Maximum Betti Numbers of Čech Complexes

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── Abstract ──

The Upper Bound Theorem for convex polytopes implies that the p-th Betti number of the Čech complex of any set of N points in $\mathbb{R}^d$ and any radius satisfies $\beta_p = O(N^m)$, with $m = \min\{p+1, \lceil d/2 \rceil\}$. We construct sets in even and odd dimensions, which prove that this upper bound is asymptotically tight. For example, we describe a set of $N = 2(n+1)$ points in $\mathbb{R}^3$ and two radii such that the first Betti number of the Čech complex at one radius is $(n+1)^2 - 1$, and the second Betti number of the Čech complex at the other radius is n^2. In particular, there is an arrangement of n contruent balls in $\mathbb{R}^3$ that enclose a quadratic number of voids, which answers a long-standing open question in computational geometry.

1 Introduction

Given a finite set of points A in $\mathbb{R}^d$ and a radius, their *Čech complex* is the collection of subsets of the points whose closed balls have a nonempty common intersection. This is an abstract simplicial complex isomorphic to the nerve of the balls, and by the Nerve Theorem [5], it has the same homotopy type as the union of the balls. This property is the reason for the popularity of the Čech complex in topological data analysis; see e.g. [7, 9]. Of particular interest are the *Betti numbers*, which may be interpreted as the numbers of holes of different dimensions. These are intrinsic properties, but for a space embedded in $\mathbb{R}^d$, they describe the connectivity of the space as well as that of its complement. Most notably, the (reduced) zero-th Betti number, β_0, is one less than the number of *connected components*, and the last possibly non-zero Betti number, β_{d-1}, is the number of *voids* (bounded components of the complement). Spaces that have the same homotopy type – such as a union of balls and the corresponding Čech complex – have identical Betti numbers. While the Čech complex is not necessarily embedded in $\mathbb{R}^d$, the corresponding union of balls is, which implies that also the
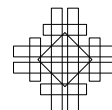
Čech complex has no non-zero Betti numbers beyond dimension $d - 1$. To gain insight into the statistical behavior of the Betti numbers of Čech complexes, it is useful to understand how large the numbers can get, and this is the question we study in this paper.

The question of maximum Betti numbers lies at the crossroads of computational topology and discrete geometry. Originally inspired by problems in the theory of polytopes [20, 28], optimization [22], robotics, motion planning [23], and molecular modeling [21], many interesting and surprisingly difficult questions were asked about the complexity of the union of n geometric objects, as n tends to infinity. For a survey, consult [1]. Particular attention was given to estimating the number of voids among N simply shaped bodies, e.g., for the translates of a fixed convex body in $\mathbb{R}^d$. In the plane, the answer is typically linear in N (for instance, for disks or other fat objects), but for $d = 3$, the situation is more delicate. The maximum number of voids among N translates of a convex polytope with a constant number of faces is $\Theta(N^2)$, but this number reduces to linear for the cube and other simple shapes [3]. It was conjectured for a long time that similar bounds hold for the translates of a convex shape that is not necessarily a polytope. However, this turned out to be false: Aronov, Cheung, Dobbins and Goaoc [2] constructed a convex body in $\mathbb{R}^3$ for which the number of voids is $\Omega(N^3)$. This is the largest possible order of magnitude for any arrangement of convex bodies that are not necessarily translates of a fixed one [19]. It is an outstanding open problem whether there exists a *centrally symmetric* convex body with this property.

For the special case where the convex body is the *unit ball* in $\mathbb{R}^3$, the maximum number of voids in a union of N translates is $O(N^2)$. This can be easily derived from the Upper Bound Theorem for 4-dimensional convex polytopes. It has been open for a long time whether this bound can be attained. Our main theorem answers this question in the affirmative, in a more general sense.

▶ **Main Theorem.** *For every $d \geq 1$, $0 \leq p \leq d - 1$, and $N \geq 1$, there is a set of N points in $\mathbb{R}^d$ and a radius such that the p-th Betti number of the Čech complex of the points and the radius is $\beta_p = \Theta(N^m)$, with $m = \min\{p + 1, \lceil d/2 \rceil\}$.*

For $d = 3$, the maximum second Betti number is $\beta_2 = \Theta(N^2)$ in $\mathbb{R}^3$, which is equivalent to the maximum number of voids being $\Theta(N^2)$. In addition to the Čech complex, the proof of the Main Theorem makes use of three complexes defined for a set of N points, $A \subseteq \mathbb{R}^d$, in which the third also depends on a radius $r \geq 0$:

- the *Voronoi domain* of a point $a \in A$, denoted $\mathrm{dom}(a, A)$, contains all points $x \in \mathbb{R}^d$ that are at least as close to a as to any other point in A, and the *Voronoi tessellation* of A, denoted $\mathrm{Vor}(A)$, is the collection of $\mathrm{dom}(a, A)$ with $a \in A$ [26];
- the *Delaunay mosaic* of A, denoted $\mathrm{Del}(A)$, contains the convex hull of $\Sigma \subseteq A$ if the common intersection of the $\mathrm{dom}(a, A)$, with $a \in \Sigma$, is non-empty, and no other Voronoi domain contains this common intersection [8]; it is closed under taking faces and, therefore, is a polyhedral complex;
- the *Alpha complex* of A and r, denoted $\mathrm{Alf}(A, r)$, is the subcomplex of the Delaunay mosaic that contains the convex hull of Σ if the common intersection of the $\mathrm{dom}(a, A)$, with $a \in \Sigma$, contains a point at distance at most r from the points in Σ; see [10, 11].

The Delaunay mosaic is also known as the *dual* of the Voronoi tessellation, or the *Delaunay triangulation* of A. Note that $\mathrm{Alf}(A, r) \subseteq \mathrm{Alf}(A, R)$ whenever $r \leq R$, and that for sufficiently large radius, the Alpha complex is the Delaunay mosaic. Similar to the Čech complex, the Alpha complex has the same homotopy type as the union of balls with radius r centered at the points in A, and thus the same Betti numbers. It is instructive to increase r from 0 to ∞ and to consider the *filtration* or nested sequence of Alpha complexes. The difference

between an Alpha complex, K, and the next Alpha complex in the filtration, L, consists of one or more cells. If it is a single cell of dimension p, then either $\beta_p(L) = \beta_p(L) + 1$ or $\beta_{p-1}(L) = \beta_{p-1}(L) - 1$, and all other Betti numbers are the same. In the first case, we say the cell gives *birth* to a p-cycle, while in the second case, it gives *death* to a $(p-1)$-cycle, and in both cases we say it is *critical*. If there are two or more cells in the difference, this may be a generic event or accidental due to non-generic position of the points. In the simplest generic case, we simultaneously add two cells (one a face of the other), and the addition is an anti-collapse, which does not affect the homotopy type of the complex. More elaborate anti-collapses, such as the simultaneous addition of an edge, two triangles, and a tetrahedron, can arise generically. The cells in an interval of size 2 or larger cancel each other's effect on the homotopy type, so we say these cells are *non-critical*. We refer to [4] for more details.

With these notions, it is not difficult to prove the upper bounds in the Main Theorem. As mentioned above, the Čech and alpha complexes for radius r have the same Betti numbers. Since a p-cycle is given birth to by a p-cell in the filtration of Alpha complexes, and every p-cell gives birth to at most one p-cycle, the number of p-cells is an upper bound on the number of p-cycles, which are counted by the p-th Betti number. The number of p-cells in the Alpha complex is at most that number in the Delaunay mosaic, which is, by the Upper Bound Theorem for convex polytopes [20, 28], at most $O(N^m)$, with $m = \min\{p+1, \lceil d/2 \rceil\}$.

By comparison, to come up with constructions that prove matching lower bounds is delicate and the main contribution of this paper. Our constructions are multipartite and inspired by Lenz' constructions related to Erdős's celebrated question on repeated distances [14]: what is the largest number of point pairs in an N-element set in $\mathbb{R}^d$ that are at distance 1 apart? Lenz noticed that in 4 (and higher) dimensions, this maximum is $\Theta(N^2)$. To see this, take two circles of radius $\sqrt{2}/2$ centered at the origin, lying in two orthogonal planes, and place $\lceil N/2 \rceil$ and $\lfloor N/2 \rfloor$ points on them. By Pythagoras' theorem, any two points on different circles are at distance 1 apart, so the number of unit distances is roughly $N^2/4$, which is nearly optimal. For $d = 2$ and 3, we are far from knowing asymptotically tight bounds. The current best constructions give $\Omega(N^{1+c/\log\log N})$ unit distance pairs in the plane [6, page 191] and $\Omega(N^{4/3}\log\log N)$ in $\mathbb{R}^3$, while the corresponding upper bounds are $O(N^{4/3})$ and $O(N^{3/2})$; see [25] and [18, 27]. Even the following, potentially simpler, bipartite analogue of the repeated distance question is open in $\mathbb{R}^3$: given N red points and N blue points in $\mathbb{R}^3$, such that the minimum distance between a red and a blue point is 1, what is the largest number of red-blue point pairs that determine a unit distance? The best known upper bound, due to Edelsbrunner and Sharir [13] is $O(N^{4/3})$, but we have no superlinear lower bound. This last question is closely related to the subject of our present paper.

It is not difficult to see that the upper bounds in the Main Theorem also hold for the Betti numbers of the union of N *not necessarily congruent* balls in $\mathbb{R}^d$. This requires the use of weighted versions of the Voronoi tessellation and the Upper Bound Theorem. In the lower bound constructions, much of the difficulty stems from the fact that we insist on using congruent balls. This suggests the analogy to the problem of repeated distances.

Outline.   Section 2 proves the Main Theorem for sets in *even* dimensions. Starting with Lenz' constructions, we partition the Delaunay mosaic into finitely many groups of *congruent* simplices. We compute the radii of their circumspheres and obtain the Betti numbers by straightforward counting. In Section 3, we establish the Main Theorem for sets in three dimensions. The situation is more delicate now, because the simplices of the Delaunay mosaic no longer fall into a small number of distinct congruence classes. Nevertheless, they can be divided into groups of nearly congruent simplices, which will be sufficient to carry out the counting argument. The generalization to odd dimensions larger than 3 can be found in the `arXiv` version of this paper [12]. Section 4 contains concluding remarks and open questions.

2   Even dimensions

In this section, we give an answer to the maximum Betti number question for Čech complexes in even dimensions. To state the result, let n_k be the minimum integer such that the edges of a regular n_k-gon inscribed in a circle of radius $1/\sqrt{2}$ are strictly shorter than $\sqrt{2/k}$. For example, if $k = 2$, we have $n_2 = 5$, as the side length of an inscribed square is equal to 1.

▶ **Theorem 1** (Maximum Betti Numbers in $\mathbb{R}^{2k}$). *For every $2k \geq 2$ and $n \geq n_k$, there exist a set A of $N = kn$ points in $\mathbb{R}^{2k}$ and radii $\rho_0 < \rho_1 < \ldots < \rho_{2k-2}$ such that*

$$\beta_p(\check{C}ech(A, \rho_p)) = \binom{k}{p+1} \cdot n^{p+1} \pm O(1), \quad \text{for } 0 \leq p \leq k-1; \tag{1}$$

$$\beta_p(\check{C}ech(A, \rho_p)) = \binom{k-1}{p+1-k} \cdot n^k \pm O(1), \quad \text{for } k \leq p \leq 2k-2. \tag{2}$$

For $p = 2k - 1$, there exist $N = k(n + 1) + 2$ points in $\mathbb{R}^{2k}$ and a radius such that the p-th Betti number of the Čech complex is $n^k \pm O(n^{k-1})$.

The reason for the condition $n \geq n_k$ will become clear in the proof of Lemma 5, which establishes a particular ordering of the circumradii of the cells in the Delaunay mosaic. The proof of the cases $0 \leq p \leq 2k - 2$ is not difficult using elementary computations, the results of which will be instrumental for establishing the more challenging odd-dimensional statements in Section 3 as well as Section 4 in [12]. The proof consists of four steps presented in four subsections: the construction of the point set in Section 2.1, the geometric analysis of the simplices in the Delaunay mosaic in Section 2.2, the ordering of the circumradii in Section 2.3, and the final counting in Section 2.4. The proof of the case $p = 2k - 1$ in $\mathbb{R}^{2k}$ readily follows the case $p = 2k - 2$ in $\mathbb{R}^{2k-1}$, as described in [12].

2.1   Construction

Let $d = 2k$. We construct a set $A = A_{2k}(n)$ of $N = kn$ points in $\mathbb{R}^d$ using k concentric circles in mutually orthogonal coordinate planes: for $0 \leq \ell \leq k - 1$, the circle C_ℓ with center at the origin, $0 \in \mathbb{R}^d$, is defined by $x_{2\ell+1}^2 + x_{2\ell+2}^2 = \frac{1}{2}$ and $x_i = 0$ for all $i \neq 2\ell + 1, 2\ell + 2$. On each of the k circles, we choose $n \geq 3$ points that form a regular n-gon. The length of the edges of these n-gons will be denoted by $2s$. Obviously, we have $s = \frac{\sqrt{2}}{2} \sin \frac{\pi}{n}$. Assuming $k \geq 2$, the condition $n \geq n_k$ implies that the Euclidean distance between consecutive points along the same circle is less than 1, and by Pythagoras' theorem, the distance between any two points on different circles is 1. It follows that for $r = \frac{1}{2}$, neighboring balls centered on the same circle overlap, while the balls centered on different circles only touch. Correspondingly, the first Betti number of the Čech complex for a radius slightly less than $\frac{1}{2}$ is $\beta_1 = k$. To get the first Betti number for $r = \frac{1}{2}$, we add all edges of length 1, of which $k - 1$ connect the k circles into a single connected component, while the others increase the first Betti number to $\beta_1 = k + \binom{k}{2}n^2 - (k - 1) = \binom{k}{2}n^2 + 1$.

To generalize the analysis beyond the first Betti number, we consider the Delaunay mosaic and two radii defined for each of its cells. The *circumsphere* of a p-cell is the unique $(p - 1)$-sphere that passes through its vertices, and we call its center and radius the *circumcenter* and the *circumradius* of the cell. To define the second radius, we call a $(d - 1)$-sphere *empty* if all points of A lie on or outside the sphere. The *radius function* on the Delaunay mosaic, $\mathrm{Rad} \colon \mathrm{Del}(A) \to \mathbb{R}$, maps each cell to the radius of the smallest empty $(d - 1)$-sphere that passes through the vertices of the cell. By construction, each Alpha complex is a sublevel set of this function: $\mathrm{Alf}(A, r) = \mathrm{Rad}^{-1}[0, r]$. The two radii of a cell may be different, but they agree for the critical cells as defined in terms of their topological effect in the introduction. It will be convenient to work with the corresponding geometric characterization of criticality; see also [24]:

▶ **Definition 2** (Critical Cell). *A critical cell of* $\mathrm{Rad}\colon \mathrm{Del}(A) \to \mathbb{R}$ *is a cell* $\Sigma \in \mathrm{Del}(A)$ *that (1) contains the circumcenter in its interior, and (2) the* $(d-1)$-*sphere centered at the circumcenter that passes through the vertices of* Σ *is empty and the vertices of* Σ *are the only points of* A *on this sphere.*

There are two conditions for a cell to be critical for a reason. The first guarantees that its topological effect is not canceled by one of its faces, and the second guarantees that it does not cancel the topological effect of one of the cells it is a face of. As proved in [4], the radius function of a generic set, $A \subseteq \mathbb{R}^d$, is *generalized discrete Morse*; see Forman [15] for background on discrete Morse functions. This means that each level set of Rad is a union of disjoint combinatorial intervals, and a simplex is critical iff it is the only simplex in its interval. Our set A is not generic because the $(d-1)$-sphere with center $0 \in \mathbb{R}^{2k}$ and radius $\sqrt{2}/2$ passes through all its points. Indeed, $\mathrm{Del}(A)$ is really a $2k$-dimensional convex polytope, namely the convex hull of A and all its faces. Nevertheless, the distinction between critical and non-critical cells is still meaningful, and all cells in the Delaunay mosaic of our construction will be seen to be critical.

The value of the $2k$-polytope under the radius function is $\sqrt{2}/2$, while the values of its proper faces are strictly smaller than $\sqrt{2}/2$. Let $\Sigma_{\ell,j}$ be such a face, in which $\ell+1$ is the number of circles that contain one or two of its vertices, and $j+1$ is the number of circles that contain two. Specifically, $\Sigma_{\ell,j}$ has $j+1$ disjoint *short* edges of length $2s$, while the remaining *long* edges all have unit length. Indeed, the geometry of the simplex is determined by ℓ and j and does not depend on the circles from which we pick the vertices or where along these circles we pick them, as long as two vertices from the same circle are consecutive along this circle. For example, $\Sigma_{1,-1}$, $\Sigma_{1,0}$, and $\Sigma_{1,1}$ are the unit length edge, the isosceles triangle with one short and two long edges, and the tetrahedron with two disjoint short and four long edges, respectively. We call the $\Sigma_{\ell,j}$ *ideal simplices*. In even dimensions they are *precisely* the simplices in the Delaunay mosaic of our construction. However, in odd dimensions, the cells in the Delaunay mosaic only *converge* to the ideal simplices. This will be explained in detail in Section 3 as well as in Section 4 of the `arXiv` version of this paper [12].

2.2 Circumradii of ideal simplices

In this section, we compute the sizes of some ideal simplices, beginning in four dimensions. The *ideal* 2-*simplex* or *triangle*, denoted $\Sigma_{1,0}$, is the isosceles triangle with one short and two long edges. We write $h(s)$ for the *height* of $\Sigma_{1,0}$ (the distance between the midpoint of the short edge and the opposite vertex), and $r(s)$ for the circumradius. There is a unique way to glue four such triangles to form the boundary of a tetrahedron: the two short edges are disjoint and their endpoints are connected by four long edges. This is the *ideal* 3-*simplex* or *tetrahedron*, denoted $\Sigma_{1,1}$. We write $H(s)$ for its *height* (the distance between the midpoints of the two short edges), and $R(s)$ for its circumradius.

▶ **Lemma 3** (Ideal Triangle and Tetrahedron). *The squared heights and circumradii of the ideal triangle and the ideal tetrahedron in* $\mathbb{R}^4$ *satisfy*

$$h^2(s) = 1 - s^2, \qquad 4r^2(s) = \frac{1}{1-s^2}, \tag{3}$$

$$H^2(s) = 1 - 2s^2, \qquad 4R^2(s) = 1 + 2s^2. \tag{4}$$

Proof. By Pythagoras' theorem, the squared height of the ideal triangle is $h^2 = 1 - s^2$. If we glue the two halves of a scaled copy of the ideal triangle to the two halves of the short edge, we get a quadrangle inscribed in the circumcircle of the triangle. One of its diagonals passes through the center, and its squared length satisfies $4r^2 = 1 + (s/h)^2 = 1 + \frac{s^2}{1-s^2}$.

By Pythagoras' theorem, the squared height of the ideal tetrahedron is $H^2 = h^2 - s^2 = 1 - 2s^2$. Hence, the squared diameter of the circumsphere is $4R^2 = H^2 + (2s)^2 = 1 + 2s^2$. ◄

To generalize the analysis beyond the ideal simplices in four dimensions, we write $r_{\ell,j}(s)$ for the circumradius of $\Sigma_{\ell,j}$, so $r_{1,-1}(s) = \frac{1}{2}$, $r_{1,0}(s) = r(s)$, and $r_{1,1}(s) = R(s)$. For two kinds of ideal simplices, the circumradii are particularly easy to compute, namely for the $\Sigma_{\ell,-1}$ and the $\Sigma_{\ell,\ell}$, and we will see that knowing their circumradii will be sufficient for our purposes.

▶ **Lemma 4** (Further Ideal Simplices). *For $\ell \geq 0$, the squared circumradii of $\Sigma_{\ell,-1}$ and $\Sigma_{\ell,\ell}$ satisfy $r_{\ell,-1}^2(s) = \ell/(2\ell + 2)$ and $r_{\ell,\ell}^2(s) = (\ell + 2s^2)/(2\ell + 2)$.*

Proof. Consider the standard ℓ-simplex, which is the convex hull of the endpoints of the $\ell + 1$ unit coordinate vectors in $\mathbb{R}^{\ell+1}$. Its squared circumradius is the squared distance between the barycenter and any one of the vertices, which is easy to compute. By comparison, the squared circumradius of the regular ℓ-simplex with unit length edges is half that of the standard ℓ-simplex:

$$R_\ell^2 = \frac{1}{2}\left[\frac{\ell^2}{(\ell+1)^2} + \frac{1}{(\ell+1)^2} + \ldots + \frac{1}{(\ell+1)^2}\right] = \frac{\ell}{2(\ell+1)}, \tag{5}$$

Since $r_{\ell,-1}^2(s) = R_\ell^2$, this proves the first equation in the lemma. Note that the convex hull of the midpoints of the $\ell + 1$ short edges of $\Sigma_{\ell,\ell}$ is a regular ℓ-simplex with edges of squared length $H^2(s) = 1 - 2s^2$. The short edges are orthogonal to this ℓ-simplex, which implies

$$r_{\ell,\ell}^2 = H^2(s) \cdot R_\ell^2 + s^2 = R_\ell^2 + (1 - 2R_\ell^2)s^2 = \frac{\ell + 2s^2}{2\ell + 2}, \tag{6}$$

which proves the second equation in the lemma. ◄

2.3 Ordering the radii

In this subsection, we show that the radii of the circumspheres of the ideal simplices increase with increasing ℓ and j:

▶ **Lemma 5** (Ordering of Radii in $\mathbb{R}^{2k}$). *Let $0 < s < 1/\sqrt{2k}$. Then the ideal simplices satisfy $r_{\ell,\ell}(s) < r_{\ell+1,-1}(s)$ for $0 \leq \ell \leq k - 2$, and $r_{\ell,j}(s) < r_{\ell,j+1}(s)$ for $-1 \leq j < \ell \leq k - 1$.*

Proof. To prove the first inequality, we use Lemma 4 to compute the difference between the two squared radii:

$$r_{\ell+1,-1}^2(s) - r_{\ell,\ell}^2(s) = \frac{\ell+1}{2(\ell+2)} - \frac{\ell+2s^2}{2(\ell+1)} = \frac{1 - 2s^2(\ell+2)}{2(\ell+2)(\ell+1)}. \tag{7}$$

Hence, $r_{\ell,\ell}^2(s) < r_{\ell+1,-1}^2(s)$ iff $s^2 < 1/(2\ell + 4)$. We need this inequality for $0 \leq \ell \leq k - 2$, so $s^2 < 1/(2k)$ is sufficient, but this is guaranteed by the assumption.

We prove the second inequality geometrically, without explicit computation of the radii. Fix an ideal simplex, $\Sigma_{\ell,j}$, and let S^{d-1} be the $(d-1)$-sphere whose center and radius are the circumcenter and circumradius of $\Sigma_{\ell,j}$. Assume w.l.o.g. that the circles C_0 to C_j contain two vertices of $\Sigma_{\ell,j}$ each, and the circles C_{j+1} to C_ℓ contain one vertex of $\Sigma_{\ell,j}$ each. For $0 \leq i \leq k - 1$, write P_i for the 2-plane that contains C_i and x_i for the projection of the center of S^{d-1} onto P_i. Note that $\|x_i\|^2$ is the squared distance to the origin, and for $0 \leq i \leq \ell$ write r_i^2 for the squared distance between x_i and the one or two vertices of $\Sigma_{\ell,j}$ in P_i. Fixing

i between 0 and ℓ, the squared radius of S^{d-1} is r_i^2 plus the squared distance of the center of S^{d-1} from P_i, which is the sum of the squared norms other than $\|x_i\|^2$. Taking the sum for $0 \leq i \leq \ell$ and dividing by $\ell + 1$, we get

$$r_{\ell,j}^2(s) = \frac{1}{\ell+1}\left[\sum_{i=0}^{\ell} r_i^2 + \ell \cdot \sum_{i=0}^{\ell} \|x_i\|^2 + (\ell+1) \cdot \sum_{i=\ell+1}^{k-1} \|x_i\|^2\right]. \tag{8}$$

By construction, $r_{\ell,j}^2(s)$ is the minimum squared radius of any $(d-1)$-sphere that passes through the vertices of $\Sigma_{\ell,j}$. Hence, also the right-hand side of (8) is a minimum, but since the 2-planes are pairwise orthogonal, we can minimize in each 2-plane independently of the other. For $\ell + 1 \leq i \leq k - 1$, this implies $\|x_i\|^2 = 0$, so we can drop the last sum in (8). For $j + 1 \leq i \leq \ell$, x_i lies on the line passing through the one vertex in P_i and the origin. This implies that S^{d-1} touches C_i at this vertex, and all other points of the circle lie strictly outside S^{d-1}. For $0 \leq i \leq j$, x_i lies on the bisector line of the two vertices, which passes through the origin. The contribution to (8) for an index between 0 and j is thus strictly larger than for an index between $j + 1$ and ℓ. This finally implies $r_{\ell,j}^2(s) < r_{\ell,j+1}^2(s)$ and completes the proof of the second inequality. ◀

Recall that $2s$ is the edge length of a regular n-gon inscribed in a circle of radius $1/\sqrt{2}$. By the definition of n_k, the condition $s < 1/\sqrt{2k}$ in the lemma holds, whenever $n \geq n_k$.

For the counting argument in the next subsection, we need the ordering of the radii as defined by the radius function, but it is now easy to see that they are the same as the circumradii, so Lemma 5 applies. Indeed, $\mathrm{Rad}(\Sigma_{\ell,j}) = r_{\ell,j}(s)$ if $\Sigma_{\ell,j}$ is a critical simplex of Rad. To realize that it is, we note that the circumcenter of $\Sigma_{\ell,j}$ lies in its interior. To see that also the second condition for criticality in Definition 2 is satisfied, we recall that S^{d-1} is the $(d-1)$-sphere whose center and radius are the circumcenter and circumradius of $\Sigma_{\ell,j}$. By the argument in the proof of Lemma 5, S^{d-1} is empty, and all points of A other than the vertices of $\Sigma_{\ell,j}$ lie strictly outside this sphere.

2.4 Counting the cycles

To compute the Betti numbers, we make essential use of the structure of the Delaunay mosaic of A, which consists of as many groups of congruent ideal simplices as there are different values of the radius function. For each $0 \leq \ell \leq k - 1$, we have $\ell + 2$ groups of simplices that touch exactly $\ell + 1$ of the k circles. In addition, we have a single $2k$-cell, conv A, with radius $\sqrt{2}/2$, which gives $1 + 2 + \ldots + (k+1) = \binom{k+2}{2}$ groups. We write $\mathcal{A}_{\ell,j} = \mathrm{Rad}^{-1}[0, r_{\ell,j}]$ for the Alpha complex that consists of all simplices with circumradii up to $r_{\ell,j} = r_{\ell,j}(s)$. We prove Theorem 1 in two steps, first the relations (1) for $0 \leq p \leq k - 1$ and second the relations (2) for $k \leq p \leq 2k - 2$. The case $p = 2k - 1$ is settled in Section 4 of [12]. To begin, we study the Alpha complexes whose simplices touch at most $\ell + 1$ of the k circles.

▶ **Lemma 6** (Constant Homology in $\mathbb{R}^{2k}$). *Let k be a constant, $A = A_{2k}(n) \subseteq \mathbb{R}^{2k}$, and $0 \leq \ell \leq k - 1$. Then $\beta_p(\mathcal{A}_{\ell,\ell}) = O(1)$ for every $0 \leq p \leq 2k - 1$.*

Proof. Fix ℓ and a subset of $\ell + 1$ circles. The full subcomplex of $\mathcal{A}_{\ell,\ell}$ defined by the points of A on these $\ell + 1$ circles consists of all cells in $\mathrm{Del}(A)$ whose vertices lie on these and not any of the other circles. Its homotopy type is that of the join of $\ell + 1$ circles or, equivalently, that of the $(2\ell + 1)$-sphere; see [17, pages 9 and 19]. This sphere has only one non-zero (reduced) Betti number, which is $\beta_{2\ell+1} = 1$. There are $\binom{k}{\ell+1}$ such full subcomplexes. The common intersection of any number of these subcomplexes is a complex of similar type, namely the full subcomplex of $\mathrm{Del}(A)$ defined by the points on the common circles, which

has the homotopy type of the $(2i + 1)$-sphere, with $i \leq \ell$. By repeated application of the Mayer–Vietoris sequence [17, page 149], this implies that the Betti numbers of $\mathcal{A}_{\ell,\ell}$ are bounded by a function of k and are, thus, independent of n. Since we assume that k is a constant, we have $\beta_p(\mathcal{A}_{\ell,\ell}) = O(1)$ for every p. ◀

Now we are ready to complete the proof of Theorem 1 for $p \leq 2k - 2$. To establish relation (1), fix p between 0 and $k - 1$ and consider $\mathcal{A}_{p,-1} = \mathrm{Rad}^{-1}[0, r_{p,-1}]$, which is the Alpha complex consisting of all simplices that touch p or fewer circles, together with all simplices that touch $p + 1$ circles but each circle in only one point. In other words, $\mathcal{A}_{p,-1}$ is $\mathcal{A}_{p-1,p-1}$ together with all the $\binom{k}{p+1} n^{p+1}$ p-simplices that have no short edges. By Lemma 6, $\mathcal{A}_{p-1,p-1}$ has only a constant number of $(p - 1)$-cycles. Hence, only a constant number of the p-simplices can give death to $(p - 1)$-cycles, while the remaining p-simplices give birth to p-cycles. This is because every p-simplex either gives birth or death, so if it cannot give death to a $(p - 1)$-cycle, then it gives birth to a p-cycle. Hence, $\beta_p(\mathcal{A}_{p,-1}) = \binom{k}{p+1} n^{p+1} \pm O(1)$, as claimed. The proof of relation (2) is similar but inductive. The induction hypothesis is

$$\beta_p(\mathcal{A}_{k-1,p-k}) = \binom{k-1}{p-k+1} \cdot n^k \pm O(1). \tag{9}$$

For $p = k - 1$, it claims $\beta_{k-1}(\mathcal{A}_{k-1,-1}) = n^k \pm O(1)$, which is what we just proved. In other words, relation (1) furnishes the base case at $p = k - 1$. A single inductive step takes us from $\mathcal{A}_{k-1,p-k}$ to $\mathcal{A}_{k-1,p-k+1}$; that is: we add all simplices that touch all k circles and $p - k + 2$ of them in two vertices to $\mathcal{A}_{k-1,p-k}$. The number of such simplices is the number of ways we can pick a pair of consecutive vertices from $p - k + 2$ circles and a single vertex from the remaining $2k - p - 2$ circles. Since there are equally many vertices as there are consecutive pairs, this number is $\binom{k}{p-k+2} n^k$. The dimension of these simplices is $(k - 1) + (p - k + 1) + 1 = p + 1$. Some of these $(p + 1)$-simplices give death to p-cycles, while the others give birth to $(p + 1)$-cycles in $\mathcal{A}_{k-1,p-k+1}$. By the induction hypothesis, there are $\binom{k-1}{p-k+1} \cdot n^k \pm O(1)$ p-cycles in $\mathcal{A}_{k-1,p-k}$, so this is also the number of $(p + 1)$-simplices that give death. Since $\binom{k}{p-k+2} - \binom{k-1}{p-k+1} = \binom{k-1}{p-k+2}$, this implies

$$\beta_p(\mathcal{A}_{k-1,p-k+1}) = \binom{k-1}{p-k+2} \cdot n^k \pm O(1), \tag{10}$$

as required to finish the inductive argument.

3 Three dimensions

In this section, we answer the maximum Betti number question for Čech complexes in the smallest odd dimension in which it is non-trivial:

▶ **Theorem 7** (Maximum Betti Numbers in $\mathbb{R}^3$). *For every $n \geq 2$, there exist $N = 2n + 2$ points in $\mathbb{R}^3$ such that the Čech complex for a radius has first Betti number $\beta_1 = (n + 1)^2 - 1$ and for another radius has second Betti number $\beta_2 = n^2$.*

The proof consists of four steps: the construction of the set in Section 3.1, the analysis of the circumradii in Section 3.2, the argument that all simplices in the Delaunay mosaic are critical in Section 3.3, and the final counting of the tunnels and voids in Section 3.4.

3.1 Construction

Given n and $0 < \Delta < 1$, we construct the point set, $A = A_3(n, \Delta)$, using two linked circles in $\mathbb{R}^3$: C_z with center $v_z = (-\frac{1}{2}, 0, 0)$ in the xy-plane defined by $(-\frac{1}{2} + \cos\varphi, \sin\varphi, 0)$ for $0 \leq \varphi < 2\pi$, and C_y with center $v_y = (\frac{1}{2}, 0, 0)$ in the xz-plane defined by $(\frac{1}{2} - \cos\psi, 0, \sin\psi)$

for $0 \le \psi < 2\pi$; see Figure 1. On each circle, we choose $n+1$ points close to the center of the other circle. To be specific, take the points $(0, -\Delta, 0)$ and $(0, \Delta, 0)$, and project them to C_z along the x-axis. The resulting points are denoted by $a_0 = (-\frac{1}{2} + \sqrt{1 - \Delta^2}, -\Delta, 0)$ and $a_n = (-\frac{1}{2} + \sqrt{1 - \Delta^2}, \Delta, 0)$. Divide the arc between them into n equal pieces by the points $a_1, a_2, \ldots, a_{n-1}$. Symmetrically, project the points $(0, 0, -\Delta)$ and $(0, 0, \Delta)$ to $b_0 = (\frac{1}{2} - \sqrt{1 - \Delta^2}, 0, -\Delta)$ and $b_n = (\frac{1}{2} - \sqrt{1 - \Delta^2}, 0, \Delta)$ lying on C_y, and place $n-1$ points $b_1, b_2, \ldots, b_{n-1}$ on the arc between them, dividing it into n equal pieces. Let $\varepsilon = \varepsilon(n, \Delta)$ be the half-length of the (straight) edge connecting two consecutive points of either sequence. Clearly, ε is a function of n and Δ, and it is easy to see that

$$\Delta/n < \varepsilon < \tfrac{\pi}{2}\Delta/n \quad \text{and} \quad \varepsilon \xrightarrow{\Delta \to 0} \Delta/n. \tag{11}$$
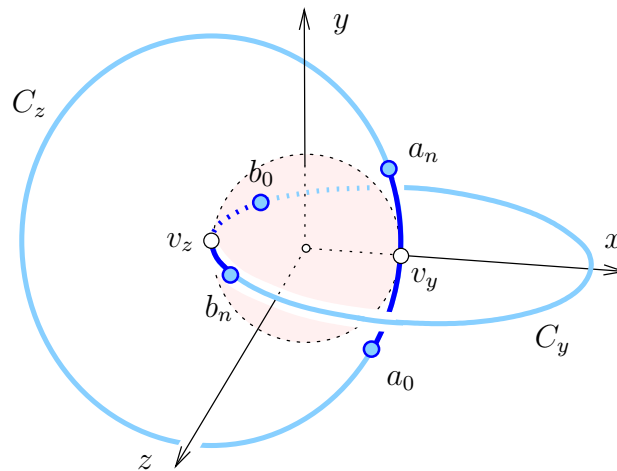
Figure 1 Two linked unit circles in orthogonal coordinate planes of $\mathbb{R}^3$, each touching the shaded sphere centered at the origin and each passing through the center of the other circle. There are $n+1$ points on each circle, on both sides and near the center of the other circle.

A sphere that does not contain a circle intersects it in at most two points. It follows that the sphere that passes through four points of A is empty if and only if two of the four points are consecutive on one circle and the other two are consecutive on the other. This determines the Delaunay mosaic: its $N = 2n + 2$ vertices are the points a_i and b_j, its $2n + (n+1)^2$ edges are of the forms $a_i a_{i+1}$, $b_j b_{j+1}$, and $a_i b_j$, its $2n(n+1)$ triangles are of the forms $a_i a_{i+1} b_j$ and $a_i b_j b_{j+1}$, and its n^2 tetrahedra of the form $a_i a_{i+1} b_j b_{j+1}$. Keeping with the terminology introduced in Section 2, we call the edges $a_i b_j$ *long* and the edges $a_i a_{i+1}$ and $b_j b_{j+1}$ *short*. Hence, every triangle in the Delaunay mosaic has one short and two long edges, and every tetrahedron has two short and four long edges.

3.2 Divergence from the ideal

The simplices in $\mathrm{Del}(A)$ are not quite ideal, in the sense of Section 2. We, therefore, need upper and lower bounds on their sizes, as quantified by their circumradii. We will make repeated use of the following two inequalities, which both hold for $x > -1$:

$$\sqrt{1+x} \le 1 + \tfrac{x}{2}, \tag{12}$$
$$\sqrt{1+x} \ge 1 + \tfrac{x}{2+x}. \tag{13}$$

For example, we will obtain some bounds on the radii of the triangle and tetrahedron in Lemma 3, avoiding the use of square roots. For the triangle, we rewrite (3) to $4r^2(s) = 1 + x$ with $x = s^2/(1 - s^2)$, and for the tetrahedron, we have $4R^2(s) = 1 + x$ with $x = 2s^2$:

$$1 + \frac{1}{2}s^2 < 1 + \frac{s^2/(1 - s^2)}{2 + s^2/(1 - s^2)} \le 2r(s) \le 1 + \frac{s^2}{2 - 2s^2} < 1 + \frac{10}{19}s^2, \tag{14}$$

$$1 + \frac{10}{11}s^2 \le 1 + \frac{s^2}{1 + s^2} \le 2R(s) \le 1 + s^2, \tag{15}$$

where we assume that n is large enough to imply $2 - 2s^2 > 1.9$ and therefore $1 + s^2 < 1.1$. We begin by proving bounds on the lengths of long edges.

▶ **Lemma 8** (Bounds for Long Edges in $\mathbb{R}^3$). *Let $0 < \Delta < 1$ and $A = A_3(n, \Delta) \subseteq \mathbb{R}^3$. Then the half-length of any long edge, $E \in \text{Del}(A)$, satisfies $\frac{1}{2} \le R_E \le \frac{1}{2}(1 + \Delta^4)$.*

Proof. To verify the lower bound, let $a \in C_z$ and consider the sphere with unit radius centered at a. This sphere intersects the xz-plane in a circle of radius at most 1, whose center lies on the x-axis. The circle passes through $v_z \in C_y$, which implies that the rest of C_y lies on or outside the circle and, therefore, on or outside the sphere centered at a. Hence, $\|a - b\| \ge 1$ for all $b \in C_y$, which implies the required lower bound.

To establish the upper bound, observe that the distance between a and b is maximized if the two points are chosen as far as possible from the x-axis, so $4R_E^2 \le \|a_0 - b_0\|^2$. By construction, $a_0 = (-\frac{1}{2} + \sqrt{1 - \Delta^2}, -\Delta, 0)$ and $b_0 = (\frac{1}{2} - \sqrt{1 - \Delta^2}, 0, -\Delta)$. Hence,

$$4R_E^2 \le \left\| (-1 + 2\sqrt{1 - \Delta^2}, -\Delta, \Delta) \right\|^2 = 5 - 2\Delta^2 - 4\sqrt{1 - \Delta^2} \tag{16}$$

$$\le 5 - 2\Delta^2 - 4\left(1 - \frac{\Delta^2}{2 - \Delta^2}\right) = 1 + \frac{2\Delta^4}{2 - \Delta^2} \tag{17}$$

$$\le 1 + 2\Delta^4, \tag{18}$$

where we used (13) to get (17) from (16), and $\Delta^2 < 1$ to obtain the final bound. Applying (12), wet get $2R_E \le 1 + \Delta^4$, as required. ◀

Next, we estimate the circumradii of the triangles in $\text{Del}(A)$. To avoid the computation of a constant, we use the big-Oh notation for Δ, in which we assume that n is fixed.

▶ **Lemma 9** (Bounds for Triangles in $\mathbb{R}^3$). *Let $0 < \Delta < \sqrt{2}/n$, $A = A_3(n, \Delta) \subseteq \mathbb{R}^3$, and $\varepsilon = \varepsilon(n, \Delta)$. Then the circumradius of any triangle, F, satisfies $\frac{1}{2} + \frac{1}{4}\varepsilon^2 \le R_F \le \frac{1}{2} + \frac{1}{4}\varepsilon^2 + O(\Delta^4)$.*

Proof. To see the lower bound, recall that the short edge of F has length 2ε and the two long edges have lengths at least 1. We place the endpoints of the short edge on a circle of radius $r(\varepsilon)$. By the choice of the radius, there is only one point on this circle with distance at least 1 from both endpoints, and it has distance 1 from both. For any radius smaller than $r(\varepsilon)$, there is no such point, which implies that the circumradius of F satisfies $R_F \ge r(\varepsilon) \ge \frac{1}{2} + \frac{1}{4}\varepsilon^2$, where the second inequality follows from (14).

To prove the upper bound, we draw F in the plane, assuming its circumcircle is the circle with radius R_F centered at the origin. Let a, b, c be the vertices of F, where a and c are the endpoints of the short edge. We have $0 \in F$, since otherwise one of the angles at a and c is obtuse, in which case the squared lengths of the two long edges differ by at least $4\varepsilon^2$. By assumption, $\sqrt{2}\Delta^2 < 2\Delta/n \le 2\varepsilon$, in which we get the second inequality from (11). But this implies that the difference between the squared lengths of the two long edges is larger than

$2\Delta^4$, which contradicts (18). Hence, b lies between the antipodes of the other two vertices, $a' = -a$ and $c' = -c$. By construction, $\|a' - c'\| = 2\varepsilon$. Assuming $\|b - a'\| \leq \|b - c'\|$, this implies

$$\|b - a'\| \leq R_F \arcsin \tfrac{\varepsilon}{R_F} \leq \arcsin \varepsilon = \varepsilon + O(\varepsilon^3). \tag{19}$$

Here, the second inequality follows from $R_F \geq 1$, using the convexity of the arcsin function, and the final expression using the Taylor expansion $\arcsin x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \ldots$. Now consider the triangle with vertices a, a', b. By the Pythagorean theorem,

$$4R_F^2 = \|b - a\|^2 + \|b - a'\|^2 < 1 + 4\Delta^4 + \varepsilon^2 + O(\varepsilon^4) = 1 + \varepsilon^2 + O(\Delta^4), \tag{20}$$

where we used Lemma 8 and (19) to bound $\|b - a\|^2$ and $\|b - a'\|^2$, respectively. We get the final expression using $\varepsilon < \Delta$. Applying (12), we obtain $2R_F \leq 1 + \frac{1}{2}\varepsilon^2 + O(\Delta^4)$, as claimed.   ◀

Similar to the case of triangles, it is not difficult to establish that the circumradius of any tetrahedron in the Delaunay mosaic is at least the circumradius of the ideal tetrahedron.

▶ **Lemma 10** (Lower Bound for Tetrahedra in $\mathbb{R}^3$). *Let $0 < \Delta < 1$, $A = A_3(n, \Delta) \subseteq \mathbb{R}^3$, and $\varepsilon = \varepsilon(n, \Delta)$. Then the circumradius of any tetrahedron $T \in \mathrm{Del}(A)$ satisfies $\frac{1}{2} + \frac{5}{11}\varepsilon^2 \leq R_T$.*

Proof. By construction, T has two disjoint short edges, both of length 2ε. We place the endpoints of one short edge on a sphere of radius $R(\varepsilon)$. The set of points on this sphere that are at distance at least 1 from both endpoints is the intersection of two spherical caps whose centers are antipodal to the endpoints. We call this intersection a *spherical bi-gon*. Since the two caps have the same size, the two corners of the bi-gon are further apart than any other two points of the bi-gon. By choice of the radius, $R(\varepsilon)$, the edge connecting the two corners has length 2ε. Hence, these corners are the only possible choice for the remaining two vertices of T, and for a radius smaller than $R(\varepsilon)$, there is no choice. It follows that the circumradius of T is at least $R(\varepsilon)$, and we get the claimed lower bound from (15).   ◀

3.3   All simplices are critical

Since no empty sphere passes through more than four points of A, the Delaunay mosaic of A is simplicial, and the radius function on this Delaunay mosaic is a generalized discrete Morse function [4]. Furthermore, all simplices are critical; see Definition 2. The point set depends on two parameters, n and Δ, and we consider n fixed while Δ goes to zero.

▶ **Lemma 11** (All Critical in $\mathbb{R}^3$). *Let $n \geq 2$, $\Delta > 0$ sufficiently small, and $A = A_3(n, \Delta) \subseteq \mathbb{R}^3$. Then every simplex of the Delaunay mosaic of A is critical.*

Proof. It is clear that the vertices and the short edges are critical, but the other simplices in $\mathrm{Del}(A)$ require an argument. We begin with the long edges. Fix i and j, and write $S^2(i; j)$ for the smallest sphere that passes through a_i and b_j. Its center is the midpoint of the long edge and by (18) its squared diameter is between 1 and $1 + 2\Delta^4$. The distance between a_i and any a_ℓ, $\ell \neq i$, is at least 2ε. Assuming a_ℓ is on or inside $S^2(i; j)$, we thus have $\|a_\ell - b_j\|^2 \leq 1 + 2\Delta^4 - 4\varepsilon^2$, which, for sufficiently small $\Delta > 0$, is less than 1. But this contradicts the lower bound in Lemma 8, so a_ℓ lies outside $S^2(i; j)$. By a symmetric argument, all b_ℓ, $\ell \neq j$, lie outside $S^2(i; j)$. Hence, $S^2(i; j)$ is strictly empty, for all $0 \leq i, j \leq n$, which implies that all edges of $\mathrm{Del}(A)$ are critical edges of the radius function.

The fact that all edges of $\mathrm{Del}(A)$ are critical implies that all triangles are acute. Indeed, if $a_i b_j b_{j+1}$ is not acute, then the midpoint of one long edge is at least as close to the third vertex as to the endpoints of the edge. Hence, any non-acute triangle would be an obstacle to the criticality of an edge, which implies that no such triangle can exist. However, the fact that all triangles are acute does not imply that all of them are critical. To prove the criticality of the Delaunay triangles, let x be the circumcenter of $a_i b_j b_{j+1}$, let $S^2(i; j, j+1)$ be centered at x and pass through a_i, b_j, b_{j+1}, and let a be the point other than a_i in which $S^2(i; j, j+1)$ intersects C_z. Since $a_i b_j b_{j+1}$ is acute, x lies in the interior of the triangle. It remains to show that the sphere is strictly empty. To this end, let x' and x'' be the centers of $S^2(i; j)$ and $S^2(i; j+1)$, let a' and a'' be the points other than a_i in which the two spheres intersect C_z, and consider the lines that pass through x and x' and through x and x'', respectively. Note that x lies in the angle between the half-lines from a_i that pass through x' and x''. This implies that a is between a' and a''. Since $S^2(i; j)$ and $S^2(i; j+1)$ are strictly empty, a' and a'' lie strictly between a_{i-1} and a_{i+1}, and so does a. Hence, $S^2(i; j, j+1)$ is strictly empty, which implies that all triangles of $\mathrm{Del}(A)$ are critical triangles of the radius function.

Since all triangles are critical, all tetrahedra of $\mathrm{Del}(A)$ must also be critical. One can argue in two ways. Combinatorially: the radius function pairs non-critical tetrahedra with non-critical triangles, but there are no such triangles. Geometrically: since every triangle has a non-empty intersection with its dual Voronoi edge, every tetrahedron must contain its dual Voronoi vertex.   ◀

3.4   Counting the tunnels and voids

Before counting the tunnels and voids, we recall that $\mathrm{Rad}\colon \mathrm{Del}(A) \to \mathbb{R}$ maps each simplex to the radius of its smallest empty sphere that passes through its vertices. By Lemma 11, all simplices of $\mathrm{Del}(A)$ are critical, so $\mathrm{Rad}(E)$ is equal to the circumradius of E, for every edge $E \in \mathrm{Del}(A)$, and similarly for every triangle and every tetrahedron.

▶ **Corollary 12** (Ordering of Radii in $\mathbb{R}^3$). *Let $\Delta > 0$ be sufficiently small, let $A = A_3(n, \Delta) \subseteq \mathbb{R}^3$, and let $\mathrm{Rad}\colon \mathrm{Del}(A) \to \mathbb{R}$ be the radius function. Then $\mathrm{Rad}(E) < \mathrm{Rad}(F) < \mathrm{Rad}(T)$ for every edge E, triangle F, and tetrahedron T in $\mathrm{Del}(A)$.*

Proof. Using Lemma 8 for the edges, Lemma 9 for the triangles, and Lemma 10 for the tetrahedra in the Delaunay mosaic of A, we get

$$\mathrm{Rad}(E) = R_E < \tfrac{1}{2} + O(\Delta^4), \tag{21}$$

$$\tfrac{1}{2} + \tfrac{1}{4}\varepsilon^2 \leq \mathrm{Rad}(F) = R_F < \tfrac{1}{2} + \tfrac{1}{4}\varepsilon^2 + O(\Delta^4), \tag{22}$$

$$\tfrac{1}{2} + \tfrac{5}{11}\varepsilon^2 \leq \mathrm{Rad}(T) = R_T, \tag{23}$$

so for sufficiently small $\Delta > 0$, the edges precede the triangles, and the triangles precede the tetrahedra in the filtration of the simplices.   ◀

For the final counting, choose ρ_1 to be any number strictly between the maximum radius of any edge and the minimum radius of any triangle. The existence of such a number is guaranteed by Corollary 12. The corresponding Čech complex is the 1-skeleton of the Delaunay mosaic. It is connected, with $N = 2n + 2$ vertices and $2n + (n+1)^2$ edges. The number of independent cycles is the difference plus 1, which implies $\beta_1(\mathrm{Čech}(A, \rho_1)) = (n+1)^2 - 1$, as claimed. Similarly, choose ρ_2 between the maximum radius of any triangle and the minimum radius of any tetrahedron, which is again possible, by Corollary 12. The corresponding Čech complex is the 2-skeleton of the Delaunay mosaic. The number of independent 2-cycles is the number of missing tetrahedra. This implies $\beta_2(\mathrm{Čech}(A, \rho_2)) = n^2$, as claimed.

4 Discussion

In this paper, we give asymptotically tight bounds for the maximum p-th Betti number of the Čech complex of N points in $\mathbb{R}^d$. These bounds also apply to the related Alpha complex and the dual union of equal-size balls in $\mathbb{R}^d$. They do not apply to the Vietoris–Rips complex, which is the flag complex that shares the 1-skeleton with the Čech complex for the same data. In other words, the Vietoris–Rips complex can be constructed by adding all 2- and higher-dimensional simplices whose complete set of edges belongs the 1-skeleton of the Čech complex. This implies $\beta_1(\mathrm{Rips}(A, r)) \leq \beta_1(\check{\mathrm{C}}\mathrm{ech}(A, r))$, since adding a triangle may lower but cannot increase the first Betti number.

As proved by Goff [16], the 1-st Betti number of the Vietoris–Rips complex of N points is $O(N)$, for all radii and in all dimensions, so also in $\mathbb{R}^3$. Compare this with the quadratic lower bound for Čech complexes proved in this paper. This implies that the first homology group has a basis in which all but $O(N)$ generators are tri-gons; that is: the three edges of a triangle. The circumradius of a tri-gon is less than $\sqrt{2}$ times the half-length of its longest edge, which implies that the corresponding generators exist only for a short range of radii. In the language of persistent homology [9], most points in the 1-dimensional persistence diagram represent 1-cycles with small persistence. This argument generalizes to the cases $p \leq m = \min\{p + 1, \lceil d/2 \rceil\}$, when the maximum number of p-cycles in the Vietoris–Rips complex is $o(N^p)$ [16], which is asymptotically less than $\Theta(N^m)$ for the Čech complex.

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