Polychromatic Colorings of Geometric Hypergraphs via Shallow Hitting Sets

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Abstract
A range family $\mathcal{R}$ is a family of subsets of $\mathbb{R}^d$, like all halfplanes, or all unit disks. Given a range family $\mathcal{R}$, we consider the $m$-uniform range capturing hypergraphs $\mathcal{H}(V, \mathcal{R}, m)$ whose vertex-sets $V$ are finite sets of points in $\mathbb{R}^d$ with any $m$ vertices forming a hyperedge $e$ whenever $e = V \cap R$ for some $R \in \mathcal{R}$. Given additionally an integer $k \geq 2$, we seek to find the minimum $m = m_R(k)$ such that every $\mathcal{H}(V, \mathcal{R}, m)$ admits a polychromatic $k$-coloring of its vertices, that is, where every hyperedge contains at least one point of each color. Clearly, $m_R(k) \geq k$ and the gold standard is an upper bound $m_R(k) = O(k)$ that is linear in $k$.

A $t$-shallow hitting set in $\mathcal{H}(V, \mathcal{R}, m)$ is a subset $S \subseteq V$ such that $1 \leq |e \cap S| \leq t$ for each hyperedge $e$; i.e., every hyperedge is hit at least once but at most $t$ times by $S$. We show for several range families $\mathcal{R}$ the existence of $t$-shallow hitting sets in every $\mathcal{H}(V, \mathcal{R}, m)$ with $t$ being a constant only depending on $\mathcal{R}$. This in particular proves that $m_R(k) \leq tk = O(k)$ in such cases, improving previous polynomial bounds in $k$. Particularly, we prove this for the range families of all axis-aligned strips in $\mathbb{R}^d$, all bottomless and topless rectangles in $\mathbb{R}^2$, and for all unit-height axis-aligned rectangles in $\mathbb{R}^2$.

1 Introduction
We investigate polychromatic colorings of geometric hypergraphs defined by a finite set of points $V \subset \mathbb{R}^d$ and a family $\mathcal{R}$ of subsets of $\mathbb{R}^d$, called a range family. Possible range families include for example all unit balls, all axis-aligned boxes, all halfplanes, or all translates of a fixed polygon. In this paper we prove results for the following range families:

- the family $\mathcal{R}_{ST} = \mathcal{R}_{ST}^1 \cup \cdots \cup \mathcal{R}_{ST}^d$ of all axis-aligned strips in $\mathbb{R}^d$ with $\mathcal{R}_{ST}^i = \{(x_i, \ldots, x_d) \in \mathbb{R}^d \mid a \leq x_i \leq b \} \mid a, b \in \mathbb{R}$ for $i = 1, \ldots, d$,
- the family $\mathcal{R}_{BL} = \{(a, b) \times (-\infty, c) \mid a, b, c \in \mathbb{R}\}$ of all bottomless rectangles in $\mathbb{R}^2$,
- the family $\mathcal{R}_{TL} = \{(a, b) \times [c, \infty) \mid a, b, c \in \mathbb{R}\}$ of all topless rectangles in $\mathbb{R}^2$, and
- the family $\mathcal{R}_{UH} = \{(a, b) \times [c, c+1] \mid a, b, c \in \mathbb{R}\}$ of all unit-height rectangles in $\mathbb{R}^2$.

For a fixed range family $\mathcal{R}$ and any finite point set $V \subset \mathbb{R}^d$, the corresponding range capturing hypergraph $H = \mathcal{H}(V, \mathcal{R})$ has vertex set $V(H) = V$, and a subset $e \subseteq V$ is a hyperedge in $E(H)$ whenever there exists a range $R \in \mathcal{R}$ with $e = V \cap R$. In this case, we say that $e$ is captured by the range $R$. That is, we have points in $\mathbb{R}^d$ and a subset of points forms a hyperedge whenever these vertices and no other vertices are captured by a range. For example, a set $e$ of points in $V \subset \mathbb{R}^d$ forms a hyperedge in $\mathcal{H}(V, \mathcal{R}_{ST})$ if and only if in at least one of the $d$ coordinates, the points in $e$ are consecutive in $V$. (We assume throughout that points in $V$ lie in general position, i.e., have pairwise different coordinates.)
Polychromatic Colorings of Geometric Hypergraphs via Shallow Hitting Sets

For a positive integer $k$, a $k$-coloring $c : V \to \{1, \ldots, k\}$ of the vertices of a hypergraph $H = (V, E)$ is called proper if each hyperedge $e \in E$ contains at least two colors, i.e., $|\{c(v) \mid v \in e\}| \geq 2$, and polychromatic if each hyperedge $e \in E$ contains all $k$ colors, i.e., $|\{c(v) \mid v \in e\}| = k$. Hence, proper 2-colorings and polychromatic 2-colorings are the same concept. However, if $k \geq 3$, then every polychromatic $k$-coloring is also a proper $k$-coloring but the converse is not true in general. In fact, for polychromatic colorings we always seek to maximize the number of colors, as each polychromatic $k$-coloring, $k \geq 2$, also gives a polychromatic $(k-1)$-coloring by merging two color classes into one.

For polychromatic colorings of range capturing hypergraphs with respect to a given point set $V \subset \mathbb{R}^d$ and range family $\mathcal{R}$, we are particularly interested in the $m$-uniform subhypergraph $\mathcal{H}(V, \mathcal{R}, m)$ that consists of all hyperedges in $\mathcal{H}(V, \mathcal{R})$ of size exactly $m$. Instead of fixing $m$ and then maximizing the $k$ for which polychromatic $k$-colorings of $\mathcal{H}(V, \mathcal{R}, m)$ exist, one usually considers the equivalent setup of fixing $k$ and minimizing $m$.

**Definition 1.** For a range family $\mathcal{R}$ and integer $k > 0$, let $m = m_{\mathcal{R}}(k)$ be the smallest integer such that $\mathcal{H}(V, \mathcal{R}, m)$ admits a polychromatic $k$-coloring for every finite set $V \subset \mathbb{R}^d$.

Clearly, $m_{\mathcal{R}}(k) \geq k$ since every hyperedge must contain $k$ different colors. Moreover, we have $m_{\mathcal{R}}(2) \leq m_{\mathcal{R}}(3) \leq \cdots$. But note that it is also possible that $m_{\mathcal{R}}(k) = \infty$ for some $k$. Namely this happens if for every positive integer $m$ there exists a finite set of points $V \subset \mathbb{R}^d$ such that the corresponding hypergraph $\mathcal{H}(V, \mathcal{R}, m)$ has no polychromatic $k$-coloring. In fact, throughout the over 40 years since their introduction by Pach [18, 19], we always observe the following surprising phenomenon for polychromatic $k$-colorings of geometric range spaces and the quantity $m_{\mathcal{R}}(k)$ as a function of $k$: Either we already have that $m_{\mathcal{R}}(2) = \infty$, or the best known lower bounds are of the form $m_{\mathcal{R}}(k) = \Omega(k)$ for all $k \geq 2$.

**Question 2.** Is there a geometric range family $\mathcal{R}$ with $m_{\mathcal{R}}(2) < \infty$ and $m_{\mathcal{R}}(k) = \omega(k)$?

So, if the answer to Question 2 is ‘No’, then we always have either $m_{\mathcal{R}}(2) = \infty$ or $m_{\mathcal{R}}(k) = O(k)$. With this paper, we make progress on Question 2 by improving the upper bounds on $m_{\mathcal{R}}(k)$ in several further cases from superlinear to $m_{\mathcal{R}}(k) = O(k)$. We do so by proving a stronger statement, namely the existence of so-called $t$-shallow hitting sets with $t = O(1)$; see Sections 1.1 and 1.2 below for the formal definition and a detailed discussion.

### 1.1 Related work

There is a rich literature on numerous range families $\mathcal{R}$, polychromatic colorings of their range capturing hypergraphs, and upper and lower bounds on $m_{\mathcal{R}}(k)$ in terms of $k$ [2–4, 7–11, 13–17, 20, 21, 23, 24, 29]. Let us mention just a few here, while the interested reader is invited to have a look at the slightly outdated survey article [21] and the excellent website [1].

**(Some) known range families $\mathcal{R}$ with $m_{\mathcal{R}}(k) < \infty$ for all $k \geq 2$.**

1. For axis-aligned strips $\mathcal{R}_{ST}$ in $\mathbb{R}^d$ it is known that $m_{\mathcal{R}_{ST}}(k) = O_d(k \log k)$ [3] and for $d = 2$ it is known that $3k/2 - 1 \leq m_{\mathcal{R}_{ST}}(k) \leq 2k - 1$ [3].
2. For bottomless rectangles $\mathcal{R}_{BL}$ in $\mathbb{R}^2$ it is known that $1.67k \leq m_{\mathcal{R}_{BL}}(k) \leq 3k - 2$ [4].
3. For halfplanes $\mathcal{R}$ in $\mathbb{R}^2$ it is known that $m_{\mathcal{R}}(k) = 2k - 1$ [29].
4. For axis-aligned squares $\mathcal{R}$ in $\mathbb{R}^2$ it is known that $m_{\mathcal{R}}(k) = O(k^{8.75})$ [2].

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1 A hypergraph $H = (V, E)$ is $m$-uniform if every hyperedge $e \in E$ has size $|e| = m$. So 2-uniform hypergraphs are just graphs (without loops), while $m$-uniform hypergraphs are also called $m$-graphs.
(5) For bottomless and topless rectangles $R_{BL} \cup R_{TL}$ it is known that $m_R(k) = O(k^{0.75})$ [10].
(6) For translates of a convex polygon $R$ in $\mathbb{R}^2$ it is known that $m_R(k) = O(k)$ [12].
(7) For homothets of a triangle $R$ in $\mathbb{R}^2$ it is known that $m_R(k) = O(k^{0.09})$ [8, 14].
(8) For translates of an octant $R$ in $\mathbb{R}^3$ it is known that $m_R(k) = O(k^{0.99})$ [8, 14].

\textbf{(Some) known range families $R$ with $m_R(k) = \infty$ for all $k \geq 2$.}

(9) For unit disks $R$ in $\mathbb{R}^2$ it is known that $m_R(2) = \infty$ [20].
(10) For strips $R$ in any direction in $\mathbb{R}^2$ it is known that $m_R(2) = \infty$ [21].
(11) For axis-aligned rectangles $R$ in $\mathbb{R}^2$ it is known that $m_R(2) = \infty$ [11].
(12) For bottomless rect. and horizontal strips $R_{BL} \cup R_{ST}$ we have $m_{R_{BL} \cup R_{ST}}(2) = \infty$ [10].

Crucially, let us mention again, that in each of (1)–(8) the best known lower bound on $m_R(k)$ is linear in $k$, and it might be (in the light of Question 2) that in fact $m_R(k) = O(k)$ holds.

One tool to prove for a range family $R$ that $m_R(k) = O(k)$ are shallow hitting sets. For a hypergraph $H = (V,E)$ and integer $t > 0$, a set $X \subseteq V$ of vertices is a $t$-shallow hitting set if

$$1 \leq |e \cap X| \leq t \quad \text{for every } e \in E.$$ 

That is, $X$ contains at least one vertex of each hyperedge ($X$ is hitting) but at most $t$ vertices of each hyperedge ($X$ is $t$-shallow). Shallow hitting sets for polychromatic colorings of range capturing hypergraphs have been used implicitly in [29], while being developed as a general tool in [7, 10, 15]. Clearly, for the $m$-uniform hypergraph $H(V,R,m)$ taking $X = V$ would be an $m$-shallow hitting set. But the challenge is to find $t$-shallow hitting sets with $t = O(1)$ being a constant independent of $m$. If we succeed, this implies $m_R(k) = O(k)$.

\textbf{Lemma 3 (Keszegh and Pálvölgyi [15]).} \textit{If for a shrinkable range family $R$ there exists a constant $t \geq 1$ such that for every $m \geq 1$ every hypergraph $H(V,R,m)$ admits a $t$-shallow hitting set, then $m_R(k) \leq t(k - 1) + 1 = O(k)$.}

Here, a range family $R$ is \textit{shrinkable} if for every finite set of points $V$, every positive integer $m$ and every hyperedge $e$ in $H(V,R,m)$ there exists a hyperedge $e'$ in $H(V,R,m-1)$ with $e' \subseteq e$. Intuitively, we “decrease the size” of a range $R \in R$ with $R \cap V = e$ until the first point of $V$ drops out of the range. In fact, all range families mentioned in this paper, except the translates of a convex polygon (6) and unit disks (9), are shrinkable.

Smorodinsky and Yuditsky [29] prove that every $H(V,R,m)$ admits 2-shallow hitting sets for $R$ being all halfplanes (3), which implies $m_R(k) \leq 2k - 1$ in this case. This is extended to so-called ABA-free hypergraphs in [15] and unions of hypergraphs in [10]. On the other hand, for the family $R_{BL}$ of all bottomless rectangles (2) the bound $m_{R_{BL}}(k) \leq 3k - 2$ is not proven [4] by shallow hitting sets, and in fact it was asked [10, 15] whether these exist in this case. For $R$ being all translates of a fixed convex polygon (6) the proof [12] for $m_R(k) = O(k)$ also involves shallow hitting sets, even though these are not explicitly stated as such, and this range family is not shrinkable anyways. Finally, the family $R$ of all translates of an octant in $\mathbb{R}^3$ (8) is the only case for which shallow hitting sets are known not to exist [7], which follows from a certain dual problem for bottomless rectangles.

\footnote{Recall that $m = m_R(k) \geq k$ is a growing function in $k$.}
1.2 Our results

We consider the range families mentioned at the beginning of Section 1 of all axis-aligned strips $\mathcal{R}_{ST}$ in $\mathbb{R}^d$, all bottomless $\mathcal{R}_{BL}$ and all topless $\mathcal{R}_{TL}$ rectangles in $\mathbb{R}^2$, as well as all unit-height rectangles $\mathcal{R}_{UH}$ in $\mathbb{R}^2$. We remark that for the axis-aligned strips $\mathcal{R}_{ST}$ we could assume without loss of generality that these have unit-width. In this sense, unit-height rectangles are a generalization of horizontal strips. Additionally, unit-height rectangles are a generalization of bottomless and topless rectangles by “choosing the unit very large”. Thus, we can observe that $m_{\mathcal{R}_{UH}}(k) \geq m_{\mathcal{R}_{ST}}^2(k)$ and $m_{\mathcal{R}_{UH}}(k) \geq m_{\mathcal{R}_{BL} \cup \mathcal{R}_{TL}}(k)$ hold for all $k$.

Our main results are the following:

Section 2. The family $\mathcal{R}_{ST}$ of all axis-aligned strips (1) in $\mathbb{R}^d$ allows for $t$-shallow hitting sets for some $t = t(d) = O(d)$ (Theorem 5). This gives $m_{\mathcal{R}_{ST}}(k) = O_d(k)$, improving the $O_d(k \log k)$-bound in [3].

We complement this with a lower bound construction giving $m_{\mathcal{R}_{ST}}(k) \geq \Omega(k \log d)$ (Theorem 10). This greatly improves the $m_{\mathcal{R}_{ST}}(k) \geq 2\left\lfloor \frac{2d-1}{2d} \right\rfloor k$ lower bound in [3].

Section 3. The family $\mathcal{R}_{BL}$ of all bottomless rectangles (2) in $\mathbb{R}^2$ allows for 10-shallow hitting sets (Theorem 12). This answers a question of Keszegh and Pálvölgyi [15], as well as Chekan and Ueckerdt [10], and provides a new proof that $m_{\mathcal{R}_{BL}}(k) = O(k)$.

The family $\mathcal{R}_{UH}$ of all unit-height rectangles allows for 63-shallow hitting sets, which already gives $m_{\mathcal{R}_{UH}}(k) = O(k)$ but can be improved to $m_{\mathcal{R}_{UH}}(k) \leq 12k - 7$ (Theorem 19).

Remark. Most recently, we learnt about an unpublished manuscript of Rok, Schwartz, and Smorodinsky [28] concerning axis-aligned strips in $\mathbb{R}^d$. They prove an upper bound of $m_{\mathcal{R}_{ST}}(k) = O(k d)$, which is better than the $O_d(k \log k)$-bound in [3], but worse than our $O(k \log d)$-bound in Section 2, as well as the same lower bound of $m_{\mathcal{R}_{ST}}(k) \geq \Omega(k \log d)$ as in Section 2. Apparently, these results also appear in the PhD thesis of Alexandre Rok [27].

Remark. Let us also mention that Keszegh and Pálvölgyi [15] define a $k$-coloring $c: V \rightarrow \{1, \ldots, k\}$ of a hypergraph $H = (V, E)$ to be $t$-balanced if for any two colors $i, j \in \{1, \ldots, k\}$ and any hyperedge $e \in E$ we have $|\{v \in e \mid c(v) = i\}| \leq t \cdot (|\{v \in e \mid c(v) = j\}| + 1)$, i.e., any two colors appear roughly equally often in each hyperedge. They show that if a (shrinkable) range family admits $t$-shallow hitting sets then it also allows for $t$-balanced $k$-colorings for every $k$. And conversely, if we have $t$-balanced $k$-colorings for every $k$, then we have $t^2$-shallow hitting sets. Thus, Theorem 12 for example gives that every range capturing hypergraph $\mathcal{H}(V, \mathcal{R}_{BL}, m)$ for bottomless rectangles admits a $10$-balanced $k$-coloring for every $k \geq 2$.

Notation. For an integer $n \geq 1$ we sometimes use $[n] = \{1, \ldots, n\}$ for the set of the first $n$ positive integers. Also, throughout this paper a hypergraph is a tuple $H = (V, E)$ consisting of a finite set $V$ of vertices (or points) and a finite multiset $E$ of hyperedges, each being a subset of $V$. That is, hypergraphs may contain parallel hyperedges forming the same subset of vertices, sometimes called multiedges or hyperedges of multiplicity $x$ for some $x \geq 2$.

2 Polychromatic Colorings for Axis-Aligned Strips

For a shorthand notation, let us define $m_d(k) = m_{\mathcal{R}_{ST}}(k)$ for the range family $\mathcal{R}_{ST}$ of all axis-aligned strips in $\mathbb{R}^d$, $d \geq 2$. As [3] pointed out, the problem of determining $m_d(k)$ for $\mathcal{R}_{ST}$ can be seen purely combinatorial. That is, the problem of determining $m_d(k)$ is equivalent to the following problem. Given a finite set $V$ of size $n$ and $d$ bijections $\pi_1, \ldots, \pi_d: \{1, \ldots, n\} \rightarrow V$, we can observe that $m_{\mathcal{R}_{UH}}(k) \geq m_{\mathcal{R}_{ST}}^2(k)$ and $m_{\mathcal{R}_{UH}}(k) \geq m_{\mathcal{R}_{BL} \cup \mathcal{R}_{TL}}(k)$ hold for all $k$.

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we have to color the set $V$ in $k$ colors such that for each bijection $\pi_i$, every $m_d(k)$ consecutive elements contain an element of each color. More formally, let $k$ and $d$ be positive integers. Then, $m_d(k)$ is the least integer such that for any finite set $V$ of size $n$ and any $d$ bijections $\pi_1, \ldots, \pi_d: \{1, \ldots, n\} \to V$, there exists a coloring $c$ of $V$ with $k$ colors such that

$$\forall x \in [k] \forall i \in [d] \forall a \in [n - m_d(k) + 1] \exists b \in [m_d(k)]: c(\pi_i(a + b - 1)) = x.$$ 

First, we list some known results for $m_d(k)$.

- For $d = 1$ it is obvious that $m_d(k) = m_1(k) = k$ for all $k$.
- For $d = 2$ it holds that $3k/2 - 1 \leq m_d(k) = m_2(k) \leq 2k - 1$ for all $k$ [3].
- For any $d \geq 2$ it holds that $m_d(k) \leq k(4 \ln k + \ln d)$ for all $k$ [3]. Thus, if $d$ is a constant, then $m_d(k) \leq O(k \log k)$.
- In [3] it is also proven that $m_d(k) \geq 2 \cdot \left\lceil \frac{2d-1}{2d} \cdot k \right\rceil - 1$, while in [22] it is proven that for every $k$ we have $m_d(k) \to \infty$ as $d \to \infty$.

In this section, we show the existence of $O(d)$-shallow hitting sets for axis-aligned strips in $\mathbb{R}^d$. Moreover, we show an upper bound $m_d(k) \leq O(k \log d)$, improving the result from [3], and provide a lower bound of $m_d(k) \geq \Omega(k \log d)$.

### 2.1 Upper Bounds

Our upper bound uses a recent result about shallow hitting edge sets in regular uniform hypergraphs. For a vertex $v \in V$ in a hypergraph $H = (V, E)$, the set of incident hyperedges at $v$ is denoted by $\text{Inc}(v) = \{e \in E \mid v \in e\}$. Hypergraph $H$ is regular if $|\text{Inc}(v)|$, the degree of $v$, is the same for all vertices $v \in V$. For an integer $t \geq 1$, a subset $M \subseteq E$ of hyperedges is a $t$-shallow hitting edge set in $H = (V, E)$ if we have

$$1 \leq |M \cap \text{Inc}(v)| \leq t \quad \text{for every } v \in V.$$ 

That is, 1-shallow hitting edge sets are exactly perfect matchings, while $t$-shallow hitting edge sets for $t \geq 2$ still cover each vertex at least once, but only at most $t$ times. It turns out, that all regular $r$-uniform hypergraphs admit $t$-shallow hitting edge sets with $t$ only depending on the uniformity $r$, and not on the number of vertices or their degree. Crucially, this result even holds for $r$-uniform hypergraphs with multiedges, i.e., where two or more hyperedges can correspond to the same set of vertices.

**Theorem 4 (Planken and Ueckerdt [26]).** Every $r$-uniform regular hypergraph $H$ (with possibly multiedges) has a $(t(r))$-shallow hitting edge set with $t(r) = cr(1 + o(1))$.

Here, $c = 2.71828 \ldots$ denotes Euler’s number.

Having Theorem 4, we find shallow hitting sets for axis-aligned strips as follows.

**Theorem 5.** Let $\mathcal{R}_{ST}$ be the range family of all axis-aligned strips in $\mathbb{R}^d$ and $m$ be a positive integer. Then, for every finite point set $V \subset \mathbb{R}^d$, the hypergraph $\mathcal{H}(V, \mathcal{R}_{ST}, m)$ admits a $t(d)$-shallow hitting set, where $t(d) = 3ed(1 + o(1))$.

**Proof.** Let $V$ be a set of $n$ points in $\mathbb{R}^d$ and let $H = (V, E) = \mathcal{H}(V, \mathcal{R}_{ST}, m)$ be the corresponding $m$-uniform range capturing hypergraph induced by axis-aligned strips in $\mathbb{R}^d$. We shall show that $H$ has a $t$-shallow hitting set, where $t = 3ed(1 + o(1))$. Set $r = \lfloor m/2 \rfloor$. We want to ensure that $n$ is a multiple of $r$. To this end, if $n \not\equiv l \pmod r$ for some $l \neq 0$, then we add a set $A$ of $r - l$ new points, all of whose coordinates are larger than the coordinates in $V$. Observe that if $X'$ is a $t$-shallow hitting set in $\mathcal{H}(V \cup A, \mathcal{R}_{ST}, m)$, then $X = X' \cap V$ is a $t$-shallow hitting set in $H$, since every $e \in E$ is also a hyperedge in $\mathcal{H}(V \cup A, \mathcal{R}_{ST}, m)$.
Thus, we may assume that $n = |V|$ and $r = \lfloor m/2 \rfloor$ divides $n$. For $i = 1, \ldots, d$, let $\pi_i : \{1, \ldots, n\} \to V$ be the ordering of the points along the $i$-th coordinate axis. That is, $\pi_i(1) \in V$ is the point in $V$ with the lowest $i$-coordinate, $\pi_i(n) \in V$ is the point with the highest $i$-coordinate, and $\pi_i(1) < \cdots < \pi_i(n)$. Then, for each hyperedge $e$ in $H(V, \mathcal{R}_\text{ST}, m)$, the vertices in $e$ are $m$ consecutive elements in $\pi_i$. For $i = 1, \ldots, d$ and $j = 0, \ldots, n/r - 1$, we define $W_{i,j}$ and $W_i$ to be

$$W_{i,j} = \{\pi_i(rj + 1), \ldots, \pi_i(rj + 1)\} \quad \text{and} \quad W_i = \{W_{i,j} \mid j = 0, \ldots, n/r - 1\}.$$

In other words, each $W_i$ is a partition of the point set $V$ into $n/r$ parts of $r$ points with consecutive $i$-coordinates each. Thus, the hypergraph $H' = (V, E')$ with $E' = \bigcup_{i=1}^d W_i$ is $r$-uniform and $d$-regular. Let $H^*$ be the dual\(^3\) hypergraph of $H'$. Then, $H^*$ is $d$-uniform and $r$-regular, with the hyperedges of $H^*$ corresponding to the vertices of $H'$, hence the points in $V$. By Theorem 4, $H^*$ has a $t'$-shallow hitting edge set, where $t' = t'(d) = ed(1 + o(1))$. Then, the corresponding set of vertices $X$ of $H'$ is a $t'$-shallow hitting set in $H'$. With $t = 3t'$, all that remains to show is that $X$ is a $3t'$-shallow hitting set in $H = H(V, \mathcal{R}_\text{ST}, m)$.

Let $e$ be any hyperedge in $H$. Since $|e| = m$, and since every hyperedge in $H'$ has size $r = \lfloor m/2 \rfloor$, there exists a hyperedge $e'$ in $H'$ with $e' \subseteq e$. Since $X$ is hitting in $H'$, it is also hitting in $H$. Moreover, for every hyperedge $e$ in $H$ we can find three hyperedges $e_1', e_2', e_3'$ in $H'$ with $e \subseteq e_1' \cup e_2' \cup e_3'$. Thus, since $X$ is $t'$-shallow in $H'$, it is $3t'$-shallow in $H$.

\[\text{Theorem 6 (Bollobás, Pritchard, Rothvoss and Scott \cite{5}).} \quad \text{Every } r\text{-uniform } \Delta\text{-regular hypergraph (with possibly multiedges) has a polychromatic } k\text{-edge-coloring with } k \geq \Delta/(\ln r + O(\ln \ln r)).\]

\[\text{Corollary 7. For the range family } \mathcal{R}_\text{ST} \text{ of all axis-aligned strips in } \mathbb{R}^d \text{ and every integer } k \geq 2 \text{ we have } m_{\mathcal{R}_\text{ST}}(k) = m_d(k) \leq 2k(\ln d + O(\ln \ln d)).\]

\[\text{Proof. Let } V \text{ be a set of } n \text{ points in } \mathbb{R}^d. \text{ Let } r = \lfloor k(\ln d + O(\ln \ln d)) \rfloor \text{ and } m = 2r. \text{ We show that the } m\text{-uniform range capturing hypergraph } H = H(V, \mathcal{R}_\text{ST}, m) \text{ induced by axis-aligned strips in } \mathbb{R}^d \text{ admits a polychromatic } k\text{-coloring.}\]

We construct the $r$-uniform $d$-regular hypergraph $H'$ as in the proof of Theorem 5 and consider its ($d$-uniform and $r$-regular) dual hypergraph $H^*$. By Theorem 6, $H^*$ admits a polychromatic $k'$-edge-coloring with $k' \geq r/(\ln d + O(\ln \ln d)) \geq k$, i.e., every vertex of $H^*$ is incident to an edge of every color. Thus, its dual $H'$ admits a polychromatic $k$-coloring $\psi$.

It remains to show that $\psi$ is a polychromatic $k$-coloring of $H$. Let $e$ be any hyperedge in $H$. Since $|e| = m$ and since every hyperedge in $H'$ has size $r = m/2$, there exists a hyperedge $e'$ in $H'$ with $e' \subseteq e$. Since $e'$ is colored polychromatically, so is $e$.

\[\text{2.2 Lower Bounds}\]

We seek to give a lower bound on $m_d(k) = m_{\mathcal{R}_\text{ST}}(k)$ for the range family $\mathcal{R}_\text{ST}$ of all axis-aligned strips in $\mathbb{R}^d$. That is, for every $d, k \geq 1$ we construct a point set $V = V_{d,k}$ in $\mathbb{R}^d$ such that for some (hopefully large) $m$ the range capturing hypergraph $H(V, \mathcal{R}_\text{ST}, m)$ admits no polychromatic $k$-coloring. Then it follows that $m_d(k) \geq m + 1$. As a first step towards the desired point sets, we present a construction of $r$-uniform $r$-partite\(^4\) $t$-regular hypergraphs with $t$ being relatively large in terms of $r$, which admit no $(t - 1)$-shallow hitting edge sets.

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\(^3\) For a hypergraph $H = (V, E)$ its dual is the hypergraph $H^* = (V^*, E^*)$ with vertex-set $V^* = E$ and edge-set $E^* = \{\text{Inc}(v) \mid v \in V\}$. Note that $H^*$ may have parallel hyperedges.

\(^4\) A hypergraph $H = (V, E)$ is $r$-partite if there exists a partition $V = V_1 \cup \cdots \cup V_r$ such that for every $e \in E$ and every $i \in [r]$ we have $|e \cap V_i| \leq 1$. The sets $V_1, \ldots, V_r$ are then called the parts of $H$. 
Theorem 8. Let \( t \geq 2 \) be an integer. There exists an \( r \)-uniform \( r \)-partite \( t \)-regular hypergraph with parts of size two that has no \((t-1)\)-shallow hitting edge set, where \( r = \binom{2t}{t}/2 \leq 4^t \), i.e., \( t \geq \log_4(r) \).

Proof. Let \( H = (V,E) \) be the hypergraph with \( V = \{1, \ldots, 2t\} \) and \( E = \binom{V}{t} \), i.e., the hyperedges are all \( t \)-element subsets of \( V \). Observe that \( H \) is \( t \)-uniform and \( r \)-regular with \( r = \binom{2t}{t}/2 \). Moreover \( H \) is the union of \( r \) perfect matchings, each of the form \( A,B \in \binom{V}{t} \) with \( B = V - A \).

First, we show that \( H \) has no \((t-1)\)-shallow hitting (vertex) set. To this end let \( X \subseteq V \) be any set of vertices in \( H \). If \(|X| \leq t\), then \(|V - X| \geq t\) and there exists a hyperedge \( e \subseteq V - X \) which is not covered by \( X \). In this case, \( X \) is not hitting. If \(|X| \geq t\), then there exists a hyperedge \( e \subseteq X \). Since \( e \) has size \( t \), the set \( X \) is not \((t-1)\)-shallow.

Now consider the dual hypergraph \( H^* \) of \( H \). Then, \( H^* \) is an \( r \)-uniform \( r \)-partite \( t \)-regular hypergraph. Two vertices \( v \) and \( v' \) in \( H^* \) (recall that \( v,v' \) are \( t \)-subsets of \( \{1, \ldots, 2t\} \)) are in the same part if and only if \( v' = \{1, \ldots, 2t\} - v \). Since \( H \) has no \((t-1)\)-shallow hitting (vertex) set, \( H^* \) has no \((t-1)\)-shallow hitting edge set.

In the next theorem, we seek to find lower bounds for \( m_d(k) \) for axis-aligned strips in \( \mathbb{R}^d \).

Lemma 9. For every \( d \) and \( k \) we have \( m_d(k) \geq m'_d(k) \).

Proof. Let \( m = m_d(k) \). Then every range capturing hypergraph \( H = \mathcal{H}(V,R_{ST},m) \) (with \( V \subset \mathbb{R}^d \) finite) admits a polychromatic \( k \)-coloring of its vertices. Let \( H' = (V',E') \) be any \( d \)-uniform \( d \)-partite \( m \)-regular hypergraph with parts \( V'_1, \ldots, V'_d \) of size \( n \) and \( V'_i = \{v_{i,1}, \ldots, v_{i,n}\} \) for \( i = 1, \ldots, d \). We deduce from \( H' \) the following finite point set \( V \subset \mathbb{R}^d \), which defines the range capturing hypergraph \( H = \mathcal{H}(V,R_{ST},m) \). For each part \( V'_i \) of \( H' \), let \( \pi_i : E \rightarrow \{1, \ldots, mn\} \) be a bijection that satisfies the following condition. For two hyperedges \( e \) and \( e' \) with \( e \cap V'_i = \{v_{i,j}\} \) and \( e' \cap V'_i = \{v_{i,j'}\} \) and \( j < j' \) it holds that \( \pi(e) < \pi(e') \). Now, let the point set be \( V = \{(\pi_1(e), \ldots, \pi_d(e)) \mid e \in E \} \subset \mathbb{R}^d \). Note that for every vertex \( v_{i,j} \) in \( V'_i \), its incident hyperedges \( \text{Inc}(v_{i,j}) \subset E' \) correspond to points in \( V \) that are consecutive in the \( i \)-th dimension (by the definition of \( \pi_i \)). Recall that \( H \) admits a polychromatic \( k \)-coloring of its vertices, i.e., each \( m \)-set of points that are consecutive in some dimension \( i \) contains points of all \( k \) colors. Then it follows that \( H' \) admits a polychromatic \( k \)-coloring of its hyperedges.

Having Lemma 9, it remains to prove a lower bound for \( m'_d(k) \).

Theorem 10. \( m_d(k) \geq m'_d(k) > \frac{1}{2} (\log_2 d - 1) \cdot [k/2] \).
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Proof. Let $k$ and $d$ be positive integers. Let $t$ be the largest integer such that \( \binom{k}{t}/2 \leq d \). Let $d_0 = \binom{k}{t}/2 \leq 4t/2$ and observe that $d_0 \leq d \leq 4d_0$. Let $H_0$ be the $d_0$-uniform $d_0$-partite $t$-regular hypergraph with two vertices per part from Theorem 8. Observe that if $M$ is any subset of hyperedges in $H_0$ that together contain all vertices of $H_0$, called a hitting edge set, then $M$ has size at least $t + 1$.

We construct the hypergraph $H$ by replacing each hyperedge of $H_0$ by a multiedge of multiplicity $\lfloor k/2 \rfloor$. Then, $H$ is a $d_0$-uniform $d_0$-partite $(t|k/2\rangle)$-regular hypergraph and each hitting edge set of $H$ has size at least $t + 1$. Observe that $|E(H)| = 2t\lfloor k/2 \rfloor \leq tk$, since each part of $H$ has size 2.

Assume for a contradiction that $H$ admits a polychromatic $k$-coloring of its hyperedges, i.e., a $k$-coloring of the hyperedges such that every vertex is incident to at least one hyperedge of each color. Since each color class is a hitting edge set, each color class contains at least $t + 1$ hyperedges. Thus, the number of hyperedges is $|E(H)| \geq (t + 1)k > tk$, a contradiction.

With $d_0 \leq d \leq 4d_0$, we conclude

$$m_d'(k) \geq m_{d_0}(k) > \frac{k}{2} \geq \frac{1}{2} \log_2(2d_0) \left\lfloor \frac{k}{2} \right\rfloor \geq \frac{1}{2} \log_2(d/2) \left\lfloor \frac{k}{2} \right\rfloor = \frac{1}{2} \left( \log_2 d - 1 \right) \left\lfloor \frac{k}{2} \right\rfloor .$$

Remark. In [26] there is a more sophisticated (compared to Theorem 8) construction of $r$-uniform $r$-partite regular hypergraphs with two vertices per part that have no $(t - 1)$-shallow hitting edge set with a slightly better bound for $t$, namely with $t = \log_2(r + 1)$. Using this construction instead, an analogous proof as in Theorem 10 then gives that $m_d(k) \geq m_{d_0}(k) > \frac{1}{2} ( (\log_2 d - 1) \cdot k - d )$, which is better by a factor of 2 as long as $k > d$.

3 Bottomless Rectangles

For the range family $\mathcal{R}_{BL}$ of all bottomless rectangles in $\mathbb{R}^2$ we have $m_{\mathcal{R}_{BL}}(k) = O(k)$ [4].

Theorem 11 (Asinowski et al. [4]). For the range family $\mathcal{R}_{BL}$ of all bottomless rectangles in $\mathbb{R}^2$ we have $m_{\mathcal{R}_{BL}}(k) \leq 3k - 2$.

However, the proof in [4] does not go via shallow hitting sets, and it is also not clear how to adjust it to give shallow hitting sets. In fact, Keszegh and Pálvölgyi [15] ask whether there exists a constant $t$ such that for every $V$ the hypergraph $\mathcal{H}(V, \mathcal{R}_{BL}, m)$ admits a $t$-shallow hitting set. We answer this question in the positive.

Theorem 12. Let $\mathcal{R}_{BL}$ be the range family of all bottomless rectangles in $\mathbb{R}^2$ and $m$ be a positive integer. Then for any finite point set $V \subset \mathbb{R}^2$ the hypergraph $\mathcal{H}(V, \mathcal{R}_{BL}, m)$ admits a 10-shallow hitting set $X \subseteq V$.

Proof. Let $V \subset \mathbb{R}^2$ be any finite point set and let $V = \{ p_1, \ldots, p_n \}$ with $y(p_1) < \cdots < y(p_n)$. (Recall that $y(p)$ denotes the $y$-coordinate of a point $p \in \mathbb{R}^2$.) Let $w = \lfloor (m + 3)/4 \rfloor$ and note that $4w - 3 \leq m \leq 4w$. We can assume that $m > 10$ since for $m \leq 10$, the point set $X = V$ is a 10-shallow hitting set of $\mathcal{H}(V, \mathcal{R}_{BL}, m)$. Moreover, we can assume that $|V| \geq m \geq 4w - 3$ since otherwise $\mathcal{H}(V, \mathcal{R}_{BL}, m)$ has no hyperedges.

We shall perform a sweep-line algorithm that goes through the points in order of increasing $y$-coordinates and builds the desired 10-shallow hitting set by selecting one by one points to be included in $X$, without ever revoking such decision. Such an algorithm is called semi-online as its choices will be independent of the points above the current sweep-line (with larger $y$-coordinates). During the sweep we consider the $x$-coordinates of the points below the sweep-line. Note that if $m$ points have consecutive $x$-coordinates among those
below the sweep-line, then these \( m \) points form a hyperedge in \( H(V, R_{BL}, m) \), as verified by a bottomless rectangle whose top side lies on the sweep-line. And conversely, if some \( m \) points of \( V \) form a hyperedge in \( H(V, R_{BL}, m) \), then these have consecutive X-coordinates among those below the sweep-line at the time that the sweep-line contains the top side of a corresponding bottomless rectangle.

We start the sweep-line algorithm with step \( j = w \). In step \( j, j \geq w \), we consider the points \( V_j = \{p_1, \ldots, p_j\} \), i.e., the \( j \) points with the lowest \( y \)-coordinates. We construct a set \( X_j \subseteq V_j \) of black points (points that are definitely in the final set \( X \)) and a set of white points \( W_j \subseteq V_j \) (points that are definitely not in the final set \( X \)) such that (for \( j > w \)) we have \( X_{j-1} \subseteq X_j \) and \( W_{j-1} \subseteq W_j \) and \( X_j \cap W_j = \emptyset \). We refer to the points that are neither white nor black as uncolored points. Additionally, we maintain a partition \( \mathbb{R} = A_{j,1} \cup \cdots \cup A_{j,l} \) of the real line \( \mathbb{R} \) into \( l \), for some \( l \), pairwise disjoint intervals \( A_{j,i} \) with \( A_{j,1} = (-\infty, a_1), A_{j,2} = [a_1, a_2], \ldots, A_{j,l} = [a_{l-1}, \infty) \) with \( -\infty < a_1 < a_2 < \cdots < a_{l-1} < \infty \). We define \( V_j \) to be the set of points \( p \in V_j \) with x-coordinate \( x(p) \in A_{j,i} \). During the sweep-line algorithm, we maintain the following invariants:

- Each \( V_{j,i} \) contains exactly one black point and \( w-1 \) white points, i.e., \( |V_{j,i} \cap X_j| = 1 \) and \( |V_{j,i} \cap W_j| = w-1 \).
- Each \( V_{j,i} \) has size \( w \leq |V_{j,i}| \leq 2w-1 \).

We start with step \( j = w \) as follows. The set of black points is \( X_w = \{p_1\} \), the set of white points is \( W_w = \{p_2, \ldots, p_w\} \) and \( \mathbb{R} = (-\infty, \infty) \) is the partition of \( \mathbb{R} \) into one set. Clearly, all conditions are satisfied.

Now, suppose that \( X_j, W_j, \) and the partition \( \mathbb{R} = A_{j,1} \cup \cdots \cup A_{j,l} \) are given as the result of step \( j \). In the next step \( j+1 \), we consider the set \( V_{j+1} = V_j \cup \{p_{j+1}\} \). Let \( A_{j,i} \) be the interval with \( x(p_{j+1}) \in A_{j,i} \). We distinguish two cases. If \( |V_{j,i}| < 2w-1 \), then we set \( X_{j+1} = X_j, W_{j+1} = W_j \) and \( A_{j+1,i} = A_{j,i} \) for all \( i = 1, \ldots, l \) for the next step \( j+1 \). Then, \( |V_{j+1,i}| \leq 2w-1 \) and all conditions are again satisfied. Otherwise, assume that \( |V_{j,i}| = 2w-1 \). Let \( q_1, \ldots, q_{2w} \) be the points in \( V_{j,i} \cup \{p_{j+1}\} \) ordered by their x-coordinate, i.e., \( x(q_1) < \cdots < x(q_{2w}) \), and define \( a' = x(q_{w+1}) \). Then, we define the partition

\[
\mathbb{R} = A_{j+1,1} \cup \cdots \cup A_{j+1,l+1} \\
= (-\infty, a_1) \cup \cdots \cup [a_{w-1}, a') \cup [a', a_w) \cup \cdots \cup [a_{l-1}, \infty).
\]

That is, we split the interval \( A_{j,i} \) into two intervals \( [a_{w-1}, a') \) and \( [a', a_w) \). Observe that \( |V_{j,i+1}| = w = |V_{j+1,i+1}| \). Since there is exactly one black point in \( V_{j,i} \) (i.e., \( |X_j \cap V_{j,i}| = 1 \)), there is exactly one black point of \( X_j \) in \( V_{j+1,i} \cup V_{j+1,i+1} \). By symmetry, assume that this black point is contained in \( V_{j+1,i} \) and therefore, \( V_{j+1,i+1} \) has no black point in \( X_j \). Now we color all uncolored points in \( V_{j+1,i} \) white. Then, \( V_{j+1,i} \) contains exactly one black and \( w-1 \) white points. Since \( V_{j,i} \) has at most \( w-1 \) white points and \( V_{j+1,i+1} \subseteq V_{j,i} \cup \{p_{j+1}\} \), the set \( V_{j+1,i+1} \) has at most \( w-1 \) white points of \( W_j \), too. Thus, there exists an uncolored point \( q \) in \( V_{j+1,i+1} \). We color \( q \) black and all other uncolored points in \( V_{j+1,i+1} \) white. Then, \( V_{j+1,i+1} \) contains exactly one black and \( w-1 \) white points. This completes step \( j+1 \). Note that both invariants are again satisfied.

After step \( n = |V| \), we have considered all points in \( V \). Let \( X_n = X_n \) be the set of black points after the last step. We show that \( X \) is a 10-shallow hitting set of \( H(V, R_{BL}, m) \).

\begin{claim}
\( X \) is hitting in \( H(V, R_{BL}, m) \).
\end{claim}

\textbf{Proof.} Let \( R = [a, b] \times (-\infty, c] \) be a bottomless rectangle that contains \( m \) points of \( V \), i.e., \( |R \cap V| = m \). Let \( p \) be the topmost point in \( R \cap V \) and consider the state of the sweep-line algorithm right after \( p \) is inserted, that is, \( p = p_j \) for some \( j \) and step \( j \) is finished. Then, the
points in $V_j$ with $x$-coordinate in the interval $[a, b]$ are exactly the points in $R \cap V$. Since we have $|R \cap V| = m \geq 4w - 3$ and each $V_{j,i}$ has size at most $2w - 1$, there exists a $V_{j,i'}$ with $V_{j,i'} \subseteq R \cap V$. Since $V_{j,i'}$ contains a black point ($V_{j,i'} \cap X \neq \emptyset$), $R \cap V$ contains a black point too ($R \cap X \neq \emptyset$) and $X$ is hitting.

Claim 14. $|X \cap V_{j,i}| \leq 2$ for every $V_{j,i}$.

Proof. By the invariants above it holds that $|X_j \cap V_{j,i}| = 1$ (Be aware of the difference between $X \cap V_{j,i}$ and $X_j \cap V_{j,i}$) and $|W_j \cap V_{j,i}| = w - 1$. Moreover, observe that whenever an uncolored point $p \in V_{j,i}$ is colored black, all other uncolored points in $V_{j,i}$ are colored white. Since white points are definitely not contained in $X$, the claim follows.

Claim 15. $X$ is 10-shallow in $H(V, R_{BL}, m)$.

Proof. Again, let $R$ be a bottomless rectangle that contains $m$ points of $V$, i.e., $|R \cap V| = m$, and let $p$ be the topmost point in $R \cap V$. Consider the state of the sweep-line algorithm after $p$ is inserted, that is, $p = p_j$ for some $j$ and step $j$ is finished. We have $|R \cap V| = m \leq 4w$ and each $V_{j,i}$ contains at least $w$ points. Therefore, there exist at most five sets $V_{j,i}$ with $V_{j,i} \cap R \neq \emptyset$. By Claim 14, each $V_{j,i}$ contains at most two points of $X$. Therefore, $|R \cap X| \leq 5 \cdot 2 = 10$ and $X$ is 10-shallow.

By Claims 13 and 15, $X$ is a 10-shallow hitting set in $H(V, R_{BL}, m)$.

Remark. The procedure in the proof of Theorem 12 can be modified to directly get a polychromatic coloring of $H(V, R_{BL}, m)$. Let $w = \lfloor (m + 3)/4 \rfloor$ be as in the proof of Theorem 12. Instead of carrying black and white sets, we carry a partial $w$-coloring (i.e., a $w$-coloring of some vertices on the sweep-line) such that in each step $j \geq 1$, every set $V_{j,i}$ of points contains every color exactly once. At the end of the algorithm, we get a partial $w$-coloring of all vertices. We complete this to a $w$-coloring by assigning colors to the uncolored vertices such that every $V_{n,i}$ contains every color at most twice. Note that every color class is a 10-shallow hitting set in $H(V, R_{BL}, m)$. By setting $w = k$, one can observe that this $k$-coloring is polychromatic in $H(V, R_{BL}, m)$, which gives a proof of $m_{R_{BL}}(k) \leq 4k - 3$. Moreover, if $e$ is an edge in $H(V, R_{BL})$, not necessarily of size $m$, and $n_1, n_2$ denote the size of two color classes in $e$ then it holds that $n_1 \leq 4 + 2n_2 \leq 4(n_2 + 1)$. Therefore, this $k$-coloring is $4$-balanced in $H(V, R_{BL})$.

Let us also remark that it was proven recently [6] that for every $m \geq 12$ there are finite point sets $V$ in $\mathbb{R}^2$ such that $H(V, R_{BL}, m)$ admits no $3$-shallow hitting sets, which also shows that one cannot achieve the bound $m_{R_{BL}}(k) \leq 3k - 2$ by using shallow hitting sets.

## 4 Bottomless and Topless Rectangles

Chekan and Ueckerdt [10] showed that $m_{R_{BL} \cup R_{TL}}(k) \leq O(k^{8.75})$ for the range family $R_{BL} \cup R_{TL}$ of bottomless and topless rectangles by a reduction to the family $R$ of all axis-aligned squares, and using that $m_R(k) = O(k^{8.75})$ in this case [2]. We improve the upper bound on $m(k)$ for the case $R_{BL} \cup R_{TL}$ to $O(k)$ in the following theorem, by a simple reduction to the case $R_{BL}$ of just all bottomless rectangles, and the case $R_{TL}$ of just all topless rectangles. Observe that we clearly have $m_{R_{BL}}(k) = m_{R_{TL}}(k)$ for all $k$, and recall that $m_{R_{BL}}(k) \leq 3k - 2$ according to [4] (see Theorem 11).

Theorem 16. For $R_{BL} \cup R_{TL}$ the range family of all bottomless and topless rectangles in $\mathbb{R}^2$, we have $m_{R_{BL} \cup R_{TL}}(k) \leq 2m_{R_{BL}}(k) + 1 \leq 6k - 3$. 
Proof. Let \( m' = m_{BL}(k) = m_{BL}(k) \) and \( m = 2m' + 1 \). Let \( V = \{p_1, \ldots, p_n\} \subset \mathbb{R}^2 \) be a finite point set with \( x(p_1) < \cdots < x(p_n) \). (Recall that \( x(p) \) denotes the \( x \)-coordinate of a point \( p \) in \( \mathbb{R}^2 \).) We partition the set \( V \) into two sets \( A \) and \( B \). For each pair \( \{p_{2i-1}, p_{2i}\} \), we put the vertex with the lower \( y \)-coordinate into set \( A \) and the point with the larger \( y \)-coordinate into set \( B \), see Figure 1(a).

There are polychromatic \( k \)-colorings \( c_1: A \to \{1, \ldots, k\} \) of the hypergraph \( \mathcal{H}(A, R_{BL}, m') \) and \( c_2: B \to \{1, \ldots, k\} \) of the hypergraph \( \mathcal{H}(B, R_{TL}, m') \). As \( V = A \cup B \), this naturally defines a \( k \)-coloring \( c: V \to \{1, \ldots, k\} \) of \( \mathcal{H}(V, R_{BL} \cup R_{TL}, m) \). To see that coloring \( c \) is polychromatic, let \( e \) be a hyperedge in \( \mathcal{H}(V, R_{BL} \cup R_{TL}, m) \) induced by a bottomless or topless rectangle \( R \in R_{BL} \cup R_{TL} \). If \( R \in R_{BL} \), then \( R \) contains at least \( \lceil (m - 2)/2 \rceil = \lceil (2m' - 1)/2 \rceil = m' \) points from \( A \), see Figure 1(b). Thus, \( e \cap A \) is colored polychromatically in \( \mathcal{H}(A, R_{BL}, m') \) and hence \( e \) is colored polychromatically in \( \mathcal{H}(V, R_{BL} \cup R_{TL}, m) \). Symmetrically, if \( R \in R_{TL} \), then \( R \) contains at least \( m' \) points from \( B \), thus \( R \cap B \) contains all \( k \) colors under \( c_2 \), and thus \( e = R \cap V \supseteq R \cap B \) contains all \( k \) colors under \( c \).

According to Theorem 16 we have \( m_{BL} \cup R_{BL}(k) = O(k) \). However, the proof relies on the polychromatic coloring from [4] and thus does not give shallow hitting sets, which (up to the constants) is the stronger statement. In fact, even if we had a shallow hitting set \( X \) for \( \mathcal{H}(A, R_{BL}, m') \) and a shallow hitting set \( Y \) for \( \mathcal{H}(B, R_{TL}, m') \) (\( A \) and \( B \) as in the proof above), their union \( X \cup Y \) would be hitting, but not necessarily shallow.

Recall that a subset \( X \) of vertices of a hypergraph \( \mathcal{H} = (V, E) \) is hitting if \( |X \cap e| \geq 1 \) for every \( e \in E \), and \( t \)-shallow if \( |X \cap e| \leq t \) for every \( e \in E \). In order to prove the existence of shallow hitting sets for \( R_{BL} \cup R_{TL} \), we shall first find a shallow hitting set for \( R_{BL} \), which is also shallow (but not necessarily hitting) for \( R_{TL} \). A similar approach has been done in [10].

**Lemma 17.** Let \( V \subset \mathbb{R}^2 \) be a finite point set and \( m \) be a positive integer. Then, there exists a set \( X \subset V \) such that
- \( X \) is a 14-shallow hitting set of \( \mathcal{H}(V, R_{BL}, m) \) and
- \( X \) is a 7-shallow set of \( \mathcal{H}(V, R_{TL}, m) \).

From Lemma 17, proven in the full version [25], we can quickly derive the full theorem.

**Theorem 18.** Let \( R_{BL} \cup R_{TL} \) be the range family of all bottomless and topless rectangles in \( \mathbb{R}^2 \) and \( m \) be a positive integer. Then for any finite point set \( V \subset \mathbb{R}^2 \) the hypergraph \( \mathcal{H}(V, R_{BL} \cup R_{TL}, m) \) admits a 21-shallow hitting set \( X \subset V \).

\[ \text{SoCG 2024} \]
Theorem 19. Let $\mathcal{R}_{UH}$ be the range family of all unit-height axis-aligned rectangles in $\mathbb{R}^2$ and $m$ be a positive integer. Then, for every finite point set $V \subset \mathbb{R}^2$ the hypergraph $\mathcal{H}(V, \mathcal{R}_{UH}, m)$ admits a 63-shallow hitting set $X \subseteq V$. Moreover, $m_{\mathcal{R}_{UH}}(k) \leq 2m_{\mathcal{R}_{BL} \cup \mathcal{R}_{TL}}(k) - 1 \leq 12k - 7$ for the range family $\mathcal{R}_{UH}$.

Proof. Let $m$ be a positive integer and $V \subset \mathbb{R}^2$ be a finite point set. Define $m' = \lfloor m/2 \rfloor$. For every integer $a \in \mathbb{Z}$, let $H_a = \mathcal{H}(V_a, \mathcal{R}_{BL} \cup \mathcal{R}_{TL}, m')$ be the range capturing hypergraph induced by the range family of all bottomless and topless rectangles, where $V_a$ is the set of all points $p$ in $V$ with $a \leq y(p) < a + 1$. By Theorem 18, every $H_a$ admits a 21-shallow hitting set $X_a$. Then, $X = \bigcup_{a \in \mathbb{Z}} X_a$ is a 63-shallow hitting set in $\mathcal{H}(V, \mathcal{R}_{UH}, m)$, which can be seen as follows. Every unit-height rectangle induces a topless rectangle $R_t$ in $H_a$ and a bottomless rectangle $R_b$ in $H_{a+1}$ (for some $a$). Then, at least one of $R_t$ and $R_b$ contains at least $\lfloor m/2 \rfloor = m'$ points of $V$, without loss of generality $R_t$. Therefore, $R_t$ contains a point of $X_a$ and hence, $X$ is hitting. Since $m \leq 2m'$, the topless rectangle $R_t$ can be covered with at most two topless rectangles of size $m'$ of $H_a$, and $R_b$ can be covered with at most one bottomless rectangle of size $m'$ of $H_{a+1}$. As each of these three rectangles contains at most 21 points of $X$, we conclude that $X$ is $t$-shallow for $t = 3 \cdot 21 = 63$.

Using the same argument, it is not difficult to see that $m_{\mathcal{R}_{UH}}(k) \leq 2m_{\mathcal{R}_{BL} \cup \mathcal{R}_{TL}}(k) - 1$. Let $m = 2m_{\mathcal{R}_{BL} \cup \mathcal{R}_{TL}}(k) - 1$ and let $H = \mathcal{H}(V, \mathcal{R}_{UH}, m)$ be the range capturing hypergraph induced by all unit-height rectangles. Let $m' = \lfloor m/2 \rfloor = m_{\mathcal{R}_{BL} \cup \mathcal{R}_{TL}}(k)$. For every $a \in \mathbb{Z}$, color each $H_a = \mathcal{H}(V_a, \mathcal{R}_{BL} \cup \mathcal{R}_{TL}, m')$ polychromatically with $k$ colors with respect to bottomless and topless rectangles $\mathcal{R}_{BL} \cup \mathcal{R}_{TL}$. This polychromatic coloring exists by Theorem 16 and since $m' = m_{\mathcal{R}_{BL} \cup \mathcal{R}_{TL}}(k)$. Then, every unit-height rectangle $R$ induces a topless rectangle $R_t$ in $H_a$ of size at least $m'$ or a bottomless rectangle $R_b$ in $H_{a+1}$ of size at

### Table 1
Shallow hitting sets and polychromatic colorings for range capturing hypergraphs.

<table>
<thead>
<tr>
<th>Range Family $\mathcal{R}$</th>
<th>t-Shell Hitting Sets Exist</th>
<th>$m_{\mathcal{R}}(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) axis-aligned strips in $\mathbb{R}^d$</td>
<td>Yes for $t \geq 3ed(1 + o(1))$ (Theorem 5)</td>
<td>$O_d(k)$ (Corollary 7)</td>
</tr>
<tr>
<td>(2) bottomless rectangles in $\mathbb{R}^2$</td>
<td>Yes for $t \geq 10$ (Theorem 12)</td>
<td>$\leq 3k - 2$ [4]</td>
</tr>
<tr>
<td>(3) half-planes in $\mathbb{R}^2$</td>
<td>Yes for $t \geq 2$ [29]</td>
<td>$\leq 2k - 1$ [29]</td>
</tr>
<tr>
<td>(4) axis-aligned squares in $\mathbb{R}^2$</td>
<td>Open</td>
<td>$O(k^{8.75})$ [2]</td>
</tr>
<tr>
<td>(5) bottomless and topless rectangles in $\mathbb{R}^2$</td>
<td>Yes for $t \geq 21$ (Theorem 18)</td>
<td>$\leq 6k - 3$</td>
</tr>
<tr>
<td>(6) translates of a convex polygon in $\mathbb{R}^2$</td>
<td>Open</td>
<td>$O(k)$ [12]</td>
</tr>
<tr>
<td>(7) homothets of a triangle in $\mathbb{R}^2$</td>
<td>Open</td>
<td>$O(k^{4.09})$ [14]</td>
</tr>
<tr>
<td>(8) translates of octants in $\mathbb{R}^3$</td>
<td>No [7]</td>
<td>$O(k^{5.09})$ [14]</td>
</tr>
</tbody>
</table>
least \( m' \) (for some \( a \in \mathbb{Z} \)). Since \( R_0 \) (respectively \( R_b \)) contains points of all colors, so does \( R \). Therefore, each unit-height rectangle with \( m \) points contains points of all colors and we have found a polychromatic \( k \)-coloring of \( H(V, R_{UH}, m) \).

5 Conclusions

In this paper, we extended the list of range families \( R \) for which the corresponding uniform range capturing hypergraphs admit shallow hitting sets. This in particular implies that \( m_R(k) = O(k) \) for that family \( R \), while \( m_R(k) \geq k \) always holds. In view of Question 2, it would be interesting to investigate further range families \( R \) for which \( m_R(k) < \infty \) is known, as to whether they admit shallow hitting sets. The current state of the art (for a selection of range families) is summarized in Table 1.

References

1 The geometric hypergraph zoo. URL: https://coge.elte.hu/cogezoo.html.


