Dimensionality of Hamming Metrics and Rademacher Type

Alexandros Eskenazis
CNRS, Institut de Mathématiques de Jussieu, Sorbonne Université, France
Trinity College, University of Cambridge, UK

Abstract

Let X be a finite-dimensional normed space. We prove that if the Hamming cube $\{-1,1\}^n$ embeds into X with bi-Lipschitz distortion at most $D \geq 1$, then

$$\dim(X) \gtrsim \sup_{p\in[1,2]} \frac{n^p}{D^p \mathsf{T}_p(X)^p},$$

where $\mathsf{T}_p(X)$ is the Rademacher type p constant of X. This estimate yields a mutual refinement of distortion lower bounds which follow from works of Oleszkiewicz (1996) and Ivanisvili, van Handel and Volberg (2020). The proof relies on a combination of semigroup techniques on the biased hypercube with the Borsuk–Ulam theorem from algebraic topology.

2012 ACM Subject Classification Mathematics of computing → Functional analysis; Theory of computation → Random projections and metric embeddings

Keywords and phrases Hamming cube, Rademacher type, metric embeddings, Borsuk–Ulam theorem

Digital Object Identifier 10.4230/LIPIcs.SoCG.2024.55

Related Version *Full Version*: `https://arxiv.org/abs/2310.01868`

Acknowledgements I wish to thank Florent Baudier, Paata Ivanisvili and Assaf Naor for their constructive feedback on this work.

1 Introduction

If $(\mathcal{M}, d_{\mathcal{M}})$ is a metric space and $(Y, \|\cdot\|_Y)$ is a normed space, we say that $\mathcal{M}$ embeds into Y with bi-Lipschitz distortion at most $D \in [1,\infty)$, if there exists a mapping $f : \mathcal{M} \to Y$ satisfying the condition

$$\forall\, p,q \in \mathcal{M}, \qquad d_{\mathcal{M}}(p,q) \leq \|f(p)-f(q)\|_Y \leq D d_{\mathcal{M}}(p,q). \tag{1}$$

The least $D \geq 1$ for which such an embedding exists will be denoted by $\mathsf{c}_Y(\mathcal{M})$. The rapidly growing field of metric dimension reduction aims to uncover conditions under which given families of metric spaces admit (or do not admit) embeddings into low-dimensional normed spaces with prescribed properties. Without attempting to survey this vast area, we note that important contributions have been made on low-dimensional embeddings of finite subsets of Hilbert space [22], arbitrary finite metric spaces [23, 3, 33, 34], discrete hypercubes [43, 27], diamond graphs [10, 28, 40], Laakso graphs [17, 27], ultrametric spaces [7], series-parallel graphs [11], recursive cycle graphs [1], Heisenberg-type metrics [25, 42], ℓ_p variants of thin Laakso structures [6, 8] and expander graphs [37]. We refer to the survey [38] for more bibliographic information and to [19, 31, 50, 2] for a sample of applications to algorithms.

Let $\{-1,1\}^n$ be the n-dimensional discrete hypercube equipped with the Hamming metric

$$\forall\, x,y \in \{-1,1\}^n, \qquad \rho(x,y) = \frac{1}{2}\sum_{i=1}^n |x(i)-y(i)|, \tag{2}$$

40th International Symposium on Computational Geometry (SoCG 2024).
Editors: Wolfgang Mulzer and Jeff M. Phillips; Article No. 55; pp. 55:1–55:13
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

where $x = (x(1), \ldots, x(n))$ and $y = (y(1), \ldots, y(n))$. The purpose of the present paper is to investigate embeddability properties of this discrete metric space into low-dimensional normed spaces. The first to study the bi-Lipschitz embeddability of hypercubes into normed spaces was Enflo. In the seminal work [13], he introduced the notion of *roundness* of a metric space and used it to show that any embedding of the Hamming cube $\{-1,1\}^n$ into an $L_p(\mu)$ space, where $p \in [1,2]$, incurs bi-Lipschitz distortion at least $n^{1-1/p}$ (see also [12, 14] for additional early results along these lines). More specifically, Enflo proved that if $p \in [1,2]$, then any mapping $f : \{-1,1\}^n \to L_p(\mu)$ satisfies the estimate

$$\int_{\{-1,1\}^n} \big\|f(x) - f(-x)\big\|_{L_p(\mu)}^p \, \mathrm{d}\sigma_n(x) \\ \leq \sum_{i=1}^n \int_{\{-1,1\}^n} \big\|f(x) - f\big(x(1), \ldots, -x(i), \ldots, x(n)\big)\big\|_{L_p(\mu)}^p \, \mathrm{d}\sigma_n(x), \tag{3}$$

where σ_n is the uniform probability on $\{-1,1\}^n$. This readily implies that if f has bi-Lipschitz distortion D, then $D \geq n^{1-1/p}$. In the follow-up work [14], he raised an influential problem by asking for which normed spaces $(X, \|\cdot\|_X)$, inequality (3) is satisfied for X-valued functions f up to a multiplicative constant T, independent of the choice of f or the dimension n. Restricting our requirement to linear functions $f(x) = \sum_{i=1}^n x_i v_i$, we recover the necessary condition

$$\int_{\{-1,1\}^n} \Big\| \sum_{i=1}^n x_i v_i \Big\|_X^p \, \mathrm{d}\sigma_n(x) \leq T^p \sum_{i=1}^n \|v_i\|_X^p, \tag{4}$$

which must be satisfied for every $n \in \mathbb{N}$ and vectors $v_1, \ldots, v_n \in X$. If a normed space X satisfies (4), we say that X has *Rademacher type* p and the least constant T is denoted by $\mathsf{T}_p(X)$. After decades of substantial efforts (see [9, 48, 41, 18, 15]), Ivanisvili, van Handel and Volberg resolved Enflo's problem in the breakthrough work [20] by proving the sufficiency of this condition, namely that any normed space of Rademacher type p also has Enflo's nonlinear type p. Consequently, any bi-Lipschitz embedding of $\{-1,1\}^n$ into a normed space X of Rademacher type p incurs distortion at least a constant multiple of $\mathsf{T}_p(X)^{-1} n^{1-1/p}$. We note in passing that, conversely, a classical theorem of Pisier [44] implies that if X does not have type p for any $p > 1$, then $\{-1,1\}^n$ embeds into X with bi-Lipschitz distortion at most $1 + \varepsilon$, for any $\varepsilon > 0$.

Independently of this line of research, the beautiful (but perhaps overlooked) work [43] of Oleszkiewicz established a nonembeddability result for discrete hypercubes in the context of dimensionality reduction. Following Ball, Carlen and Lieb [4], we say that a normed space is *p-uniformly smooth*, where $p \in [1,2]$, if there exists a constant $S > 0$ such that

$$\forall\, x, y \in X, \qquad \frac{\|x\|_X^p + \|y\|_X^p}{2} \leq \Big\| \frac{x+y}{2} \Big\|_X^p + S^p \Big\| \frac{x-y}{2} \Big\|_X^p; \tag{5}$$

the least such constant S is denoted by $\mathsf{S}_p(X)$. A well-known tensorization argument due to Pisier [46] shows that $\mathsf{T}_p(X) \leq \mathsf{S}_p(X)$, yet there exist examples of normed spaces X for which $\mathsf{T}_p(X) < \infty$ whereas $\mathsf{S}_p(X) = \infty$ for $p \in (1,2]$, see [47, 21, 49]. The main result of Oleszkiewicz's paper [43] asserts that if $\{-1,1\}^n$ embeds into a finite-dimensional normed space X with bi-Lipschitz distortion at most $D \geq 1$, then

$$\dim(X) \geq \sup_{p \in [1,2]} \frac{n^p}{D^p \mathsf{S}_p(X)^p} \tag{6}$$

(see also [5] for a precursor of this result for linear embeddings). Viewed differently, the result of [43] asserts that if X is a d-dimensional normed space, then

$$\mathsf{c}_X(\{-1,1\}^n) \geq \sup_{p \in [1,2]} \frac{n}{\mathsf{S}_p(X) d^{1/p}} \tag{7}$$

which substantially improves the bound $\mathsf{c}_X(\{-1,1\}^n) \gtrsim \mathsf{T}_p(X)^{-1}n^{1-1/p}$ which follows from [20] for spaces X with $\mathsf{S}_p(X) \asymp 1$ and dimensions $\dim(X) \ll n$.

The main purpose of the present paper is to revisit the technique used for Oleszkiewicz's nonembeddability theorem [43], in particular proving the following mutual refinement of this result with the recent work of Ivanisvili, van Handel and Volberg [20].

▶ **Theorem 1.** *Fix $D \geq 1$ and let $(X, \|\cdot\|_X)$ be a finite-dimensional normed space. If the Hamming cube $\{-1,1\}^n$ embeds into X with bi-Lipschitz distortion at most D, then*

$$\dim(X) \gtrsim \sup_{p\in[1,2]} \frac{n^p}{D^p \mathsf{T}_p(X)^p}. \tag{8}$$

We emphasize that the bound obtained in Theorem 1 is sharp. Indeed, if $X = \ell_q^n$ and $1 \leq q \leq p \leq 2$, then Enflo's theorem [13] combined with simple manipulations implies that $\mathsf{T}_p(\ell_q^n) = n^{1/q-1/p}$ and $D = n^{1-1/q}$, so the two sides of the inequality become equal. On the other hand, in contrast to Oleszkiewicz's bound (6), Theorem 1 also captures more accurately the nonembeddability of the hypercube into (finite-dimensional subspaces of) normed spaces which have Rademacher type p but are not r-smooth for any $r \in (1,2]$, see [21, 49].

1.1 About the proof

Theorem 1 is proven by a combination of semigroup tools with a clever topological trick of [43]. More specifically, let $f : \{-1,1\}^n \to X$ be a function, where X is a d-dimensional normed space and $d < n$. An application of the Borsuk–Ulam theorem [35] for the unique multilinear extension of f implies that there exists a subset $\sigma \subseteq \{1,\ldots,n\}$ with $|\sigma| = d$, a *product* measure ν on $\{-1,1\}^\sigma$ and a point $w \in \{-1,1\}^{\sigma^c}$ such that

$$\int_{\{-1,1\}^\sigma} f(x,w)\,\mathrm{d}\nu(x) = \int_{\{-1,1\}^\sigma} f(-x,-w)\,\mathrm{d}\nu(x). \tag{9}$$

Then, a Poincaré inequality à la Enflo for the product measure ν (instead of the uniform measure σ_d) on the d-dimensional subcubes $\{-1,1\}^\sigma \times \{w\}$ and $\{-1,1\}^\sigma \times \{-w\}$ yields the distortion bounds of Theorem 1 (see Theorem 9 and also equation (39) below).

In the case of p-uniformly smooth spaces, Oleszkiewicz [43] used (9) and a bootstrap argument for the Lipschitz constant of f, based on the two-point inequality (5), to obtain (6). In our case, the biased Poincaré inequality which will yield (8) is an extension of an inequality for the uniform measure that was proven in [20]. The key technical contribution of [20] was a novel representation of the time derivative of the heat flow on $\{-1,1\}^n$. Instead, we consider a Markov process having the product measure ν as stationary measure (see Section 2) and prove a formula for the time derivative of the corresponding semigroup (see Proposition 8) which extends the formula of [20] (see also (38) below). Due to the fact that our product measure ν is no longer the stationary measure of the random walk on a group, the resulting identity lacks some homogeneity properties that were used in [20], but nevertheless it is sufficient for the proof of the biased Poincaré inequality which is needed for our geometric application.

2 Preliminaries

2.1 Probability

In this section, we outline the basics of analysis on the biased hypercube, with an emphasis on the underlying semigroup structure.

The biased measure

For $\alpha \in (0,1)$, consider the α-biased probability measure μ_α on $\{-1,1\}$ given by $\mu_\alpha\{1\} = \alpha$ and $\mu_\alpha\{-1\} = 1-\alpha$. Moreover, if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (0,1)^n$, then we shall denote by $\mu_{\boldsymbol{\alpha}}$ the product measure $\mu_{\alpha_1} \otimes \cdots \otimes \mu_{\alpha_n}$ on the hypercube $\{-1,1\}^n$.

The Markov process

For $\alpha \in (0,1)$, consider the transition matrices $\{p_t^\alpha\}_{t\geq 0}$ on $\{-1,1\}$ given by

$$\forall\, t \geq 0, \qquad \begin{pmatrix} p_t^\alpha(1,1) & p_t^\alpha(1,-1) \\ p_t^\alpha(-1,1) & p_t^\alpha(-1,-1) \end{pmatrix} = \begin{pmatrix} 1-(1-e^{-t})(1-\alpha) & (1-e^{-t})(1-\alpha) \\ (1-e^{-t})\alpha & 1-(1-e^{-t})\alpha \end{pmatrix} \tag{10}$$

Moreover, for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (0,1)^n$ consider the corresponding tensor products $\{p_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$ on $\{-1,1\}^n$ given by

$$\forall\, x, y \in \{-1,1\}^n, \qquad p_t^{\boldsymbol{\alpha}}(x,y) = \prod_{i=1}^n p_t^{\alpha_i}\big(x(i), y(i)\big). \tag{11}$$

As each $p_t^{\alpha_i}$ is a row-stochastic 2×2 matrix with nonnegative entries, the same holds also for the $2^n \times 2^n$ matrices $p_t^{\boldsymbol{\alpha}}$. Therefore, $\{p_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$ is the transition kernel of a time-homogeneous Markov chain $\{X_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$ on $\{-1,1\}^n$, that is

$$\forall\, t, s \geq 0, \qquad \mathbb{P}\big\{X_{t+s}^{\boldsymbol{\alpha}} = y \;\big|\; X_s^{\boldsymbol{\alpha}} = x\big\} = p_t^{\boldsymbol{\alpha}}(x,y), \tag{12}$$

where $x, y \in \{-1,1\}^n$. We shall need the following simple facts for this process.

▶ **Lemma 2.** *Fix $\boldsymbol{\alpha} \in (0,1)^n$ and let $\{X_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$ be a Markov process on $\{-1,1\}^n$ with transition kernels $\{p_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$. Then, $\{X_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$ is stationary and reversible with respect to $\mu_{\boldsymbol{\alpha}}$.*

Proof. Due to the product structure of the Markov chain, it suffices to consider the case $n = 1$, that is, to prove that for $\alpha \in (0,1)$,

$$\forall\, x, y \in \{-1,1\}, \qquad \mu_\alpha(x) p_t^\alpha(x,y) = \mu_\alpha(y) p_t^\alpha(y,x). \tag{13}$$

This follows automatically by the expression (10) for the transition matrix. The simple fact that reversibility implies stationarity is well-known [30, Proposition 1.20]. ◀

The stationary Markov process $\{X_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$ has a simple probabilistic interpretation which we shall now describe. For $i = 1, \ldots, n$, let $\{N_t(i)\}_{t\geq 0}$ be n independent Poisson processes of unit rate and suppose that $X_0^{\boldsymbol{\alpha}}$ is sampled from $\mu_{\boldsymbol{\alpha}}$ independently of $\{N_t\}_{t\geq 0}$. Then, at any time $t > 0$ for which the process $N_t(i)$ jumps for some $i \in \{1, \ldots, n\}$, the corresponding value $X_t^{\boldsymbol{\alpha}}(i)$ is updated independently from μ_{α_i}. An explicit calculation shows that this probabilistic construction gives rise exactly to the transition kernel of (10) and (11).

The corresponding semigroup

Fix $\boldsymbol{\alpha} \in (0,1)^n$ and let $\{P_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$ be the Markov semigroup associated to the process $\{X_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$. Concretely, if X is a vector space, then for every function $f : \{-1,1\}^n \to X$ and $t \geq 0$, we denote by

$$\forall\, x \in \{-1,1\}^n, \qquad P_t^{\boldsymbol{\alpha}} f(x) = \mathbb{E}\big[f\big(X_t^{\boldsymbol{\alpha}}\big) \;\big|\; X_0^{\boldsymbol{\alpha}} = x\big]. \tag{14}$$

In view of the above interpretation of $\{X_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$ by means of a Poisson process, the action of the semigroup $\{P_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$ can be computed via the identity

$$\begin{aligned} P_t^{\boldsymbol{\alpha}} f &= \sum_{S\subseteq\{1,\ldots,n\}} \mathbb{P}\big\{N_t(i)>0 \text{ for } i\in S \text{ and } N_t(i)=0 \text{ for } i\notin S\big\} \int_{\{-1,1\}^S} f \,\mathrm{d}\prod_{i\in S}\mu_{\alpha_i} \\ &= \sum_{S\subseteq\{1,\ldots,n\}} (1-e^{-t})^{|S|} e^{-t(n-|S|)} \int_{\{-1,1\}^S} f\,\mathrm{d}\prod_{i\in S}\mu_{\alpha_i}. \end{aligned} \tag{15}$$

▶ **Lemma 3.** *Fix $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_n)\in(0,1)^n$ and let $\{P_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$ be the semigroup (14). Then, the action of its generator $\mathcal{L}_{\boldsymbol{\alpha}}$ on a function $f:\{-1,1\}^n\to X$, where X is a vector space, is given by*

$$\forall\, x\in\{-1,1\}^n, \qquad \mathcal{L}_{\boldsymbol{\alpha}} f(x) = -\sum_{i=1}^n \partial_i^{\alpha_i} f(x) \tag{16}$$

where $\partial_i^{\beta} f(x) = f(x) - \int_{\{-1,1\}} f(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n)\,\mathrm{d}\mu_\beta(y)$ for $\beta\in(0,1)$.

Proof. The claim follows from the expression (15) of the semigroup and the definition

$$\forall\, x\in\{-1,1\}^n, \qquad \mathcal{L}_{\boldsymbol{\alpha}} f(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P_t^{\boldsymbol{\alpha}} f(x). \qquad ◀$$

2.2 Topology

Apart from the probabilistic elements from analysis on biased hypercubes, the proof of Theorem 1 also has a crucial topological component, following an idea of [43].

The Borsuk–Ulam theorem

While the Poincaré-type inequality of Enflo for X-valued functions on $\{-1,1\}^n$ cannot capture the dimension of the target space X, a key part of the argument towards Theorem 1 is to show that there exists a $\dim(X)$-dimensional subcube of $\{-1,1\}^n$ along with a bias vector $\boldsymbol{\alpha}$ for which the $\boldsymbol{\alpha}$-biased Poincaré inequalities (see Theorem 9 below) on this subcube and its antipodal yield much better distortion lower bounds. This will be done using the Borsuk–Ulam theorem from algebraic topology, see [35].

▶ **Theorem 4** (Borsuk–Ulam). *For every continuous function $g:\mathbb{S}^d\to\mathbb{R}^d$, where $d\in\mathbb{N}$, there exists a point $w\in\mathbb{S}^d$ such that $g(w)=g(-w)$.*

Multilinear extension and low-dimensional faces of the cube

Every function $f:\{-1,1\}^n\to X$ admits a unique multilinear extension on the solid cube $[-1,1]^n$, given by

$$\forall\, y\in[-1,1]^n, \qquad F(y) \stackrel{\mathrm{def}}{=} \sum_{S\subseteq\{-1,1\}^n} \Big(\frac{1}{2^n}\sum_{x\in\{-1,1\}^n} f(x)w_S(x)\Big) w_S(y), \tag{17}$$

where $w_S(a)=\prod_{i\in S} a_i$, which is usually referred to as the Fourier–Walsh expansion of f. Extending f to the continuous cube allows for the use of topological methods. In what follows, we will exploit the fact that the cube $[-1,1]^n$ is equipped with a canonical CW complex structure. Concretely, for $d\in\{1,\ldots,n\}$, consider the subsets

$$\mathcal{C}_d^n \stackrel{\mathrm{def}}{=} \big\{x\in[-1,1]^n:\ \text{there exists } \sigma\subseteq\{1,\ldots,n\} \text{ with } |\sigma|\geq n-d \text{ and } |x(i)|=1,\ \forall\, i\in\sigma\big\}$$

consisting of all ℓ-dimensional faces of $[-1,1]^n$ for $\ell \le d$, so that $\mathcal{C}_n^n = [-1,1]^n$ and $\mathcal{C}_0^n = \{-1,1\}^n$. We shall use the following elementary topological fact (see [43, Lemma 1]).

▶ **Lemma 5.** *If $d < n$, there exists a continuous map $h_d : \mathbb{S}^d \to \mathcal{C}_d^n$ with $h_d(-x) = -h_d(x)$, $\forall\, x \in \mathbb{S}^d$.*

Combining this and the Borsuk–Ulam theorem, we deduce the following useful lemma.

▶ **Lemma 6.** *If $n, d \in \mathbb{N}$ with $d < n$, then for every continuous function $F : \mathcal{C}_d^n \to \mathbb{R}^d$ there exists a point $z \in \mathcal{C}_d^n$ such that $F(z) = F(-z)$.*

Proof. Consider the function $g \overset{\text{def}}{=} F \circ h_d : \mathbb{S}^d \to \mathbb{R}^d$, where h_d is the function of Lemma 5. By the Borsuk–Ulam theorem and the oddness of h_d, there exists a point $w \in \mathbb{S}^d$ such that

$$F(h_d(w)) = g(w) = g(-w) = F(h_d(-w)) = F(-h_d(w)) \tag{18}$$

and the conclusion follows by choosing $z = h_d(w) \in \mathcal{C}_d^n$. ◀

3 Proof of Theorem 1

We are now ready to proceed to the main part of the proof. The main analytic component is a biased version of the key formula of [20] for the time derivative of the heat flow on $\{-1,1\}^n$. Given $t \ge 0$, $\alpha \in (0,1)$ and an auxiliary parameter $\theta \in \mathbb{R}$, consider the matrix $\eta_t^\alpha(\cdot,\cdot;\theta)$ given by

$$\forall\, t \ge 0, \qquad \begin{pmatrix} \eta_t^\alpha(1,1;\theta) & \eta_t^\alpha(1,-1;\theta) \\ \eta_t^\alpha(-1,1;\theta) & \eta_t^\alpha(-1,-1;\theta) \end{pmatrix} = \begin{pmatrix} \frac{e^{-t}-\theta}{p_t^\alpha(1,1)} & \frac{-\theta}{p_t^\alpha(-1,1)} \\ \frac{\theta-e^{-t}}{p_t^\alpha(1,-1)} & \frac{\theta}{p_t^\alpha(-1,-1)} \end{pmatrix}. \tag{19}$$

For future reference, we record the following straightforward properties of $\eta_t^\alpha(\cdot,\cdot;\theta)$.

▶ **Lemma 7.** *Fix $t \ge 0$ and $\alpha \in (0,1)$. Then,*

$$\forall\, x \in \{-1,1\},\ \theta \in \mathbb{R}, \qquad p_t^\alpha(x,1)\eta_t^\alpha(1,x;\theta) + p_t^\alpha(x,-1)\eta_t^\alpha(-1,x;\theta) = 0 \tag{20}$$

and

$$\begin{aligned} &\min_{\theta\in\mathbb{R}} \max_{x\in\{-1,1\}} \Big\{ p_t^\alpha(x,1)\eta_t^\alpha(1,x;\theta)^2 + p_t^\alpha(x,-1)\eta_t^\alpha(-1,x;\theta)^2 \Big\} \\ &\qquad = \frac{e^{-t}}{(e^t-1)\big(\sqrt{\alpha p_t^\alpha(-1,-1)} + \sqrt{(1-\alpha)p_t^\alpha(1,1)}\big)^2} \le \frac{1}{e^t-1}. \end{aligned} \tag{21}$$

Proof. The centering condition (20) can be checked easily using the explicit formulas (10) and (19) of the matrices. For (21), we compute that for any $\theta \in \mathbb{R}$,

$$\begin{aligned} &\max_{x\in\{-1,1\}} \Big\{ p_t^\alpha(x,1)\eta_t^\alpha(1,x;\theta)^2 + p_t^\alpha(x,-1)\eta_t^\alpha(-1,x;\theta)^2 \Big\} \\ &\qquad = \max\Big\{ \frac{(e^{-t}-\theta)^2}{p_t^\alpha(1,1)p_t^\alpha(1,-1)}, \frac{\theta^2}{p_t^\alpha(-1,1)p_t^\alpha(-1,-1)} \Big\}. \end{aligned} \tag{22}$$

As this is the maximum of two quadratic functions in θ, its minimum is attained at the point θ^* where they intersect in the interval $(0, e^{-t})$, namely at

$$\theta^* = \frac{e^{-t}\sqrt{\alpha p_t^\alpha(-1,-1)}}{\sqrt{\alpha p_t^\alpha(-1,-1)} + \sqrt{(1-\alpha)p_t^\alpha(1,1)}}. \tag{23}$$

The first equality in (21) is immediate, whereas for the inequality we compute

$$\begin{aligned}\frac{e^{-t}}{(e^t-1)(\sqrt{\alpha p_t^\alpha(-1,-1)}+\sqrt{(1-\alpha)p_t^\alpha(1,1)})^2} &\leq \frac{e^{-t}}{(e^t-1)(\alpha p_t^\alpha(-1,-1)+(1-\alpha)p_t^\alpha(1,1))} \\ &= \frac{e^{-t}}{(e^t-1)\big(1-(1-e^{-t})(\alpha^2+(1-\alpha)^2)\big)} \\ &\leq \frac{1}{e^t-1},\end{aligned}$$

where both inequalities follow from the convexity of $x \mapsto x^2$. ◀

The key technical ingredient in the proof of Theorem 1 is the following identity.

▶ **Proposition 8.** *Fix $n \in \mathbb{N}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (0,1)^n$, $t \geq 0$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$. Then, for every function $f : \{-1,1\}^n \to X$, where X is a vector space, we have*

$$\forall\ x \in \{-1,1\}^n, \qquad \mathcal{L}_{\boldsymbol{\alpha}} P_t^{\boldsymbol{\alpha}} f(x) = -\mathbb{E}\Big[\sum_{i=1}^n \eta_t^{\alpha_i}\big(x(i), X_t^{\boldsymbol{\alpha}}(i);\theta_i\big)\partial_i^{\alpha_i} f(X_t^{\boldsymbol{\alpha}}) \ \Big|\ X_0^{\boldsymbol{\alpha}} = x\Big]. \tag{24}$$

Proof. In view of (16) and the product structure of the process $\{X_t^{\boldsymbol{\alpha}}\}_{t\geq 0}$, it suffices to check the claim for $n = 1$, namely that for every $\beta \in (0,1)$, $\theta \in \mathbb{R}$ and $f : \{-1,1\} \to X$,

$$\forall\ x \in \{-1,1\}, \qquad e^{-t}\partial^\beta f(x) = P_t^\beta \partial^\beta f(x) = \mathbb{E}\big[\eta_t^\beta(x, X_t^\beta;\theta)\partial^\beta f(X_t^\beta)\ \big|\ X_0^\beta = x\big], \tag{25}$$

where the first equality follows from the probabilistic representation (15). Taking into account that $\beta\partial^\beta f(1) + (1-\beta)\partial^\beta f(-1) = 0$, this amounts to the system of equations

$$\begin{cases} e^{-t} = p_t^\beta(1,1)\eta_t^\beta(1,1;\theta) - \frac{\beta}{1-\beta}p_t^\beta(1,-1)\eta_t^\beta(1,-1;\theta) \\ e^{-t} = -\frac{1-\beta}{\beta}p_t^\beta(-1,1)\eta_t^\beta(-1,1;\theta) + p_t^\beta(-1,-1)\eta_t^\beta(-1,-1;\theta)\end{cases} \tag{26}$$

which can be easily verified by direct computation. ◀

▶ **Theorem 9.** *Fix $p \in [1,2]$ and let $(X, \|\cdot\|_X)$ be a normed space of Rademacher type p. Then, for any $n \in \mathbb{N}$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in (0,1)^n$, every function $f : \{-1,1\}^n \to X$ satisfies the inequality*

$$\int_{\{-1,1\}^n} \Big\| f(x) - \int_{\{-1,1\}^n} f \, \mathrm{d}\mu_{\boldsymbol{\alpha}} \Big\|_X^p \, \mathrm{d}\mu_{\boldsymbol{\alpha}}(x) \leq \big(2\pi \mathsf{T}_p(X)\big)^p \sum_{i=1}^n \int_{\{-1,1\}^n} \big\|\partial_i^{\alpha_i} f(x)\big\|_X^p \, \mathrm{d}\mu_{\boldsymbol{\alpha}}(x).$$

Proof. Writing

$$f(x) - \int_{\{-1,1\}^n} f \, \mathrm{d}\mu_{\boldsymbol{\alpha}} = P_0^{\boldsymbol{\alpha}} f(x) - P_\infty^{\boldsymbol{\alpha}} f(x) = -\int_0^\infty \mathcal{L}_{\boldsymbol{\alpha}} P_t^{\boldsymbol{\alpha}} f(x) \, \mathrm{d}t \tag{27}$$

and using Jensen's inequality and Proposition 8, we see that for $\theta_1(t), \ldots, \theta_n(t) \in \mathbb{R}$,

$$\begin{aligned}&\bigg(\int_{\{-1,1\}^n} \Big\| f(x) - \int_{\{-1,1\}^n} f\, \mathrm{d}\mu_{\boldsymbol{\alpha}}\Big\|_X^p \, \mathrm{d}\mu_{\boldsymbol{\alpha}}(x)\bigg)^{1/p} \\ &\leq \int_0^\infty \bigg(\int_{\{-1,1\}^n} \Big\|\mathcal{L}_{\boldsymbol{\alpha}} P_t^{\boldsymbol{\alpha}} f(x)\Big\|_X^p \, \mathrm{d}\mu_{\boldsymbol{\alpha}}(x)\bigg)^{1/p} \mathrm{d}t \\ &= \int_0^\infty \bigg(\int_{\{-1,1\}^n} \bigg\|\mathbb{E}\Big[\sum_{i=1}^n \eta_t^{\alpha_i}\big(x(i), X_t^{\boldsymbol{\alpha}}(i);\theta_i(t)\big)\partial_i^{\alpha_i} f(X_t^{\boldsymbol{\alpha}})\ \Big|\ X_0^{\boldsymbol{\alpha}} = x\Big]\bigg\|_X^p \, \mathrm{d}\mu_{\boldsymbol{\alpha}}(x)\bigg)^{1/p} \mathrm{d}t \\ &\leq \int_0^\infty \bigg(\mathbb{E}\Big\|\sum_{i=1}^n \eta_t^{\alpha_i}\big(X_0^{\boldsymbol{\alpha}}(i), X_t^{\boldsymbol{\alpha}}(i);\theta_i(t)\big)\partial_i^{\alpha_i} f(X_t^{\boldsymbol{\alpha}})\Big\|_X^p\bigg)^{1/p} \mathrm{d}t,\end{aligned} \tag{28}$$

where in the last expectation $X_0^{\boldsymbol{\alpha}}$ is distributed according to $\mu_{\boldsymbol{\alpha}}$. Now, by the reversibility of the chain, this expectation can be written as

$$\mathbb{E}\Big\|\sum_{i=1}^n \eta_t^{\alpha_i}\big(X_0^{\boldsymbol{\alpha}}(i), X_t^{\boldsymbol{\alpha}}(i);\theta_i(t)\big)\partial_i^{\alpha_i} f(X_t^{\boldsymbol{\alpha}})\Big\|_X^p$$
$$= \int_{\{-1,1\}^n} \sum_{y\in\{-1,1\}^n} p_t^{\boldsymbol{\alpha}}(x,y)\Big\|\sum_{i=1}^n \eta_t^{\alpha_i}\big(y(i), x(i);\theta_i(t)\big)\partial_i^{\alpha_i} f(x)\Big\|_X^p \,\mathrm{d}\mu_{\boldsymbol{\alpha}}(x). \quad (29)$$

Fixing $x\in\{-1,1\}^n$, equation (20) asserts that each $\eta_t^{\alpha_i}(y(i),x(i);\theta_i(t))$ is a centered random variable when $y(i)$ is distributed according to $p_t^{\alpha_i}(x(i),\cdot)$. Therefore, as $p_t^{\boldsymbol{\alpha}}(x,\cdot)$ is a product measure, the Rademacher type condition for sums of centered independent random vectors (see [26, Proposition 9.11]) yields the bound

$$\int_{\{-1,1\}^n} \sum_{y\in\{-1,1\}^n} p_t^{\boldsymbol{\alpha}}(x,y)\Big\|\sum_{i=1}^n \eta_t^{\alpha_i}\big(y(i), x(i);\theta_i(t)\big)\partial_i^{\alpha_i} f(x)\Big\|_X^p \,\mathrm{d}\mu_{\boldsymbol{\alpha}}(x)$$
$$\leq \big(2\mathsf{T}_p(X)\big)^p \int_{\{-1,1\}^n} \sum_{i=1}^n \sum_{y(i)\in\{-1,1\}} p_t^{\alpha_i}\big(x(i),y(i)\big)\ \big|\eta_t^{\alpha_i}\big(y(i),x(i);\theta_i(t)\big)\big|^p \big\|\partial_i^{\alpha_i} f(x)\big\|_X^p \,\mathrm{d}\mu_{\boldsymbol{\alpha}}(x)$$
$$\leq \big(2\mathsf{T}_p(X)\big)^p \sum_{i=1}^n \int_{\{-1,1\}^n} \Big(\sum_{y(i)\in\{-1,1\}} p_t^{\alpha_i}\big(x(i),y(i)\big)\ \big|\eta_t^{\alpha_i}\big(y(i),x(i);\theta_i(t)\big)\big|^2\Big)^{p/2} \big\|\partial_i^{\alpha_i} f(x)\big\|_X^p \,\mathrm{d}\mu_{\boldsymbol{\alpha}}(x), \quad (30)$$

where we also used that $p\leq 2$. Now, choosing the $\theta_i(t)$ which minimize the quantity in the left-hand side of (21) with bias α_i, and combining (28), (29) and (30), we conclude that

$$\bigg(\int_{\{-1,1\}^n} \Big\|f(x) - \int_{\{-1,1\}^n} f\,\mathrm{d}\mu_{\boldsymbol{\alpha}}\Big\|_X^p \,\mathrm{d}\mu_{\boldsymbol{\alpha}}(x)\bigg)^{1/p}$$
$$\leq 2\mathsf{T}_p(X)\int_0^\infty \bigg(\sum_{i=1}^n \int_{\{-1,1\}^n} \big\|\partial_i^{\alpha_i} f(x)\big\|_X^p \,\mathrm{d}\mu_{\boldsymbol{\alpha}}(x)\bigg)^{1/p} \frac{\mathrm{d}t}{\sqrt{e^t-1}}, \quad (31)$$

which is precisely the desired estimate. ◀

Equipped with the biased Poincaré inequality of Theorem 9, we can now conclude the proof.

Proof of Theorem 1. Let $X=(\mathbb{R}^d,\|\cdot\|_X)$ be a d-dimensional normed space and suppose that $f:\{-1,1\}^n\to X$ is a function such that

$$\forall\, x,y\in\{-1,1\}^n, \qquad \rho(x,y)\leq \|f(x)-f(y)\|_X\leq D\rho(x,y) \quad (32)$$

for some $D\geq 1$. The conclusion of the theorem follows from [20] when $d\geq n$ so we shall assume that $d<n$. Let $F:[-1,1]^n\to X$ be the multilinear extension of f given by (17). Then, F is clearly continuous as a polynomial and therefore, by Lemma 6, there exists a point $z\in\mathcal{C}_d^n$ such that $F(z)=F(-z)$. As z has at least $n-d$ coordinates equal to 1 in absolute value we shall assume without loss of generality that $|z(d+1)|=\ldots=|z(n)|=1$ and consider the functions $h_+,h_-:\{-1,1\}^d\to X$ which are defined as

$$\forall\, x\in\{-1,1\}^d, \qquad h_\pm(x) = f\big(\pm x(1),\ldots,\pm x(d),\pm z(d+1),\ldots,\pm z(n)\big). \quad (33)$$

Consider also the bias vector $\boldsymbol{\alpha}_z = \big(\frac{1+z(1)}{2}, \ldots, \frac{1+z(d)}{2}\big) \in (0,1)^d$ and notice that, by the multilinearity of F, we have the identity

$$\int_{\{-1,1\}^d} h_+(x)\, \mathrm{d}\mu_{\boldsymbol{\alpha}_z}(x) = F(z) = F(-z) = \int_{\{-1,1\}^d} h_-(x)\, \mathrm{d}\mu_{\boldsymbol{\alpha}_z}(x). \tag{34}$$

Therefore, by the triangle inequality and Theorem 9 we get

$$\begin{aligned}\int_{\{-1,1\}^d} &\big\|h_+(x) - h_-(x)\big\|_X^p \,\mathrm{d}\mu_{\boldsymbol{\alpha}_z}(x)\\ &\le 2^{p-1} \int_{\{-1,1\}^d} \big\|h_+(x) - F(z)\big\|_X^p + \big\|h_-(x) - F(-z)\big\|_X^p \,\mathrm{d}\mu_{\boldsymbol{\alpha}_z}(x)\\ &\le 2^{2p-1}\big(\pi \mathsf{T}_p(X)\big)^p \sum_{i=1}^d \int_{\{-1,1\}^d} \big\|\partial_i^{\frac{1+z(i)}{2}} h_+(x)\big\|_X^p + \big\|\partial_i^{\frac{1+z(i)}{2}} h_-(x)\big\|_X^p \,\mathrm{d}\mu_{\boldsymbol{\alpha}_z}(x). \end{aligned} \tag{35}$$

Now, in view of the lower Lipschitz condition (32), we clearly have

$$\big\|h_+(x) - h_-(x)\big\|_X = \big\|f(x,\bar{z}) - f(-x,-\bar{z})\big\|_X \ge n \tag{36}$$

for every $x \in \{-1,1\}^d$, where $\bar{z} = (z(d+1), \ldots, z(n))$. On the other hand, for a fixed $i \in \{1,\ldots,d\}$ and $\beta = \frac{1+z(i)}{2}$, we have

$$\begin{aligned}\int_{\{-1,1\}} &\big\|\partial_i^\beta h_+(x)\big\|_X^p \,\mathrm{d}\mu_\beta(x(i))\\ &= \beta\|\partial_i^\beta h_+(x(1),\ldots,1,\ldots,x(d))\|_X^p + (1-\beta)\|\partial_i^\beta h_+(x(1),\ldots,-1,\ldots,x(d))\|_X^p\\ &= \big(\beta(1-\beta)^p + (1-\beta)\beta^p\big)\ \big\|h_+(x(1),\ldots,1,\ldots,x(d)) - h_+(x(1),\ldots,-1,\ldots,x(d))\big\|_X^p\\ &\le \frac{D^p}{2^p},\end{aligned}$$

where in the last equality we used that $p \le 2$ along with the upper Lipschitz condition (32). The same bound also holds for h_-. Integrating the last two inequalities and combining them with (35), we deduce that

$$n^p \le (2\pi \mathsf{T}_p(X))^p d D^p, \tag{37}$$

which completes the proof of the theorem. ◀

4 Concluding remarks

1. The identity of Proposition 8 in the case of the uniform measure σ_n (which was obtained in [20]) is simpler. Let $\xi_1(t), \ldots, \xi_n(t)$ be i.i.d. random variables distributed according to $\mu_{\beta(t)}$, where $\beta(t) = \frac{1+e^{-t}}{2}$. Then, for any point $x \in \{-1,1\}^n$, the corresponding unbiased process $\{X_t(i)\}_{t\ge 0}$ with $X_0 = x$ has distribution equal to $x(i)\xi_i(t)$ at time t. Thus applying formula (24) with $\alpha_i = \frac{1}{2}$ and $\theta_i(t) = \frac{e^{-t}}{2}$, we recover the usual identity

$$\forall\ x \in \{-1,1\}^n, \qquad \mathscr{L}P_t f(x) = -\mathbb{E}\Big[\sum_{i=1}^n \frac{\xi_i(t) - e^{-t}}{e^t - e^{-t}} \cdot \partial_i f\big(x\xi(t)\big)\Big], \tag{38}$$

where $x\xi(t) = (x(1)\xi_1(t), \ldots, x(n)\xi_n(t))$, as was proven in [20].

2. The proof of Theorem 1 in fact implies a Poincaré-type inequality for *restrictions* of functions $f : \{-1,1\}^n \to X$ if $\dim(X) < n$, which in turn yields the refined distortion lower bounds. An inspection of the argument reveals that for every such f there exists a subset $\sigma \subseteq \{1,\ldots,n\}$ with $|\sigma| \le \dim(X)$, a point $w \in \{-1,1\}^{\sigma^c}$ and a bias vector $\boldsymbol{\alpha} = (\alpha_i)_{i\in\sigma} \in (0,1)^\sigma$ such that

$$\int_{\{-1,1\}^\sigma} \big\|f(x,w) - f(-x,-w)\big\|_X^p \,\mathrm{d}\mu_{\boldsymbol{\alpha}}(x) \\ \le 2^{2p-1}\big(\pi \mathsf{T}_p(X)\big)^p \sum_{i\in\sigma} \int_{\{-1,1\}^\sigma} \big\|\partial_i^{\alpha_i} f(x,w)\big\|_X^p + \big\|\partial_i^{\alpha_i} f(-x,-w)\big\|_X^p \,\mathrm{d}\mu_{\boldsymbol{\alpha}}(x). \tag{39}$$

3. Such refinements of Poincaré-type inequalities for topological reasons had not been exploited since Oleszkiewicz's original work [43]. The last decades have seen the development of many metric inequalities on graphs which yield nonembeddability results into normed spaces. We believe that investigating whether the distortion estimates which one obtains this way can be further improved assuming upper bounds for the dimension of the target space is a very worthwhile research program. As examples, we mention the nonembeddability of graphs with large girth into uniformly smooth spaces [32, 39], of ℓ_∞-grids into spaces of finite cotype [36] and of trees [29] and diamond graphs [24] into uniformly convex spaces.

4. The results of [20] in fact imply that any Lipschitz embedding of $\{-1,1\}^n$ into a normed space of Rademacher type p incurs *p-average* distortion at least a constant multiple of $\mathsf{T}_p(X)^{-1} n^{1-1/p}$. It would be interesting to understand whether the bound of Theorem 1 can be extended to average distortion embeddings beyond bi-Lipschitz ones.

5. In the full version [16] of this paper we will discuss some further applications of heat flow methods. In particular, we shall present a biased version of Pisier's inequality [48] and derive a nonlinear version of the classical isomorphic invariant of stable type [45].

References

1 Alexandr Andoni, Moses S. Charikar, Ofer Neiman, and Huy L. Nguyen. Near linear lower bound for dimension reduction in ℓ_1. In *2011 IEEE 52nd Annual Symposium on Foundations of Computer Science – FOCS 2011*, pages 315–323. IEEE Computer Soc., Los Alamitos, CA, 2011. doi:10.1109/FOCS.2011.87.

2 Alexandr Andoni, Piotr Indyk, and Ilya Razenshteyn. Approximate nearest neighbor search in high dimensions. In *Proceedings of the International Congress of Mathematicians – Rio de Janeiro 2018. Vol. IV. Invited lectures*, pages 3287–3318. World Sci. Publ., Hackensack, NJ, 2018.

3 Juan Arias-de-Reyna and Luis Rodríguez-Piazza. Finite metric spaces needing high dimension for Lipschitz embeddings in Banach spaces. *Israel J. Math.*, 79(1):103–111, 1992. doi:10.1007/BF02764804.

4 Keith Ball, Eric A. Carlen, and Elliott H. Lieb. Sharp uniform convexity and smoothness inequalities for trace norms. *Invent. Math.*, 115(3):463–482, 1994. doi:10.1007/BF01231769.

5 Imre Bárány and Victor S. Grinberg. On some combinatorial questions in finite-dimensional spaces. *Linear Algebra Appl.*, 41:1–9, 1981. doi:10.1016/0024-3795(81)90085-9.

6 Yair Bartal, Lee-Ad Gottlieb, and Ofer Neiman. On the impossibility of dimension reduction for doubling subsets of ℓ_p. In *Computational geometry (SoCG'14)*, pages 60–66. ACM, New York, 2014.

7 Yair Bartal and Manor Mendel. Dimension reduction for ultrametrics. In *Proceedings of the Fifteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 664–665. ACM, New York, 2004.

8 Florent Baudier, Krzysztof Swieçicki, and Andrew Swift. No dimension reduction for doubling subsets of ℓ_q when $q > 2$ revisited. *J. Math. Anal. Appl.*, 504(2):Paper No. 125407, 18, 2021. doi:10.1016/j.jmaa.2021.125407.

9 Jean Bourgain, Vitali Milman, and Haim Wolfson. On type of metric spaces. *Trans. Amer. Math. Soc.*, 294(1):295–317, 1986. doi:10.2307/2000132.

10 Bo Brinkman and Moses Charikar. On the impossibility of dimension reduction in l_1. *J. ACM*, 52(5):766–788, 2005. doi:10.1145/1089023.1089026.

11 Bo Brinkman, Adriana Karagiozova, and James R. Lee. Vertex cuts, random walks, and dimension reduction in series-parallel graphs. In *STOC'07 – Proceedings of the 39th Annual ACM Symposium on Theory of Computing*, pages 621–630. ACM, New York, 2007. doi:10.1145/1250790.1250882.

12 Per Enflo. On a problem of Smirnov. *Ark. Mat.*, 8:107–109, 1969. doi:10.1007/BF02589550.

13 Per Enflo. On the nonexistence of uniform homeomorphisms between L_p-spaces. *Ark. Mat.*, 8:103–105, 1969. doi:10.1007/BF02589549.

14 Per Enflo. On infinite-dimensional topological groups. In *Séminaire sur la Géométrie des Espaces de Banach (1977–1978)*, pages Exp. No. 10–11, 11. École Polytech., Palaiseau, 1978.

15 Alexandros Eskenazis. On Pisier's inequality for UMD targets. *Canad. Math. Bull.*, 64(2):282–291, 2021. doi:10.4153/S0008439520000442.

16 Alexandros Eskenazis. Some geometric applications of the discrete heat flow. Preprint available at https://arxiv.org/abs/2310.01868, 2023.

17 Anupam Gupta, Robert Krauthgamer, and James R. Lee. Bounded geometries, fractals, and low-distortion embeddings. In *44th Annual IEEE Symposium on Foundations of Computer Science, 2003. Proceedings.*, pages 534–543, 2003. doi:10.1109/SFCS.2003.1238226.

18 Tuomas Hytönen and Assaf Naor. Pisier's inequality revisited. *Studia Math.*, 215(3):221–235, 2013. doi:10.4064/sm215-3-2.

19 Piotr Indyk. Algorithmic applications of low-distortion geometric embeddings. In *42nd IEEE Symposium on Foundations of Computer Science (Las Vegas, NV, 2001)*, pages 10–33. IEEE Computer Soc., Los Alamitos, CA, 2001.

20 Paata Ivanisvili, Ramon van Handel, and Alexander Volberg. Rademacher type and Enflo type coincide. *Ann. of Math. (2)*, 192(2):665–678, 2020. doi:10.4007/annals.2020.192.2.8.

21 Robert C. James. Nonreflexive spaces of type 2. *Israel J. Math.*, 30(1-2):1–13, 1978. doi:10.1007/BF02760825.

22 William B. Johnson and Joram Lindenstrauss. Extensions of Lipschitz mappings into a Hilbert space. In *Conference in modern analysis and probability (New Haven, Conn., 1982)*, volume 26 of *Contemp. Math.*, pages 189–206. Amer. Math. Soc., Providence, RI, 1984. doi:10.1090/conm/026/737400.

23 William B. Johnson, Joram Lindenstrauss, and Gideon Schechtman. On Lipschitz embedding of finite metric spaces in low-dimensional normed spaces. In *Geometrical aspects of functional analysis (1985/86)*, volume 1267 of *Lecture Notes in Math.*, pages 177–184. Springer, Berlin, 1987. doi:10.1007/BFb0078145.

24 William B. Johnson and Gideon Schechtman. Diamond graphs and super-reflexivity. *J. Topol. Anal.*, 1(2):177–189, 2009. doi:10.1142/S1793525309000114.

25 Vincent Lafforgue and Assaf Naor. A doubling subset of L_p for $p > 2$ that is inherently infinite dimensional. *Geom. Dedicata*, 172:387–398, 2014. doi:10.1007/s10711-013-9924-4.

26 Michel Ledoux and Michel Talagrand. *Probability in Banach spaces*, volume 23 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991. Isoperimetry and processes. doi:10.1007/978-3-642-20212-4.

27 James R. Lee, Manor Mendel, and Assaf Naor. Metric structures in L_1: dimension, snowflakes, and average distortion. *European J. Combin.*, 26(8):1180–1190, 2005. doi:10.1016/j.ejc.2004.07.002.

28 James R. Lee and Assaf Naor. Embedding the diamond graph in L_p and dimension reduction in L_1. *Geom. Funct. Anal.*, 14(4):745–747, 2004. doi:10.1007/s00039-004-0473-8.

29 James R. Lee, Assaf Naor, and Yuval Peres. Trees and Markov convexity. In *Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1028–1037. ACM, New York, 2006. doi:10.1145/1109557.1109671.

30 David A. Levin and Yuval Peres. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, second edition, 2017. With contributions by Elizabeth L. Wilmer, With a chapter on "Coupling from the past" by James G. Propp and David B. Wilson. doi:10.1090/mbk/107.

31 Nathan Linial. Finite metric-spaces – Combinatorics, geometry and algorithms. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pages 573–586. Higher Ed. Press, Beijing, 2002.

32 Nathan Linial, Avner Magen, and Assaf Naor. Girth and Euclidean distortion. *Geom. Funct. Anal.*, 12(2):380–394, 2002. doi:10.1007/s00039-002-8251-y.

33 Jiří Matoušek. Note on bi-Lipschitz embeddings into normed spaces. *Comment. Math. Univ. Carolin.*, 33(1):51–55, 1992.

34 Jiří Matoušek. On the distortion required for embedding finite metric spaces into normed spaces. *Israel J. Math.*, 93:333–344, 1996. doi:10.1007/BF02761110.

35 Jiří Matoušek. *Using the Borsuk-Ulam theorem*. Universitext. Springer-Verlag, Berlin, 2003. Lectures on topological methods in combinatorics and geometry, Written in cooperation with Anders Björner and Günter M. Ziegler.

36 Manor Mendel and Assaf Naor. Metric cotype. *Ann. of Math. (2)*, 168(1):247–298, 2008. doi:10.4007/annals.2008.168.247.

37 Assaf Naor. A spectral gap precludes low-dimensional embeddings. In *33rd International Symposium on Computational Geometry*, volume 77 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 50, 16. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2017.

38 Assaf Naor. Metric dimension reduction: a snapshot of the Ribe program. In *Proceedings of the International Congress of Mathematicians – Rio de Janeiro 2018. Vol. I. Plenary lectures*, pages 759–837. World Sci. Publ., Hackensack, NJ, 2018.

39 Assaf Naor, Yuval Peres, Oded Schramm, and Scott Sheffield. Markov chains in smooth Banach spaces and Gromov-hyperbolic metric spaces. *Duke Math. J.*, 134(1):165–197, 2006. doi:10.1215/S0012-7094-06-13415-4.

40 Assaf Naor, Gilles Pisier, and Gideon Schechtman. Impossibility of dimension reduction in the nuclear norm [extended abstract]. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1345–1352. SIAM, Philadelphia, PA, 2018. doi:10.1137/1.9781611975031.88.

41 Assaf Naor and Gideon Schechtman. Remarks on non linear type and Pisier's inequality. *J. Reine Angew. Math.*, 552:213–236, 2002. doi:10.1515/crll.2002.092.

42 Assaf Naor and Robert Young. Foliated corona decompositions. *Acta Math.*, 229(1):55–200, 2022. doi:10.4310/acta.2022.v229.n1.a2.

43 Krzysztof Oleszkiewicz. On a discrete version of the antipodal theorem. *Fund. Math.*, 151(2):189–194, 1996.

44 Gilles Pisier. Sur les espaces de Banach qui ne contiennent pas uniformément de l_n^1. *C. R. Acad. Sci. Paris Sér. A-B*, 277:A991–A994, 1973.

45 Gilles Pisier. "Type" des espaces normés. In *Séminaire Maurey-Schwartz 1973–1974: Espaces L^p, applications radonifiantes et géométrie des espaces de Banach, Exp. No. 3,*, pages 12 pp. (errata, p. E.1). „ 1974.

46 Gilles Pisier. Martingales with values in uniformly convex spaces. *Israel J. Math.*, 20(3-4):326–350, 1975. doi:10.1007/BF02760337.

47 Gilles Pisier. Un exemple concernant la super-réflexivité. In *Séminaire Maurey-Schwartz 1974–1975: Espaces L^p applications radonifiantes et géométrie des espaces de Banach, Annexe No. 2,*, page 12. „ 1975.

48 Gilles Pisier. Probabilistic methods in the geometry of Banach spaces. In *Probability and analysis (Varenna, 1985)*, volume 1206 of *Lecture Notes in Math.*, pages 167–241. Springer, Berlin, 1986. doi:10.1007/BFb0076302.

49 Gilles Pisier and Quanhua Xu. Random series in the real interpolation spaces between the spaces v_p. In *Geometrical aspects of functional analysis (1985/86)*, volume 1267 of *Lecture Notes in Math.*, pages 185–209. Springer, Berlin, 1987. doi:10.1007/BFb0078146.

50 Santosh S. Vempala. *The random projection method*, volume 65 of *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*. American Mathematical Society, Providence, RI, 2004. With a foreword by Christos H. Papadimitriou. doi:10.1090/dimacs/065.