] \to [k].$$

Note that we can equivalently think of $V_{n,k}$ as the set of k-partitions on $[n]$

$$\big(\varphi^{-1}(1), \dots, \varphi^{-1}(k)\big).$$

For a family of sets $F = \{C_1, \dots, C_n\}$ and $\varphi \in V_{n,k}$, we write φF to denote the k-partition

$$\big(F^{(1)}, \dots, F^{(k)}\big)$$

where $F^{(i)} = \{C_j \ : \ j \in \varphi^{-1}(i)\}$.

We define $K_{n,k}$ to be the simplicial complex on the vertex set $V_{n,k}$ whose faces consists of subsets $\sigma = \{\varphi_1, \dots, \varphi_r\} \subset V_{n,k}$ such that

$$X_i = \varphi_1^{-1}(i) \cap \cdots \cap \varphi_r^{-1}(i) \neq \emptyset \tag{1}$$

for every $i \in [k]$. Note that the facets of $K_{n,k}$ correspond to a choice of representative from each part of a k-partition, and the vertices of a facet correspond to all the ways of extending this choice of representatives to a k-partition.

The symmetric group S_k acts (freely) on $V_{n,k}$ by permuting the parts of the partition. That is, for $g \in S_k$ and $\varphi = \big(\varphi^{-1}(1), \dots, \varphi^{-1}(k)\big) \in V_{n,k}$ we have

$$g\varphi = \big(\varphi^{-1}(g(1)), \dots, \varphi^{-1}(g(k))\big),$$

which means that S_k acts freely on $K_{n,k}$.

In order to apply Volovikov's theorem, we need a lower bound on the connectedness of $K_{n,k}$. This is given by

▶ **Lemma 8.** *For all $n > k \geq 1$, the simplicial complex $K_{n,k}$ is $(n - k - 1)$-connected.*

We give the proof of this lemma in section 3. For now we proceed with the proof of Theorem 1 assuming the bound on the connectedness of $K_{n,k}$. We will construct an equivariant map on the m-fold join $K_{n,k}^{*m} = K_{n,k} * \cdots * K_{n,k}$.

2.2 Sarkaria's criterion

Here we demonstrate one of the standard methods for proving Tverberg's theorem, Sarkaria's tensor method. The method is usually applied for collections of points, but here we apply it to families of convex sets, similar to the approach taken in [16].

For each $i \in [k]$, define the vector $v_i \in \mathbb{R}^k$ as

$$v_i = e_i - \tfrac{1}{k}\mathbf{1},$$

where e_i is the ith standard unit vector and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^k$. Observe that the v_i satisfy only one linear dependency up to a scalar multiple, which is

$$v_1 + \cdots + v_k = 0. \tag{2}$$

Next, define the map

$$\begin{array}{rccc} L_i : & \mathbb{R}^d & \to & \mathbb{R}^{(d+1)\times k} \\ & x & \mapsto & \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \otimes v_i \end{array}$$

where $\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ denotes the vector in $\mathbb{R}^{d+1}$ obtained from x by appending an additional coordinate and setting this equal to 1. For a given $x \in \mathbb{R}^d$, we will regard the image $L_i(x)$ as a $(d+1) \times k$ matrix. Observe that since each v_i is orthogonal to $\mathbf{1} \in \mathbb{R}^k$, it follows that $L_i(x)$ belongs to the subspace

$$Y = \{\, [w_1 \ \cdots \ w_k] \ : \ w_1 + \cdots + w_k = \mathbf{0}\} \subset \mathbb{R}^{(d+1)\times k}.$$

We note that the symmetric group S_k acts on the subspace Y by permuting the columns, and for $g \in S_k$ and $L_i(x) \in Y$, we have

$$gL_i(x) = g(\left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \otimes v_i) = \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right] \otimes v_{g(i)} = L_{g(i)}(x).$$

Observe that the action is not free, but it is *fixed-point free* on $Y \setminus \{0\}$.

For a convex set $C \subset \mathbb{R}^d$, we write $L_i C$ to denote the set

$$L_i C = \{L_i(x) \ : \ x \in C\},$$

which is a convex subset of Y. The crucial step of the Sarkaria method is the following

▶ **Observation 9.** *Let $F = \{C_1, \dots, C_n\}$ be a family of convex sets in $\mathbb{R}^d$ and $\varphi \in V_{n,k}$. If*

$$0 \in \operatorname{conv}\left(L_{\varphi(1)}C_1 \cup \cdots \cup L_{\varphi(n)}C_n\right),$$

then φF is a Tverberg k-partition.

Indeed, suppose $0 = \alpha_1 L_1(x_1) + \cdots + \alpha_k L_k(x_k)$ is a convex combination, where

$$L_i(x_i) = \left[\begin{smallmatrix} x_i \\ 1 \end{smallmatrix}\right] \otimes v_i \in \operatorname{conv}\left(\bigcup_{j\in\varphi^{-1}(i)} L_i C_j\right).$$

Then in each coordinate, the $\alpha_i \left[\begin{smallmatrix} x_i \\ 1 \end{smallmatrix}\right]$ are the coefficients of a linear dependency of the v_i, and since (2) is the unique linear dependency up to scalar multiple, we have

$$\alpha_1 = \cdots = \alpha_k \quad \text{and} \quad x_1 = \cdots = x_k.$$

We also have $x_i \in \operatorname{conv}\left(\bigcup_{j\in\varphi^{-1}(i)} C_j\right)$, so φF is a Tverberg k-partition.

2.3 The test map f

Consider a single family $F_1 = \{C_1, \dots, C_n\}$ of compact convex sets in $\mathbb{R}^d$ which *does not* have a Tverberg k-partition. We show how this gives us an equivariant map

$$f_1 : K_{n,k} \to Y.$$

For $\varphi \in V_{n,k}$, consider the k-partition φF_1. By hypothesis, this is not a Tverberg k-partition, so by Observation 9, there is a vector $a_\varphi \in Y$ which defines an open halfspace

$$H_\varphi = \{y \in Y : a_\varphi \cdot y > 0\}$$

such that

$$L_{\varphi(1)} C_1 \cup \cdots \cup L_{\varphi(n)} C_n \subset H_\varphi. \tag{3}$$

It is important that the vectors a_φ are chosen such that

$$g a_\varphi = a_{g\varphi}$$

for every $g \in S_k$. This can be done by first choosing one vector a_φ in each S_k orbit, and then allowing the rest of the vectors in that orbit to be defined accordingly.

To verify that such a choice is valid, suppose that a_φ has been chosen such that the containment (3) holds and consider $g \in S_k$. Since $gL_i(x) = L_{g(i)}(x)$, we get

$$\begin{array}{rcccl} L_{g\varphi(i)} C_i & = & g(L_{\varphi(i)} C_i) & \subset & g(H_\varphi) \\ & & & = & \{g(y) : a_\varphi \cdot y > 0\} \\ & & & = & \{z : a_\varphi \cdot g^{-1}(z) > 0\} \\ & & & = & \{z : g a_\varphi \cdot z > 0\} \\ & & & = & \{z : a_{g\varphi} \cdot z > 0\} = H_{g\varphi}. \end{array}$$

The equivariant map $f_1 : K_{n,k} \to Y$ is defined by affine extension of $a_\bullet$ by setting

$$f_1(\sigma) = \operatorname{conv}\{a_\varphi : \varphi \in \sigma\},$$

for every face $\sigma \in K_{n,k}$.

Now consider the setting of Theorem 1 where we have families $F_1, \dots, F_m$ of compact convex sets in $\mathbb{R}^d$, each of size n, and suppose none of them have a Tverberg k-partition. For every $1 \le i \le m$, the family F_i gives us an equivariant map $f_i : K_{n,k} \to Y$ as defined above. By taking joins, we get an equivariant map

$$f = f_1 * f_2 * \cdots * f_m : K_{n,k}^{*m} \to Y.$$

(Note that the symmetric group S_k acts freely on $K_{n,k}^{*m}$ by applying the group action to each component of the join.)

2.4 Using colorful intersections to avoid a subspace

We now want to determine the range of f, and for this we use the assumption that the families F_i satisfy the colorful intersection property.

First we note that the facets of $K_{n,k}$ are in one-to-one correspondence with *injective* functions

$$\rho : [k] \to [n].$$

In particular, each facet $\sigma \in K_{n,k}$ corresponds to the unique injection $\rho : [k] \to [n]$ that has σ as its set of left inverses. That is,

$$\varphi \in \sigma \iff \varphi \circ \rho = \mathrm{id}_{[k]}.$$

Consider a facet $\sigma = \sigma_1 * \sigma_2 * \cdots * \sigma_m \in K_{n,k}^{*m}$ with the corresponding injections ρ_i satisfying $\varphi \circ \rho_i = \mathrm{id}_{[k]}$ for all $\varphi \in \sigma_i$. For each $1 \le i \le m$, we apply ρ_i to select a k-tuple of distinct convex sets

$$C^{(i)}_{\rho_i(1)}, C^{(i)}_{\rho_i(2)}, \ldots, C^{(i)}_{\rho_i(k)} \in F_i.$$

The colorful intersection property now guarantees that, for every $1 \le j \le k$, we can select a point $x_j \in \mathbb{R}^d$ which satisfies

$$x_j \in C^{(1)}_{\rho_1(j)} \cap C^{(2)}_{\rho_2(j)} \cap \cdots \cap C^{(m)}_{\rho_m(j)}.$$

Now consider a vertex $\varphi \in \sigma_i$ and the halfspace $H_\varphi = \{y \in Y : a_\varphi \cdot y > 0\}$ from the definition of the test map f. Since $\varphi \circ \rho_i = \mathrm{id}_{[k]}$, we get

$$L_1 C^{(i)}_{\rho_i(1)} \cup L_2 C^{(i)}_{\rho_i(2)} \cup \cdots \cup L_k C^{(i)}_{\rho_i(k)} \subset H_\varphi,$$

by the containment (3), which in turn implies

$$\big\{L_1(x_1), L_2(x_2), \ldots, L_k(x_k)\big\} \subset H_\varphi.$$

Thus, for every vertex $\varphi \in \sigma$ and every $1 \le j \le k$, we have $a_\varphi \cdot L_j(x_j) > 0$, which gives us

$$f(\sigma) = \mathrm{conv}\{a_\varphi : \varphi \in \sigma\} \subset \{y \in Y : L_j(x_j) \cdot y > 0\}. \tag{4}$$

We claim that $f(\sigma)$ does not intersect the subspace

$$B := e_{d+1} \otimes \mathbb{R}^k.$$

For the sake of contradiction, suppose there were a vector $c = (c_1, c_2, \ldots, c_k) \in \mathbb{R}^k$ such that

$$b = e_{d+1} \otimes c \in f(\sigma) \cap B.$$

Let $j \in [k]$ be a coordinate such that c_j is minimized. Then

$$v_j \cdot c = \tfrac{k-1}{k} c_j - \tfrac{1}{k} \textstyle\sum_{i \neq j} c_i \le 0,$$

and so

$$L_j(x_j) \cdot b = \left(\left[\begin{smallmatrix} x_j \\ 1 \end{smallmatrix}\right] \otimes v_j\right) \cdot (e_{d+1} \otimes c) = \left(\left[\begin{smallmatrix} x_j \\ 1 \end{smallmatrix}\right] \cdot e_{d+1}\right) (v_j \cdot c) \le 0,$$

but $L_j(x_j) \cdot b > 0$ by (4) since $b \in f(\sigma)$, so $f(\sigma)$ cannot intersect B.

We conclude that the range of f is contained in $Y \setminus B$, so we have an equivariant map

$$f : K_{n,k}^{*m} \to Y \setminus B.$$

2.5 Finishing the proof

By Lemma 8, it follows that the complex $K_{n,k}^{*m}$ is $(m(n-k+1)-2)$-connected (see e.g. [13, Proposition 4.4.3]), and since $n > (\frac{d}{m}+1)(k-1)$, we get

$$m(n-k+1)-2 > d(k-1)-2.$$

Therefore $K_{n,k}^{*m}$ is at least $(d(k-1)-1)$-connected.

The symmetric group S_k acts on $Y \setminus B$ by permuting columns, which makes it a fixed-point free action. If $k = p^r$, then the action is also fixed-point free with respect to the subgroup $G = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$. The subspace $Y \cap B^{\perp}$ has dimension $d(k-1)$ as it consists of the matrices in Y whose $(d+1)$st row is equal to the 0-vector, and therefore $Y \setminus B$ is homotopy equivalent to $S^{d(k-1)-1}$. Consequently, the existence of the map f contradicts Volovikov's theorem. ◀

3 Proof of Lemma 8

Here we show that the simplicial complex $K_{n,k}$ is $(n-k-1)$-connected. The proof is in two steps. First we define a polyhedral complex $C_{n,k}$ whose cells correspond to *partial surjective functions* $\pi : [n] \to [k]$, and use Quillen's fiber lemma to show that $C_{n,k}$ is homotopy equivalent to $K_{n,k}$. We then bound the connectedness of $C_{n,k}$ using discrete Morse theory.

3.1 The complex of partial surjections

Given integers $n > k \geq 1$, we let $C_{n,k}$ denote the set of *partial surjective functions*

$$\eta : [n] \to [k].$$

Equivalently, we can think of $C_{n,k}$ as the set of k-partitions

$$\big(\eta^{-1}(1), \eta^{-1}(2), \ldots, \eta^{-1}(k)\big)$$

of *subsets* of $[n]$. Furthermore, we may identify an element $\eta \in C_{n,k}$ with the product of simplices

$$\Delta_\eta := 2^{\eta^{-1}(1)} \times 2^{\eta^{-1}(2)} \times \cdots \times 2^{\eta^{-1}(k)},$$

whose geometric realization is a convex polytope of dimension $|\eta^{-1}([k])| - k$.

Observe that for $\eta, \gamma \in C_{n,k}$, if $\eta^{-1}([k]) \subset \gamma^{-1}([k])$ and $\eta(x) = \gamma(x)$ for all $x \in \eta^{-1}([k])$, then Δ_η is a face of Δ_γ. Consequently, $C_{n,k}$ has the structure of a polyhedral complex (see Figure 1). Note that $C_{n,k}$ may also be viewed as a poset where the faces are ordered by inclusion. We need the following.

▶ **Lemma 10.** *For all integers $n > k \geq 1$, we have a homotopy equivalence $K_{n,k} \simeq C_{n,k}$.*

Proof. Let $K = K_{n,k}$ and $C = C_{n,k}$. Our goal is to construct an order-reversing map

$$g : K \to C$$

such that $g^{-1}(C_{\succeq \eta})$ is contractible for every $\eta \in C$, where $C_{\succeq \eta} = \{\gamma \in C : \Delta_\gamma \supset \Delta_\eta\}$. This will establish the desired homotopy equivalence by Quillen's fiber lemma [4, Theorem 10.5].

To this end, consider a face $\sigma = \{\varphi_1, \varphi_2, \ldots, \varphi_{|\sigma|}\} \in K_{n,k}$ and define the k-tuple

$$X = (X_1, X_2, \ldots, X_k),$$

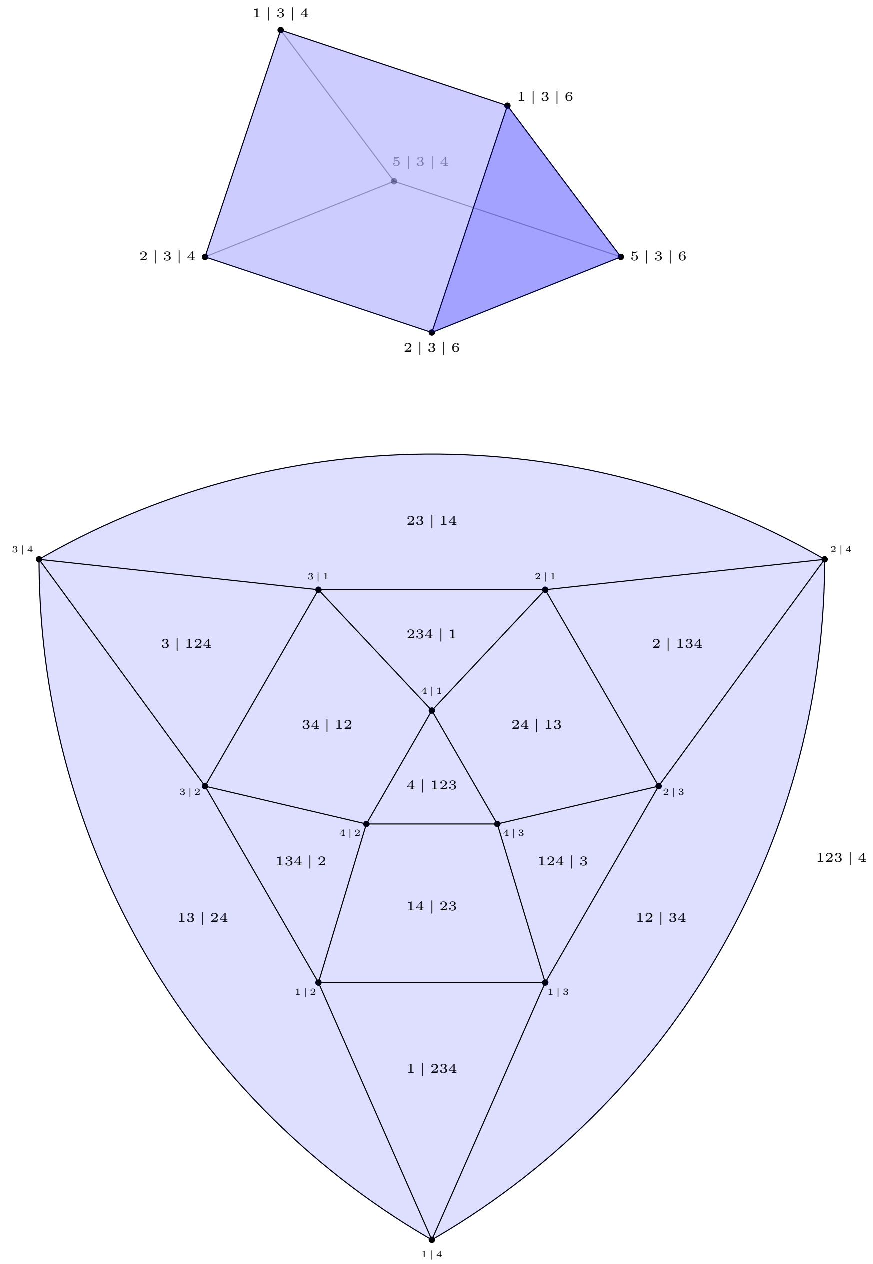

Figure 1 Above: The polyhedral cell of $C_{6,3}$ corresponding to the ordered partition $\big(\{1,2,5\},\{3\},\{4,6\}\big)$, which we express more succinctly by $(125 \mid 3 \mid 46)$. Below: The cell complex $C_{4,2}$.

where $X_j = \varphi_1^{-1}(j) \cap \varphi_2^{-1}(j) \cap \cdots \cap \varphi_{|\sigma|}^{-1}(j)$ as in (1). By definition, each X_j is nonempty, and so X is a k-partition of a subset of $[n]$. We may therefore identify X with a partial surjective function $\eta_\sigma : [n] \to [k]$ given by $\eta_\sigma(i) = j$ when $\varphi(i) = j$ for each $\varphi \in \sigma$, and $\eta_\sigma(i)$ is undefined otherwise, or equivalently $\eta_\sigma^{-1}(j) = X_j$. By setting

$$g(\sigma) := \eta_\sigma,$$

we have

$$\begin{aligned} \tau \subset \sigma &\implies \eta_\sigma^{-1}(j) \subset \eta_\tau^{-1}(j) \text{ for each } j \in [k] \\ &\implies \Delta_{g(\sigma)} \subset \Delta_{g(\tau)}, \end{aligned}$$

so we obtain a surjective map $g : K \to C$, which is order-reversing. The upper set $C_{\succeq \eta}$ is the set of partial functions γ such that $\gamma^{-1}(j) \supset \eta^{-1}(j)$ for each j, that is, $C_{\succeq \eta}$ is the set of extensions of η, so

$$g^{-1}(C_{\succeq \eta}) = 2^{\sigma_\eta}$$

where

$$\sigma_\eta := \big\{\varphi \in K_{n,k} : \varphi(i) = \eta(i) \text{ if } \eta(i) \text{ is defined}\big\}.$$

Thus the fibers are simplices, and the associated complexes $K_{n,k}$ and $C_{n,k}$ are homotopy equivalent by Quillen's fiber lemma. ◀

3.2 Acyclic matchings

We now apply discete Morse theory [7] to determine the connectedness of $C_{n,k}$. This means that we now view $C_{n,k}$ as a poset, and it contains the *empty face* as a unique minimal element.

Recall that a *matching* in a poset P is a matching in the underlying graph of the Hasse diagram of P. In other words, a matching M in P is a collection of pairs

$$M = \big\{\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_t, b_t\}\big\},$$

where the a_i and b_j are all distinct, and a_i is an immediate predecessor of b_i for every i. The matching M is called *cyclic* if there is a subsequence of indices $i_1, i_2, \ldots, i_s \in [t]$ such that

$$\begin{aligned} a_{i_2} &\prec b_{i_1} \\ a_{i_3} &\prec b_{i_2} \\ &\vdots \\ a_{i_s} &\prec b_{i_{s-1}} \\ a_{i_1} &\prec b_{i_s} \end{aligned}$$

If no such subsequence exists, then the matching M is called *acyclic*.

In the case when P is the face lattice of a polyhedral complex and M is an acyclic matching, then the unmatched elements of P are called the *critical cells* of the matching. One of the fundamental theorems of discrete Morse theory [9, Theorem 4.7] (see also [10, Theorem 11.13]) asserts that if all the critical cells of M have dimension at least d, then the polyhedral complex is $(d-1)$-connected. Our goal is therefore to show the following

▶ **Lemma 11.** *For all $n > k \geq 1$, there is an acyclic matching M on $C_{n,k}$ whose critical cells (if there are any) have dimension $n - k$.*

In order to find such an acyclic matching we need two basic tools from the toolbox of discrete Morse theory. The first one (referred to as an *element matching* [9, Lemma 4.1]) describes a particular acyclic matching which we use repeatedly. Let $(\sigma - x, \sigma + x)$ denote $(\sigma \setminus \{x\}, \sigma \cup \{x\})$.

▶ **Lemma** (Element matching). *Let X be a finite set, $P \subseteq 2^X$ is ordered by inclusion, and for a fixed element $x \in X$, let*

$$P_x = \{\sigma : \sigma - x, \sigma + x \in P\} \tag{5}$$
$$M_x = \{\{\sigma - x, \sigma + x\} : \sigma \in P_x\}. \tag{6}$$

Then M_x is an acyclic matching on P.

The second tool allows us to combine a collection of acyclic matchings into a single one (see [9, Lemma 4.2] or [10, Theorem 11.10]).

▶ **Lemma** (Patchwork lemma). *Let P and Q be finite posets, and let $h : P \to Q$ be an order-preserving map. Assume we have acyclic matchings M_q on each of the subposets $h^{-1}(q)$. Then, $M = \bigcup_{q \in Q} M_q$ is an acyclic matching on P.*

Proof of Lemma 11. We proceed by induction on k. For $k = 1$, observe that $C_{n,1}$ is isomorphic to $2^{[n]}$, and so the element matching

$$\big\{\{\sigma - n, \sigma + n\} : \sigma \in 2^{[n]}\big\}$$

is a complete acyclic matching (i.e. there are no critical cells).

Now, assume $k > 1$ and that the lemma holds for $C_{n',k-1}$ for all $n' > k - 1$. We denote the cells of $C_{n,k}$ as k-tuples

$$X = (X_1, X_2, \ldots, X_k),$$

where the X_i are either nonempty, pairwise disjoint subsets of $[n]$ (corresponding to a nonempty cell of $C_{n,k}$), or they satisfy $X_1 = X_2 = \cdots = X_k = \emptyset$ (corresponding to the unique empty cell). Note that the dimension of a nonempty cell equals $\sum_{i=1}^{k}(|X_i| - 1)$, and so our goal is to find an acyclic matching in $C_{n,k}$ whose critical cells (if there are any) satisfy $X_1 \cup X_2 \cup \cdots \cup X_k = [n]$.

Let $\{a_1, a_2, \ldots, a_{k-1}\}$ be an antichain and let $a_k \prec a_i$ for all $1 \leq i < k$. Then a map $h_1 : C_{n,k} \to \{a_1, a_2, \ldots, a_k\}$, defined by

$$h_1(X_1, X_2, \ldots, X_k) = \begin{cases} a_i & \text{if } n \in X_i \text{ and } i < k \\ a_k & \text{otherwise} \end{cases}$$

is order-preserving. Our goal is therefore to apply the patchwork lemma by finding appropriate acyclic matchings on each of the subposets $A_i := h_1^{-1}(a_i)$.

Fix $1 \leq i < k$ and consider the projection map $p_i : A_i \to (2^{[n-1]})^{\times(k-1)}$ which forgets the ith component

$$p_i(X_1, X_2, \ldots, X_k) = (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k).$$

This is an order preserving map when the range of p_i is ordered by (componentwise) inclusion. For a given $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k)$ in the range of p_i, we observe that $p_i^{-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k)$ is isomorphic to 2^{Y_i}, where

$$Y_i := [n-1] \setminus (X_1 \cup \cdots \cup X_{i-1} \cup X_{i+1} \cup \cdots \cup X_k).$$

Therefore, if $Y_i \neq \emptyset$, then $p_i^{-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k)$ has a complete element matching. Otherwise, we have

$$p_i^{-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k) = (X_1, \ldots, X_{i-1}, \{n\}, X_{i+1}, \ldots, X_k),$$

which will be one of our critical cells of dimension $n-k$.

It remains to find an appropriate acyclic matching on A_k. Define an order-preserving map $h_2 : A_k \to \{a \prec b\}$ where

$$h_2(X_1, X_2, \ldots, X_k) = \begin{cases} a & \text{if } X_k = \{n\} \text{ or } X_k = \emptyset, \\ b & \text{otherwise.} \end{cases}$$

Set $A := h_2^{-1}(a)$ and $B := h_2^{-1}(b)$. Note that the empty cell belongs to A, and A is isomorphic to $C_{n-1,k-1}$. By induction, there is an acyclic matching on A such that all critical cells (if there are any) have dimension $n-k$.

Consider the projection map $p_k : B \to (2^{[n-1]})^{\times(k-1)}$ which forgets the kth component

$$p_i(X_1, X_2, \ldots, X_k) = (X_1, X_2, \ldots, X_{k-1}).$$

This is an order preserving map when the range of p_k is ordered by (componentwise) inclusion. For a given $(X_1, X_2, \ldots, X_{k-1})$ in the range of p_k, we observe that $p_i^{-1}(X_1, X_2, \ldots, X_{k-1})$ is isomorphic to $2^{Y_k} \setminus \{\emptyset, \{n\}\}$, where

$$Y_k := [n] \setminus (X_1 \cup X_2 \cup \cdots \cup X_{k-1}).$$

By definition, $Y_k \neq \emptyset$, and so $p_i^{-1}(X_1, X_2, \ldots, X_{k-1})$ has a complete element matching of the form $\{\sigma - n, \sigma + n\}$. ◀

References

1 Imre Bárány. A generalization of Carathéodory's theorem. *Discrete Mathematics*, 40(2-3):141–152, 1982.

2 Imre Bárány. *Combinatorial convexity*, volume 77. American Mathematical Soc., 2021.

3 Imre Bárány and Gil Kalai. Helly-type problems. *Bulletin of the American Mathematical Society*, 59(4):471–502, 2022.

4 Anders Björner. Topological methods. *Handbook of combinatorics*, 2:1819–1872, 1995.

5 Jesús De Loera, Xavier Goaoc, Frédéric Meunier, and Nabil Mustafa. The discrete yet ubiquitous theorems of Carathéodory, Helly, Sperner, Tucker, and Tverberg. *Bulletin of the American Mathematical Society*, 56(3):415–511, 2019.

6 Jürgen Eckhoff. Helly, Radon, and Carathéodory type theorems. In *Handbook of convex geometry*, pages 389–448. Elsevier, 1993.

7 Robin Forman. A discrete Morse theory for cell complexes. In *Geometry, topology, and physics, Conf. Proc. Lecture Notes Geom. Topology, IV*, pages 112–125, 1995.

8 Andreas Holmsen and Rephael Wenger. Helly-type theorems and geometric transversals. In *Handbook of discrete and computational geometry*, pages 91–123. Chapman and Hall/CRC, 2017.

9 Jakob Jonsson. *Simplicial complexes of graphs*, volume 1928. Springer, 2008.

10 Dimitry Kozlov. *Combinatorial algebraic topology*, volume 21. Springer Science & Business Media, 2008.

11 Leonardo Martínez-Sandoval, Edgardo Roldán-Pensado, and Natan Rubin. Further consequences of the colorful Helly hypothesis. *Discrete & Computational Geometry*, 63(4):848–866, 2020.

12 Jiri Matousek. *Lectures on discrete geometry*, volume 212. Springer Science & Business Media, 2013.

13 Jiří Matoušek, Anders Björner, Günter M Ziegler, et al. *Using the Borsuk-Ulam theorem: lectures on topological methods in combinatorics and geometry*, volume 2003. Springer, 2003.

14 L Montejano. Transversals, topology and colorful geometric results. *Geometry—Intuitive, Discrete, and Convex: A Tribute to László Fejes Tóth*, pages 205–218, 2013.

15 Luis Montejano and Roman N Karasev. Topological transversals to a family of convex sets. *Discrete & Computational Geometry*, 46(2):283–300, 2011.

16 Sherry Sarkar and Pablo Soberón. Tolerance for colorful Tverberg partitions. *European Journal of Combinatorics*, 103:103527, 2022.

17 Helge Tverberg. A generalization of Radon's theorem. *Journal of the London Mathematical Society*, 1(1):123–128, 1966.

18 A Yu Volovikov. On a topological generalization of the Tverberg theorem. *Mathematical Notes*, 59(3):324–326, 1996.