A Structure Theorem for Pseudo-Segments and Its Applications

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Abstract

We prove a far-reaching strengthening of Szemerédi's regularity lemma for intersection graphs of pseudo-segments. It shows that the vertex set of such graphs can be partitioned into a bounded number of parts of roughly the same size such that almost all of the bipartite graphs between pairs of parts are complete or empty. We use this to get an improved bound on disjoint edges in simple topological graphs, showing that every n-vertex simple topological graph with no k pairwise disjoint edges has at most $n(\log n)^{O(\log k)}$ edges.

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1 Introduction

Given a set of curves $\mathcal{C}$ in the plane, we say that $\mathcal{C}$ is a collection of *pseudo-segments* if any two members in $\mathcal{C}$ have at most one point in common, and no three members in $\mathcal{C}$ have a point in common. The intersection graph of a collection $\mathcal{C}$ of sets has vertex set $\mathcal{C}$ and two sets in $\mathcal{C}$ are adjacent if and only if they a have nonempty intersection.

A partition of a set is an *equipartition* if each pair of parts in the partition differ in size by at most one. Szemerédi's celebrated regularity lemma roughly says that the vertex set of any graph has an equipartition such that the bipartite graph between almost all pairs of parts is random-like. Our main result is a strengthening of Szemerédi's regularity lemma for intersection graphs of pseudo-segments. It replaces the condition that the bipartite graphs between almost all pairs of parts is random-like to being complete or empty.

▶ **Theorem 1.** *For each $\varepsilon > 0$ there is $K = K(\varepsilon)$ such that for every finite collection $\mathcal{C}$ of pseudo-segments in the plane, there is an equipartition of $\mathcal{C}$ into K parts $\mathcal{C}_1, \ldots, \mathcal{C}_K$ such that for all but at most εK^2 pairs $\mathcal{C}_i$, $\mathcal{C}_j$ of parts, either every curve in $\mathcal{C}_i$ crosses every curve in $\mathcal{C}_j$, or every curve in $\mathcal{C}_i$ is disjoint from every curve in $\mathcal{C}_j$.*

Pach and Solymosi [18] proved the special case of Theorem 1 where $\mathcal{C}$ is a collection of segments in the plane, and this result was later extended to semi-algebraic graphs [2] and hypergraphs [6] of bounded description complexity. However, the techniques used to prove these results heavily rely on the algebraic structure. In fact, while it follows from the Milnor-Thom theorem that there are only $2^{O(n \log n)}$ graphs on n vertices which are semialgebraic of bounded description complexity (see [19, 2, 21]) there are many more (namely $2^{\Omega(n^{4/3})}$) graphs on n vertices which are intersection graphs of pseudo-segments [7].

Next, we discuss an application of Theorem 1 in graph drawing.

Disjoint edges in simple topological graphs. A *topological graph* is a graph drawn in the plane such that its vertices are represented by points and its edges are represented by nonself-intersecting arcs connecting the corresponding points. The edges are allowed to intersect, but they may not intersect vertices apart from their endpoints. Furthermore, no two edges are tangent, i.e., if two edges share an interior point, then they must properly cross at that point in common. A topological graph is *simple* if every pair of its edges intersect at most once. Two edges of a topological graph *cross* if their interiors share a point, and are *disjoint* if they neither share a common vertex nor cross.

Determining the maximum number of edges in a simple topological graph with no k pairwise disjoint edges seems to be a difficult task. When $k = 2$, a linear upper bound is known [17, 3, 11, 12]. When $k \geq 3$, Pach and Tóth [20] showed that every n-vertex simple topological graph with no k pairwise disjoint edges has $O(n \log^{4k-8} n)$ edges. They conjectured that for every fixed k, the number of edges in such graphs is at most $O_k(n)$. Our next result substantially improves the upper bound for large k.

▶ **Theorem 2.** *If $G = (V, E)$ is an n-vertex simple topological graph with no k pairwise disjoint edges, then $|E(G)| \leq n(\log n)^{O(\log k)}$.*

The proof of Theorem 2 follows the arguments in [20, 22], and is by double induction on n and k. We consider the cases when there are many or few disjoint pairs of edges in G. In the former case, it was used in [20] that there is an edge which is disjoint from many other edges (so, among these edges, no $k-1$ are pairwise disjoint), and the argument was completed by induction on k. Instead, we can apply a variant of Theorem 1 to get two large subsets of edges that are disjoint from each other (so, at least one of these subsets has no $k/2$ pairwise disjoint edges), and again use induction on k. In the second case, where there are few disjoint pairs of edges in G, we apply a bisection width result due to Pach and Tóth [20] and induction on n. See [8] for more details. In [10], Fox and Sudakov showed that every dense n-vertex simple topological graph contains $\Omega(\log^{1+\delta} n)$ pairwise disjoint edges, where $\delta \approx 1/40$. As an immediate corollary to Theorem 2, we improve this bound to nearly polynomial under a much weaker assumption.

▶ **Corollary 3.** *Let $\varepsilon > 0$, and let $G = (V, E)$ be an n-vertex simple topological graph with at least $2n^{1+\varepsilon}$ edges. Then G has $n^{\Omega(\varepsilon / \log \log n)}$ pairwise disjoint edges.*

For complete n-vertex simple topological graphs, Aichholzer et al. [1] showed that one can always find $\Omega(n^{1/2})$ pairwise disjoint edges.

The proofs of the above theorems heavily rely on the following bipartite Ramsey-type result for intersection graphs of pseudo-segments. As shown in [8], the main result in this paper, Theorem 1, is equivalent to the following.

▶ **Theorem 4.** *Let $\mathcal{R}$ be a set of n red curves, and $\mathcal{B}$ be a set of n blue curves in the plane such that $\mathcal{R} \cup \mathcal{B}$ is a collection of pseudo-segments. Then there are subsets $\mathcal{R}' \subset \mathcal{R}$ and $\mathcal{B}' \subset \mathcal{B}$, where $|\mathcal{R}'|, |\mathcal{B}'| = \Omega(n)$, such that either every curve in $\mathcal{R}'$ crosses every curves in $\mathcal{B}'$, or every curve in $\mathcal{R}'$ is disjoint from every curve in $\mathcal{B}'$.*

The rest of this paper is devoted to proving Theorem 4. In the next section, we recall that any finite collection of pseudo-segments in the plane contains a linear-sized subset with the property that only a small fraction of pairs in the subset are crossing, or nearly all of them cross. In Section 3, we prove Theorem 4 in the special case where one of the families is double grounded. Building on these results, in Section 4, we establish our bipartite Ramsey-type theorem (Theorem 4) for any two families of pseudo-segments with the property that for each family, only a small fraction of pairs are crossing, or nearly all of them cross. Finally, in Section 5, we prove Theorem 4 in its full generality.

2 Tools

We say that a graph G is *ε-homogeneous* if the edge density in G is less than ε or greater than $1 - \varepsilon$. For the proof of Theorem 4, we need the following result from [5].

▶ **Theorem 5** ([5]). *There is an constant $c' > 0$ such that the following holds. Let $\mathcal{C}$ be a collection of n pseudo-segments in the plane with at least εn^2 crossing pairs. Then there are subsets $\mathcal{C}_1, \mathcal{C}_2 \subset \mathcal{C}$, each of size $c'\varepsilon n$, such that every curve in $\mathcal{C}_1$ crosses every curve in $\mathcal{C}_2$.*

Given a collection $\mathcal{C}$ of curves in the plane, let $G(\mathcal{C})$ denote the intersection graph of $\mathcal{C}$. In [9], Fox, Pach, and Tóth showed that pseudo-segments has the strong Erdős-Hajnal property, which implies the following.

▶ **Corollary 6** ([9]). *The family of intersection graphs of pseudo-segments has the polynomial Rödl property. That is, there is an absolute constant $c_1 > 0$ such that the following holds. Let $\varepsilon > 0$ and $\mathcal{C}$ be a collection of n pseudo-segments in the plane. Then there is a subset $\mathcal{C}' \subset \mathcal{C}$ of size $\varepsilon^{c_1} n$ whose intersection graph $G(\mathcal{C}')$ is ε-homogeneous.*

We will frequently use the following simple lemma in this paper. See [8] for the proof.

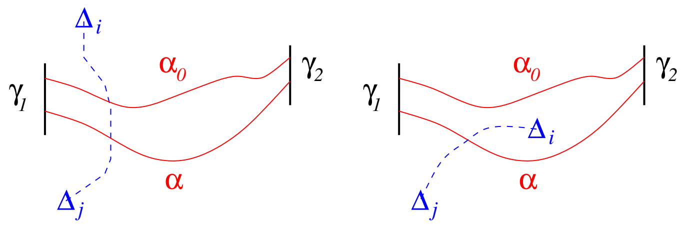
▶ **Lemma 7.** *Let $G = (V, E)$ be a graph on n vertices. If the edge density of G is at most ε, then any induced subgraph on δn vertices has edge density at most $2\varepsilon/\delta^2$. Likewise, if the edge density of G is at least $1 - \varepsilon$, then any induced subgraph on δn vertices has edge density at least $1 - 2\varepsilon/\delta^2$.*

3 Proof of Theorem 4 – for double grounded red curves

Given a collection of curves $\mathcal{C}$ in the plane, we say that $\mathcal{C}$ is *double grounded* if there are two distinct curves γ_1 and γ_2 such that for each curve $\alpha \in \mathcal{C}$, α has one endpoint on γ_1 and the other on γ_2, and the interior of α is disjoint from γ_1 and γ_2. Throughout this paper, for simplicity, we will always assume that both endpoints of each of our curves have distinct x-coordinates. We refer to the endpoint of a curve with the smaller (larger) x-coordinate as its *left (right) endpoint*. The aim of this section is to prove Theorem 4 in the special case where one of the color classes (the red one, say) consists of double grounded curves.

A curve in the plane is called *x-monotone* if every vertical line intersects it in at most one point. We start by considering double grounded x-monotone curves, and at the end of this section, we will remove the x-monotone condition. We will need the following result, known as the cutting-lemma for x-monotone curves. See, for example, Proposition 2.11 in [15].

▶ **Lemma 8** (The Cutting Lemma). *Let $\mathcal{C}$ be a collection of n double grounded x-monotone curves, whose grounds are disjoint vertical segments γ_1 and γ_2, and let $r > 1$ be a parameter. Then $\mathbb{R}^2 \setminus (\gamma_1 \cup \gamma_2)$ can be subdivided into t connected regions $\Delta_1, \ldots, \Delta_t$, such that the interior of each Δ_i is intersected by at most n/r curves from $\mathcal{C}$, and we have $t = O(r^2)$.*



(a) Case 1.a. **(b)** Case 1.b.

Figure 1 Case 1, α_0 and α are disjoint.

Throughout the paper, we will implicitly use the Jordan curve theorem.

▶ **Lemma 9.** *Let $\mathcal{R}$ be a set of n red double grounded x-monotone curves, whose grounds are disjoint vertical segments γ_1 and γ_2. Let $\mathcal{B}$ be a set of n blue curves (not necessarily x-monotone) such that every blue curve in $\mathcal{B}$ is disjoint from grounds γ_1 and γ_2, and suppose that $\mathcal{R} \cup \mathcal{B}$ is a collection of pseudo-segments. Then there are subsets $\mathcal{R}' \subset \mathcal{R}$ and $\mathcal{B}' \subset \mathcal{B}$ such that $|\mathcal{R}'|, |\mathcal{B}'| = \Omega(n)$, and either every curve in $\mathcal{R}'$ crosses every curve in $\mathcal{B}'$, or every curve in $\mathcal{R}'$ is disjoint from every curve in $\mathcal{B}'$.*

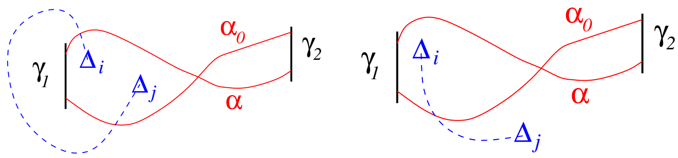
Proof. Let P be the set of left-endpoints of the curves in $\mathcal{B}$. We apply Lemma 8 to $\mathcal{R}$ with parameter $r = 4$ to obtain a subdivision $\mathbb{R}^2 \setminus (\gamma_1 \cup \gamma_2) = \Delta_1 \cup \cdots \cup \Delta_t$, such that for each Δ_i, the interior of Δ_i intersects at most $n/4$ members in $\mathcal{R}$, and $t \leq c_0 4^2$ where c_0 is an absolute constant from Lemma 8. By the pigeonhole principle, there is a region Δ_i such that Δ_i contains at least $n/c_0 4^2$ points from P. Let $\mathcal{B}_0 \subset \mathcal{B}$ be the set of blue curves whose left endpoints are in Δ_i. Hence $|\mathcal{B}_0| = \Omega(n)$.

Let Q be the right endpoints of the curves in $\mathcal{B}_0$. Using the same subdivision described above, there is a region Δ_j such that Δ_j contains at least $|Q|/(c_0 4^2) \geq n/(c_0 4^2)^2$ points from Q. Let $\mathcal{B}_1 \subset \mathcal{B}_0$ be the set of blue curves with their left endpoint in Δ_i and right endpoint in Δ_j. Let $\mathcal{R}_1 \subset \mathcal{R}$ consists of all red curves that do not intersect the interior of Δ_i and Δ_j. Lemma 8 implies that $|\mathcal{R}_1| \geq n - \frac{2n}{4} = \frac{n}{2}$, and $|\mathcal{B}_1| = \Omega(n)$. Recall that each blue curve in $\mathcal{B}_1$ does not intersect the grounds γ_1 nor γ_2. Fix an arbitrary curve $\alpha_0 \in \mathcal{R}_1$. The proof now falls into the following cases.

Case 1. Suppose at least $|\mathcal{R}_1|/2$ curves in $\mathcal{R}_1$ are disjoint from α_0. Let $\mathcal{R}_2 \subset \mathcal{R}_1$ be the set of red curves disjoint from α_0. For each $\alpha \in \mathcal{R}_2$, $\mathbb{R}^2 \setminus (\gamma_1 \cup \gamma_2 \cup \alpha_0 \cup \alpha)$, consists of two connected components, one bounded and the other unbounded.

Case 1.a. Suppose for at least $|\mathcal{R}_2|/2$ red curves $\alpha \in \mathcal{R}_2$, both Δ_i and Δ_j lie in the same connected component of $\mathbb{R}^2 \setminus (\gamma_1 \cup \gamma_2 \cup \alpha_0 \cup \alpha)$. See Figure 1a. Let $\mathcal{R}_3 \subset \mathcal{R}_2$ be the collection of such red curves. Then for each $\alpha \in \mathcal{R}_3$, each blue curve $\beta \in \mathcal{B}_1$ crosses α if and only if β crosses α_0. Hence, there is a subset $\mathcal{B}_2 \subset \mathcal{B}_1$ of size at least $\Omega(n)$, such that either every blue curve in $\mathcal{B}_2$ crosses every red curve in $\mathcal{R}_3$, or every blue curve in $\mathcal{B}_2$ is disjoint from every red curve in $\mathcal{R}_3$. Moreover, $|\mathcal{R}_3| = \Omega(n)$ and we are done.

Case 1.b. Suppose for at least $|\mathcal{R}_2|/2$ red curves $\alpha \in \mathcal{R}_2$, regions Δ_i and Δ_j lie in different connected component of $\mathbb{R}^2 \setminus (\gamma_1 \cup \gamma_2 \cup \alpha_0 \cup \alpha)$. See Figure 1b. Similar to above, let $\mathcal{R}_3 \subset \mathcal{R}_2$ be the collection of such red curves. By the pseudo-segment condition, for each $\alpha \in \mathcal{R}_3$, each blue curve $\beta \in \mathcal{B}_1$ crosses α if and only if β is disjoint from α_0. Hence, there is a subset $\mathcal{B}_2 \subset \mathcal{B}_1$ of size $\Omega_r(n)$, such that either every blue curve in $\mathcal{B}_2$ crosses every red curve in $\mathcal{R}_3$, or every blue curve in $\mathcal{B}_2$ is disjoint from every red curve in $\mathcal{R}_3$. Moreover, $|\mathcal{R}_3| = \Omega(n)$ and we are done.



(a) Case 2.a. **(b)** Case 2.c.

Figure 2 Case 2, α_0 and α cross.

Case 2. Suppose at least $|\mathcal{R}_1|/2$ curves in $\mathcal{R}_1$ cross α_0. Let $\mathcal{R}_2 \subset \mathcal{R}_1$ be the set of red curves that crosses α_0. For each $\alpha \in \mathcal{R}_2 \setminus \{\alpha_0\}$, $\mathbb{R}^2 \setminus (\gamma_1 \cup \gamma_2 \cup \alpha_0 \cup \alpha)$ consists of three connected components, two of which are bounded and the other unbounded.

Case 2.a. Suppose for at least $|\mathcal{R}_2|/3$ red curves $\alpha \in \mathcal{R}_2$, Both Δ_i and Δ_j lie in the same connected component of $\mathbb{R}^2 \setminus (\gamma_1 \cup \gamma_2 \cup \alpha_0 \cup \alpha)$. See Figure 2a. Let $\mathcal{R}_3 \subset \mathcal{R}_2$ be the collection of such red curves. By the pseudo-segment condition, for each $\alpha \in \mathcal{R}_3$, each blue curve $\beta \in \mathcal{B}_1$ crosses α if and only if β crosses α_0. Hence, there is a subset $\mathcal{B}_2 \subset \mathcal{B}_1$ of size at least $\Omega(n)$, such that either every blue curve in $\mathcal{B}_2$ crosses every red curve in $\mathcal{R}_3$, or every blue curve in $\mathcal{B}_2$ is disjoint from every red curve in $\mathcal{R}_3$. Moreover, $|\mathcal{R}_3| = \Omega(n)$.

Case 2.b. Suppose for at least $|\mathcal{R}_2|/3$ red curves $\alpha \in \mathcal{R}_2$, regions Δ_i and Δ_j lie in different bounded connected components of $\mathbb{R}^2 \setminus (\gamma_1 \cup \gamma_2 \cup \alpha_0 \cup \alpha)$. Let $\mathcal{R}_3 \subset \mathcal{R}_2$ be the collection of such red curves. Then for each $\alpha \in \mathcal{R}_3$, every blue curve $\beta \in \mathcal{B}_1$ crosses α. Since $|\mathcal{R}_3| = \Omega(n)$, we have $|\mathcal{B}_1| = \Omega(n)$.

Case 2.c. Suppose for at least $|\mathcal{R}_2|/3$ red curves $\alpha \in \mathcal{R}_2$, regions Δ_i and Δ_j lie in different connected components of $\mathbb{R}^2 \setminus (\gamma_1 \cup \gamma_2 \cup \alpha_0 \cup \alpha)$, one of which is bounded and the other unbounded. See Figure 2b. Let $\mathcal{R}_3 \subset \mathcal{R}_2$ be the collection of such red curves. By the pseudo-segment condition, for each $\alpha \in \mathcal{R}_3$, each blue curve $\beta \in \mathcal{B}_1$ crosses α if and only if β is disjoint from α_0. Hence, there is a subset $\mathcal{B}_2 \subset \mathcal{B}_1$ of size $\Omega(n)$, such that either every blue curve in $\mathcal{B}_2$ crosses every red curve in $\mathcal{R}_3$, or every blue curve in $\mathcal{B}_2$ is disjoint from every red curve in $\mathcal{R}_3$. Moreover, $|\mathcal{R}_3| = \Omega(n)$, and we are done. ◀

Recall that a *pseudoline* is an unbounded arc in $\mathbb{R}^2$, whose complement is disconnected. An *arrangement of pseudolines* is a set of pseudolines such that every pair meets exactly once, and no three members have a point in common. A classic result of Goodman [13] states that every arrangement of pseudolines is isomorphic to an arrangement of *wiring diagram* (bi-infinite x-monotone curves). Moreover, Goodman and Pollack showed the following.

▶ **Theorem 10** ([14]). *Every arrangement of pseudolines can be continuously deformed (through isomorphic arrangements) to a wiring diagram.*

We also need the following simple lemma.

▶ **Lemma 11.** *Given a finite linearly ordered set whose elements are colored red or blue, we can select half of the red elements and half of the blue elements such that all of the selected elements of one color come before all of the selected elements of the other color.*

We are now ready to establish the main result of this section.

▶ **Theorem 12.** *Let $\mathcal{R}$ be a set of n red double grounded curves with grounds γ_1 and γ_2, where γ_1 and γ_2 cross each other. Let $\mathcal{B}$ be a set of n blue curves such that $\mathcal{R} \cup \mathcal{B} \cup \{\gamma_1, \gamma_2\}$ is a collection of pseudo-segments. Then there are subsets $\mathcal{R}' \subset \mathcal{R}$ and $\mathcal{B}' \subset \mathcal{B}$ such that $|\mathcal{R}'|, |\mathcal{B}'| = \Omega(n)$, and either every curve in $\mathcal{R}'$ crosses every curve in $\mathcal{B}'$, or every curve in $\mathcal{R}'$ is disjoint from every curve in $\mathcal{B}'$.*

Proof. By passing to linear-sized subsets of $\mathcal{R}$ and $\mathcal{B}$ and subcurves of γ_1 and γ_2, we will reduce the problem to the setting of Lemma 9. Let us assume that γ_1 and γ_2 cross at point p. Hence, $(\gamma_1 \setminus \gamma_2) \cup (\gamma_2 \setminus \gamma_1)$ consists of four connected components. By the pigeonhole principle, there is a subset $\mathcal{R}_1 \subset \mathcal{R}$ of size $n/4$ such that every curve in $\mathcal{R}_1$ has an endpoint on one of the connected components of $\gamma_1 \setminus \gamma_2$, and all of the other endpoints lie on one of the connected components of $\gamma_2 \setminus \gamma_1$. Let $\gamma_i' \subset \gamma_i$, for $i = 1, 2$, be these connected components so that they have a common endpoint at p and their interiors are disjoint.

For each $\alpha \in \mathcal{R}_1$, the sequence of curves $(\gamma_1', \gamma_2', \alpha)$ appear either in *clockwise* or *counter-clockwise* order along the unique simple closed curve that lies in $\gamma_1' \cup \gamma_2' \cup \alpha$. Without loss of generality, we can assume that there is a subset $\mathcal{R}_2 \subset \mathcal{R}_1$, where $|\mathcal{R}_2| = \Omega(n)$, such that for every curve $\alpha \in \mathcal{R}_2$, the sequence $(\gamma_1', \gamma_2', \alpha)$ appears in clockwise order, since a symmetric argument would follow otherwise.

We define the *orientation* of each curve $\alpha \in \mathcal{R}_2$ as the sequence of turns, either *left-left*, *left-right*, *right-left*, or *right-right*, made by starting at p and moving along γ_1' in the arrangement $\gamma_1' \cup \gamma_2' \cup \alpha$, until we return back to p. More precisely, starting at p we move along γ_1' until we reach the endpoint of α. We then turn either *left* or *right* to move along α towards γ_2'. Once we've reached γ_2', we either turn *left* or *right* in order to move along γ_2' and reach p again. By the pigeonhole principle, there is a subset $\mathcal{R}_3 \subset \mathcal{R}_2$ of size at least $\Omega(n)$ such that all curves in $\mathcal{R}_3$ have the same orientation. Without loss of generality, we can assume that the orientation is *left-left*, since a symmetric argument would follow otherwise.

Starting at p and moving along γ_1' towards its other endpoint, let us consider the sequence of curves from $\mathcal{R}_3 \cup \mathcal{B}$ intersecting γ_1'. Then, by Lemma 11, there are subsets $\mathcal{R}_4 \subset \mathcal{R}_3$ and $\mathcal{B}_1 \subset \mathcal{B}$, where $|\mathcal{R}_4| \geq |\mathcal{R}_3|/2$ and $|\mathcal{B}_1| \geq |\mathcal{B}|/2$, such that either all of the curves in $\mathcal{R}_4$ appear before all of the curves in $\mathcal{B}_1$ that intersect γ_1' in this sequence, or all of the curves in $\mathcal{R}_4$ appear after all of the curves in $\mathcal{B}_1$ in this sequence. Note that $\mathcal{B}_1$ consists of the blue curves in $\mathcal{B}$ that are disjoint to γ_1' and at least half of the curves in $\mathcal{B}$ that intersect γ_1' found by the application of Lemma 11. Hence, there is a subcurve $\gamma_1'' \subset \gamma_1'$ such that γ_1'' is one of the grounds for $\mathcal{R}_4$, and is disjoint from every curve in $\mathcal{B}_1$. We apply the same argument to $\mathcal{R}_4 \cup \mathcal{B}_1$ and γ_2', and obtain subsets $\mathcal{R}_5 \subset \mathcal{R}_4$, $\mathcal{B}_2 \subset \mathcal{B}_1$, and a subcurve $\gamma_2'' \subset \gamma_2'$, such that $|\mathcal{R}_5|, |\mathcal{B}_2| = \Omega(n)$, and $\mathcal{R}_5$ is double grounded with disjoint grounds γ_1'' and γ_2'', and every curve in $\mathcal{B}_2$ is disjoint from γ_1'' and γ_2''.

For $i \in \{1, 2\}$, let p_i be the endpoint of γ_i'' that lies closest to p along γ_i'. Starting at p_i and moving along γ_i'', let π_i be the sequence of curves in $\mathcal{R}_5$ that appear on γ_i''. Since every curve in $\mathcal{R}_5$ has the same left-left orientation, and appears clockwise order with respect to γ_1' and γ_2', two curves $\alpha, \alpha' \in \mathcal{R}_5$ cross if and only if the order in which they appear in π_1 and π_2 changes. Let γ_3'' be a curve very close to γ_2'' such that γ_3'' has the same endpoints as γ_2'', and is disjoint from all curves in $\mathcal{R}_5 \cup \mathcal{B}_2$. Hence, $\gamma_2'' \cup \gamma_3''$ makes an empty lens in the arrangement $\mathcal{R}_5 \cup \mathcal{B}_2$. We slightly extend each curve $\alpha \in \mathcal{R}_5$ through this lens to γ_3'' so that the resulting curve, α' properly crosses γ_2'' and has its new endpoint on γ_3''. Moreover, the extension will be made in such a way that the sequence π_3 of curves in $\mathcal{R}_5$ appearing along γ_3'' starting from p_2 will appear in the opposite order of π_1. Let $\mathcal{R}_5' = \{\alpha' : \alpha \in \mathcal{R}_5\}$. Thus, every pair of curves in $\mathcal{R}_5'$ will cross exactly once.

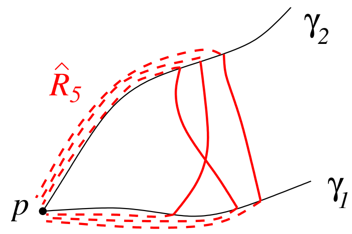


Figure 3 The resulting extension $\hat{\mathcal{R}}_5$.

For each curve $\alpha' \in \mathcal{R}'_5$, we further extend α' by moving both endpoints towards p along γ_1 and γ_2, so that we do not create any additional crossings within $\mathcal{R}'_5$. Let $\hat{\alpha}$ be the resulting extension, where both endpoints of $\hat{\alpha}$ lie arbitrarily close to p. Set $\hat{\mathcal{R}}_5 = \{\hat{\alpha} : \alpha' \in \mathcal{R}'_5\}$. See Figure 3. Furthermore, we can assume that p lies in the unbounded face of the arrangement $\hat{\mathcal{R}}_5$, since otherwise we could project the arrangement $\hat{\mathcal{R}}_5$ onto a sphere, and then project it back to the plane so that p lies in the unbounded face, without creating or removing any crossing. Therefore, $\hat{\mathcal{R}}_5$ can be extended to a family of pseudolines. By Theorem 10, we can apply a continuous deformation of the plane so that $\hat{\mathcal{R}}_5$ becomes a collection of unbounded x-monotone curves. Hence, after the deformation, the original set $\mathcal{R}_5$ becomes a collection of double grounded x-monotone curves, with grounds γ''_1, γ''_2, such that every curve in $\mathcal{B}_2$ is disjoint from the grounds γ''_1 and γ''_2, the crossing pattern in the arrangement $\mathcal{R}_5 \cup \mathcal{B}_2$ is the same as before. Moreover, γ''_1 and γ''_2 will be disjoint vertical segments. We apply Lemma 9 to $\mathcal{R}_5$ and $\mathcal{B}_2$ and obtain subsets $\mathcal{R}_6 \subset \mathcal{R}_5$ and $\mathcal{B}_3 \subset \mathcal{B}_2$, each of size $\Omega(n)$, such that either every curve in $\mathcal{R}_6$ crosses every curve in $\mathcal{B}_3$, or every curve in $\mathcal{R}_6$ is disjoint from every curve in $\mathcal{B}_3$. This completes the proof. ◀

By combining Theorem 12 with a variant of Szemerédi's regularity lemma due to Komlós [16], we have the following (see [8] for more details).

▶ **Theorem 13.** *There is a constant $c' > 0$ such that the following holds. Let $\mathcal{R}$ be a collection of n red double grounded curves with grounds γ_1 and γ_2, such that γ_1 and γ_2 cross. Let $\mathcal{B}$ be a collection of n blue curves such that $\mathcal{R} \cup \mathcal{B} \cup \{\gamma_1, \gamma_2\}$ is a collection of pseudo-segments. If there are at least εn^2 crossing pairs in $\mathcal{R} \times \mathcal{B}$, then there are subsets $\mathcal{R}' \subset \mathcal{R}, \mathcal{B}' \subset \mathcal{B}$, where $|\mathcal{R}'|, |\mathcal{B}'| \geq \varepsilon^{c'} n$, such that every curve in $\mathcal{R}'$ crosses every curve in $\mathcal{B}'$.*

An analogous theorem holds in the case there are at least εn^2 disjoint pairs.

4 Proof of Theorem 4 – for ε-homogeneous families

The aim of this section is to prove Theorem 4, the main result of this paper, in the special case where the edge density of the intersection graph of the red curves is nearly 0 or nearly 1, and the same is true for the intersection graph of the blue curves. This will easily imply Theorem 4 in its full generality, as shown in the next section.

4.1 Low versus low density

By Corollary 6, we can reduce to the case that the intersection graphs $G(\mathcal{R})$ and $G(\mathcal{B})$ are both ε-homogeneous, where $\varepsilon > 0$ is a small absolute constant. Below, we first consider the cases when both $G(\mathcal{R}), G(\mathcal{B})$ have edge density less than ε.

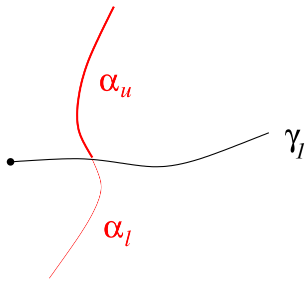


Figure 4 Partitioning of the red curve $\alpha = \alpha_u \cup \alpha_\ell$.

▶ **Theorem 14.** *There is an absolute constant $\varepsilon_1 > 0$ such that the following holds. Let $\mathcal{R}$ be a set of n red curves and $\mathcal{B}$ be a set of n blue curves in the plane such that $\mathcal{R} \cup \mathcal{B}$ is a collection of pseudo-segments. If the edge densities of the intersection graphs $G(\mathcal{R})$ and $G(\mathcal{B})$ are both less than ε_1, then there are subsets $\mathcal{R}' \subset \mathcal{R}$ and $\mathcal{B}' \subset \mathcal{B}$, each of size $\Omega(n)$, such that every red curve in $\mathcal{R}'$ crosses every blue curve in $\mathcal{B}'$, or every red curve in $\mathcal{R}'$ is disjoint from every blue curve in $\mathcal{B}'$.*

The proof of Theorem 14 is a simple application of a separator theorem from [4] (see [8]).

4.2 High versus low edge density

In this subsection, we consider the case when the intersection graph $G(\mathcal{R})$ has edge density at least $1-\varepsilon$, and $G(\mathcal{B})$ has edge density less than ε. Since the edge density in the intersection graph $G(\mathcal{R})$ is at least $1-\varepsilon$, we can further reduce to the case when there is a red curve γ_1 that crosses every member in $\mathcal{R}$ exactly once.

▶ **Lemma 15.** *For each integer $t \geq 1$, there is a constant $\varepsilon'_t > 0$ such that the following holds. Let $\mathcal{R}$ be a set of n red curves in the plane, all crossed by a curve γ_1 exactly once, and $\mathcal{B}$ be a set of n blue curves in the plane such that $\mathcal{R} \cup \mathcal{B} \cup \{\gamma_1\}$ is a collection of pseudo-segments. Suppose that the intersection graph $G(\mathcal{B})$ has edge density less than ε'_t, and $G(\mathcal{R})$ has edge density at least $1-\varepsilon'_t$. Then there are subsets $\hat{\mathcal{R}} \subset \mathcal{R}$, $\hat{\mathcal{B}} \subset \mathcal{B}$, each of size $\Omega_{\varepsilon'_t}(n)$, such that either every red curve in $\hat{\mathcal{R}}$ crosses every blue curve in $\hat{\mathcal{B}}$, or every red curve in $\hat{\mathcal{R}}$ is disjoint from every blue curve in $\hat{\mathcal{B}}$, or each curve $\alpha \in \hat{\mathcal{R}}$ has a partition into two connected parts $\alpha = \hat{\alpha}_u \cup \hat{\alpha}_\ell$, such that for*

$$\hat{\mathcal{U}} = \{\hat{\alpha}_u : \alpha \in \hat{\mathcal{R}}, \alpha = \hat{\alpha}_u \cup \hat{\alpha}_\ell\} \quad \text{and} \quad \hat{\mathcal{L}} = \{\hat{\alpha}_\ell : \alpha \in \hat{\mathcal{R}}, \alpha = \hat{\alpha}_u \cup \hat{\alpha}_\ell\},$$

every curve in $\hat{\mathcal{L}}$ is disjoint to every curve in $\hat{\mathcal{B}}$, and the edge density of $G(\hat{\mathcal{U}})$ is less than 2^{-t}.

Proof. Each curve $\alpha \in \mathcal{R}$ is partitioned into two connected parts by γ_1, say an upper and lower part. More precisely, we have the partition $\alpha = \alpha_u \cup \alpha_\ell$, where the parts α_u and α_ℓ are defined, as follows. We start at the left endpoint of γ_1 and move along γ_1 until we reach $\alpha \cap \gamma_1$. At this point, we turn left along α to obtain α_u and right to obtain α_ℓ. See Figure 4. Let $\mathcal{U}$ ($\mathcal{L}$) be the upper (lower) part of each curve in $\mathcal{R}$, that is,

$$\mathcal{U} = \{\alpha_u : \alpha \in \mathcal{R}, \alpha = \alpha_\ell \cup \alpha_u\} \quad \text{and} \quad \mathcal{L} = \{\alpha_\ell : \alpha \in \mathcal{R}, \alpha = \alpha_\ell \cup \alpha_u\}.$$

In what follows, for every integer $t \geq 1$, we will obtain subsets $\mathcal{R}^{(t)} \subset \mathcal{R}$, $\mathcal{B}^{(t)} \subset \mathcal{B}$, each of size $\Omega_{\varepsilon'_t}(n)$, such that either every red curve in $\mathcal{R}^{(t)}$ crosses every blue curve in $\mathcal{B}^{(t)}$, or every red curve in $\mathcal{R}^{(t)}$ is disjoint from every blue curve in $\mathcal{B}^{(t)}$, or each curve $\alpha \in \mathcal{R}^{(t)}$ has a new partition into upper and lower parts $\alpha = \alpha'_u \cup \alpha'_\ell$, such that the following holds.

1. We have $\alpha'_u \subset \alpha_u$, that is, the upper part α'_u is a subcurve of the previous upper part α_u.
2. The lower part α'_ℓ of each curve in $\mathcal{R}^{(t)}$ is disjoint from each blue curve in $\mathcal{B}^{(t)}$.
3. There is an equipartition $\mathcal{R}^{(t)} = \mathcal{R}_1^{(t)} \cup \cdots \cup \mathcal{R}_{2^t}^{(t)}$ into 2^t parts such that for $1 \leq i < j \leq 2^{t-1}$, the upper part α'_u of each curve $\alpha \in \mathcal{R}_i^{(t)}$ is disjoint from the upper part β'_u of each curve $\beta \in \mathcal{R}_j^{(t)}$.

Hence, the lemma follows from the statement above by setting $\hat{\mathcal{B}} = \mathcal{B}^{(t)}$, $\hat{\mathcal{R}} = \mathcal{R}^{(t)}$.

We proceed by induction on t. The bulk of the argument below is actually for the base case $t = 1$, since we will just repeat the entire argument for the inductive step with parameter ε'_t. Let ε'_1 be a small positive constant that will be determined later such that $\varepsilon'_1 < \varepsilon_1$, where ε_1 is from Theorem 14. Thus, $G(\mathcal{R})$ has edge density at least $1 - \varepsilon'_1$ and $G(\mathcal{B})$ has edge density less than ε'_1.

Let $\delta > 0$ also be a sufficiently small constant determined later, such that $\varepsilon'_1 < \delta < \varepsilon_1$. We apply Corollary 6 to $\mathcal{L}$ with parameter δ and obtain a subset $\mathcal{L}_1 \subset \mathcal{L}$ such that $\mathcal{L}_1$ is δ-homogeneous and $|\mathcal{L}_1| = \Omega_\delta(n)$. Let $\mathcal{R}_1 \subset \mathcal{R}$ be the red curves in $\mathcal{R}$ corresponding to the curves in $\mathcal{L}_1$, and let $\mathcal{U}_1 \subset \mathcal{U}$ be the curves in $\mathcal{U}$ that corresponds to the red curves in $\mathcal{R}_1$.

Without loss of generality, we can assume that the intersection graph $G(\mathcal{L}_1)$ has edge density less than δ. Indeed, otherwise if $G(\mathcal{L}_1)$ has edge density greater than $1 - \delta$, by the pseudo-segment condition, the intersection graph $G(\mathcal{U}_1)$ must have edge density less than δ and a symmetric argument would follow. In order to apply Theorem 14, we need two subsets of equal size. By averaging, there is a subset $\mathcal{B}' \subset \mathcal{B}$ with $|\mathcal{B}'| = |\mathcal{L}_1|$ such that the edge density of $G(\mathcal{B}')$ is at most that of $G(\mathcal{B})$. Since $G(\mathcal{L}_1)$ has edge density less than δ and $G(\mathcal{B}')$ has edge density less than ε'_1, by setting $\varepsilon'_1 < \delta < \varepsilon_1$, we can apply Theorem 14 to $\mathcal{L}_1$ and $\mathcal{B}'$ and obtain subsets $\mathcal{L}_2 \subset \mathcal{L}_1$ and $\mathcal{B}_1 \subset \mathcal{B}'$, each of size $\Omega_\delta(n)$, such that every curve in $\mathcal{L}_2$ crosses every blue curve in $\mathcal{B}_1$, or every curve in $\mathcal{L}_2$ is disjoint from every blue curve in $\mathcal{B}_1$. If we are in the former case, then we are done. Hence, we can assume that we are in the latter case. Let $\mathcal{R}_2 \subset \mathcal{R}_1$ be the red curves that corresponds to $\mathcal{L}_2$, and let $\mathcal{U}_2 \subset \mathcal{U}_1$ be the curves in $\mathcal{U}_1$ that corresponds to $\mathcal{R}_2$. We apply Corollary 6 to $\mathcal{U}_2$ with parameter δ and obtain a subset $\mathcal{U}_3 \subset \mathcal{U}_2$ such that $\mathcal{U}_3$ is δ-homogeneous and $|\mathcal{U}_3| = \Omega_\delta(n)$. Let $\mathcal{R}_3$ be the red curves in $\mathcal{R}$ corresponding to $\mathcal{U}_3$, and let $\mathcal{L}_3$ be the curves in $\mathcal{L}_2$ that corresponds to $\mathcal{R}_3$.

Suppose that the intersection graph $G(\mathcal{U}_3)$ has edge density less than δ. Since $|\mathcal{B}_1| = \delta_0 n$, where $\delta_0 = \delta_0(\delta, \varepsilon_1)$, by Lemma 7, the intersection graph $G(\mathcal{B}_1)$ has edge density at most $2\varepsilon'_1/\delta_0^2$. Thus, we set δ and ε'_1 sufficiently small so that $\delta < \varepsilon_1$ and $2\varepsilon'_1/\delta_0^2 < \varepsilon_1$. By averaging, we can find subsets of $\mathcal{U}_3$ and $\mathcal{B}_1$, each of size $\min(|\mathcal{U}_3|, |\mathcal{B}_1|)$ and with densities less than ε_1, and apply Theorem 14 to these subsets and obtain subsets $\mathcal{U}_4 \subset \mathcal{U}_3$ and $\mathcal{B}_2 \subset \mathcal{B}_1$, each of size $\Omega_\delta(n)$, such that every curve in $\mathcal{U}_4$ crosses every blue curve in $\mathcal{B}_2$, or every curve in $\mathcal{U}_4$ is disjoint from every blue curve in $\mathcal{B}_2$. In both cases, we are done since every curve in $\mathcal{L}_3$ is disjoint from every curve in $\mathcal{B}_2$. Therefore, we can assume that $G(\mathcal{U}_3)$ has edge density greater than $1 - \delta$.

For each curve $\alpha \in \mathcal{U}_3$, let $N(\alpha)$ denote the set of curves in $\mathcal{U}_3$ that intersects α, and let $d(\alpha) = |N(\alpha)|$. We label the curves $\beta \in N(\alpha)$ with integers 0 to $d(\alpha) - 1$ according to their closest intersection point to the ground γ_1 along α, that is, the label $f_\alpha(\beta)$ of $\beta \in N(\alpha)$ is the number of curves in $\mathcal{U}_3$ that intersects the portion of α strictly between γ_1 and $\alpha \cap \beta$. Since $\sum\limits_{\alpha \in \mathcal{U}_3} d(\alpha) - 1 \geq 2(1-\delta)\binom{|\mathcal{U}_3|}{2} - |\mathcal{U}_3|$, by Jensen's inequality, we have

$$\sum_{\alpha \in \mathcal{U}_3} \sum_{\beta \in N(\alpha)} f_\alpha(\beta) = \sum_{\alpha \in \mathcal{U}_3} \binom{d(\alpha)}{2} \geq |\mathcal{U}_3| \binom{\frac{\sum_{\alpha \in \mathcal{U}_3} d(\alpha)}{|\mathcal{U}_3|}}{2} \geq \frac{|\mathcal{U}_3|^3}{4}.$$

Let the weight $w(\beta)$ of a curve $\beta \in \mathcal{U}_3$ be the sum of its labels, that is, $w(\beta) = \sum_{\alpha:\beta\in N(\alpha)} f_\alpha(\beta)$. Hence, the weight $w(\beta)$ is the total number of crossing points along curves α strictly between γ_1 and β, where α crosses both γ_1 and β. By averaging, there is a curve $\gamma_2 \in \mathcal{U}_3$ whose weight is at least $|\mathcal{U}_3|^2/4$.

Using γ_2, we partition each curve $\alpha \in \mathcal{U}_3 \setminus \{\gamma_2\}$ that crosses γ_2 into two connected parts, $\alpha = \alpha_w \cup \alpha_m$, where α_m is the connected subcurve with endpoints on γ_1 and γ_2, and α_w is the other connected part. Set

$$\mathcal{W}_3 = \{\alpha_w : \alpha \in \mathcal{U}_3 \setminus \{\gamma_2\}, \alpha \cap \gamma_2 \neq \emptyset\} \quad \text{and} \quad \mathcal{M}_3 = \{\alpha_m : \alpha \in \mathcal{U}_3 \setminus \{\gamma_2\}, \alpha \cap \gamma_2 \neq \emptyset\}.$$

Since γ_2 has weight at least $|\mathcal{U}_3|^2/4$, by the pigeonhole principle, there are at least $|\mathcal{U}_3|^2/8$ intersecting pairs in $\mathcal{M}_3 \times \mathcal{M}_3$, or at least $|\mathcal{U}_3|^2/8$ intersecting pairs in $\mathcal{M}_3 \times \mathcal{W}_3$.

Case 1. Suppose there are at least $|\mathcal{U}_3|^2/8$ pairs in $\mathcal{M}_3 \times \mathcal{W}_3$ that cross. The set $\mathcal{M}_3$ is double grounded with grounds γ_1 and γ_2 that cross exactly once, and every curve in $\mathcal{W}_3$ is disjoint from γ_1 and γ_2. As $|\mathcal{M}_3|, |\mathcal{W}_3| \leq |\mathcal{U}_3|$, the density of edges in the bipartite intersection graph of $\mathcal{M}_3$ and $\mathcal{W}_3$ is at least $1/8$. By averaging, we can find subsets of $\mathcal{M}_3$ and $\mathcal{W}_3$ each of size $\min(|\mathcal{M}_3|, |\mathcal{W}_3|)$ such that the density of edges in the bipartite intersection graph of these subsets is at least $1/8$. By setting $\delta > 0$ sufficiently small, we can apply Theorem 13 to these subsets of $\mathcal{M}_3$ and $\mathcal{W}_3$ and obtain subsets $\mathcal{M}_4 \subset \mathcal{M}_3$ and $\mathcal{W}'_4 \subset \mathcal{W}_3$, each of size $\Omega_\delta(n)$, such that each curve in $\mathcal{M}_4$ crosses each curve in $\mathcal{W}'_4$. Moreover, by the pseudo-segment condition, each curve in $\mathcal{M}_4 \cup \mathcal{W}'_4$ corresponds to a unique curve in $\mathcal{U}_3$. Let $\mathcal{U}_4 \subset \mathcal{U}_3$ be the curves that corresponds to $\mathcal{M}_4$ and let $\mathcal{U}'_4 \subset \mathcal{U}_3$ be the curves that corresponds to $\mathcal{W}'_4$. Hence, we set

$$\mathcal{W}_4 = \{\alpha_m : \alpha \in \mathcal{U}_4, \alpha = \alpha_w \cup \alpha_m\} \quad \text{and} \quad \mathcal{M}'_4 = \{\alpha_m : \alpha \in \mathcal{U}'_4, \alpha = \alpha_w \cup \alpha_m\}.$$

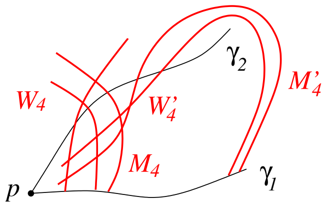
See Figure 5a. We apply Theorem 12 to arbitrary subsets of $\mathcal{M}_4$ and $\mathcal{B}_2$, each of size $\min(|\mathcal{M}_4|, |\mathcal{B}_2|)$, and obtain subsets $\mathcal{M}_5 \subset \mathcal{M}_4$ and $\mathcal{B}_3 \subset \mathcal{B}_2$, each of size $\Omega_\delta(n)$, such that either every red curve in $\mathcal{M}_5$ crosses every blue curve in $\mathcal{B}_3$, or every red curve in $\mathcal{M}_5$ is disjoint from every blue curve in $\mathcal{B}_3$. In the former case, we are done. Hence, we can assume that we are in the latter case.

We again apply Theorem 12 to arbitrary subsets of $\mathcal{M}'_4$ and $\mathcal{B}_3$, each of size $\min(|\mathcal{M}'_4|, |\mathcal{B}_3|)$, to obtain subsets $\mathcal{M}'_5 \subset \mathcal{M}'_4$ and $\mathcal{B}_4 \subset \mathcal{B}_3$, each of size $\Omega_\delta(n)$, such that either every red curve in $\mathcal{M}'_5$ crosses every blue curve in $\mathcal{B}_4$, or every red curve in $\mathcal{M}'_5$ is disjoint from every blue curve in $\mathcal{B}_4$. Again, if we are in the former case, we are done. Hence, we can assume that we are in the latter case. Let

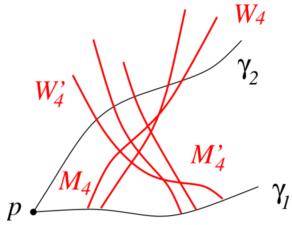
$$\mathcal{W}_5 = \{\alpha_w : \alpha = \alpha_w \cup \alpha_m, \alpha_m \in \mathcal{M}_5\} \quad \text{and} \quad \mathcal{W}'_5 = \{\alpha_w : \alpha = \alpha_w \cup \alpha_m, \alpha_m \in \mathcal{M}'_5\},$$

and recall that every element in $\mathcal{M}_5$ crosses every element in $\mathcal{W}'_5$. By the pseudo-segment condition, every element in $\mathcal{W}_5$ is disjoint from every element in $\mathcal{W}'_5$.

Let $\mathcal{R}_5$ be the red curves in $\mathcal{R}$ that corresponds to $\mathcal{W}_5$, and let $\mathcal{R}'_5$ be the red curves in $\mathcal{R}$ that corresponds to $\mathcal{W}'_5$. We have $|\mathcal{R}_5|, |\mathcal{R}'_5| = \Omega_\delta(n)$, and moreover, we can assume that $|\mathcal{R}_5| = |\mathcal{R}'_5|$. For each curve $\alpha \in \mathcal{R}_5 \cup \mathcal{R}'_5$, and its original partition $\alpha = \alpha_u \cup \alpha_\ell$ defined by γ_1, we have a new partition $\alpha = \alpha'_u \cup \alpha'_\ell$ defined by γ_2, where $\alpha'_u = \alpha_w$ and $\alpha'_\ell = \alpha_m \cup \alpha_\ell$. By setting $\mathcal{R}^{(1)} = \mathcal{R}_5 \cup \mathcal{R}'_5$, and $\mathcal{B}^{(1)} = \mathcal{B}_4$, where each curve $\alpha \in \mathcal{R}^{(1)}$ is equipped with the partition $\alpha = \alpha'_u \cup \alpha'_\ell$, we satisfy the base case of the statement.



(a) Case 1.



(b) Case 2.

Figure 5 In both cases, $\mathcal{W}_4$ is disjoint to $\mathcal{W}'_4$.

Case 2. The argument is essentially the same as Case 1. Suppose we have at least $|\mathcal{U}_3|^2/8$ crossing pairs in $\mathcal{M}_3 \times \mathcal{M}_3$. Then by Theorem 5, there are subsets $\mathcal{M}_4, \mathcal{M}'_4 \subset \mathcal{M}_3$, each of size $\Omega_\delta(n)$, such that every curve in $\mathcal{M}_4$ crosses every curve in $\mathcal{M}'_4$. Let $\mathcal{U}_4 \subset \mathcal{U}$ be the curves that corresponds to $\mathcal{M}_4$ and let $\mathcal{U}'_4 \subset \mathcal{U}$ be the curves that corresponds to $\mathcal{M}'_4$. Set

$$\mathcal{W}_4 = \{\alpha_w : \alpha \in \mathcal{U}_4, \alpha = \alpha_w \cup \alpha_m\} \quad \text{and} \quad \mathcal{W}'_4 = \{\alpha_w : \alpha \in \mathcal{U}'_4, \alpha = \alpha_w \cup \alpha_m\}.$$

See Figure 5b. Hence, by the pseudo-segment condition, every curve in $\mathcal{W}_4$ is disjoint from every curve in $\mathcal{W}'_4$. By taking arbitrary subsets of $\mathcal{M}_4$ and $\mathcal{B}_2$ of size $\min(|\mathcal{M}_4|, |\mathcal{B}_2|)$, we can apply Theorem 12 to these subsets and obtain subsets $\mathcal{M}_5 \subset \mathcal{M}_4$ and $\mathcal{B}_3 \subset \mathcal{B}_2$, each of size $\Omega_\delta(n)$, such that either every red curve in $\mathcal{M}_5$ crosses every blue curve in $\mathcal{B}_3$, or every red curve in $\mathcal{M}_5$ is disjoint from every blue curve in $\mathcal{B}_3$. In the former case, we are done. Hence, we can assume that we are in the latter case.

Again, we take an arbitrary subset of $\mathcal{M}'_4$ and $\mathcal{B}_3$ of size $\min(|\mathcal{M}'_4|, |\mathcal{B}_3|)$ and apply Theorem 12 to $\mathcal{M}'_4$ and $\mathcal{B}_3$, to obtain subsets $\mathcal{M}'_5 \subset \mathcal{M}'_4$ and $\mathcal{B}_4 \subset \mathcal{B}_3$, each of size $\Omega_\delta(n)$, such that either every red curve in $\mathcal{M}'_5$ crosses every blue curve in $\mathcal{B}_4$, or every red curve in $\mathcal{M}'_5$ is disjoint from every blue curve in $\mathcal{B}_4$. Again, if we are in the former case, we are done. Hence, we can assume that we are in the latter case. Set $\mathcal{R}_5$ be the red curves in $\mathcal{R}$ that corresponds to $\mathcal{M}_5$, and let $\mathcal{R}'_5$ be the red curves in $\mathcal{R}$ that corresponds to $\mathcal{M}'_5$.

We have $|\mathcal{R}_5|, |\mathcal{R}'_5| = \Omega_\delta(n)$, and moreover, we can assume that $|\mathcal{R}_5| = |\mathcal{R}'_5|$. For each curve $\alpha \in \mathcal{R}_5 \cup \mathcal{R}'_5$, and its original partition $\alpha = \alpha_u \cup \alpha_\ell$ defined by γ_1, we have a new partition $\alpha = \alpha'_u \cup \alpha'_\ell$ defined by γ_2, where $\alpha'_u = \alpha_w$ and $\alpha'_\ell = \alpha_m \cup \alpha_\ell$. By setting $\mathcal{R}^{(1)} = \mathcal{R}_5 \cup \mathcal{R}'_5$, and $\mathcal{B}^{(1)} = \mathcal{B}_4$, where each curve $\alpha \in \mathcal{R}^{(1)}$ is equipped with the partition $\alpha = \alpha'_u \cup \alpha'_\ell$, we satsify the base case of the statement.

For the inductive step, suppose we have obtained constants $\varepsilon'_{t-1} < \cdots < \varepsilon'_1$ such that the statement follows. Let ε'_t be a small constant that will be determined later such that $\varepsilon'_t < \varepsilon'_{t-1}$. Let $\mathcal{R}$ be a set of n red curves in the plane, all crossed by a curve γ_1 exactly once, and $\mathcal{B}$ be a set of n blue curves in the plane such that $\mathcal{R} \cup \mathcal{B} \cup \{\gamma_1\}$ is a collection of pseudo-segments. Moreover, $G(\mathcal{R})$ has edge density at least $1 - \varepsilon'_t$ and $G(\mathcal{B})$ has edge density less than ε'_t. We set $\delta' < 0$ to be a small constant such that $\varepsilon'_t < \delta' < \varepsilon_{t-1}$. We repeat the entire argument above, replacing ε'_1 with ε'_t and δ with δ', to obtain subsets $\mathcal{R}_5, \mathcal{R}'_5 \subset \mathcal{R}$ and $\mathcal{B}_4 \subset \mathcal{B}$, each of size $\Omega_{\delta'}(n)$, such that each $\alpha \in \mathcal{R}_5 \cup \mathcal{R}'_5$ is equipped with the partition $\alpha = \alpha'_u \cup \alpha'_\ell$, and α'_ℓ is disjoint to every blue curve in $\mathcal{B}_4$. Moreover, for $\alpha \in \mathcal{R}_5$ and $\beta \in \mathcal{R}'_5$, where $\alpha = \alpha'_u \cup \alpha'_\ell$ and $\beta = \beta'_u \cup \beta'_\ell$, α'_u is disjoint to β'_u.

Since $|\mathcal{R}_5|, |\mathcal{B}_4| \geq \delta_1 n$, where δ_1 depends only on δ', by Theorem 7, $G(\mathcal{R}_5)$ has edge density at least $1 - 2\varepsilon'_t/\delta_1^2$ and $G(\mathcal{B}_4)$ has edge density less than $2\varepsilon'_t/\delta_1^2$. By setting ε'_t sufficiently small, $G(\mathcal{R}_5)$ has edge density at least $1 - \varepsilon'_{t-1}$, and $G(\mathcal{B}_4)$ has edge density less than ε'_{t-1}. By averaging, we can find subsets of $\mathcal{R}_5$ and $\mathcal{B}_4$, each of size $\min(|\mathcal{R}_5|, |\mathcal{B}_4|)$ and with densities at least $1-\varepsilon'_{t-1}$ and less than ε'_{t-1} respectively, and apply induction to these subsets with parameter $t'=t-1$, and obtain subsets $\mathcal{R}^{(t-1)}\subset\mathcal{R}_5$, $\mathcal{B}^{(t-1)}\subset\mathcal{B}_4$, each of size $\Omega_{\varepsilon'_{t-1}}(n)$, with the desired properties. If every red curve in $\mathcal{R}^{(t-1)}$ is disjoint from every blue curve in $\mathcal{B}^{(t-1)}$, or if every red curve in $\mathcal{R}^{(t-1)}$ crosses every blue curve in $\mathcal{B}^{(t-1)}$, then we are done. Hence, we can assume that each curve $\alpha\in\mathcal{R}^{(t-1)}$ has a partition $\alpha=\alpha''_u\cup\alpha''_\ell$ such that α''_u is a subcurve of α'_u, α''_ℓ is disjoint from every blue curve in $\mathcal{B}^{(t-1)}$, and there is an equipartition $\mathcal{R}^{(t-1)}=\mathcal{R}^{(t-1)}_1\cup\cdots\cup\mathcal{R}^{(t-1)}_{2^{t-1}}$, such that for $1\le i<j\le 2^{t-1}$, the upper part α''_u of each curve $\alpha\in\mathcal{R}^{(t-1)}_i$ is disjoint the upper part β''_u of each curve $\beta\in\mathcal{R}^{(t-1)}_j$.

Finally, since $|\mathcal{R}'_5|,|\mathcal{B}^{(t-1)}|\ge\delta_2 n$, where δ_2 depends only on δ', by Theorem 7, $G(\mathcal{R}'_5)$ has edge density at least $1-2\varepsilon'_t/\delta_2^2$ and $G(\mathcal{B}^{(t-1)})$ has edge density less than $2\varepsilon'_t/\delta_2^2$. By setting ε'_t sufficiently small, $G(\mathcal{R}'_5)$ has edge density at least $1-\varepsilon'_{t-1}$, and $G(\mathcal{B}^{(t-1)})$ has edge density less than ε'_{t-1}. By averaging, we can find subsets of $\mathcal{R}'_5$ and $\mathcal{B}^{(t-1)}$, each of size $\min(|\mathcal{R}'_5|,|\mathcal{B}^{(t-1)}|)$ and with densities at least $1-\varepsilon'_{t-1}$ and less than ε'_{t-1} respectively, and apply induction to these subsets parameter $t'=t-1$, and obtain subsets $\mathcal{S}^{(t-1)}\subset\mathcal{R}'_5$, $\mathcal{B}^{(t)}\subset\mathcal{B}^{(t-1)}$, each of size $\Omega_{\varepsilon'_{t-1}}(n)$, with the desired properties. If every red curve in $\mathcal{S}^{(t-1)}$ is disjoint from every blue curve in $\mathcal{B}^{(t)}$, or if every red curve in $\mathcal{S}^{(t-1)}$ crosses every blue curve in $\mathcal{B}^{(t)}$, then we are done. Hence, we can assume that each curve $\alpha\in\mathcal{S}^{(t-1)}$ has a partition $\alpha=\alpha''_u\cup\alpha''_\ell$ such that α''_u is a subcurve of α'_u, α''_ℓ is disjoint from every blue curve in $\mathcal{B}^{(t-1)}$, and there is an equipartition $\mathcal{S}^{(t-1)}=\mathcal{S}^{(t-1)}_1\cup\cdots\cup\mathcal{S}^{(t-1)}_{2^{t-1}}$, such that for $1\le i<j\le 2^{t-1}$, the upper part α''_u of each curve $\alpha\in\mathcal{S}^{(t-1)}_i$ is disjoint the upper part β''_u of each curve $\beta\in\mathcal{S}^{(t-1)}_j$. We then (arbitrarily) remove curves from each part in $\mathcal{R}^{(t-1)}_i$ and $\mathcal{S}^{(t-1)}_j$ such that the resulting parts all have the same size and for

$$\mathcal{R}^{(t)}=\mathcal{R}^{(t-1)}_1\cup\cdots\cup\mathcal{R}^{(t-1)}_{2^{t-1}}\cup\mathcal{S}^{(t-1)}_1\cup\cdots\cup\mathcal{S}^{(t-1)}_{2^{t-1}},$$

we have $|\mathcal{R}^{(t)}|=\Omega_{\varepsilon'_{t-1}}(n)$. Then $\mathcal{R}^{(t)}$ and $\mathcal{B}^{(t)}$ has the desired properties. ◀

We now prove the following.

▶ **Theorem 16.** *There is an absolute constant $\varepsilon_3>0$ such that the following holds. Let $\mathcal{R}$ be a set of n red curves in the plane and $\mathcal{B}$ be a set of n blue curves in the plane such that $\mathcal{R}\cup\mathcal{B}$ is a collection of pseudo-segments, and the intersection graph $G(\mathcal{B})$ has edge density less than ε_3, and $G(\mathcal{R})$ has edge density at least $1-\varepsilon_3$. Then there are subsets $\mathcal{R}'\subset\mathcal{R}$, $\mathcal{B}'\subset\mathcal{B}$, each of size $\Omega(n)$, such that either every red curve in $\mathcal{R}$ crosses every blue curve in $\mathcal{B}$, or every red curve in $\mathcal{R}$ is disjoint from every blue curve in $\mathcal{B}$.*

Proof. Let t be a fixed large integer such that $2^{-t}<\varepsilon_1$, where ε_1 is defined in Theorem 14. Let ε_3 be a small constant determined later such that $\varepsilon_3<\varepsilon'_t$, where ε'_t is defined in Lemma 15. Recall that $\varepsilon'_t<\varepsilon_1$. Since $G(\mathcal{R})$ has edge density at least $1-\varepsilon_3$, there is a curve $\gamma_1\in\mathcal{R}$ such that γ_1 crosses at least $n/2$ red curves in $\mathcal{R}$. Let $\mathcal{R}_0\subset\mathcal{R}$ be the red curves that crosses γ_1. By Lemma 7, $G(\mathcal{R}_0)$ has edge density at least $1-8\varepsilon_3$. By averaging, we can find a subset $\mathcal{B}'\subset\mathcal{B}$ of size $|\mathcal{R}_0|$ whose edge density is less than ε_3. By setting ε_3 sufficiently small so that $8\varepsilon_3<\varepsilon'_t$, we can apply Lemma 15 to $\mathcal{R}_0$ and $\mathcal{B}'$ with parameter t, and obtain subsets $\hat{\mathcal{R}}\subset\mathcal{R}_0$, $\hat{\mathcal{B}}\subset\mathcal{B}$, each of size $\Omega_{\varepsilon'_t}(n)$, with the desired properties. If every red curve in $\hat{\mathcal{R}}$ crosses every blue curve in $\hat{\mathcal{B}}$, or every red curve in $\hat{\mathcal{R}}$ is disjoint from every blue curve in $\hat{\mathcal{B}}$, then we are done. Therefore, we can assume that each curve $\alpha\in\hat{\mathcal{R}}$ has a partition into two parts $\alpha=\alpha'_u\cup\alpha'_\ell$ with the properties described in Lemma 15. Set

$$\mathcal{U}=\{\alpha'_u:\alpha\in\hat{\mathcal{R}},\alpha=\alpha'_u\cup\alpha'_\ell\}\quad\text{and}\quad\mathcal{L}=\{\alpha'_\ell:\alpha\in\hat{\mathcal{R}},\alpha=\alpha'_u\cup\alpha'_\ell\}.$$

Hence, every curve in $\mathcal{L}$ is disjoint from every curve in $\hat{\mathcal{B}}$, and $G(\mathcal{U})$ has edge density at most $2^{-t} < \varepsilon_1$. Since $|\hat{\mathcal{B}}| \geq \delta n$, where δ depends only on ε'_t, by Lemma 7, $G(\hat{\mathcal{B}})$ has edge density at most $2\varepsilon_3/\delta^2$. By setting ε_3 sufficiently small so that $2\varepsilon_3/\delta_0^2 < \varepsilon_1$, $G(\hat{\mathcal{B}})$ has edge density at most ε_1. By averaging, we can find subsets of $\mathcal{U}$ and $\hat{\mathcal{B}}$, each of size $\min(|\mathcal{U}|, |\hat{\mathcal{B}}|)$ and with densities at most ε_1, and apply Theorem 14 to these subsets to obtain subsets $\mathcal{U}' \subset \mathcal{U}$ and $\mathcal{B}' \subset \hat{\mathcal{B}}$, each of size $\Omega_{\varepsilon_3}(n)$, such that every curve in $\mathcal{U}'$ is disjoint from every curve in $\mathcal{B}'$, or every curve in $\mathcal{U}'$ crosses every curve in $\mathcal{B}'$. By setting $\mathcal{R}'$ to be the red curves in $\mathcal{R}$ corresponding to $\mathcal{U}'$, every red curve in $\mathcal{R}'$ is disjoint from every blue curve in $\mathcal{B}'$, or every red curve in $\mathcal{R}'$ crosses every blue curve in $\mathcal{B}'$, and each subset has size $\Omega_{\varepsilon_3}(n)$. ◀

4.3 High versus high edge density

Finally, we consider the case when the intersection graphs $G(\mathcal{R})$ and $G(\mathcal{B})$ both have edge densities at least $1-\varepsilon$. By copying the proof of Theorem 16, except using Theorem 16 (high versus low density) instead of Theorem 14 (low versus low density), we obtain the following.

▶ **Theorem 17.** *There is an absolute constant $\varepsilon_4 > 0$ such that the following holds. Let $\mathcal{R}$ be a set of n red curves in the plane and $\mathcal{B}$ be a set of n blue curves in the plane such that $\mathcal{R} \cup \mathcal{B}$ is a collection of pseudo-segments, and the intersection graphs $G(\mathcal{B})$ and $G(\mathcal{R})$ both have edge density at least $1-\varepsilon_4$. Then there are subsets $\mathcal{R}' \subset \mathcal{R}$, $\mathcal{B}' \subset \mathcal{B}$, each of size $\Omega(n)$, such that either every red curve in $\mathcal{R}$ crosses every blue curve in $\mathcal{B}$, or every red curve in $\mathcal{R}$ is disjoint from every blue curve in $\mathcal{B}$.*

5 Proof of Theorem 4

Let $\mathcal{R}$ be a set of n red curves in the plane, and $\mathcal{B}$ be a set of n blue curves in the plane such that $\mathcal{R} \cup \mathcal{B}$ is a collection of pseudo-segments. Let ε be a sufficiently small constant such that $\varepsilon < \varepsilon_4 < \varepsilon_3 < \varepsilon_1$, where ε_1 is from Theorem 14, ε_3 is from Theorem 16, and ε_4 is from Theorem 17. We apply Corollary 6 to both $\mathcal{R}$ and $\mathcal{B}$ and obtain subsets $\mathcal{R}_1 \subset \mathcal{R}$ and $\mathcal{B}_1 \subset \mathcal{B}$ such that both $G(\mathcal{R}_1)$ and $G(\mathcal{B}_1)$ are ε-homogeneous. Moreover, we can assume that $|\mathcal{R}_1| = |\mathcal{B}_1|$. If both $G(\mathcal{R}_1)$ and $G(\mathcal{B}_1)$ have edge densities less than ε, then, since ε is sufficiently small, we can apply Theorem 14 to obtain subsets $\mathcal{R}_2 \subset \mathcal{R}_1$ and $\mathcal{B}_2 \subset \mathcal{B}_1$, each of size $\Omega_\varepsilon(n)$, such that either every red curve in $\mathcal{R}_2$ is disjoint from every blue curve in $\mathcal{B}_2$, or every red curve in $\mathcal{R}_2$ crosses every blue curve in $\mathcal{B}_2$. If one of the graphs $G(\mathcal{R}_1)$ and $G(\mathcal{B}_1)$ has edge density less than ε, and the other has edge density greater than $1-\varepsilon$, then we apply Theorem 16 to $\mathcal{R}_1$ and $\mathcal{B}_1$ to obtain subsets $\mathcal{R}_2 \subset \mathcal{R}_1$ and $\mathcal{B}_2 \subset \mathcal{B}_1$, each of size $\Omega_\varepsilon(n)$, such that either every red curve in $\mathcal{R}_2$ is disjoint from every blue curve in $\mathcal{B}_2$, or every red curve in $\mathcal{R}_2$ crosses every blue curve in $\mathcal{B}_2$. Finally, if both $G(\mathcal{R}_1)$ and $G(\mathcal{B}_1)$ have edge densities at least $1-\varepsilon$, then, since ε is sufficiently small, we can apply Theorem 17 to obtain subsets $\mathcal{R}_2 \subset \mathcal{R}_1$ and $\mathcal{B}_2 \subset \mathcal{B}_1$, each of size $\Omega_\varepsilon(n)$, such that either every red curve in $\mathcal{R}_2$ is disjoint from every blue curve in $\mathcal{B}_2$, or every red curve in $\mathcal{R}_2$ crosses every blue curve in $\mathcal{B}_2$.

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