Polychromatic Colorings of Geometric Hypergraphs via Shallow Hitting Sets

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Abstract

A range family $\mathcal{R}$ is a family of subsets of $\mathbb{R}^d$, like all halfplanes, or all unit disks. Given a range family $\mathcal{R}$, we consider the m-uniform range capturing hypergraphs $\mathcal{H}(V, \mathcal{R}, m)$ whose vertex-sets V are finite sets of points in $\mathbb{R}^d$ with any m vertices forming a hyperedge e whenever $e = V \cap R$ for some $R \in \mathcal{R}$. Given additionally an integer $k \geq 2$, we seek to find the minimum $m = m_{\mathcal{R}}(k)$ such that every $\mathcal{H}(V, \mathcal{R}, m)$ admits a polychromatic k-coloring of its vertices, that is, where every hyperedge contains at least one point of each color. Clearly, $m_{\mathcal{R}}(k) \geq k$ and the gold standard is an upper bound $m_{\mathcal{R}}(k) = O(k)$ that is linear in k.

A t-shallow hitting set in $\mathcal{H}(V, \mathcal{R}, m)$ is a subset $S \subseteq V$ such that $1 \leq |e \cap S| \leq t$ for each hyperedge e; i.e., every hyperedge is hit at least once but at most t times by S. We show for several range families $\mathcal{R}$ the existence of t-shallow hitting sets in every $\mathcal{H}(V, \mathcal{R}, m)$ with t being a constant only depending on $\mathcal{R}$. This in particular proves that $m_{\mathcal{R}}(k) \leq tk = O(k)$ in such cases, improving previous polynomial bounds in k. Particularly, we prove this for the range families of all axis-aligned strips in $\mathbb{R}^d$, all bottomless and topless rectangles in $\mathbb{R}^2$, and for all unit-height axis-aligned rectangles in $\mathbb{R}^2$.

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1 Introduction

We investigate polychromatic colorings of geometric hypergraphs defined by a finite set of points $V \subset \mathbb{R}^d$ and a family $\mathcal{R}$ of subsets of $\mathbb{R}^d$, called a *range family*. Possible range families include for example all unit balls, all axis-aligned boxes, all halfplanes, or all translates of a fixed polygon. In this paper we prove results for the following range families:

- the family $\mathcal{R}_{\text{ST}} = \mathcal{R}^1_{\text{ST}} \cup \cdots \cup \mathcal{R}^d_{\text{ST}}$ of all axis-aligned strips in $\mathbb{R}^d$ with $\mathcal{R}^i_{\text{ST}} = \{\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid a \leq x_i \leq b\} \mid a, b \in \mathbb{R}\}$ for $i = 1, \ldots, d$,
- the family $\mathcal{R}_{\text{BL}} = \{[a, b] \times (-\infty, c] \mid a, b, c \in \mathbb{R}\}$ of all bottomless rectangles in $\mathbb{R}^2$,
- the family $\mathcal{R}_{\text{TL}} = \{[a, b] \times [c, \infty) \mid a, b, c \in \mathbb{R}\}$ of all topless rectangles in $\mathbb{R}^2$, and
- the family $\mathcal{R}_{\text{UH}} = \{[a, b] \times [c, c + 1] \mid a, b, c \in \mathbb{R}\}$ of all unit-height rectangles in $\mathbb{R}^2$.

For a fixed range family $\mathcal{R}$ and any finite point set $V \subset \mathbb{R}^d$, the corresponding *range capturing hypergraph* $H = \mathcal{H}(V, \mathcal{R})$ has vertex set $V(H) = V$, and a subset $e \subseteq V$ is a hyperedge in $E(H)$ whenever there exists a range $R \in \mathcal{R}$ with $e = V \cap R$. In this case, we say that e is *captured* by the range R. That is, we have points in $\mathbb{R}^d$ and a subset of points forms a hyperedge whenever these vertices and no other vertices are captured by a range. For example, a set e of points in $V \subset \mathbb{R}^d$ forms a hyperedge in $\mathcal{H}(V, \mathcal{R}_{\text{ST}})$ if and only if in at least one of the d coordinates, the points in e are consecutive in V. (We assume throughout that points in V lie in general position, i.e., have pairwise different coordinates.)

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For a positive integer k, a k-coloring $c\colon V \to \{1,\dots,k\}$ of the vertices of a hypergraph $H = (V,E)$ is called *proper* if each hyperedge $e \in E$ contains at least two colors, i.e., $|\{c(v) \mid v \in e\}| \geq 2$, and *polychromatic* if each hyperedge $e \in E$ contains all k colors, i.e., $|\{c(v) \mid v \in e\}| = k$. Hence, proper 2-colorings and polychromatic 2-colorings are the same concept. However, if $k \geq 3$, then every polychromatic k-coloring is also a proper k-coloring but the converse is not true in general. In fact, for polychromatic colorings we always seek to *maximize* the number of colors, as each polychromatic k-coloring, $k \geq 2$, also gives a polychromatic $(k-1)$-coloring by merging two color classes into one.

For polychromatic colorings of range capturing hypergraphs with respect to a given point set $V \subset \mathbb{R}^d$ and range family $\mathcal{R}$, we are particularly interested in the m-uniform[1] subhypergraph $\mathcal{H}(V,\mathcal{R},m)$ that consists of all hyperedges in $\mathcal{H}(V,\mathcal{R})$ of size exactly m. Instead of fixing m and then maximizing the k for which polychromatic k-colorings of $\mathcal{H}(V,\mathcal{R},m)$ exist, one usually considers the equivalent setup of fixing k and minimizing m.

▶ **Definition 1.** *For a range family $\mathcal{R}$ and integer $k > 0$, let $m = m_{\mathcal{R}}(k)$ be the smallest integer such that $\mathcal{H}(V,\mathcal{R},m)$ admits a polychromatic k-coloring for every finite set $V \subset \mathbb{R}^d$.*

Clearly, $m_{\mathcal{R}}(k) \geq k$ since every hyperedge must contain k different colors. Moreover, we have $m_{\mathcal{R}}(2) \leq m_{\mathcal{R}}(3) \leq \cdots$. But note that it is also possible that $m_{\mathcal{R}}(k) = \infty$ for some k. Namely this happens if for every positive integer m there exists a finite set of points $V \subset \mathbb{R}^d$ such that the corresponding hypergraph $\mathcal{H}(V,\mathcal{R},m)$ has no polychromatic k-coloring. In fact, throughout the over 40 years since their introduction by Pach [18, 19], we always observe the following surprising phenomenon for polychromatic k-colorings of geometric range spaces and the quantity $m_{\mathcal{R}}(k)$ as a function of k: Either we already have that $m_{\mathcal{R}}(2) = \infty$, or the best known lower bounds are of the form $m_{\mathcal{R}}(k) = \Omega(k)$ for all $k \geq 2$.

▶ **Question 2.** *Is there a geometric range family $\mathcal{R}$ with $m_{\mathcal{R}}(2) < \infty$ and $m_{\mathcal{R}}(k) = \omega(k)$?*

So, if the answer to Question 2 is 'No', then we always have either $m_{\mathcal{R}}(2) = \infty$ or $m_{\mathcal{R}}(k) = O(k)$. With this paper, we make progress on Question 2 by improving the upper bounds on $m_{\mathcal{R}}(k)$ in several further cases from superlinear to $m_{\mathcal{R}}(k) = O(k)$. We do so by proving a stronger statement, namely the existence of so-called t-shallow hitting sets with $t = O(1)$; see Sections 1.1 and 1.2 below for the formal definition and a detailed discussion.

1.1 Related work

There is a rich literature on numerous range families $\mathcal{R}$, polychromatic colorings of their range capturing hypergraphs, and upper and lower bounds on $m_{\mathcal{R}}(k)$ in terms of k [2–4, 7–11, 13–17, 20, 21, 23, 24, 29]. Let us mention just a few here, while the interested reader is invited to have a look at the slightly outdated survey article [21] and the excellent website [1].

(Some) known range families $\mathcal{R}$ with $m_{\mathcal{R}}(k) < \infty$ for all $k \geq 2$.

(1) For axis-aligned strips $\mathcal{R}_{\mathrm{ST}}$ in $\mathbb{R}^d$ it is known that $m_{\mathcal{R}_{\mathrm{ST}}}(k) = O_d(k \log k)$ [3] and for $d = 2$ it is known that $3k/2 - 1 \leq m_{\mathcal{R}_{\mathrm{ST}}}(k) \leq 2k - 1$ [3].
(2) For bottomless rectangles $\mathcal{R}_{\mathrm{BL}}$ in $\mathbb{R}^2$ it is known that $1.67k \leq m_{\mathcal{R}_{\mathrm{BL}}}(k) \leq 3k - 2$ [4].
(3) For halfplanes $\mathcal{R}$ in $\mathbb{R}^2$ it is known that $m_{\mathcal{R}}(k) = 2k - 1$ [29].
(4) For axis-aligned squares $\mathcal{R}$ in $\mathbb{R}^2$ it is known that $m_{\mathcal{R}}(k) = O(k^{8.75})$ [2].

[1] A hypergraph $H = (V,E)$ is *m-uniform* if every hyperedge $e \in E$ has size $|e| = m$. So 2-uniform hypergraphs are just graphs (without loops), while m-uniform hypergraphs are also called m-graphs.

(5) For bottomless and topless rectangles $\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}$ it is known that $m_{\mathcal{R}}(k) = O(k^{8.75})$ [10].
(6) For translates of a convex polygon $\mathcal{R}$ in $\mathbb{R}^2$ it is known that $m_{\mathcal{R}}(k) = O(k)$ [12].
(7) For homothets of a triangle $\mathcal{R}$ in $\mathbb{R}^2$ it is known that $m_{\mathcal{R}}(k) = O(k^{4.09})$ [8, 14].
(8) For translates of an octant $\mathcal{R}$ in $\mathbb{R}^3$ it is known that $m_{\mathcal{R}}(k) = O(k^{5.09})$ [8, 14].

(Some) known range families $\mathcal{R}$ with $m_{\mathcal{R}}(k) = \infty$ for all $k \geq 2$.
(9) For unit disks $\mathcal{R}$ in $\mathbb{R}^2$ it is known that $m_{\mathcal{R}}(2) = \infty$ [20].
(10) For strips $\mathcal{R}$ in any direction in $\mathbb{R}^2$ it is known that $m_{\mathcal{R}}(2) = \infty$ [21].
(11) For axis-aligned rectangles $\mathcal{R}$ in $\mathbb{R}^2$ it is known that $m_{\mathcal{R}}(2) = \infty$ [11].
(12) For bottomless rect. and horizontal strips $\mathcal{R}_{\text{BL}} \cup \mathcal{R}^2_{\text{ST}}$ we have $m_{\mathcal{R}_{\text{BL}} \cup \mathcal{R}^2_{\text{ST}}}(2) = \infty$ [10].

Crucially, let us mention again, that in each of (1)–(8) the best known lower bound on $m_{\mathcal{R}}(k)$ is linear in k, and it might be (in the light of Question 2) that in fact $m_{\mathcal{R}}(k) = O(k)$ holds.

One tool to prove for a range family $\mathcal{R}$ that $m_{\mathcal{R}}(k) = O(k)$ are shallow hitting sets. For a hypergraph $H = (V, E)$ and integer $t > 0$, a set $X \subseteq V$ of vertices is a *t-shallow hitting set* if

$$1 \leq |e \cap X| \leq t \qquad \text{for every } e \in E.$$

That is, X contains at least one vertex of each hyperedge (X is hitting) but at most t vertices of each hyperedge (X is t-shallow). Shallow hitting sets for polychromatic colorings of range capturing hypergraphs have been used implicitly in [29], while being developed as a general tool in [7, 10, 15]. Clearly, for the m-uniform hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ taking $X = V$ would be an m-shallow hitting set. But the challenge is to find t-shallow hitting sets with $t = O(1)$ being a constant[2] independent of m. If we succeed, this implies $m_{\mathcal{R}}(k) = O(k)$.

▶ **Lemma 3** (Keszegh and Pálvölgyi [15]). *If for a shrinkable range family $\mathcal{R}$ there exists a constant $t \geq 1$ such that for every $m \geq 1$ every hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ admits a t-shallow hitting set, then $m_{\mathcal{R}}(k) \leq t(k-1) + 1 = O(k)$.*

Here, a range family $\mathcal{R}$ is *shrinkable* if for every finite set of points V, every positive integer m and every hyperedge e in $\mathcal{H}(V, \mathcal{R}, m)$ there exists a hyperedge e' in $\mathcal{H}(V, \mathcal{R}, m-1)$ with $e' \subseteq e$. Intuitively, we "decrease the size" of a range $R \in \mathcal{R}$ with $R \cap V = e$ until the first point of V drops out of the range. In fact, all range families mentioned in this paper, except the translates of a convex polygon (6) and unit disks (9), are shrinkable.

Smorodinsky and Yuditsky [29] prove that every $\mathcal{H}(V, \mathcal{R}, m)$ admits 2-shallow hitting sets for $\mathcal{R}$ being all halfplanes (3), which implies $m_{\mathcal{R}}(k) \leq 2k - 1$ in this case. This is extended to so-called ABA-free hypergraphs in [15] and unions of hypergraphs in [10]. On the other hand, for the family $\mathcal{R}_{\text{BL}}$ of all bottomless rectangles (2) the bound $m_{\mathcal{R}_{\text{BL}}}(k) \leq 3k - 2$ is not proven [4] by shallow hitting sets, and in fact it was asked [10, 15] whether these exist in this case. For $\mathcal{R}$ being all translates of a fixed convex polygon (6) the proof [12] for $m_{\mathcal{R}}(k) = O(k)$ also involves shallow hitting sets, even though these are not explicitly stated as such, and this range family is not shrinkable anyways. Finally, the family $\mathcal{R}$ of all translates of an octant in $\mathbb{R}^3$ (8) is the only case for which shallow hitting sets are known *not* to exist [7], which follows from a certain dual problem for bottomless rectangles.

[2] Recall that $m = m_{\mathcal{R}}(k) \geq k$ is a growing function in k.

1.2 Our results

We consider the range families mentioned at the beginning of Section 1 of all axis-aligned strips $\mathcal{R}_{\mathrm{ST}}$ in $\mathbb{R}^d$, all bottomless $\mathcal{R}_{\mathrm{BL}}$ and all topless $\mathcal{R}_{\mathrm{TL}}$ rectangles in $\mathbb{R}^2$, as well as all unit-height rectangles $\mathcal{R}_{\mathrm{UH}}$ in $\mathbb{R}^2$. We remark that for the axis-aligned strips $\mathcal{R}_{\mathrm{ST}}$ we could assume without loss of generality that these have unit-width. In this sense, unit-height rectangles are a generalization of horizontal strips. Additionally, unit-height rectangles are a generalization of bottomless and topless rectangles by "choosing the unit very large". Thus, we can observe that $m_{\mathcal{R}_{\mathrm{UH}}}(k) \geq m_{\mathcal{R}^2_{\mathrm{ST}}}(k)$ and $m_{\mathcal{R}_{\mathrm{UH}}}(k) \geq m_{\mathcal{R}_{\mathrm{BL}} \cup \mathcal{R}_{\mathrm{TL}}}(k)$ hold for all k.

Our main results are the following:

Section 2. The family $\mathcal{R}_{\mathrm{ST}}$ of all axis-aligned strips (1) in $\mathbb{R}^d$ allows for t-shallow hitting sets for some $t = t(d) = O(d)$ (Theorem 5). This gives $m_{\mathcal{R}_{\mathrm{ST}}}(k) = O_d(k)$, improving the $O_d(k \log k)$-bound in [3].
We complement this with a lower bound construction giving $m_{\mathcal{R}_{\mathrm{ST}}}(k) \geq \Omega(k \log d)$ (Theorem 10). This greatly improves the $m_{\mathcal{R}_{\mathrm{ST}}}(k) \geq 2\left\lceil \frac{2d-1}{2d} \cdot k \right\rceil - 1$ lower bound in [3].

Section 3. The family $\mathcal{R}_{\mathrm{BL}}$ of all bottomless rectangles (2) in $\mathbb{R}^2$ allows for 10-shallow hitting sets (Theorem 12). This answers a question of Keszegh and Pálvölgyi [15], as well as Chekan and Ueckerdt [10], and provides a new proof that $m_{\mathcal{R}_{\mathrm{BL}}}(k) = O(k)$.

Section 4. The family $\mathcal{R}_{\mathrm{BL}} \cup \mathcal{R}_{\mathrm{TL}}$ of all bottomless and topless rectangles (5) in $\mathbb{R}^2$ allows for 21-shallow hitting sets (Theorem 18). This already proves that $m_{\mathcal{R}_{\mathrm{BL}} \cup \mathcal{R}_{\mathrm{TL}}}(k) = O(k)$, which we improve to $m_{\mathcal{R}_{\mathrm{BL}} \cup \mathcal{R}_{\mathrm{TL}}}(k) \leq 6k - 3$ (Theorem 16).
The family $\mathcal{R}_{\mathrm{UH}}$ of all unit-height rectangles allows for 63-shallow hitting sets, which already gives $m_{\mathcal{R}_{\mathrm{UH}}}(k) = O(k)$ but can be improved to $m_{\mathcal{R}_{\mathrm{UH}}}(k) \leq 12k - 7$ (Theorem 19).

▶ Remark. Most recently, we learnt about an unpublished manuscript of Rok, Schwartz, and Smorodinsky [28] concerning axis-aligned strips in $\mathbb{R}^d$. They prove an upper bound of $m_{\mathcal{R}_{\mathrm{ST}}}(k) = O(kd)$, which is better than the $O_d(k \log k)$-bound in [3], but worse than our $O(k \log d)$-bound in Section 2, as well as the same lower bound of $m_{\mathcal{R}_{\mathrm{ST}}}(k) \geq \Omega(k \log d)$ as in Section 2. Apparently, these results also appear in the PhD thesis of Alexandre Rok [27].

▶ Remark. Let us also mention that Keszegh and Pálvölgyi [15] define a k-coloring $c\colon V \to \{1, \ldots, k\}$ of a hypergraph $H = (V, E)$ to be *t-balanced* if for any two colors $i, j \in \{1, \ldots, k\}$ and any hyperedge $e \in E$ we have $|\{v \in e \mid c(v) = i\}| \leq t \cdot (|\{v \in e \mid c(v) = j\}| + 1)$, i.e., any two colors appear roughly equally often in each hyperedge. They show that if a (shrinkable) range family admits t-shallow hitting sets then it also allows for t-balanced k-colorings for every k. And conversely, if we have t-balanced k-colorings for every k, then we have t^2-shallow hitting sets. Thus, Theorem 12 for example gives that every range capturing hypergraph $\mathcal{H}(V, \mathcal{R}_{\mathrm{BL}}, m)$ for bottomless rectangles admits a 10-balanced k-coloring for every $k \geq 2$.

Notation. For an integer $n \geq 1$ we sometimes use $[n] = \{1, \ldots, n\}$ for the set of the first n positive integers. Also, throughout this paper a hypergraph is a tuple $H = (V, E)$ consisting of a finite set V of vertices (or points) and a finite multiset E of hyperedges, each being a subset of V. That is, hypergraphs may contain *parallel* hyperedges forming the same subset of vertices, sometimes called *multiedges* or *hyperedges of multiplicity x* for some $x \geq 2$.

2 Polychromatic Colorings for Axis-Aligned Strips

For a shorthand notation, let us define $m_d(k) = m_{\mathcal{R}_{\mathrm{ST}}}(k)$ for the range family $\mathcal{R}_{\mathrm{ST}}$ of all axis-aligned strips in $\mathbb{R}^d$, $d \geq 2$. As [3] pointed out, the problem of determining $m_d(k)$ for $\mathcal{R}_{\mathrm{ST}}$ can be seen purely combinatorial. That is, the problem of determining $m_d(k)$ is equivalent to the following problem. Given a finite set V of size n and d bijections $\pi_1, \ldots, \pi_d\colon \{1, \ldots, n\} \to V$, we have to color the set V in k colors such that for each bijection π_i, every $m_d(k)$ consecutive elements contain an element of each color. More formally, let k and d be positive integers. Then, $m_d(k)$ is the least integer such that for any finite set V of size n and any d bijections $\pi_1, \dots, \pi_d \colon \{1, \dots, n\} \to V$, there exists a coloring c of V with k colors such that

$$\forall x \in [k]\ \forall i \in [d]\ \forall a \in [n - m_d(k) + 1]\ \exists b \in [m_d(k)] \colon c(\pi_i(a + b - 1)) = x.$$

First, we list some known results for $m_d(k)$.

- For $d = 1$ it is obvious that $m_d(k) = m_1(k) = k$ for all k.
- For $d = 2$ it holds that $3k/2 - 1 \le m_d(k) = m_2(k) \le 2k - 1$ for all k [3].
- For any $d \ge 2$ it holds that $m_d(k) \le k(4 \ln k + \ln d)$ for all k [3]. Thus, if d is a constant, then $m_d(k) \le O(k \log k)$.
- In [3] it is also proven that $m_d(k) \ge 2 \cdot \lceil \frac{2d-1}{2d} \cdot k \rceil - 1$, while in [22] it is proven that for every k we have $m_d(k) \to \infty$ as $d \to \infty$.

In this section, we show the existence of $O(d)$-shallow hitting sets for axis-aligned strips in $\mathbb{R}^d$. Moreover, we show an upper bound $m_d(k) \le O(k \log d)$, improving the result from [3], and provide a lower bound of $m_d(k) \ge \Omega(k \log d)$.

2.1 Upper Bounds

Our upper bound uses a recent result about shallow hitting edge sets in regular uniform hypergraphs. For a vertex $v \in V$ in a hypergraph $H = (V, E)$, the set of incident hyperedges at v is denoted by $\mathrm{Inc}(v) = \{e \in E \mid v \in e\}$. Hypergraph H is *regular* if $|\mathrm{Inc}(v)|$, the *degree* of v, is the same for all vertices $v \in V$. For an integer $t \ge 1$, a subset $M \subseteq E$ of hyperedges is a *t-shallow hitting edge set* in $H = (V, E)$ if we have

$$1 \le |M \cap \mathrm{Inc}(v)| \le t \qquad \text{for every } v \in V.$$

That is, 1-shallow hitting edge sets are exactly perfect matchings, while t-shallow hitting edge sets for $t \ge 2$ still cover each vertex at least once, but only at most t times. It turns out, that all regular r-uniform hypergraphs admit t-shallow hitting edge sets with t only depending on the uniformity r, and not on the number of vertices or their degree. Crucially, this result even holds for r-uniform hypergraphs with multiedges, i.e., where two or more hyperedges can correspond to the same set of r vertices.

▶ **Theorem 4** (Planken and Ueckerdt [26]). *Every r-uniform regular hypergraph H (with possibly multiedges) has a $t(r)$-shallow hitting edge set with $t(r) = \mathrm{e}r(1 + o(1))$.*

Here, $\mathrm{e} = 2.71828\dots$ denotes Euler's number.

Having Theorem 4, we find shallow hitting sets for axis-aligned strips as follows.

▶ **Theorem 5.** *Let $\mathcal{R}_{ST}$ be the range family of all axis-aligned strips in $\mathbb{R}^d$ and m be a positive integer. Then, for every finite point set $V \subset \mathbb{R}^d$, the hypergraph $\mathcal{H}(V, \mathcal{R}_{ST}, m)$ admits a $t(d)$-shallow hitting set, where $t(d) = 3\mathrm{e}d(1 + o(1))$.*

Proof. Let V be a set of n points in $\mathbb{R}^d$ and let $H = (V, E) = \mathcal{H}(V, \mathcal{R}_{\mathrm{ST}}, m)$ be the corresponding m-uniform range capturing hypergraph induced by axis-aligned strips in $\mathbb{R}^d$. We shall show that H has a t-shallow hitting set, where $t = 3\mathrm{e}d(1 + o(1))$. Set $r = \lfloor m/2 \rfloor$. We want to ensure that n is a multiple of r. To this end, if $n = l \pmod r$ for some $l \neq 0$, then we add a set A of $r - l$ new points, all of whose coordinates are larger than the coordinates in V. Observe that if X' is a t-shallow hitting set in $\mathcal{H}(V \cup A, \mathcal{R}_{\mathrm{ST}}, m)$, then $X = X' \cap V$ is a t-shallow hitting set in H, since every $e \in E$ is also a hyperedge in $\mathcal{H}(V \cup A, \mathcal{R}_{\mathrm{ST}}, m)$.

Thus, we may assume that $n = |V|$ and $r = \lfloor m/2 \rfloor$ divides n. For $i = 1, \ldots, d$, let $\pi_i \colon \{1, \ldots, n\} \to V$ be the ordering of the points along the i-th coordinate axis. That is, $\pi_i(1) \in V$ is the point in V with the lowest i-coordinate, $\pi_i(n) \in V$ is the point with the highest i-coordinate, and $\pi_i(1)_i < \cdots < \pi_i(n)_i$. Then, for each hyperedge e in $\mathcal{H}(V, \mathcal{R}^i_{\mathrm{ST}}, m)$, the vertices in e are m consecutive elements in π_i. For $i = 1, \ldots, d$ and $j = 0, \ldots, n/r - 1$, we define $W_{i,j}$ and $\mathcal{W}_i$ to be

$$W_{i,j} = \{\pi_i(rj+1), \ldots, \pi_i(r(j+1))\} \quad \text{and}$$
$$\mathcal{W}_i = \{W_{i,j} \mid j = 0, \ldots, n/r - 1\} .$$

In other words, each $\mathcal{W}_i$ is a partition of the point set V into n/r parts of r points with consecutive i-coordinates each. Thus, the hypergraph $H' = (V, E')$ with $E' = \bigcup_{i=1}^d \mathcal{W}_i$ is r-uniform and d-regular. Let H^* be the dual[3] hypergraph of H'. Then, H^* is d-uniform and r-regular, with the hyperedges of H^* corresponding to the vertices of H', hence the points in V. By Theorem 4, H^* has a t'-shallow hitting edge set, where $t' = t'(d) = \mathrm{e}d(1 + o(1))$. Then, the corresponding set of vertices X of H' is a t'-shallow hitting set in H'. With $t = 3t'$, all that remains to show is that X is a $3t'$-shallow hitting set in $H = \mathcal{H}(V, \mathcal{R}_{\mathrm{ST}}, m)$.

Let e be any hyperedge in H. Since $|e| = m$, and since every hyperedge in H' has size $r = \lfloor m/2 \rfloor$, there exists a hyperedge e' in H' with $e' \subseteq e$. Since X is hitting in H', it is also hitting in H. Moreover, for every hyperedge e in H we can find three hyperedges e'_1, e'_2, e'_3 in H' with $e \subseteq e'_1 \cup e'_2 \cup e'_3$. Thus, since X is t'-shallow in H', it is $3t'$-shallow in H. ◀

▶ **Theorem 6** (Bollobás, Pritchard, Rothvoss and Scott [5]). *Every r-uniform Δ-regular hypergraph (with possibly multiedges) has a polychromatic k-edge-coloring with $k \geq \Delta/(\ln r + O(\ln \ln r))$.*

▶ **Corollary 7.** *For the range family $\mathcal{R}_{ST}$ of all axis-aligned strips in $\mathbb{R}^d$ and every integer $k \geq 2$ we have $m_{\mathcal{R}_{ST}}(k) = m_d(k) \leq 2k(\ln d + O(\ln \ln d))$.*

Proof. Let V be a set of n points in $\mathbb{R}^d$. Let $r = \lceil k(\ln d + O(\ln \ln d)) \rceil$ and $m = 2r$. We show that the m-uniform range capturing hypergraph $H = \mathcal{H}(V, \mathcal{R}_{\mathrm{ST}}, m)$ induced by axis-aligned strips in $\mathbb{R}^d$ admits a polychromatic k-coloring.

We construct the r-uniform d-regular hypergraph H' as in the proof of Theorem 5 and consider its (d-uniform and r-regular) dual hypergraph H^*. By Theorem 6, H^* admits a polychromatic k'-edge-coloring with $k' \geq r/(\ln d + O(\ln \ln d)) \geq k$, i.e., every vertex of H^* is incident to an edge of every color. Thus, its dual H' admits a polychromatic k-coloring ψ.

It remains to show that ψ is a polychromatic k-coloring of H. Let e be any hyperedge in H. Since $|e| = m$ and since every hyperedge in H' has size $r = m/2$, there exists a hyperedge e' in H' with $e' \subseteq e$. Since e' is colored polychromatically, so is e. ◀

2.2 Lower Bounds

We seek to give a lower bound on $m_d(k) = m_{\mathcal{R}_{\mathrm{ST}}}(k)$ for the range family $\mathcal{R}_{\mathrm{ST}}$ of all axis-aligned strips in $\mathbb{R}^d$. That is, for every $d, k \geq 1$ we construct a point set $V = V_{d,k}$ in $\mathbb{R}^d$ such that for some (hopefully large) m the range capturing hypergraph $\mathcal{H}(V, \mathcal{R}_{\mathrm{ST}}, m)$ admits no polychromatic k-coloring. Then it follows that $m_d(k) \geq m + 1$. As a first step towards the desired point sets, we present a construction of r-uniform r-partite[4] t-regular hypergraphs with t being relatively large in terms of r, which admit no $(t-1)$-shallow hitting edge sets.

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[3] For a hypergraph $H = (V, E)$ its *dual* is the hypergraph $H^* = (V^*, E^*)$ with vertex-set $V^* = E$ and edge-set $E^* = \{\mathrm{Inc}(v) \mid v \in V\}$. Note that H^* may have parallel hyperedges.

[4] A hypergraph $H = (V, E)$ is *r-partite* if there exists a partition $V = V_1 \dot\cup \cdots \dot\cup V_r$ such that for every $e \in E$ and every $i \in [r]$ we have $|e \cap V_i| \leq 1$. The sets $V_1, \ldots, V_r$ are then called the *parts* of H.

▶ **Theorem 8.** *Let $t \geq 2$ be an integer. There exists an r-uniform r-partite t-regular hypergraph with parts of size two that has no $(t-1)$-shallow hitting edge set, where $r = \binom{2t}{t}/2 \leq 4^t$, i.e., $t \geq \log_4(r)$.*

Proof. Let $H = (V, E)$ be the hypergraph with $V = \{1, \ldots, 2t\}$ and $E = \binom{V}{t}$, i.e., the hyperedges are all t-element subsets of V. Observe that H is t-uniform and r-regular with $r = \binom{2t}{t}/2$. Moreover H is the union of r perfect matchings, each of the form $A, B \in \binom{V}{t}$ with $B = V - A$.

First, we show that H has no $(t-1)$-shallow hitting (vertex) set. To this end let $X \subseteq V$ be any set of vertices in H. If $|X| \leq t$, then $|V - X| \geq t$ and there exists a hyperedge $e \subseteq V - X$ which is not covered by X. In this case, X is not hitting. If $|X| \geq t$, then there exists a hyperedge $e \subseteq X$. Since e has size t, the set X is not $(t-1)$-shallow.

Now consider the dual hypergraph H^* of H. Then, H^* is an r-uniform r-partite t-regular hypergraph. Two vertices v and v' in H^* (recall that v, v' are t-subsets of $\{1, \ldots, 2t\}$) are in the same part if and only if $v' = \{1, \ldots, 2t\} - v$. Since H has no $(t-1)$-shallow hitting (vertex) set, H^* has no $(t-1)$-shallow hitting edge set. ◀

In the next theorem, we seek to find lower bounds for $m_d(k)$ for axis-aligned strips in $\mathbb{R}^d$. For that, we use the constructions in Theorem 8. We reduce the problem of finding lower bounds for $m_d(k)$ to the problem of finding lower bounds of $m'_d(k)$, defined as follows. Let $m'_d(k)$ be the least integer m' such that every d-uniform d-partite m'-regular hypergraph admits a *polychromatic edge-coloring* with k colors, that is, a coloring of the hyperedges such that each vertex is incident to a hyperedge of every color. Note that in a d-uniform d-partite hypergraph every hyperedge uses exactly one vertex in each part. If such a hypergraph is additionally regular, it follows that each part has the same size. Moreover, note that $m'_{d+1}(k) \geq m'_d(k)$ since one can "extend" every d-uniform d-partite m'-regular hypergraph H to a $(d+1)$-uniform $(d+1)$-partite m'-regular hypergraph H' containing H as a subgraph.

It remains to first show that $m_d(k) \geq m'_d(k)$ and then prove a lower bound for $m'_d(k)$.

▶ **Lemma 9.** *For every d and k we have $m_d(k) \geq m'_d(k)$.*

Proof. Let $m = m_d(k)$. Then every range capturing hypergraph $H = \mathcal{H}(V, \mathcal{R}_{\mathrm{ST}}, m)$ (with $V \subset \mathbb{R}^d$ finite) admits a polychromatic k-coloring of its vertices. Let $H' = (V', E')$ be any d-uniform d-partite m-regular hypergraph with parts $V'_1, \ldots, V'_d$ of size n and $V'_i = \{v_{i,1}, \ldots, v_{i,n}\}$ for $i = 1, \ldots, d$. We deduce from H' the following finite point set $V \subset \mathbb{R}^d$, which defines the range capturing hypergraph $H = \mathcal{H}(V, \mathcal{R}_{\mathrm{ST}}, m)$. For each part V'_i of H', let $\pi_i \colon E \to \{1, \ldots, nm\}$ be a bijection that satisfies the following condition. For two hyperedges e and e' with $e \cap V'_i = \{v_{i,j}\}$ and $e' \cap V'_i = \{v_{i,j'}\}$ and $j < j'$ it holds that $\pi_i(e) < \pi_i(e')$. Now, let the point set be $V = \{(\pi_1(e), \ldots, \pi_d(e)) \mid e \in E\} \subset \mathbb{R}^d$. Note that, for every vertex $v_{i,j}$ in V'_i, its incident hyperedges $\mathrm{Inc}(v_{i,j}) \subseteq E'$ correspond to points in V that are consecutive in the i-th dimension (by the definition of π_i). Recall that H admits a polychromatic k-coloring of its vertices, i.e., each m-set of points that are consecutive in some dimension i contains points of all k colors. Then it follows that H' admits a polychromatic k-coloring of its hyperedges. ◀

Having Lemma 9, it remains to prove a lower bound on $m'_d(k)$.

▶ **Theorem 10.** $m_d(k) \geq m'_d(k) > \frac{1}{2}(\log_2 d - 1) \cdot \lfloor k/2 \rfloor$.

Proof. Let k and d be positive integers. Let t be the largest integer such that $\binom{2t}{t}/2 \leq d$. Let $d_0 = \binom{2t}{t}/2 \leq 4^t/2$ and observe that $d_0 \leq d \leq 4d_0$. Let H_0 be the d_0-uniform d_0-partite t-regular hypergraph with two vertices per part from Theorem 8. Observe that if M is any subset of hyperedges in H_0 that together contain all vertices of H_0, called a *hitting edge set*, then M has size at least $t+1$.

We construct the hypergraph H by replacing each hyperedge of H_0 by a multiedge of multiplicity $\lfloor k/2 \rfloor$. Then, H is a d_0-uniform d_0-partite $(t\lfloor k/2 \rfloor)$-regular hypergraph and each hitting edge set of H has size at least $t+1$. Observe that $|E(H)| = 2t\lfloor k/2 \rfloor \leq tk$, since each part of H has size 2.

Assume for a contradiction that H admits a polychromatic k-coloring of its hyperedges, i.e., a k-coloring of the hyperedges such that every vertex is incident to at least one hyperedge of each color. Since each color class is a hitting edge set, each color class contains at least $t+1$ hyperedges. Thus, the number of hyperedges is $|E(H)| \geq (t+1)k > tk$, a contradiction.

With $d_0 \leq d \leq 4d_0$, we conclude

$$m'_d(k) \geq m'_{d_0}(k) > t\left\lfloor \frac{k}{2} \right\rfloor \geq \frac{1}{2}\log_2(2d_0)\left\lfloor \frac{k}{2} \right\rfloor \geq \frac{1}{2}\log_2(d/2)\left\lfloor \frac{k}{2} \right\rfloor = \frac{1}{2}\left(\log_2 d - 1\right)\left\lfloor \frac{k}{2} \right\rfloor. \qquad \blacktriangleleft$$

▶ Remark. In [26] there is a more sophisticated (compared to Theorem 8) construction of r-uniform r-partite regular hypergraphs with two vertices per part that have no $(t-1)$-shallow hitting edge set with a slightly better bound for t, namely with $t = \log_2(r+1)$. Using this construction instead, an analogous proof as in Theorem 10 then gives that $m_d(k) \geq m'_d(k) > \frac{1}{2}\left((\log_2 d - 1)\cdot k - d\right)$, which is better by a factor of 2 as long as $k > d$.

3 Bottomless Rectangles

For the range family $\mathcal{R}_{\text{BL}}$ of all bottomless rectangles in $\mathbb{R}^2$ we have $m_{\mathcal{R}_{\text{BL}}}(k) = O(k)$ [4].

▶ **Theorem 11** (Asinowski et al. [4]). *For the range family $\mathcal{R}_{BL}$ of all bottomless rectangles in $\mathbb{R}^2$ we have $m_{\mathcal{R}_{BL}}(k) \leq 3k-2$.*

However, the proof in [4] does not go via shallow hitting sets, and it is also not clear how to adjust it to give shallow hitting sets. In fact, Keszegh and Pálvölgyi [15] ask whether there exists a constant t such that for every V the hypergraph $\mathcal{H}(V, \mathcal{R}_{\text{BL}}, m)$ admits a t-shallow hitting set. We answer this question in the positive.

▶ **Theorem 12.** *Let $\mathcal{R}_{BL}$ be the range family of all bottomless rectangles in $\mathbb{R}^2$ and m be a positive integer. Then for any finite point set $V \subset \mathbb{R}^2$ the hypergraph $\mathcal{H}(V, \mathcal{R}_{BL}, m)$ admits a 10-shallow hitting set $X \subseteq V$.*

Proof. Let $V \subset \mathbb{R}^2$ be any finite point set and let $V = \{p_1, \ldots, p_n\}$ with $y(p_1) < \cdots < y(p_n)$. (Recall that $y(p)$ denotes the y-coordinate of a point $p \in \mathbb{R}^2$.) Let $w = \lfloor (m+3)/4 \rfloor$ and note that $4w - 3 \leq m \leq 4w$. We can assume that $m > 10$ since for $m \leq 10$, the point set $X = V$ is a 10-shallow hitting set of $\mathcal{H}(V, \mathcal{R}_{\text{BL}}, m)$. Moreover, we can assume that $|V| \geq m \geq 4w-3$ since otherwise $\mathcal{H}(V, \mathcal{R}_{\text{BL}}, m)$ has no hyperedges.

We shall perform a sweep-line algorithm that goes through the points in order of increasing y-coordinates and builds the desired 10-shallow hitting set by selecting one by one points to be included in X, without ever revoking such decision. Such an algorithm is called *semi-online* as its choices will be independent of the points above the current sweep-line (with larger y-coordinates). During the sweep we consider the x-coordinates of the points below the sweep-line. Note that if m points have consecutive x-coordinates among those below the sweep-line, then these m points form a hyperedge in $\mathcal{H}(V, \mathcal{R}_{\mathrm{BL}}, m)$, as verified by a bottomless rectangle whose top side lies on the sweep-line. And conversely, if some m points of V form a hyperedge in $\mathcal{H}(V, \mathcal{R}_{\mathrm{BL}}, m)$, then these have consecutive x-coordinates among those below the sweep-line at the time that the sweep-line contains the top side of a corresponding bottomless rectangle.

We start the sweep-line algorithm with step $j = w$. In step j, $j \geq w$, we consider the points $V_j = \{p_1, \ldots, p_j\}$, i.e., the j points with the lowest y-coordinates. We construct a set $X_j \subseteq V_j$ of *black points* (points that are definitely in the final set X) and a set of *white points* $W_j \subseteq V_j$ (points that are definitely *not* in the final set X) such that (for $j > w$) we have $X_{j-1} \subseteq X_j$ and $W_{j-1} \subseteq W_j$ and $X_j \cap W_j = \emptyset$. We refer to the points that are neither white nor black as *uncolored points*. Additionally, we maintain a partition $\mathbb{R} = A_{j,1} \dot\cup \cdots \dot\cup A_{j,l}$ of the real line $\mathbb{R}$ into l, for some l, pairwise disjoint intervals $A_{j,i}$ with $A_{j,1} = (-\infty, a_1), A_{j,2} = [a_1, a_2), \ldots, A_{j,l} = [a_{l-1}, \infty)$ with $-\infty < a_1 < a_2 < \cdots < a_{l-1} < \infty$. We define $V_{j,i}$ to be the set of points $p \in V_j$ with x-coordinate $x(p) \in A_{j,i}$. During the sweep-line algorithm, we maintain the following invariants:

- Each $V_{j,i}$ contains exactly one black point and $w - 1$ white points, i.e., $|V_{j,i} \cap X_j| = 1$ and $|V_{j,i} \cap W_j| = w - 1$.
- Each $V_{j,i}$ has size $w \leq |V_{j,i}| \leq 2w - 1$.

We start with step $j = w$ as follows. The set of black points is $X_w = \{p_1\}$, the set of white points is $W_w = \{p_2, \ldots, p_w\}$ and $\mathbb{R} = (-\infty, \infty)$ is the partition of $\mathbb{R}$ into one set. Clearly, all conditions are satisfied.

Now, suppose that X_j, W_j, and the partition $\mathbb{R} = A_{j,1} \dot\cup \cdots \dot\cup A_{j,l}$ are given as the result of step j. In the next step $j + 1$, we consider the set $V_{j+1} = V_j \cup \{p_{j+1}\}$. Let $A_{j,i'}$ be the interval with $x(p_{j+1}) \in A_{j,i'}$. We distinguish two cases. If $|V_{j,i'}| < 2w - 1$, then we set $X_{j+1} = X_j$, $W_{j+1} = W_j$ and $A_{j+1,i} = A_{j,i}$ for all $i = 1, \ldots, l$ for the next step $j + 1$. Then, $|V_{j+1,i'}| \leq 2w - 1$ and all conditions are again satisfied. Otherwise, assume that $|V_{j,i'}| = 2w - 1$. Let $q_1, \ldots, q_{2w}$ be the points in $V_{j,i'} \cup \{p_{j+1}\}$ ordered by their x-coordinate, i.e., $x(q_1) < \cdots < x(q_{2w})$, and define $a' = x(q_{w+1})$. Then, we define the partition

$$\begin{aligned}\mathbb{R} &= A_{j+1,1} \,\dot\cup\, \cdots \,\dot\cup\, A_{j+1,l+1} \\ &= (-\infty, a_1) \,\dot\cup\, \cdots \,\dot\cup\, [a_{i'-1}, a') \,\dot\cup\, [a', a_{i'}) \,\dot\cup\, \cdots \,\dot\cup\, [a_{l-1}, \infty) \; .\end{aligned}$$

That is, we split the interval $A_{j,i'} = [a_{i'-1}, a_{i'})$ from step j into two intervals $[a_{i'-1}, a')$ and $[a', a_{i'})$. Observe that $|V_{j+1,i'}| = w = |V_{j+1,i'+1}|$. Since there is exactly one black point in $V_{j,i'}$ (i.e., $|X_j \cap V_{j,i'}| = 1$), there is exactly one black point of X_j in $V_{j+1,i'} \cup V_{j+1,i'+1}$. By symmetry, assume that this black point is contained in $V_{j+1,i'}$ and therefore, $V_{j+1,i'+1}$ has no black point in X_j. Now we color all uncolored points in $V_{j+1,i'}$ white. Then, $V_{j+1,i'}$ contains exactly one black and $w - 1$ white points. Since $V_{j,i'}$ has at most $w - 1$ white points and $V_{j+1,i'+1} \subseteq V_{j,i'} \cup \{p_{j+1}\}$, the set $V_{j+1,i'+1}$ has at most $w - 1$ white points of W_j, too. Thus, there exists an uncolored point q in $V_{j+1,i'+1}$. We color q black and all other uncolored points in $V_{j+1,i'+1}$ white. Then, $V_{j+1,i'+1}$ contains exactly one black and $w - 1$ white points. This completes step $j + 1$. Note that both invariants are again satisfied.

After step $n = |V|$, we have considered all points in V. Let $X = X_n$ be the set of black points after the last step. We show that X is a 10-shallow hitting set of $\mathcal{H}(V, \mathcal{R}_{\mathrm{BL}}, m)$.

▷ Claim 13. X is hitting in $\mathcal{H}(V, \mathcal{R}_{\mathrm{BL}}, m)$.

Proof. Let $R = [a, b] \times (-\infty, c]$ be a bottomless rectangle that contains m points of V, i.e., $|R \cap V| = m$. Let p be the topmost point in $R \cap V$ and consider the state of the sweep-line algorithm right after p is inserted, that is, $p = p_j$ for some j and step j is finished. Then, the points in V_j with x-coordinate in the interval $[a, b]$ are exactly the points in $R \cap V$. Since we have $|R \cap V| = m \geq 4w - 3$ and each $V_{j,i}$ has size at most $2w - 1$, there exists a $V_{j,i'}$ with $V_{j,i'} \subseteq R \cap V$. Since $V_{j,i'}$ contains a black point ($V_{j,i'} \cap X \neq \emptyset$), $R \cap V$ contains a black point too ($R \cap X \neq \emptyset$) and X is hitting. ◁

▷ Claim 14. $|X \cap V_{j,i}| \leq 2$ for every $V_{j,i}$.

Proof. By the invariants above it holds that $|X_j \cap V_{j,i}| = 1$ (Be aware of the difference between $X \cap V_{j,i}$ and $X_j \cap V_{j,i}$.) and $|W_j \cap V_{j,i}| = w - 1$. Moreover, observe that whenever an uncolored point $p \in V_{j,i}$ is colored black, all other uncolored points in $V_{j,i}$ are colored white. Since white points are definitely not contained in X, the claim follows. ◁

▷ Claim 15. X is 10-shallow in $\mathcal{H}(V, \mathcal{R}_{\text{BL}}, m)$.

Proof. Again, let R be a bottomless rectangle that contains m points of V, i.e., $|R \cap V| = m$, and let p be the topmost point in $R \cap V$. Consider the state of the sweep-line algorithm after p is inserted, that is, $p = p_j$ for some j and step j is finished. We have $|R \cap V| = m \leq 4w$ and each $V_{j,i}$ contains at least w points. Therefore, there exist at most five sets $V_{j,i}$ with $V_{j,i} \cap R \neq \emptyset$. By Claim 14, each $V_{j,i}$ contains at most two points of X. Therefore, $|R \cap X| \leq 5 \cdot 2 = 10$ and X is 10-shallow. ◁

By Claims 13 and 15, X is a 10-shallow hitting set in $\mathcal{H}(V, \mathcal{R}_{\text{BL}}, m)$. ◀

▶ Remark. The procedure in the proof of Theorem 12 can be modified to directly get a polychromatic coloring of $\mathcal{H}(V, \mathcal{R}_{\text{BL}}, m)$. Let $w = \lfloor (m + 3)/4 \rfloor$ be as in the proof of Theorem 12. Instead of carrying black and white sets, we carry a partial w-coloring (i.e., a w-coloring of some vertices on the sweep-line) such that in each step $j \geq 1$, every set $V_{j,i}$ of points contains every color exactly once. At the end of the algorithm, we get a partial w-coloring of all vertices. We complete this to a w-coloring by assigning colors to the uncolored vertices such that every $V_{n,i}$ contains every color at most twice. Note that every color class is a 10-shallow hitting set in $\mathcal{H}(V, \mathcal{R}_{\text{BL}}, m)$. By setting $w = k$, one can observe that this k-coloring is polychromatic in $\mathcal{H}(V, \mathcal{R}_{\text{BL}}, m)$, which gives a proof of $m_{\mathcal{R}_{\text{BL}}}(k) \leq 4k - 3$. Moreover, if e is an edge in $\mathcal{H}(V, \mathcal{R}_{\text{BL}})$, not necessarily of size m, and n_1, n_2 denote the size of two color classes in e then it holds that $n_1 \leq 4 + 2n_2 \leq 4(n_2 + 1)$. Therefore, this k-coloring is 4-balanced in $\mathcal{H}(V, \mathcal{R}_{\text{BL}})$.

Let us also remark that it was proven recently [6] that for every $m \geq 12$ there are finite point sets V in $\mathbb{R}^2$ such that $\mathcal{H}(V, \mathcal{R}_{\text{BL}}, m)$ admits no 3-shallow hitting sets, which also shows that one cannot achieve the bound $m_{\mathcal{R}_{\text{BL}}}(k) \leq 3k - 2$ by using shallow hitting sets.

4 Bottomless and Topless Rectangles

Chekan and Ueckerdt [10] showed that $m_{\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}}(k) \leq O(k^{8.75})$ for the range family $\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}$ of bottomless and topless rectangles by a reduction to the family $\mathcal{R}$ of all axis-aligned squares, and using that $m_{\mathcal{R}}(k) = O(k^{8.75})$ in this case [2]. We improve the upper bound on $m(k)$ for the case $\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}$ to $O(k)$ in the following theorem, by a simple reduction to the case $\mathcal{R}_{\text{BL}}$ of just all bottomless rectangles, and the case $\mathcal{R}_{\text{TL}}$ of just all topless rectangles. Observe that we clearly have $m_{\mathcal{R}_{\text{BL}}}(k) = m_{\mathcal{R}_{\text{TL}}}(k)$ for all k, and recall that $m_{\mathcal{R}_{\text{BL}}}(k) \leq 3k - 2$ according to [4] (see Theorem 11).

▶ **Theorem 16.** *For $\mathcal{R}_{BL} \cup \mathcal{R}_{TL}$ the range family of all bottomless and topless rectangles in $\mathbb{R}^2$, we have $m_{\mathcal{R}_{BL} \cup \mathcal{R}_{TL}}(k) \leq 2m_{\mathcal{R}_{BL}}(k) + 1 \leq 6k - 3$.*

Proof. Let $m' = m_{\mathcal{R}_{\text{BL}}}(k) = m_{\mathcal{R}_{\text{TL}}}(k)$ and $m = 2m' + 1$. Let $V = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2$ be a finite point set with $x(p_1) < \cdots < x(p_n)$. (Recall that $x(p)$ denotes the x-coordinate of a point $p \in \mathbb{R}^2$.) We partition the set V into two sets A and B. For each pair $\{p_{2i-1}, p_{2i}\}$, we put the vertex with the lower y-coordinate into set A and the point with the larger y-coordinate into set B, see Figure 1(a).

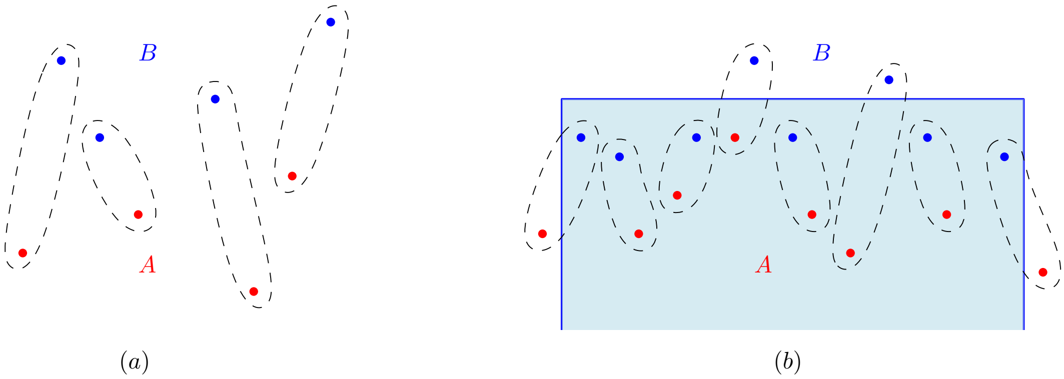

Figure 1 (a) For each pair $\{p_{2i-1}, p_{2i}\}$, the vertex with lower y-coordinate is in set A and the other in B. (b) Any bottomless rectangle with m points contains at least $\lceil (m-2)/2 \rceil$ points of A.

There are polychromatic k-colorings $c_1 \colon A \to \{1, \ldots, k\}$ of the hypergraph $\mathcal{H}(A, \mathcal{R}_{\text{BL}}, m')$ and $c_2 \colon B \to \{1, \ldots, k\}$ of the hypergraph $\mathcal{H}(B, \mathcal{R}_{\text{TL}}, m')$. As $V = A \dot{\cup} B$, this naturally defines a k-coloring $c \colon V \to \{1, \ldots, k\}$ of $\mathcal{H}(V, \mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}, m)$. To see that coloring c is polychromatic, let e be a hyperedge in $\mathcal{H}(V, \mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}, m)$ induced by a bottomless or topless rectangle $R \in \mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}$. If $R \in \mathcal{R}_{\text{BL}}$, then R contains at least $\lceil (m-2)/2 \rceil = \lceil (2m'-1)/2 \rceil = m'$ points from A, see Figure 1(b). Thus, $e \cap A$ is colored polychromatically in $\mathcal{H}(A, \mathcal{R}_{\text{BL}}, m')$ and hence e is colored polychromatically in $\mathcal{H}(V, \mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}, m)$. Symmetrically, if $R \in \mathcal{R}_{\text{TL}}$, then R contains at least m' points from B, thus $R \cap B$ contains all k colors under c_2, and thus $e = R \cap V \supseteq R \cap B$ contains all k colors under c. ◀

According to Theorem 16 we have $m_{\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}}(k) = O(k)$. However, the proof relies on the polychromatic coloring from [4] and thus does not give shallow hitting sets, which (up to the constants) is the stronger statement. In fact, even if we had a shallow hitting set X for $\mathcal{H}(A, \mathcal{R}_{\text{BL}}, m')$ and a shallow hitting set Y for $\mathcal{H}(B, \mathcal{R}_{\text{TL}}, m')$ (A and B as in the proof above), their union $X \cup Y$ would be hitting, but not necessarily shallow.

Recall that a subset X of vertices of a hypergraph $H = (V, E)$ is *hitting* if $|X \cap e| \geq 1$ for every $e \in E$, and *t-shallow* if $|X \cap e| \leq t$ for every $e \in E$. In order to prove the existence of shallow hitting sets for $\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}$, we shall first find a shallow hitting set for $\mathcal{R}_{\text{BL}}$, which is also shallow (but not necessarily hitting) for $\mathcal{R}_{\text{TL}}$. A similar approach has been done in [10].

▶ **Lemma 17.** *Let $V \subset \mathbb{R}^2$ be a finite point set and m be a positive integer. Then, there exists a set $X \subseteq V$ such that*

- *X is a 14-shallow hitting set of $\mathcal{H}(V, \mathcal{R}_{BL}, m)$ and*
- *X is a 7-shallow set of $\mathcal{H}(V, \mathcal{R}_{TL}, m)$.*

From Lemma 17, proven in the full version [25], we can quickly derive the full theorem.

▶ **Theorem 18.** *Let $\mathcal{R}_{BL} \cup \mathcal{R}_{TL}$ be the range family of all bottomless and topless rectangles in $\mathbb{R}^2$ and m be a positive integer. Then for any finite point set $V \subset \mathbb{R}^2$ the hypergraph $\mathcal{H}(V, \mathcal{R}_{BL} \cup \mathcal{R}_{TL}, m)$ admits a 21-shallow hitting set $X \subseteq V$.*

Proof. By Lemma 17, there exists a set Y that is a 14-shallow hitting set of $\mathcal{H}(V, \mathcal{R}_{\text{BL}}, m)$ and a 7-shallow set in $\mathcal{H}(V, \mathcal{R}_{\text{TL}}, m)$. Symmetrically, there exists a set Z that is a 14-shallow hitting set of $\mathcal{H}(V, \mathcal{R}_{\text{TL}}, m)$ and a 7-shallow set in $\mathcal{H}(V, \mathcal{R}_{\text{BL}}, m)$. Then, $X = Y \cup Z$ is a 21-shallow hitting set of $\mathcal{H}(V, \mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}, m)$. ◀

▶ **Theorem 19.** *Let $\mathcal{R}_{UH}$ be the range family of all unit-height axis-aligned rectangles in $\mathbb{R}^2$ and m be a positive integer. Then, for every finite point set $V \subset \mathbb{R}^2$ the hypergraph $\mathcal{H}(V, \mathcal{R}_{UH}, m)$ admits a 63-shallow hitting set $X \subseteq V$. Moreover, $m_{\mathcal{R}_{UH}}(k) \leq 2m_{\mathcal{R}_{BL} \cup \mathcal{R}_{TL}}(k) - 1 \leq 12k - 7$ for the range family $\mathcal{R}_{UH}$.*

Proof. Let m be a positive integer and $V \subset \mathbb{R}^2$ be a finite point set. Define $m' = \lceil m/2 \rceil$. For every integer $a \in \mathbb{Z}$, let $H_a = \mathcal{H}(V_a, \mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}, m')$ be the range capturing hypergraph induced by the range family of all bottomless and topless rectangles, where V_a is the set of all points p in V with $a \leq y(p) < a+1$. By Theorem 18, every H_a admits a 21-shallow hitting set X_a. Then, $X = \bigcup_{a \in \mathbb{Z}} X_a$ is a 63-shallow hitting set in $\mathcal{H}(V, \mathcal{R}_{\text{UH}}, m)$, which can be seen as follows. Every unit-height rectangle induces a topless rectangle R_t in H_a and a bottomless rectangle R_b in H_{a+1} (for some a). Then, at least one of R_t and R_b contains at least $\lceil m/2 \rceil = m'$ points of V, without loss of generality R_t. Therefore, R_t contains a point of X_a and hence, X is hitting. Since $m \leq 2m'$, the topless rectangle R_t can be covered with at most two topless rectangles of size m' of H_a, and R_b can be covered with at most one bottomless rectangle of size m' of H_{a+1}. As each of these three rectangles contains at most 21 points of X, we conclude that X is t-shallow for $t = 3 \cdot 21 = 63$.

Using the same argument, it is not difficult to see that $m_{\mathcal{R}_{\text{UH}}}(k) \leq 2m_{\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}}(k) - 1$. Let $m = 2m_{\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}}(k) - 1$ and let $H = \mathcal{H}(V, \mathcal{R}_{\text{UH}}, m)$ be the range capturing hypergraph induced by all unit-height rectangles. Let $m' = \lceil m/2 \rceil = m_{\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}}(k)$. For every $a \in \mathbb{Z}$, color each $H_a = \mathcal{H}(V_a, \mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}, m')$ polychromatically with k colors with respect to bottomless and topless rectangles $\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}$. This polychromatic coloring exists by Theorem 16 and since $m' = m_{\mathcal{R}_{\text{BL}} \cup \mathcal{R}_{\text{TL}}}(k)$. Then, every unit-height rectangle R induces a topless rectangle R_t in H_a of size at least m' or a bottomless rectangle R_b in H_{a+1} of size at

Table 1 Shallow hitting sets and polychromatic colorings for range capturing hypergraphs.

	range family $\mathcal{R}$	t-shallow hitting sets exist	$m_{\mathcal{R}}(k)$
(1)	axis-aligned strips in $\mathbb{R}^d$	**Yes** for $t \geq 3ed(1 + o(1))$ (Theorem 5)	$O_d(k)$ (Corollary 7)
(2)	bottomless rectangles in $\mathbb{R}^2$	**Yes** for $t \geq 10$ (Theorem 12)	$\leq 3k - 2$ [4]
(3)	half-planes in $\mathbb{R}^2$	**Yes** for $t \geq 2$ [29]	$\leq 2k - 1$ [29]
(4)	axis-aligned squares in $\mathbb{R}^2$	**Open**	$O(k^{8.75})$ [2]
(5)	bottomless and topless rectangles in $\mathbb{R}^2$	**Yes** for $t \geq 21$ (Theorem 18)	$\leq 6k - 3$ (Theorem 16)
(6)	translates of a convex polygon in $\mathbb{R}^2$	**Open**	$O(k)$ [12]
(7)	homothets of a triangle in $\mathbb{R}^2$	**Open**	$O(k^{4.09})$ [14]
(8)	translates of octants in $\mathbb{R}^3$	**No** [7]	$O(k^{5.09})$ [14]

least m' (for some $a \in \mathbb{Z}$). Since R_t (respectively R_b) contains points of all colors, so does R. Therefore, each unit-height rectangle with m points contains points of all colors and we have found a polychromatic k-coloring of $\mathcal{H}(V, \mathcal{R}_{\mathrm{UH}}, m)$. ◀

5 Conclusions

In this paper, we extended the list of range families $\mathcal{R}$ for which the corresponding uniform range capturing hypergraphs admit shallow hitting sets. This in particular implies that $m_{\mathcal{R}}(k) = O(k)$ for that family $\mathcal{R}$, while $m_{\mathcal{R}}(k) \geq k$ always holds. In view of Question 2, it would be interesting to investigate further range families $\mathcal{R}$ for which $m_{\mathcal{R}}(k) < \infty$ is known, as to whether they admit shallow hitting sets. The current state of the art (for a selection of range families) is summarized in Table 1.

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