Demystifying Latschev's Theorem: Manifold Reconstruction from Noisy Data

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Abstract

For a closed Riemannian manifold $\mathcal{M}$ and a metric space S with a small Gromov–Hausdorff distance to it, Latschev's theorem guarantees the existence of a sufficiently small scale $\beta > 0$ at which the Vietoris–Rips complex of S is homotopy equivalent to $\mathcal{M}$. Despite being regarded as a stepping stone to the topological reconstruction of Riemannian manifolds from a noisy data, the result is only a qualitative guarantee. Until now, it had been elusive how to quantitatively choose such a proximity scale β in order to provide sampling conditions for S to be homotopy equivalent to $\mathcal{M}$. In this paper, we prove a stronger and pragmatic version of Latschev's theorem, facilitating a simple description of β using the sectional curvatures and convexity radius of $\mathcal{M}$ as the sampling parameters.

Our study also delves into the topological recovery of a closed Euclidean submanifold from the Vietoris–Rips complexes of a Hausdorff close Euclidean subset. As already known for Čech complexes, we show that Vietoris–Rips complexes also provide topologically faithful reconstruction guarantees for submanifolds.

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1 Introduction

Given a metric space (X, d_X) and a positive proximity scale β, the *Vietoris–Rips complex* of X, denoted $\mathcal{R}_\beta(X)$, is defined to be an abstract simplicial complex having an m–simplex for every finite subset of X with cardinality $(m + 1)$ and diameter less than β.

The notion was first introduced by L. Vietoris [27], then extensively studied by E. Rips in the context of hyperbolic groups. Despite its inception in the early twentieth century, it is only the last decade that witnessed an increasing popularity of these complexes – particularly in the applied topology and topological data analysis (TDA) community. The computational simplicity makes the Vietoris–Rips complexes more palatable in applications than its traditional alternatives, e.g., the Čech complexes.

The combinatorial flexibility, however, comes at a theoretical cost. The topology of the Vietoris–Rips complex of (even a finite) metric space is generally poorly understood. Nonetheless, there have been noteworthy developments in the study of the Vietoris–Rips complexes constructed on or near Riemannian manifolds [17, 19] and metric graphs [22].

1.1 Motivation

In his pioneering work [17], Hausmann established a homotopy equivalence between a closed Riemannian manifold $\mathcal{M}$ and its Vietoris-Rips complex at a scale β smaller than the convexity radius (Definition 3) of $\mathcal{M}$. As a naive exercise [17, Problem 3.11], Hausmann asked the curious question of *finite reconstruction*: for a dense enough subset $S \subset \mathcal{M}$ and small enough scale β, is $\mathcal{M}$ also homotopy equivalent to the Vietoris–Rips complex of S?

The quest foreshadowed Latschev's remarkable result [19, Theorem 1.1]: for every closed Riemannian manifold $\mathcal{M}$, there exists a positive number ϵ_0 such that for any $0 < \beta \leq \epsilon_0$ there exists some $\delta > 0$ such that for every metric space S with Gromov–Hausdorff distance to $\mathcal{M}$ less than δ, the Vietoris–Rips complex $\mathcal{R}_\beta(S)$ is homotopy equivalent to $\mathcal{M}$. The result is only qualitative in nature, guaranteeing the existence of a sufficiently small scale β such that $\mathcal{M}$ is homotopy equivalent to the Vietoris–Rips complex of a metric space (S, d_S) that is close to $\mathcal{M}$ in the Gromov–Hausdorff distance (Definition 2). The result has been regarded as a stepping stone to the finite reconstruction of an abstract Riemannian manifold from a noisy sample. Despite the qualitative guarantee, it is not apparently clear how to quantitatively choose such an ϵ_0 for a given manifold $\mathcal{M}$.

The current paper primarily aims at presenting the first quantitative version (Theorem 12) of Latschev's theorem in order to develop a provable Vietoris–Rips inspired manifold reconstruction scheme from a noisy sample. Our sampling conditions – for a faithful reconstruction – are given based on the convexity radius (Definition 3) and the upper bound of the sectional curvatures of $\mathcal{M}$. We recognize these parameters to be very natural in the context of Riemannian manifolds – strict enough to prove the desired homotopy equivalences and flexible enough to retain practicality.

Our techniques naturally extend to the topological reconstruction of a Euclidean submanifold from a Hausdorff–close sample. For a sufficiently small scale, we prove in (Theorem 18) that a submanifold is homotopy equivalent to the Vietoris–Rips complex of a dense sample. In the Euclidean case, we describe the sampling conditions using the reach (Definition 14) of the submanifold.

1.2 Related Work

In the same vein, Majhi [22] has recently studied the Vietoris–Rips complexes near a special class of geodesic spaces: metric graphs. For the Vietoris–Rips complexes of a noisy sample, the author provides sampling conditions for a faithful topological recovery of metric graphs under both the Gromov–Hausdorff and Hausdorff noise. Our investigation revolves around the same reconstruction theme, but for Riemannian manifolds. Since graphs can generally have branches and non-smooth corners, it is worth noting that the relevance of the results of Majhi [22] on metric graphs are not subsumed by our results on manifolds.

Relevant in this context are the works of Adams et al. [1, 2], where the homotopy equivalence results of Hausmann [17] and Latschev [19] have been restated in terms of the Vietoris–Rips thickening via the theory of optimal transport. We also mention [28] and [20] for providing an alternative and much simpler proof of Hausmann's theorem and extending Latschev's result to selective Rips complexes, respectively.

In this paper, we also further our understanding of the Vietoris–Rips complexes in another fundamental direction: the recovery of a Euclidean submanifold $\mathcal{M} \subset \mathbb{R}^d$ from a noisy Euclidean sample $S \subset \mathbb{R}^d$ in its close Hausdorff proximity. There have been several approaches to recover the topology (sometimes only the homology/homotopy groups) of a submanifold using the Čech and Vietoris–Rips complexes and their filtrations, as we survey in the following paragraph.

In a landmark paper by Niyogi et al. [25], the normal injectivity radius of $\mathcal{M}$ has been recognized as the sampling parameter; for a sufficiently small scale, the Čech complexes of a dense sample $S \subset \mathcal{M}$ have been shown to be homotopy equivalent to $\mathcal{M}$. For the recovery of homology/homotopy groups of Euclidean shapes using filtrations of the Čech and Vietoris-Rips complexes, we mention the works of Chazal et al. [9, 10] using the weak feature size of compact sets and Fasy et al. [15] using the convexity radius and distortion of geodesic subspaces. For a submanifold $\mathcal{M}$, we ask a more ambitious inference question: are the Vietoris–Rips complexes of a Hausdorff–close sample homotopy equivalent to $\mathcal{M}$ for a sufficiently small scale?

Attali et al. [4, Theorem 14] and Kim et al. [18, Theorem 20] show that a Euclidean subset with a positive reach (more generally μ–reach) is homotopy equivalent to the Vietoris–Rips complexes of a noisy point-cloud. Our Theorem 18 puts forward a similar result for submanifold reconstruction in terms of the reach of $\mathcal{M}$. Although, some results in [4, 18] generalize and improve over Theorem 18.

1.3 Our Contribution

One of the major contributions of this work is to quantify the scale parameters at which the Vietoris–Rips complex of a sample S recovers (up to homotopy type) a Riemannian manifold $\mathcal{M}$, under both the Gromov–Hausdorff and Hausdorff sampling conditions. Our main homotopy equivalence results are presented in Theorem 12 and Theorem 18, respectively.

This paper is organized in the following manner. Section 2 contains definitions, notations, and facts that are frequently used throughout the paper. In Section 3, the recovery of an abstract Riemannian manifold $\mathcal{M}$ from a Gromov–Hausdorff close sample S is obtained. We present a novel proof of Latschev's theorem with a much stronger statement. The sampling parameter $\Delta(\mathcal{M})$ as defined in (1).

▶ **Theorem 12** (Manifold Reconstruction under Gromov–Hausdorff Distance)**.** *Let $(\mathcal{M}, d_{\mathcal{M}})$ be a closed, connected Riemannian manifold. Let (S, d_S) be a compact metric space and $\beta > 0$ a number such that*

$$\frac{1}{\zeta} d_{GH}(\mathcal{M}, S) < \beta < \frac{1}{1+2\zeta} \Delta(\mathcal{M})$$

for some $0 < \zeta \leq 1/14$. Then, $\left|\mathcal{R}_\beta(S)\right| \simeq \mathcal{M}$.

Section 4 is devoted to the recovery of a Euclidean submanifold $\mathcal{M} \subset \mathbb{R}^d$ from a Hausdorff close, Euclidean sample $S \subset \mathbb{R}^d$. Theorem 18 shows the homotopy equivalence between $\mathcal{M}$ and the Vietoris–Rips complex $\mathcal{R}_\beta(S)$. Here, $\tau(\mathcal{M})$ denotes the reach (Definition 14) of $\mathcal{M}$.

▶ **Theorem 18** (Submanifold Reconstruction under Hausdorff Distance)**.** *Let $\mathcal{M} \subset \mathbb{R}^d$ be a closed, connected Euclidean submanifold. Let $S \subset \mathbb{R}^d$ be a compact subset and $\beta > 0$ a number such that*

$$\frac{1}{\zeta} d_H(\mathcal{M}, S) < \beta \leq \frac{3(1+2\zeta)(1-14\zeta)}{8(1-2\zeta)^2} \tau(\mathcal{M})$$

for some $0 < \zeta < 1/14$. Then, $\left|\mathcal{R}_\beta(S)\right| \simeq \mathcal{M}$.

2 Preliminaries

In this section, we present definitions and notations that we use throughout the paper. The standard results from algebraic topology and Riemannian geometry are stated here without proof; details can be found in any standard textbook on the subjects, e.g., [24, 26] and [5, 6], respectively.

2.1 Metric Spaces

Let (X, d_X) be a metric space. When it is clear from the context, we omit the metric d_X from the notation, and denote the metric space just by X.

▶ **Definition 1** (Diameter). *The diameter, denoted* $\text{diam}_X(Y)$*, of a subset* $Y \subset X$ *is defined by the supremum of the pairwise distances in* Y*.*

$$\text{diam}_X(Y) \stackrel{def}{=} \sup_{y_1, y_2 \in Y} d_X(y_1, y_2).$$

When Y is compact, its diameter is finite. We denote by $\mathbb{B}_X(x, r)$ the metric ball of radius $r \geq 0$ centered at $x \in X$.

A *correspondence* $\mathcal{C}$ between two (non-empty) metric spaces (X, d_X) and (Y, d_Y) is defined to be a subset of $X \times Y$ such that (a) for any $x \in X$, there exists $y \in Y$ such that $(x, y) \in \mathcal{C}$, and (b) for any $y \in Y$, there exists $x \in X$ such that $(x, y) \in \mathcal{C}$. We denote the set of all correspondences between X, Y by $\mathcal{C}(X, Y)$. The *distortion* of a correspondence $\mathcal{C} \in \mathcal{C}(X, Y)$ is defined as:

$$\text{dist}(\mathcal{C}) \stackrel{\text{def}}{=} \sup_{(x_1, y_1), (x_2, y_2) \in \mathcal{C}} \left| d_X(x_1, x_2) - d_Y(y_1, y_2) \right|.$$

▶ **Definition 2** (Gromov–Hausdorff Distance). *Let* (X, d_X) *and* (Y, d_Y) *be two compact metric spaces. The* Gromov–Hausdorff *distance between* X *and* Y*, denoted by* $d_{GH}(X, Y)$*, is defined as:*

$$d_{GH}(X, Y) \stackrel{def}{=} \frac{1}{2} \left[\inf_{\mathcal{C} \in \mathcal{C}(X, Y)} \text{dist}(\mathcal{C}) \right].$$

2.2 Simplicial Complexes

An *abstract simplicial complex* $\mathcal{K}$ is a collection of finite sets such that if $\sigma \in \mathcal{K}$, then so are all its non-empty subsets. In general, elements of $\mathcal{K}$ are called *simplices* of $\mathcal{K}$. The singleton sets in $\mathcal{K}$ are called the *vertices* of $\mathcal{K}$. If a simplex $\sigma \in \mathcal{K}$ has cardinality $(m + 1)$, then it is called an *m-simplex* and is denoted by σ_m. An m–simplex σ_m is also written as $[v_0, v_1, \ldots, v_m]$, where v_i's belong to the vertex set of $\mathcal{K}$. If σ' is a (proper) subset of σ, then σ' is called a (proper) *face* of σ, written as $\sigma' \preceq \sigma$ ($\sigma' \prec \sigma$ when proper).

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be abstract simplicial complexes with vertex sets V_1 and V_2, respectively. A *vertex map* is a map between the vertex sets. Let $\phi\colon V_1 \to V_2$ be a vertex map. We say that ϕ induces a *simplicial map* $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ if for all $\sigma_m = [v_0, v_1, \ldots, v_m] \in K_1$, the image

$$\phi(\sigma_m) \stackrel{def}{=} [\phi(v_0), \phi(v_1), \ldots, \phi(v_m)]$$

is a simplex of $\mathcal{K}_2$. Two simplicial maps $\phi, \psi : \mathcal{K}_1 \to \mathcal{K}_2$ are called *contiguous* if for every simplex $\sigma_1 \in \mathcal{K}_1$, there exists a simplex $\sigma_2 \in \mathcal{K}_2$ such that $\phi(\sigma_1) \cup \psi(\sigma_1) \preceq \sigma_2$.

For an abstract simplicial complex $\mathcal{K}$ with vertex set V, one can define its *geometric complex* or *underlying topological space*, denoted by $|\mathcal{K}|$, as the space of all functions $h : V \to [0, 1]$ satisfying the following two properties:

(i) $\text{supp}(h) \stackrel{def}{=} \{v \in V \mid h(v) \neq 0\}$ is a simplex of $\mathcal{K}$, and

(ii) $\sum_{v \in V} h(v) = 1$.

For $h \in |\mathcal{K}|$ and vertex v of $\mathcal{K}$, the real number $h(v)$ is called the v–th *barycentric coordinate* of h. For a simplex σ of $\mathcal{K}$, its *closed simplex* $|\sigma|$ and *open simplex* $\langle\sigma\rangle$ are subsets of $|\mathcal{K}|$ defined as follows:

$$|\sigma| \stackrel{def}{=} \{h \in |\mathcal{K}| \mid \mathrm{supp}(h) \subseteq \sigma\}, \text{ and } \langle\sigma\rangle \stackrel{def}{=} \{h \in |\mathcal{K}| \mid \mathrm{supp}(h) = \sigma\}.$$

In this work, we use the standard metric topology on $|\mathcal{K}|$, as defined in [26]. A simplicial map $\phi : \mathcal{K}_1 \to \mathcal{K}_2$ induces a continuous (in this topology) map $|\phi| : |\mathcal{K}_1| \to |\mathcal{K}_2|$ defined by

$$|\phi|(h)(v') \stackrel{def}{=} \sum_{\phi(v)=v'} h(v), \text{ for } v' \in \mathcal{K}_2.$$

From the above definition, it follows that $|\phi|(h) \in |h(\sigma)|$ whenever $h \in \langle\sigma\rangle$.

A simplicial complex $\mathcal{K}$ is called a *pure m–complex* if every simplex of $\mathcal{K}$ is a face of an m–simplex. A simplicial complex $\mathcal{K}$ is called a *flag complex* if σ is a simplex of $\mathcal{K}$ whenever every pair of points in σ is a simplex of $\mathcal{K}$.

2.3 Barycentric Subdivision

The *barycenter*, denoted $\widehat{\sigma}_m$, of an m–simplex $\sigma_m = [v_0, v_1, \ldots, v_m]$ of $\mathcal{K}$ is the point of $\langle\sigma_m\rangle$ such that $\widehat{\sigma}_m(v_i) = \frac{1}{m+1}$ for all $0 \le i \le m$. Using linearity of simplices, a more convenient way of writing this is:

$$\widehat{\sigma}_m = \sum_{i=0}^{m} \frac{1}{m+1} v_i.$$

Let $\mathcal{K}$ be a complex. A *subdivision* of $\mathcal{K}$ is a simplicial complex $\mathcal{K}'$ such that

- **(i)** the vertices of $\mathcal{K}'$ are points of $|\mathcal{K}|$,
- **(ii)** if s' is a simplex of $\mathcal{K}'$, then there is $s \in \mathcal{K}$ such that $s' \subset |s|$, and
- **(iii)** the linear map $h : |\mathcal{K}'| \to |\mathcal{K}|$ sending each vertex of $\mathcal{K}'$ to the corresponding point of $|\mathcal{K}|$ is a homeomorphism.

For a simplicial complex $\mathcal{K}$, its *barycentric subdivision*, denoted by $\mathrm{sd}(\mathcal{K})$, is a special subdivision defined as follows. The vertices of $\mathrm{sd}(\mathcal{K})$ are the barycenters of the simplices of $\mathcal{K}$. The simplices of $\mathrm{sd}(\mathcal{K})$ are (non-empty) finite sets $[\widehat{\sigma}_0, \widehat{\sigma}_1, \ldots, \widehat{\sigma}_m]$ such that $\sigma_{i-1} \prec \sigma_i$ for $1 \le i \le m$ and $\sigma_i \in \mathcal{K}$.

2.4 Riemannian Manifolds

Let $\mathcal{M}$ be an n–dimensional Riemannian manifold, equipped with the shortest geodesic metric $d_{\mathcal{M}}$. More formally, for any two points $p, q \in \mathcal{M}$, their distance is given by

$$d_{\mathcal{M}}(x, y) \stackrel{def}{=} \inf \{\mathrm{length}(\gamma) \mid \gamma \text{ is a smooth curve in } \mathcal{M} \text{ joining } p, q\};$$

see [5, p. 174] for more details. Throughout the paper, we always assume that $\mathcal{M}$ is connected and closed (without boundary and compact). A subset $A \subset \mathcal{M}$ is called *(geodesically) convex* if for any two points $p, q \in A$, there exists a unique minimizing geodesic segment from p to q whose image lies entirely in A.

▶ **Definition 3** (Convexity Radius). *The* convexity radius *of $\mathcal{M}$, denoted $\rho(\mathcal{M})$, is defined as the infimum of the set of radii of the largest convex balls across the points of $\mathcal{M}$. Formally,*

$$\rho(\mathcal{M}) = \inf_{p \in \mathcal{M}} \sup \{r \ge 0 \mid \mathbb{B}_{\mathcal{M}}(p, s) \textit{ is convex for all } 0 < s < r\}.$$

The compactness of $\mathcal{M}$ guarantees that $\rho(\mathcal{M})$ is indeed positive; see [5, Proposition 95]. For example, the sphere $\mathbb{S}^n$ of radius R has $\rho(\mathbb{S}^n) = \pi R/2$. The following remarkable result by Hausmann states that $\mathcal{M}$ is homotopy equivalent to its Vietoris–Rips complex for a scale smaller than the convexity radius.

▶ **Theorem 4** (Hausmann's Theorem [17]). *For any $0 < \beta < \rho(\mathcal{M})$, the geometric complex of $\mathcal{R}_\beta(\mathcal{M})$ is homotopy equivalent to $\mathcal{M}$.*

▶ Remark 5. We remark that Hausmann defines the quantity $\rho(\mathcal{M})$ (denoted $r(\mathcal{M})$ by Hausmann) slightly differently; see conditions (a)–(c) in [17, Section 3]. Nonetheless, the veracity of Hausmann's original result is not compromised by the current substitution, since the implications from these conditions used by Hausmann are still obtained using the current definition of convexity radius.

The definition of sectional curvatures of an abstract manifold $\mathcal{M}$ uses a lot of machinery from Riemannian geometry. We skip the definition here, suggesting the interested reader to call upon any graduate level textbook on the subject, e.g., [6, Chapter 9]. For a point $p \in \mathcal{M}$ and (unit norm) vectors $u, v \in T_p(\mathcal{M})$ the tangent space of $\mathcal{M}$, the sectional curvature at p along the plane spanned by u, v is denoted by $\kappa_p(u, v)$. Intuitively, it measures the Gaussian curvature at p if $\mathcal{M}$ is a Euclidean surface. The (embedded) sphere $\mathbb{S}^n$ of radius R has a constant sectional curvature of $1/R^2$.

Let $\kappa(\mathcal{M}) \in \mathbb{R}$ denote the supremum of the set of sectional curvatures $\kappa_p(u, v)$ across all u, v and all p. Since $\mathcal{M}$ is compact, it can be shown that $\kappa(\mathcal{M})$ is finite; see [6, p. 166] for example. For the sake of simplifying the statements of our results, we introduce:

$$\Delta(\mathcal{M}) = \begin{cases} \rho(\mathcal{M}), & \text{if } \kappa(\mathcal{M}) \leq 0 \\ \min\left\{\rho(\mathcal{M}), \frac{\pi}{4\sqrt{\kappa(\mathcal{M})}}\right\}, & \text{if } \kappa(\mathcal{M}) > 0. \end{cases} \tag{1}$$

The quantity $\frac{1}{\Delta(\mathcal{M})}$ can be called the *condition number* of $\mathcal{M}$. The justification behind the name is that a manifold with a small condition number is well-conditioned to be reconstructed; whereas, the recovery of a manifold with a large condition number would require a large and extremely dense sample.

3 Abstract Manifold Reconstruction

This section is devoted to the study of the Vietoris–Rips complexes of a metric space (S, d_S) that is close to $\mathcal{M}$ in the Gromov–Hausdorff distance (see Definition 2). Our main homotopy equivalence result of this section is presented in Theorem 12. The proof of the result uses Jung's theorem as a very important ingredient. We first discuss the classical Jung's theorem but in the context of Riemannian manifolds.

3.1 Jung's Theorem in Riemannian Manifolds

For a compact subset $A \subset \mathcal{M}$, its diameter satisfies $\operatorname{diam}_{\mathcal{M}}(A) < \infty$. We can define a *minimal enclosing ball* to be a closed metric ball in $\mathcal{M}$ that contains A and has the smallest radius. If such a ball exists for A, we call its center a *circumcenter*, denoted $\Theta(A)$, and its radius the *circumradius*, denoted $\mathfrak{R}(A)$. For A compact, the circumradius is uniquely defined, but a circumcenter may not exist. When $\mathcal{M} = \mathbb{R}^n$, however, the circumcenter exists uniquely. Moreover, for a compact Euclidean subset $A \subset \mathbb{R}^d$, the classical Jung's theorem [11, Theorem 2.6] states that $\mathfrak{R}(A) \leq \sqrt{\frac{n}{2(n+1)}}\operatorname{diam}_{\mathcal{M}}(A)$.

The result was further extended by B. V. Dekster in [12, 14, 13] – first for compact subsets of Riemannian manifolds with constant sectional curvatures, then for Alexadrov spaces of curvature bounded above. A corollary in [13, Section 2] affirms that $\Theta(A)$ exists (possibly non-uniquely) for a compact A if:

(i) A is contained in the interior of a compact convex domain $C^n \subset \mathcal{M}$, and

(ii) $\text{diam}_{\mathcal{M}}(C^n) < 2\pi/(3\sqrt{\kappa(\mathcal{M})})$ when $\kappa(\mathcal{M}) > 0$.

▶ Remark 6. If $\text{diam}_{\mathcal{M}}(A) < \Delta(\mathcal{M})$, we note that both the above conditions are satisfied. In particular, one can choose C^n to be the closed ball $\overline{\mathbb{B}(a,r)}$ for any $a \in A$ and r with $\text{diam}_{\mathcal{M}}(A) < r < \Delta(\mathcal{M})$. Here $\Delta(\mathcal{M})$ is as defined in (1).
Moreover, $\Theta(A)$ belongs to the interior of C^n, and we have the following bound on the circumradius of A.

▶ **Theorem 7** (Extented Jung's Theorem [13]). *Let $\mathcal{M}$ be a compact, connected, n–dimensional manifold with the sectional curvatures at each point bounded above by $\kappa \in \mathbb{R}$. For any compact $A \subset \mathcal{M}$ with $\text{diam}_{\mathcal{M}}(A) < \Delta(\mathcal{M})$, its circumcenter $\Theta(A)$ exists in $\mathcal{M}$. Moreover, its diameter*

$$\text{diam}_{\mathcal{M}}(A) \geq \begin{cases} \frac{2}{\sqrt{-\kappa}} \sinh^{-1}\left(\sqrt{\frac{n+1}{2n}} \sinh\left(\sqrt{-\kappa}\,\mathfrak{R}(A)\right)\right), & \textit{for } \kappa < 0 \\ 2\mathfrak{R}(A)\sqrt{\frac{n+1}{2n}}, & \textit{for } \kappa = 0 \\ \frac{2}{\sqrt{\kappa}} \sin^{-1}\left(\sqrt{\frac{n+1}{2n}} \sin\left(\sqrt{\kappa}\,\mathfrak{R}(A)\right)\right), & \textit{for } \kappa > 0 \textit{ and} \\ & \mathfrak{R}(A) \in \left[0, \frac{\pi}{2\sqrt{\kappa}}\right] \end{cases} \tag{2}$$

Utilizing the above result, we show the following key result bounding the circumradius. See [21] for a proof.

▶ **Proposition 8** (Circumradius). *For any compact $A \subset \mathcal{M}$ with $\text{diam}_{\mathcal{M}}(A) < \Delta(\mathcal{M})$, its diameter satisfies*

$$\text{diam}_{\mathcal{M}}(A) \geq \frac{4}{3}\mathfrak{R}(A).$$

We immediately note the following important proposition, whose proof is presented in [21].

▶ **Proposition 9** (Circumcenters of Subsets). *If A is a compact subset of $\mathcal{M}$ with $\text{diam}_{\mathcal{M}}(A) < \Delta(\mathcal{M})$, then for any non-empty subset $B \subseteq A$, we have*

$$d_{\mathcal{M}}(\Theta(B), \Theta(A)) \leq \frac{3}{4}\text{diam}_{\mathcal{M}}(A).$$

3.2 Homotopy Equivalence

We now assume that (S, d_S) is a compact metric space such that the Gromov–Hausdorff distance $d_{GH}(S, \mathcal{M}) < \zeta\beta$ for some $\beta > 0$ and $0 < \zeta < 1/2$. From the definition of the Gromov–Hausdorff distance (Definition 2), then there exists a correspondence $\mathcal{C} \in \mathcal{C}(M, S)$ with $\text{dist}(\mathcal{C}) < 2\zeta\beta$. The correspondence induces a (possibly non-continuous and non-unique) vertex map $\phi : \mathcal{M} \to S$ such that $(p, \phi(p)) \in \mathcal{C}$ for all $p \in \mathcal{M}$. The vertex map ϕ extends to a simplicial map ϕ:

$$\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M}) \xrightarrow{\phi} \mathcal{R}_{\beta}(S). \tag{3}$$

To see that ϕ is a simplicial map, take an l–simplex $\sigma_l = [p_0, p_1, \dots, p_l]$ in $\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})$. By the construction of the Vietoris–Rips complex, we must have $d_{\mathcal{M}}(p_i, p_j) < (1-2\zeta)\beta$ for any $0 \le i, j \le l$. Since $\mathrm{dist}(\mathcal{C}) < 2\zeta\beta$ and $(p, \phi(p)) \in \mathcal{C}$ for all $p \in \mathcal{M}$, we have

$$d_S(\phi(p_i), \phi(p_j)) \le d_{\mathcal{M}}(p_i, p_j) + 2\zeta\beta < (1-2\zeta)\beta + 2\zeta\beta = \beta.$$

So, the image $\phi(\sigma_l) = [\phi(p_0), \phi(p_1), \dots, \phi(p_l)]$ is a simplex of $\mathcal{R}_\beta(S)$.

In the rest of the section, we show that ϕ induces a homotopy equivalence on the respective geometric complexes. First, we show in the following lemma that the simplicial map induces a surjective homomorphism on all homotopy groups.

▶ **Lemma 10** (Surjectivity). *Let (S, d_S) be a compact metric space and $\beta > 0$ a number such that*

$$\frac{1}{\zeta} d_{GH}(\mathcal{M}, S) < \beta < \frac{1}{1+2\zeta}\Delta(\mathcal{M})$$

for some $0 < \zeta \le 1/14$. Then for any $m \ge 0$, the simplicial map $\phi : \mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M}) \to \mathcal{R}_\beta(S)$ (as defined in (3)) induces a surjective homomorphism on the m–th homotopy group.

Proof. As observed in [21, Proposition A.1], both $\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})$ and $\mathcal{R}_\beta(S)$ are path-connected. So, the result holds for $m = 0$.

For $m \ge 1$, let us take an abstract simplicial complex $\mathcal{K}$ such that $|\mathcal{K}|$ is a triangulation of the m-dimensional sphere $\mathbb{S}^m$. Note that $\mathcal{K}$ is a pure m–complex. In order to show surjectivity of $|\phi|_*$ on $\pi_m\left(|\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})|\right)$, we start with a simplicial map $g : \mathcal{K} \longrightarrow \mathcal{R}_\beta(S)$, and argue that there must exist a simplicial map $\widetilde{g} : \mathrm{sd}\,(\mathcal{K}) \longrightarrow \mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} |\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})| & \xrightarrow{\;|\phi|\;} & |\mathcal{R}_\beta(S)| \\ \Big\uparrow{\scriptstyle |\widetilde{g}|} & & \Big\uparrow{\scriptstyle |g|} \\ |\mathrm{sd}\,(\mathcal{K})| & \xleftarrow{\;h^{-1}\;} & |\mathcal{K}| \end{array} \tag{4}$$

where the linear homeomorphism $h : |\mathrm{sd}\,(\mathcal{K})| \longrightarrow |\mathcal{K}|$ maps each vertex of $\mathrm{sd}\,(\mathcal{K})$ to the corresponding point of $|\mathcal{K}|$ as discussed in Subsection 2.3.

We note that each vertex of $\mathrm{sd}\,(\mathcal{K})$ is the barycenter, $\widehat{\sigma}$, of a simplex σ of $\mathcal{K}$. In order to construct the simplicial map $\widetilde{g} : \mathrm{sd}\,(\mathcal{K}) \longrightarrow \mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})$, we define it on the vertices of $\mathrm{sd}\,(\mathcal{K})$ first, and prove that the vertex map extends to a simplicial map.

Let $\sigma_l = [v_0, v_1, \dots, v_l]$ be an l–simplex of $\mathcal{K}$. Since g is a simplicial map, then the image $g(\sigma_l) = [g(v_0), g(v_1), \dots, g(v_l)]$ is a simplex of $\mathcal{R}_\beta(S)$, hence a subset of S with $\mathrm{diam}_S\,(g(\sigma_l)) < \beta$. For each $0 \le j \le l$, there exists $p_j \in \mathcal{M}$ such that $(p_j, g(v_j)) \in \mathcal{C}$. Recall that $\mathcal{C}$ is a correspondence with distortion $\mathrm{dist}(\mathcal{C}) < 2\zeta\beta$ as already fixed right above (3). We denote $\sigma'_j := [p_0, p_1, \dots, p_j]$ for $0 \le j \le l$. We note for later that the diameter of σ'_l is less than $\Delta(\mathcal{M})$:

$$\mathrm{diam}_{\mathcal{M}}\left(\sigma'_j\right) \le \mathrm{diam}_S\,(g(\sigma_j)) + 2\zeta\beta < \beta + 2\zeta\beta = (1+2\zeta)\beta < \Delta(\mathcal{M}). \tag{5}$$

We then define the vertex map

$$\widetilde{g}(\widehat{\sigma}_l) \overset{def}{=} \Theta(\sigma'_l),$$

where $\Theta(\sigma'_l) \in \mathcal{M}$ is a circumcenter of σ'_l. Due to the diameter bound in (5), Theorem 7 implies that a circumcenter of σ'_l exists.

To see that $\widetilde{g}$ extends to a simplicial map, we consider a typical l–simplex $\tau_l = [\widehat{\sigma}_0, \ldots, \widehat{\sigma}_l]$, of $\mathrm{sd}\,(\mathcal{K})$, where $\sigma_{i-1} \prec \sigma_i$ for $1 \le i \le l$ and $\sigma_i \in \mathcal{K}$. Now,

$$
\begin{aligned}
\mathrm{diam}_{\mathcal{M}}\,(\widetilde{g}(\tau_l)) &= \mathrm{diam}_{\mathcal{M}}\,([\Theta(\sigma'_0), \Theta(\sigma'_1), \ldots, \Theta(\sigma'_l)]) \\
&= \max_{0 \le i < j \le l} \{d_{\mathcal{M}}(\Theta(\sigma'_i), \Theta(\sigma'_j))\} \\
&\le \max_{0 \le j \le l} \left\{ \left(\frac{3}{4}\right) \mathrm{diam}_{\mathcal{M}}\left(\sigma'_j\right) \right\}, \\
&\qquad \text{by Proposition 9 as } \mathrm{diam}_{\mathcal{M}}\left(\sigma'_j\right) < \Delta(\mathcal{M}) \\
&= \frac{3}{4} \mathrm{diam}_{\mathcal{M}}\,(\sigma'_l) \\
&< \frac{3}{4}(1 + 2\zeta)\beta, \text{ from (5)} \\
&= (1 - 2\zeta)\beta - (1 - 14\zeta)\beta/4 \\
&\le (1 - 2\zeta)\beta, \text{ since } \zeta \le 1/14.
\end{aligned}
$$

So, $\widetilde{g}(\tau_l)$ is a simplex of $\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})$. This implies that $\widetilde{g}$ is a simplicial map.

We lastly invoke [21, Proposition A.2] to show that the diagram commutes up to homotopy. We need to argue that the simplicial maps g and $(\phi \circ \widetilde{g})$ satisfy the conditions of the proposition:

(a) For any vertex $v \in \mathcal{K}$,

$$(\phi \circ \widetilde{g})(v) = g(v).$$

(b) For any simplex $\sigma_m = [v_0, v_1, \ldots, v_m]$ of $\mathcal{K}$, we have for $0 \le j \le m$:

$$
\begin{aligned}
d_S(g(v_j), (\phi \circ \widetilde{g})(\widehat{\sigma}_m)) &= d_S(g(v_j), \phi(\Theta(\sigma'_m))) \\
&\le d_{\mathcal{M}}\,(p_j, \Theta(\sigma'_m)) + 2\zeta\beta, \text{ since } (p_j, g(v_j)) \in \mathcal{C} \\
&\le \frac{3}{4} \mathrm{diam}_{\mathcal{M}}\,(\sigma'_m) + 2\zeta\beta, \text{ by Proposition 9 as } p_j = \Theta(p_j) \\
&< \frac{3}{4}(1 + 2\zeta)\beta + 2\zeta\beta, \text{ from (5)} \\
&= \beta - (1 - 14\zeta)\beta/4 \\
&\le \beta, \text{ since } \zeta \le 1/14.
\end{aligned}
$$

So, $g(\sigma_m) \cup (\phi \circ \widetilde{g})(\widehat{\sigma}_m)$ is a simplex of $\mathcal{R}_\beta(S)$.

Therefore, [21, Proposition A.2] implies that the diagram commutes. Since $\left|\mathcal{K}\right| = \mathbb{S}^m$ and g is arbitrary, we conclude that $\left|\phi\right|$ induces a surjective homomorphism. ◀

▶ Remark 11. For the description and computation of homotopy groups, the consideration of basepoint is deliberately ignored throughout this paper. This is justified, as the considered scale parameters are such that all the Vietoris–Rips complexes used here are path-connected. We prove the claim in [21, Proposition A.1].

▶ **Theorem 12** (Manifold Reconstruction under Gromov–Hausdorff Distance)**.** *Let $(\mathcal{M}, d_{\mathcal{M}})$ be a closed, connected Riemannian manifold. Let (S, d_S) be a compact metric space and $\beta > 0$ a number such that*

$$\frac{1}{\zeta} d_{GH}(\mathcal{M}, S) < \beta < \frac{1}{1 + 2\zeta} \Delta(\mathcal{M})$$

for some $0 < \zeta \le 1/14$. Then, $\left|\mathcal{R}_\beta(S)\right| \simeq \mathcal{M}$.

Proof. Since $d_{GH}(\mathcal{M}, S) < \zeta\beta$, let us assume that $\mathcal{C} \in \mathcal{C}(\mathcal{M}, S)$ is a correspondence with $\operatorname{dist}(\mathcal{C}) < 2\zeta\beta$. As a result, we have the following chain of simplicial maps

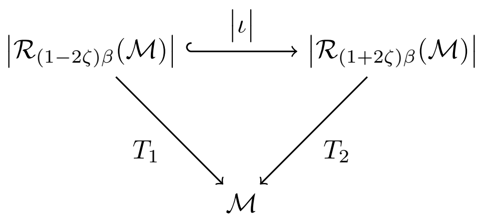
$$\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M}) \xrightarrow{\phi} \mathcal{R}_{\beta}(S) \xrightarrow{\psi} \mathcal{R}_{(1+2\zeta)\beta}(\mathcal{M}),$$

such that $(p, \phi(p)) \in \mathcal{C}$ for all $p \in \mathcal{M}$ and $(\psi(x), x) \in \mathcal{C}$ for all $x \in S$. There is also the natural inclusion $\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M}) \xhookrightarrow{\iota} \mathcal{R}_{(1+2\zeta)\beta}(\mathcal{M})$. We first claim that $(\psi \circ \phi)$ and ι are contiguous. To prove the claim, take an l–simplex $\sigma_l = [p_0, p_1, \ldots, p_l]$ in $\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})$. So, $d_{\mathcal{M}}(p_i, p_j) < (1-2\zeta)\beta$ for all $0 \leq i, j \leq l$. We then have

$$\begin{aligned} d_{\mathcal{M}}((\psi \circ \phi)(p_i), p_j) &= d_{\mathcal{M}}(\psi(\phi(p_i)), p_j) \\ &\leq d_S(\phi(p_i), \phi(p_j)) + 2\zeta\beta \\ &\leq d_{\mathcal{M}}(p_i, p_j) + 2\zeta\beta + 2\zeta\beta \\ &< (1-2\zeta)\beta + 4\zeta\beta = (1+2\zeta)\beta. \end{aligned}$$

This implies that $(\psi \circ \phi)(\sigma_l) \cup \iota(\sigma_l)$ is a simplex of $\mathcal{R}_{(1+2\zeta)\beta}(\mathcal{M})$. Since σ_l is an arbitrary simplex, the simplicial maps $(\psi \circ \phi)$ and ι are contiguous. Consequently, the maps $|\psi \circ \phi|$ and $|\iota|$ are homotopic.

Since $(1 + 2\zeta)\beta < \Delta(\mathcal{M}) \leq \rho(\mathcal{M})$, Theorem 4 implies that there exist homotopy equivalences T_1, T_2 such that the following diagram commutes (up to homotopy):



So, $|\iota|$ is also a homotopy equivalence. Hence, the induced homomorphism $|\iota|_*$ on the homotopy groups is an isomorphism. On the other hand, we already have $|\iota| \simeq |\psi \circ \phi|$. Therefore, the induced homomorphism $\left(|\psi|_* \circ |\phi|_*\right)$ is also an isomorphism, implying that $|\phi|_*$ is an injective homomorphism on $\pi_m\left(\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})\right)$. The surjectivity of $|\phi|_*$ comes from Lemma 10. For any $m \geq 0$, therefore,

$$|\phi|_* : \pi_m\left(|\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})|\right) \longrightarrow \pi_m\left(|\mathcal{R}_{\beta}(S)|\right).$$

is an isomorphism.

It follows from Whitehead's theorem that $|\phi|$ is a homotopy equivalence. Since $|\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})|$ is homotopy equivalent to $\mathcal{M}$, we conclude that $|\mathcal{R}_{\beta}(S)| \simeq \mathcal{M}$. ◀

▶ Remark 13. Whitehead's theorem requires the two spaces to admit CW-complex structures. In our case, they are the geometric realizations of Vietoris–Rips complexes on M and S. Although they can be infinite, we can always assume a total ordering on them. Consequently, the Vietoris–Rips complexes become ordered, abstract simplicial complexes; for the construction see [16]. Since an ordered simplicial complex is a simplicial set, its geometric realization must attain a CW-complex structure [23].

4 Hausdorff Reconstruction of Euclidean Submanifolds

Let $\mathcal{M} \subset \mathbb{R}^d$ be a closed, connected (smoothly embedded) Euclidean submanifold. In this section, we consider the topological reconstruction of $\mathcal{M}$ from a Euclidean subset $S \subset \mathbb{R}^d$ that is close to $\mathcal{M}$ in the Hausdorff distance. Theorem 18 is the main homotopy equivalence result of the section. To provide the sampling conditions, we use the reach of $\mathcal{M}$, which we define first.

The *medial axis* of a compact subset $X \subset \mathbb{R}^d$ is the set of points $y \in \mathbb{R}^d$ such that there are at least two (distinct) points $x_1, x_2 \in X$ with

$$\|x_1 - y\| = \|x_2 - y\| = \min_{x \in X} \|x - y\|.$$

▶ **Definition 14** (Reach). *The* reach *of X, denoted by $\tau(X)$, is the minimum of the set of distances between a point of X and a point on its medial axis.*

It can be shown that $\tau(\mathcal{M})$ is positive for a smoothly embedded closed submanifold $\mathcal{M}$. As we show in the next proposition, the reach controls both the sectional curvatures and the convexity radius of $\mathcal{M}$.

Fix a point $p \in \mathcal{M}$. Let $T_p(\mathcal{M})$ and $T_p^{\perp}(\mathcal{M})$ denote, respectively, the tangent and normal space of $\mathcal{M}$ at p. It can be shown that a symmetric, bilinear form $B(u,v) : T_p(\mathcal{M}) \times T_p(\mathcal{M}) \to T_p^{\perp}(\mathcal{M})$ exists, called the *second fundamental form* at p. More details can be found in any standard text, e.g., [7, Chapter 6]. For any orthonormal vectors $u, v \in T_p(\mathcal{M})$, the *sectional curvature* at p along the plane generated by u, v is defined as

$$\kappa(u,v) \stackrel{def}{=} \langle B(u,u), B(v,v)\rangle - \|B(u,v)\|^2, \tag{6}$$

where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the standard Euclidean inner product and norm, respectively. For any normal vector $\eta \in T_p^{\perp}(\mathcal{M})$, one can define a symmetric, bilinear form

$$B_\eta(u,v) \stackrel{def}{=} \langle \eta, B(u,v)\rangle, \quad u, v \in T_p(\mathcal{M}).$$

Let us denote by $L_\eta : T_p(\mathcal{M}) \to T_p(\mathcal{M})$ the linear, self-adjoint operator associated to the bilinear form $B_\eta(u,v)$, i.e., $B_\eta(u,v) = \big\langle u, L_\eta(v)\big\rangle$ for all $u, v \in T_p(\mathcal{M})$. Using an important result from [25] connecting the norm of L_η with the reach $\tau(\mathcal{M})$, we list the following consequences. A proof is presented in [21].

▶ **Proposition 15.** *Let $p \in \mathcal{M}$ be any point and $u, v \in T_p(\mathcal{M})$ unit norm. Then,*

(i) $\|B(u,v)\| \leq 1/\tau(\mathcal{M})$,

(ii) $-1/\tau(\mathcal{M})^2 \leq \kappa(\mathcal{M}) \leq 1/\tau(\mathcal{M})^2$,

(iii) $\rho(\mathcal{M}) \geq \pi\tau(\mathcal{M})/2$, *and*

(iv) $\Delta(\mathcal{M}) \geq \pi\tau(\mathcal{M})/4$.

Finally, we obtain the following important bound on the distortion of a pair of points on $\mathcal{M}$. See [21] for a proof.

▶ **Proposition 16.** *Let $1 < \xi < 2$ and $p, q \in \mathcal{M}$ be such that $\|p - q\| \leq 2\left(\frac{\xi-1}{\xi^2}\right)\tau(\mathcal{M})$. Then,*

$$d_{\mathcal{M}}(p,q) \leq \xi \|p - q\|.$$

We now assume that $S \subset \mathbb{R}^d$ is a compact subset and $\beta > 0$ a number such that the Hausdorff distance $d_H(\mathcal{M}, S) < \zeta\beta$ for some $0 < \zeta < 1/2$. There is a (possibly non-continuous and non-unique) vertex map $\phi : \mathcal{M} \longrightarrow S$ such that $\|p - \phi(p)\| < \zeta\beta$ for all $p \in \mathcal{M}$. The vertex map ϕ extends to a simplicial map:

$$\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M}) \xrightarrow{\ \phi\ } \mathcal{R}_\beta(S). \tag{7}$$

To see that ϕ is a simplicial map, take an l–simplex $\sigma_l = [p_0, p_1, \ldots, p_l]$ in $\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})$. By the construction of the Vietoris–Rips complex, we must have $d_{\mathcal{M}}(p_i, p_j) < (1 - 2\zeta)\beta$ for any $0 \le i, j \le l$. Using the triangle inequality, we get

$$\begin{aligned}\|\phi(p_i) - \phi(p_j)\| &\le \|\phi(p_i) - p_i\| + \|p_i - p_j\| + \|p_j - \phi(p_j)\| \\ &< \zeta\beta + d_{\mathcal{M}}(p_i, p_j) + \zeta\beta \\ &< \zeta\beta + (1 - 2\zeta)\beta + \zeta\beta = \beta.\end{aligned}$$

So, the image $\phi(\sigma_l) = [\phi(p_0), \phi(p_1), \ldots, \phi(p_l)]$ is a simplex of $\mathcal{R}_\beta(S)$.

The following lemma proves that the simplicial map ϕ defined in (7) induces a surjective homomorphism on the homotopy groups.

▶ **Lemma 17** (Surjectivity). *Let $\mathcal{M} \subset \mathbb{R}^d$ be a closed, connected submanifold, $S \subset \mathbb{R}^d$ a compact subset, and $\beta > 0$ a number such that*

$$\frac{1}{\zeta} d_H(\mathcal{M}, S) < \beta \le \frac{3(1 + 2\zeta)(1 - 14\zeta)}{8(1 - 2\zeta)^2} \tau(\mathcal{M})$$

for some $0 < \zeta < 1/14$. Then for any $m \ge 0$, the simplicial map $\phi : \mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M}) \to \mathcal{R}_\beta(S)$ (as defined in (7)) induces a surjective homomorphism on the m–th homotopy group.

Proof. Due to [21, Proposition A.1], the complexes $\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})$ and $\mathcal{R}_\beta(S)$ are path-connected. So, the result holds for $m = 0$.

For $m \ge 1$, let us take an abstract simplicial complex $\mathcal{K}$ such that $|\mathcal{K}|$ is a triangulation of the m–dimensional sphere $\mathbb{S}^m$. In order to show surjectivity of $|\phi|_*$, we start with a simplicial map $g : \mathcal{K} \to \mathcal{R}_\beta(S)$, and argue that there must exist a simplicial map $\widetilde{g} : \mathrm{sd}\,(\mathcal{K}) \longrightarrow \mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} \left|\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})\right| & \xrightarrow{\ |\phi|\ } & \left|\mathcal{R}_\beta(S)\right| \\ \Big\uparrow{\scriptstyle |\widetilde{g}|} & & \Big\uparrow{\scriptstyle |g|} \\ \left|\mathrm{sd}\,(\mathcal{K})\right| & \xleftarrow{\ h^{-1}\ } & |\mathcal{K}| \end{array} \tag{8}$$

where the linear homeomorphism $h : |\mathrm{sd}\,(\mathcal{K})| \longrightarrow |\mathcal{K}|$ maps each vertex of $\mathrm{sd}\,(\mathcal{K})$ to the corresponding point of $|\mathcal{K}|$.

We first note that each vertex of $\mathrm{sd}\,(\mathcal{K})$ is the barycenter, $\widehat{\sigma}_l$, of an l–simplex σ_l of $\mathcal{K}$. In order to construct the simplicial map $\widetilde{g} : \mathrm{sd}\,(\mathcal{K}) \longrightarrow \mathcal{R}_\beta(\mathcal{M})$, we define it on the vertices $\mathrm{sd}\,(\mathcal{K})$ first, and prove that the vertex map extends to a simplicial map.

Let $\sigma_l = [v_0, v_1, \ldots, v_l]$ be an l–simplex of $\mathcal{K}$. Since g is a simplicial map, we have that the image $g(\sigma_l) = [g(v_0), g(v_1), \ldots, g(v_l)]$ is a subset of S with $\mathrm{diam}_S\,(g(\sigma_l)) < \beta$. There is a corresponding subset $\sigma'_l = [p_0, p_1, \ldots, p_l] \subset \mathcal{M}$ with $\|p_j - g(v_j)\| < \zeta\beta$ for $0 \le j \le l$. Choose $\xi = \frac{4(1-2\zeta)}{3(1+2\zeta)}$. Since $0 < \zeta < 1/14$, we observe that $1 < \xi < 2$. Note from our assumption that

$$\|p_i - p_j\| < \beta \le \frac{3(1 + 2\zeta)(1 - 14\zeta)}{8(1 - 2\zeta)^2} \tau(\mathcal{M}) = 2\left(\frac{\xi - 1}{\xi^2}\right) \tau(\mathcal{M}).$$

By Proposition 16, we then have

$$\begin{aligned}
d_{\mathcal{M}}(p_i,p_j) &\leq \xi\,\|p_i-p_j\| \\
&= \frac{4(1-2\zeta)}{3(1+2\zeta)}\,\|p_i-p_j\| \\
&\leq \frac{4(1-2\zeta)}{3(1+2\zeta)}(\|p_i-g(v_i)\|+\|g(v_i)-g(v_j)\|+\|g(v_j)-p_j\|) \\
&< \frac{4(1-2\zeta)}{3(1+2\zeta)}(\zeta\beta+\|g(v_i)-g(v_j)\|+\zeta\beta) \\
&< \frac{4(1-2\zeta)}{3(1+2\zeta)}(\beta+2\zeta\beta)=\frac{4}{3}(1-2\zeta)\beta.
\end{aligned}$$

For any $0\leq j\leq l$, define $\sigma'_j:=[p_0,p_1,\ldots,p_j]$. Therefore, the diameter

$$\mathrm{diam}_{\mathcal{M}}\left(\sigma'_j\right) < \frac{4}{3}(1-2\zeta)\beta. \tag{9}$$

Moreover, due to our assumption on the upper bound on β:

$$\begin{aligned}
\mathrm{diam}_{\mathcal{M}}\left(\sigma'_j\right) &< \frac{4}{3}(1-2\zeta)\beta \\
&\leq \frac{4}{3}(1-2\zeta)\frac{3(1+2\zeta)(1-14\zeta)}{8(1-2\zeta)^2}\tau(\mathcal{M}) \qquad (10)\\
&= \frac{1}{2}\left(\frac{1-12\zeta-28\zeta^2}{1-2\zeta}\right)\tau(\mathcal{M}) \\
&< \frac{1}{2}\cdot 1\cdot\tau(\mathcal{M}),\ \text{since } \zeta>0 \\
&< \pi\tau(\mathcal{M})/4\leq\Delta(\mathcal{M}),\ \text{from Proposition 15.}
\end{aligned}$$

So, Proposition 8 implies that $\Theta(\sigma'_l)$ exists. We define

$$\widetilde{g}(\widehat{\sigma}_l)\overset{def}{=}\Theta(\sigma'_l).$$

To see that $\widetilde{g}$ extends to a simplicial map, consider a typical l–simplex, $\tau_l=[\widehat{\sigma}_0,\widehat{\sigma}_1,\ldots,\widehat{\sigma}_l]$, of $\mathrm{sd}\,(\mathcal{K})$, where $\sigma_i\prec\sigma_{i+1}$ for $0\leq i\leq l-1$ and $\sigma_i\in\mathcal{K}$. Now,

$$\begin{aligned}
\mathrm{diam}_{\mathcal{M}}\left(\widetilde{g}(\tau_l)\right) &= \mathrm{diam}_{\mathcal{M}}\left([\Theta(\sigma'_0),\Theta(\sigma'_1),\ldots,\Theta(\sigma'_l)]\right) \\
&= \max_{0\leq i<j\leq l}\{d_{\mathcal{M}}(\Theta(\sigma'_i),\Theta(\sigma'_j))\} \\
&\leq \max_{0\leq j\leq l}\left\{\frac{3}{4}\mathrm{diam}_{\mathcal{M}}\left(\sigma'_j\right)\right\},\ \text{by Proposition 9} \\
&\leq \frac{3}{4}\mathrm{diam}_{\mathcal{M}}\left(\sigma'_l\right) \\
&< \frac{3}{4}\cdot\frac{4}{3}(1-2\zeta)\beta,\ \text{from (9)} \\
&= (1-2\zeta)\beta.
\end{aligned}$$

So, $\widetilde{g}(\tau_l)$ is a simplex of $\mathcal{R}_{(1-2\zeta)\beta}(\mathcal{M})$. This implies that $\widetilde{g}$ is a simplicial map.

We invoke [21, Proposition A.2] to show that Diagram (8) commutes up to homotopy. We need to argue that the simplicial maps g and $(\phi\circ\widetilde{g})$ satisfy the conditions of the proposition:

(a) For any vertex $v\in\mathcal{K}$,

$$(\phi\circ\widetilde{g})(v)=g(v).$$

(b) For any simplex $\sigma_m = [v_0, v_1, \dots, v_m]$ of $\mathcal{K}$, we have for $0 \le j \le m$:

$$\begin{aligned}\|g(v_j) - (\phi \circ \widetilde{g})(\widehat{\sigma}_m)\| &< \|p_j - \Theta(\sigma'_m)\| + 2\zeta\beta \\ &\le d_{\mathcal{M}}(p_j, \Theta(\sigma'_m)) + 2\zeta\beta \\ &= \frac{3}{4}\mathrm{diam}_{\mathcal{M}}(\sigma'_m) + 2\zeta\beta, \text{ by Proposition 9 as } \Theta(p_j) = p_j \\ &< (1 - 2\zeta)\beta + 2\zeta\beta = \beta.\end{aligned}$$

So, $g(\sigma_m) \cup (\phi \circ \widetilde{g})(\Theta(\sigma_m))$ is a simplex of $\mathcal{R}_\beta(S)$.
Therefore, [21, Proposition A.2] implies that the diagram commutes. Since $|\mathcal{K}| = \mathbb{S}^m$ and g is arbitrary, we conclude that ϕ induces a surjective homomorphism on the m–th homotopy group. ◀

▶ **Theorem 18** (Submanifold Reconstruction under Hausdorff Distance). *Let $\mathcal{M} \subset \mathbb{R}^d$ be a closed, connected Euclidean submanifold. Let $S \subset \mathbb{R}^d$ be a compact subset and $\beta > 0$ a number such that*

$$\frac{1}{\zeta} d_H(\mathcal{M}, S) < \beta \le \frac{3(1+2\zeta)(1-14\zeta)}{8(1-2\zeta)^2}\tau(\mathcal{M})$$

for some $0 < \zeta < 1/14$. Then, $|\mathcal{R}_\beta(S)| \simeq \mathcal{M}$.

Since the technique of the proof is similarly to Theorem 12, we refer the reader to [21] for a proof.

5 Conclusion

The current work provides satisfactory answers to the quest of recovering a closed Riemannian manifold $\mathcal{M}$ from the Vietoris–Rips complexes of a compact metric space S close to it – both in the Gromov–Hausdorff and Hausdorff distance. The study sparks a number of intriguing future research directions. Although we provide a homotopy equivalent recovery of a Euclidean submanifold, the resulting complex $\mathcal{R}_\beta(S)$, being very high-dimensional, does not produce a natural embedding for the reconstruction. Consequently, our result for submanifold reconstruction does not lend itself well to recovering the geometry. Since S is a subset of $\mathbb{R}^d$, one may collapse the resulting Vietoris–Rips complex into something simpler [3]; or consider its shadow (as defined by Chambers et al. [8]) as a *geometric reconstruction* of $\mathcal{M}$. As pointed out in [8, Proposition 5.3], the shadow of a complex is notorious for being topologically unfaithful. When the Hausdorff distance between S and $\mathcal{M}$ is very small, however, we conjecture to have homotopy equivalent shadow of $\mathcal{R}_\beta(S)$, hence providing a homotopy equivalent reconstruction of $\mathcal{M}$ with an embedding in the same ambient space.

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