Grid Peeling of Parabolas

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Abstract

Grid peeling is the process of repeatedly removing the convex hull vertices of the grid points that lie inside a given convex curve. It has been conjectured that, for a more and more refined grid, grid peeling converges to a continuous process, the *affine curve-shortening flow*, which deforms the curve based on the curvature. We prove this conjecture for one class of curves, parabolas with a vertical axis, and we determine the value of the constant factor in the formula that relates the two processes.

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1 Introduction

In 2017, Eppstein, Har-Peled, and Nivasch [8] observed a remarkable connection between a continuous deformation of a curve in the plane, the *affine curve-shortening flow* (ACSF), and a discrete process, *grid peeling*.

In the affine curve-shortening flow, a smooth curve is deformed by moving every point toward the direction in which the curve bends, at a speed of $\kappa^{1/3}$ in the normal direction, where κ is the curvature at that point at the current point of time, see Figure 1. Figure 2 shows snapshots of this inward-growing process, starting from a semicircle. (The semicircle is not a smooth curve, but the definition of the flow can be extended to cover such curves.)

By contrast, grid peeling is a process that is discrete both in space and in time. Given a convex curve, we start by finding the convex hull of all points of a uniform square grid inside the curve. Then we iteratively remove the vertices of the convex hull, and take the convex hull of the remaining grid points.

Eppstein, Har-Peled, and Nivasch observed that, as the underlying grid is refined, grid peeling approximates the ACSF process, as can be seen in Figure 2. More specifically [8, Conjecture 1], they conjectured that, for a more and more refined grid of spacing $1/n$, the m-th convex layer of a convex curve converges to the ACSF after time T, if m is chosen as

$$m = \lfloor c_g T n^{4/3} \rfloor \qquad (1)$$

for an appropriate constant c_g, which was experimentally determined to be approximately 1.6.

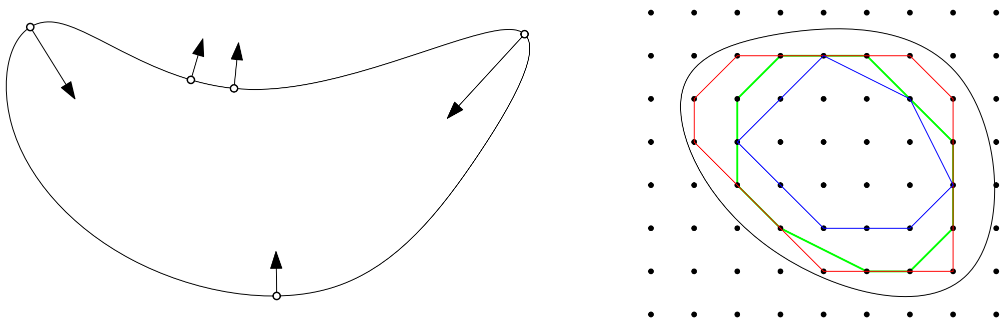

Figure 1 Left: The Affine Curve-Shortening Flow (ACSF). The velocity is indicated by arrows, whose length is proportional to $\kappa^{1/3}$. Right: A convex curve and the first three steps of grid peeling.

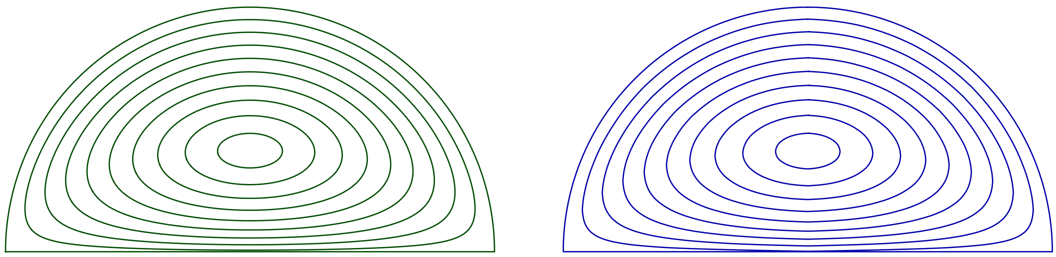

Figure 2 ACSF (left) and grid peeling (right) of a semicircle of diameter 1. The left figure shows 10 snapshots of ACSF with regular time increments; the right figure shows every 2714th convex layer for a grid of spacing 1/5000. (From [8, Figure 3], by permission from the authors).

We prove this conjecture for the special case when the curve is a parabola with vertical axis, and we find the precise value of the constant c_{g}:

$$c_{\mathrm{g}} = \sqrt[3]{\frac{\pi^2}{2\zeta(3)}} \approx 1.60120980542577, \tag{2}$$

where $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$ is the Riemann zeta function.

▶ **Theorem 1.** *For the parabola $\Pi\colon y = ax^2/2 + bx + c$, the ACSF is a vertical translation with velocity $a^{1/3}$. Thus, at time $T > 0$, it becomes the parabola $\Pi^T : y = ax^2/2 + bx + c + Ta^{1/3}$.*

If we apply grid peeling to Π with a grid of spacing $1/n$ for $m = \lfloor c_{\mathrm{g}} T n^{4/3} \rfloor$ steps, then, as $n \to \infty$, the vertical distance between the resulting grid polygon and Π^T is bounded by

$$O\left(\frac{(Ta^{2/3} + a^{-2/3}) \log \frac{n}{a}}{n^{1/3}}\right).$$

For fixed T and a, this error bound goes to 0 as $n \to \infty$.

We can extend this theorem to any parabola whose axis has a rational slope a/b: A unimodular transformation with a suitable matrix $\left(\begin{smallmatrix} a & -b \\ u & v \end{smallmatrix}\right)$ of determinant 1 will leave the grid unchanged and make the axis vertical, and then Theorem 1 can be applied.

1.1 History and background

The ACSF process was first studied in the 1990s in the area of computer vision and image processing, by Alvarez, Guichard, Lions, and Morel [1] and by Sapiro and Tannenbaum [11]. One way to understand the ACSF is to regard it as a limit of *affine erosions*, as shown by

F. Cao [6, Theorem 6.22]. An affine erosion with parameter ε removes the union of all pieces of area ε that can be cut off by a straight line. (In convex geometry, this is also called the *wet part*; it plays a role in estimating the area and the number of vertices of the convex hull of a random sample of points [4, 3].) Repeating this process makes the shape rounder and rounder, like a pebble rolling in water. Letting ε go to zero leads to the ACSF as the continuous limit.

The other process that we study is formed by the *convex layers* or *onion layers* of a point set. They have their origin in computational geometry and statistics: The innermost convex layer provides a robust estimate of the "center" of a distribution. The special case where the point set is a grid was first investigated by Har-Peled and Lidický [9], who showed that the $n \times n$ square grid has $\Theta(n^{4/3})$ convex layers. For a box in three and higher dimensions, the asymptotic number of layers is not known, see [7] and the references given there. See [2, 5, 6] for more background and references to the literature, both on the ACSF and on grid peeling.

1.1.1 Peeling with random sets

More recently, Calder and Smart [5] investigated the related process where the grid is replaced by a random point set. More precisely, the refined grid of spacing $1/n$ is replaced by a Poisson point set of density $1/n^2$. In this setting, they could prove an analogous statement:

There exists a constant $c_\mathrm{r} \approx 1.3$ such that the m-th convex layer, for $m = \lfloor c_\mathrm{r} T n^{4/3} \rfloor$, approximates the ACSF at time T. Since the underlying process is random, this statement requires some probabilistic qualification; see [5, Theorem 1.2] for the precise statement, which is quite strong and general: It is valid in arbitrary dimension, and convergence holds (with high probability) uniformly for all T.

1.1.2 Homotopic curve shortening

Avvakumov and Nivasch [2] extended peeling to nonconvex and even self-crossing curves, introducing the concept of *homotopic curve shortening*. Both for grid peeling and for random-set peeling, the observed relation with the ACSF process persists also in this setting.

1.1.3 Equivariance under affine transformations

It is easy to check that the ACSF is equivariant under area-preserving affine transformations, a property that gave rise to the term "*affine* curve-shortening flow." The relation between ACSF and grid peeling is the more surprising as grid peeling is equivariant only under a special class of affine transformations, namely those that also preserve the grid (unimodular transformations), a property that we will often use. (Peeling with random sets, on the other hand, is clearly equivariant under area-preserving affine transformations.)

1.2 Conics

As stated in Theorem 1, the ACSF for a parabola is just a translation at constant speed. This special behavior is shared, to a certain extent, by the other types of conics: They are scaled under ACSF but otherwise maintain their shape [12, Lemma 8]. More specifically,

- an ellipse (or a circle) *shrinks* toward the center, and eventually collapses to a point;
- a parabola is *translated* parallel to the axis;
- a hyperbola *expands* from its center.

Among the conics, parabolas appear most attractive for investigation, because they don't even need to be scaled. Also for the case of random-set peeling, the peeling of a parabola lies at the core of the proof of Calder and Smart [5], forming what they call the *cell problem*.

As regards experiments, the downside of parabolas, as opposed to ellipses, is that a parabola is an unbounded curve, and even the first step of grid peeling is not obvious to compute. However, as we shall see, for parabolas with rational coefficients, we can make use of a certain periodicity along the curve, which reduces grid peeling to a finite computation. Once the sequence of peelings goes into a loop, one has a complete overview of the whole infinite process.

Grid peeling has been investigated also for *hyperbolas*, in a sense. If one starts with the upper-right quadrant $\mathbb{R}_+ \times \mathbb{R}_+$, the ACSF develops into positive branches of hyperbolas $xy = c$. Eppstein, Har-Peled, and Nivasch [8, Theorem 5] investigated the convex layers of $\mathbb{N} \times \mathbb{N}$ and proved that the m-th convex layer is sandwiched between two hyperbolas:

$$c_1 m^{3/2} \le xy \le c_2 m^{3/2}, \tag{3}$$

except that the lower bound does not hold within a distance $O(\sqrt{m}\log^2 m)$ from the axes.[1]

1.3 Overview

In Section 2, we define a family of specific curves, the so-called *grid parabolas* P_t. Grid peeling reproduces them with a vertical shift after t steps (or $t+1$, depending on the parity). This is our main technical result (Theorem 2), whose proof is postponed to Section 4. Based on Theorem 2, we prove Theorem 1, our main theorem about grid peeling of parabolas, in Section 3. The quantities that arise in the construction of the grid parabola are analyzed in the full version of our paper (Appendix A and B), using arguments from elementary number theory. Appendix D of the full version reports computer experiments with grid peeling for parabolas. These experiments were the source the discoveries expressed in Theorem 2 below, and its consequence, our main Theorem 1. We also describe there some interesting phenomena beyond the ones discussed here.

2 The grid parabola

Our object of investigation is a special infinite polygonal chain P_t, which depends on a positive integer parameter t. It is defined as follows:

1. Let S_t be the set of all rational numbers $s = a/b$ with $0 < b \le t$. We call these elements the *slopes*. We will always assume that the fractions a/b representing slopes are reduced.
2. For each slope $s = a/b \in S_t$, take the longest integer vector of the form

   $$\binom{x}{y} = q\binom{b}{a} \quad (q \in \mathbb{Z})$$

   with $0 < x \le t$. Let V_t denote the set of these vectors. Figure 3 shows V_t for $t = 11$.
3. Form the chain $P = P_t$ by concatenating these vectors in order of increasing slope.

Figure 4 shows a section of the grid parabola P_5.

We can make a few simple observations: It is clear that for every vector $(x, y) \in V_t$ with a positive slope, there is a corresponding vector $(x, -y) \in V_t$ with negative slope. Thus, the curve P is symmetric with respect to a vertical axis. The lowest points on P form a horizontal edge of length t. We place the origin O of our coordinate system at the center of this edge, so that the symmetry axis becomes the y-axis. When t is odd, this implies that the vertices of P have half-integral x-coordinates. Nevertheless, we will refer to the points of the square unit grid on which the vertices of P lie (as shown in Figure 4) as the *grid*.

[1] [8, Theorem 5] claims an "exception strip" around the axes with a smaller width of $O(\sqrt{m})$; however, in the proof of that theorem, an error term of the form $\pm O(N \log N)$ from Lemma 7 is ignored.

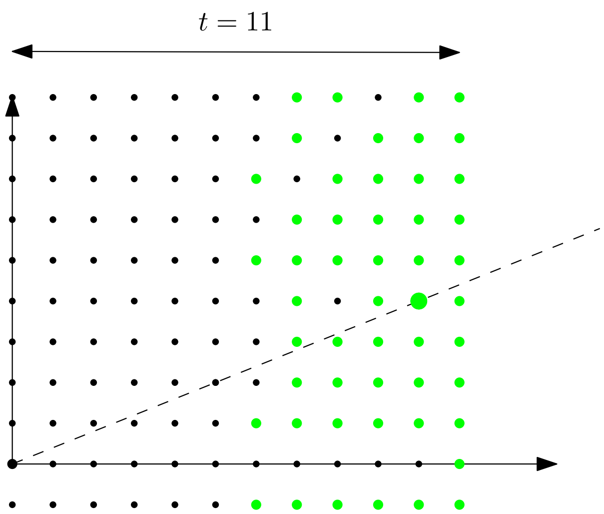

Figure 3 The set V_{11} of vectors (x, y) from which P_{11} is formed, shown as green dots. The vector with slope $s = 2/5$ is highlighted. The points of V_{11} extend indefinitely to the top and to the bottom.

Figure 5 applies a few grid peeling steps to the grid parabola P_5. We can see that P_5 reproduces itself after 5 iterations, translated up by 1 unit. From then on, the process repeats ad infinitum. Our central technical result is that this is always the case.

▶ **Theorem 2.** *For odd t, the chain P_t repeats after t peeling steps, one unit higher. For even t, the chain P_t repeats after $t + 1$ peeling steps, one unit higher.*

2.1 The horizontal period

While P is an infinite object, we will argue that it is sufficient to look at a finite section, because this section "repeats periodically" in a certain sense.

We partition the vectors $V_t = \cdots \cup V_t^{(-2)} \cup V_t^{(-1)} \cup V_t^{(0)} \cup V_t^{(1)} \cup V_t^{(2)} \cup \cdots$ according to the integral part of their slope into the sets

$$V_t^{(i)} := \{\, (x, y) \in S_t \mid i < \tfrac{y}{x} \le i + 1 \,\}$$

for $i \in \mathbb{Z}$. The vectors in $V_t^{(0)}$ lead P from the origin to an edge with vector (t, t) of slope 1. More precisely, in the way we have defined $V_t^{(0)}$, these vectors lead from the right endpoint of the horizontal edge to the upper-right endpoint of the edge (t, t). However, we prefer to select the midpoint of the edge (t, t), and we place a reference point Q at this point. We define the *horizontal period* H_t as the horizontal distance between the origin O and Q.

H_t is the sum of the x-coordinates of the vectors in $V_t^{(0)}$. (Only half of the vector $(t, t) \in V_t^{(0)}$ contributes to H_t, but this is compensated by including half of the vector $(t, 0) \notin V_t^{(0)}$.) The first values of H_t are $H_1, H_2, \ldots = 1, 4, 11, 22, 43, 64, 107, 150$, etc. In Section 3.1, we will evaluate this quantity and see that $H_t \approx 0.24\, t^3$ (Lemma 7).

We claim that the segment OQ has slope $\frac{1}{2}$, and therefore the y-coordinate of Q is $H_t/2$.

▶ **Proposition 3.** *The segment OQ has slope $\frac{1}{2}$.*

Proof. The set V_t is symmetric with respect to the shearing operation $\binom{x}{y} \leftrightarrow \binom{x}{x-y}$, which keeps the "mirror line" $y = x/2$ fixed and inverts the orientation of every vertical line, see the inset of Figure 4. Thus, every vector in V_t with slope $> \frac{1}{2}$ can be matched with a vector

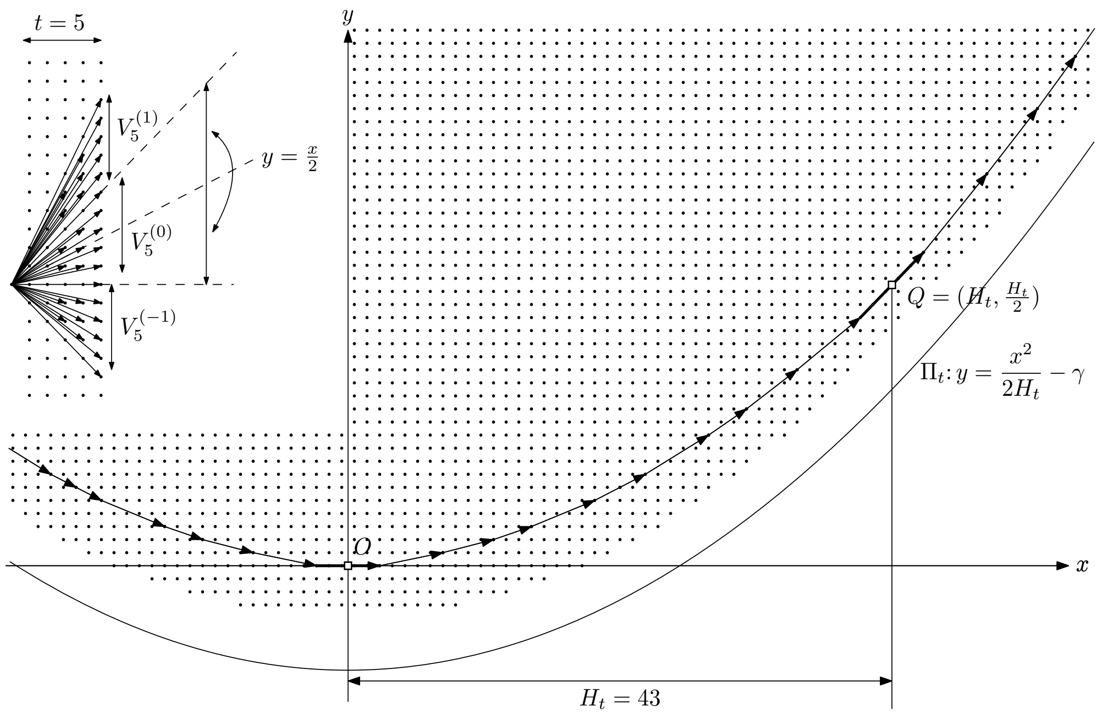

Figure 4 The grid parabola P_t for $t = 5$. The inset in the upper left corner shows some vectors of the set V_5, at a slightly enlarged scale. Π_t is the reference parabola defined by $y = \frac{x^2}{2H_t}$. Here it is shifted down by some offset γ.

with slope $< \frac{1}{2}$, so that the sum of these two vectors has slope $\frac{1}{2}$. The set $V_t^{(0)}$ is slightly asymmetric with respect to this mirror operation because it contains the edge of slope 1 but not the corresponding edge of slope 0. This asymmetry is taken care of by including half of both edges in the vector from O to Q. Thus, the slope between O and Q averages to $\frac{1}{2}$. ◀

2.1.1 Periodic continuation

The mapping $(x, y) \mapsto (x, y + ix)$ maps $V_t^{(0)}$ to $V_t^{(i)}$; hence it is sufficient to know $V_t^{(0)}$; all other sets $V_t^{(i)}$ are copies of $V_t^{(0)}$ where the slope of each vector is modified by an integer constant, leaving the x-coordinate fixed. This means that the continuation of P beyond the arc from O to Q is in some sense periodic: the same sequence of edges will appear again and again, only with modified slopes. The mapping $(x, y) \mapsto (x + H_t, y + H_t/2 + x)$ maps O to Q, and it maps the curve P to itself. The midpoints of the edges with integer slopes appear regularly at intervals of length H_t. The midpoint of the edge with slope i is $(iH_t, i^2H_t/2)$. These points lie on a parabola, which we call the *reference parabola*

$$\Pi_t \colon y = x^2/(2H_t),$$

and the polygonal chain P_t follows Π_t with bounded local deviations. We summarize these considerations in the following lemma, whose proof is straightforward.

▶ **Lemma 4.** *The affine transformation*

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + H_t \\ y + x + H_t/2 \end{pmatrix}$$

maps the grid to itself, and in addition, it maps both P_t and the reference parabola Π_t, or any vertical translate of it, to itself.

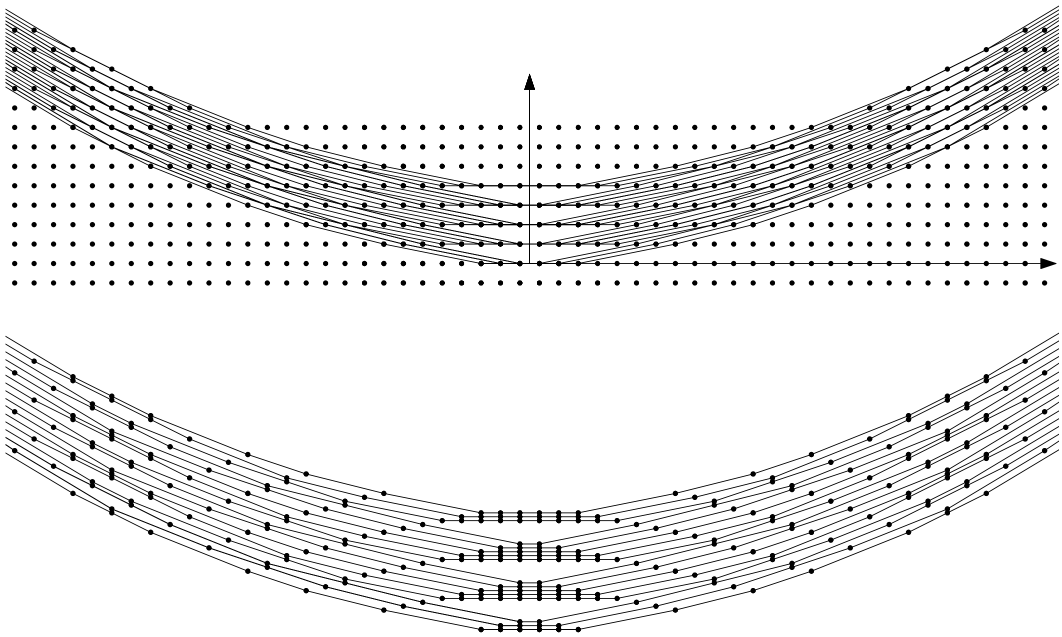

Figure 5 Consecutive peelings of P_5. Since consecutive peelings share many vertices, it is not easy to distinguish the curves. In the lower part, we have therefore vertically separated the consecutive peelings. This has the effect that some grid points appear in several copies with small vertical offsets, and horizontal grid lines get a curved appearance.

3 Grid peeling for parabolas

We start with the simple observation that grid peeling preserves inclusion:

▶ **Observation 5.** *Let $U \subseteq U' \subset \mathbb{Z}^2$ be two sets of grid points that are* upward closed*: $(x, y) \in U \implies (x, y+1) \in U$. Let f denote one peeling step. Then $f(U) \subseteq f(U')$.*

Observation 5 implies that if C and D are two convex x-monotone curves that extend from $x = -\infty$ to $x = +\infty$ (e.g. grid curve or an arbitrary smooth or piecewise smooth curve), and C lies everywhere (weakly) below D, then this relation is maintained by grid peeling.

▶ **Lemma 6.** *P and Π advance at the same limiting speed.*

Proof. As already argued, P approximates Π in a global sense, while locally, there might be deviations. This implies that we can shift Π vertically down by some integer amount γ and ensure that the shifted parabola, denoted by $\Pi - \gamma$, lies completely below P. Figure 4 shows the parabola for $\gamma = 8$, but actually, $\gamma = 1$ should already be sufficient to push Π below P.

Now imagine that we start grid peeling simultaneously with P and with $\Pi - \gamma$, or more precisely, with the convex hull of the grid points on or above $\Pi - \gamma$. Applying the monotonicity property of Observation 5, we conclude that the evolution of $\Pi - \gamma$ always remains below the evolution of P. It can never overtake P, and in particular, the limiting speed of $\Pi - \gamma$, which is the same as the limiting speed of Π, is at most the limiting speed of P.

We can push Π upward and start with a parabola $\Pi + \gamma'$ that lies above P everywhere, and argue in the same way that the evolution of P can never overtake the evolution of $\Pi + \gamma'$, and thus, the limiting speed of Π is at least the limiting speed of P. ◀

3.1 Evaluating the horizontal period H_t

We have seen that H_t is the sum of the x-coordinates of the vectors in $V_t^{(0)}$. It is thus given by the following expression:

$$H_t = \sum_{(x,y)\in V_t^{(0)}} x = \sum_{\substack{0<y\le x\le t\\ \gcd(x,y)=1}} \left\lfloor \frac{t}{x} \right\rfloor x$$

This sequence appears in the O.E.I.S. [10, A174405]. It starts with the values

$$H_1, H_2, \ldots = 1, 4, 11, 22, 43, 64, 107, 150, 211, 274, 385, 462, 619, 748, 895, 1066, 1339, \ldots$$

Sándor and Kramer [13] proved the asymptotic estimate $H_t \sim \frac{2\zeta(3)}{\pi^2} t^3 \approx 0.243587656\, t^3$. This can also be derived as a consequence of the more general Lemma 7, which is stated below. Appendix A of the full paper gives several alternative expressions for H_t.

3.2 Distance between the grid parabola and the reference parabola

We want to analyze the deviation between the grid parabola and the reference parabola. While the reference parabola has an explicit expression, the grid parabola is given by a multistep process, as described in Section 2. For getting from the origin to an arbitrary vertex of P_t, we need to sum the vectors whose slope is at most some threshold α:

$$U_t^\alpha := \sum_{\substack{0\le x\le t\\ 0<y\le \alpha x\\ \gcd(x,y)=1}} \left\lfloor \frac{t}{x} \right\rfloor \binom{x}{y}$$

More precisely, since the sum includes only vectors with $y > 0$, it measures the distance from the right endpoint of the horizontal segment of P_t and not from the origin. Lemma 7, which is proved in Appendix B of the full paper, gives an asymptotic expression for this sum:

▶ **Lemma 7.** *Let* $0 \le \alpha \le 1$. *Then*

$$U_t^\alpha = \frac{2\zeta(3)}{\pi^2} \begin{pmatrix} t^3\alpha + O(t^2 \log t) \\ t^3\alpha^2/2 + O(t^2 \log t) \end{pmatrix}, \quad \textit{and} \quad H_t = \frac{2\zeta(3)}{\pi^2} t^3 + O(t^2 \log t).$$

The second expression is obtained from the first one by setting $\alpha = 1$ and looking only at the x-coordinate.

▶ **Proposition 8.** *The vertical distance between the grid parabola* P_t *and the reference parabola* Π_t *is bounded by* $O(t^2 \log t)$.

Proof. By the periodic behavior of P_t and Π_t (Lemma 4), it suffices to look at the interval $0 \le x \le H_t$. Pick a point x_0 in this interval, see Figure 6. The corresponding point $\binom{x_0}{y_0}$ on Π_t has $y_0 = x_0^2/(2H_t)$, and the slope at this point is $\alpha := x_0/H_t \le 1$. As a first step, we find the point $\binom{x_1}{y_1}$ on P_t with the same slope α. We will show that it deviates from $\binom{x_0}{y_0}$ by at most $O(t^2 \log t)$ in each coordinate: By construction the grid parabola contains the vertex

$$\binom{x_1}{y_1} = \binom{t/2}{0} + U_t^\alpha.$$

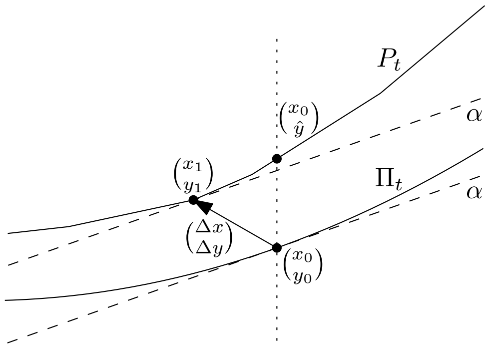

Figure 6 The vertical distance between P_t and Π_t.

The correction term $t/2$ accounts for the fact that U_t^α does not include the vector $\binom{t}{0}$ and thus measures the distance from the right endpoint of the horizontal segment of P_t and not from the origin. Applying both parts of Lemma 7, we get

$$\begin{aligned}\begin{pmatrix}x_1\\y_1\end{pmatrix} = \begin{pmatrix}t/2\\0\end{pmatrix} + U_t^\alpha &= \frac{2\zeta(3)t^3}{\pi^2}\begin{pmatrix}\alpha\\\alpha^2/2\end{pmatrix} + \begin{pmatrix}O(t^2\log t)\\O(t^2\log t)\end{pmatrix} + \begin{pmatrix}t/2\\0\end{pmatrix}\\ &= \big(H_t + O(t^2\log t)\big)\begin{pmatrix}\alpha\\\alpha^2/2\end{pmatrix} + \begin{pmatrix}O(t^2\log t)\\O(t^2\log t)\end{pmatrix}\\ &= H_t\begin{pmatrix}\alpha\\\alpha^2/2\end{pmatrix} + \begin{pmatrix}O(t^2\log t)\\O(t^2\log t)\end{pmatrix} = \begin{pmatrix}x_0\\y_0\end{pmatrix} + \begin{pmatrix}\Delta x\\\Delta y\end{pmatrix}\end{aligned}$$

with $\Delta x, \Delta y = O(t^2\log t)$. In the range $0 \le x \le H_t$, the slope of P_t is bounded by 1. So when we move from x_1 to $x_0 = x_1 + \Delta x$ on P_t, we arrive at a point $(x_0, \hat{y})$ with $|\hat{y} - y_1| \le |\Delta x|$, and thus, the vertical distance $|\hat{y} - y_0|$ between P_t and Π_t is at most $|\Delta y| + |\Delta x| = O(t^2\log t)$. ◀

Figure 20 in the full paper shows the actual difference $P_t - \Pi_t$ for several values of t.

3.3 Comparison to true parabolas

Let $y = ax^2/2 + bx + c$. We are interested in the average (vertical) speed in which the curve moves upwards. If we start grid peeling with a parabola $y = ax^2 + bx + c$ for rational coefficients a and b, we can show that, after some irregular "preperiod", it will enter a periodic behavior: After a certain number Δm of steps, the same curve reappears, translated upward by Δy. We call Δm the *time period* and Δy the *vertical period* (to be distinguished from yet another period, the horizontal period H, which was introduced at the beginning). The *average (vertical) speed* is then $\Delta y/\Delta m$. However, if a or b is irrational, we no longer have a periodic behavior. For a more general curve, like $y = e^x$, different parts of the curve will move at different speeds, and a common average speed will not exist. For this reason, we define the *lower* and *upper* average speed.

▶ **Definition 9** (average speed). *Let C be the graph of a convex function on $\mathbb{R}$. We denote by $C + \gamma$ the copy of C vertically translated by γ (upwards for $\gamma > 0$, downwards for $\gamma < 0$).*

For another such curve D, we write $C \le D$ if no point of C lies above D.

Let $f^{(m)}$ denote m steps of grid peeling.

The lower average speed *$v_- = v_-(C)$ is defined as follows:*

$$v_- = \liminf_{m\to\infty} \frac{\sup\{\,\gamma \mid C + \gamma \le f^{(m)}(C)\,\}}{m}$$

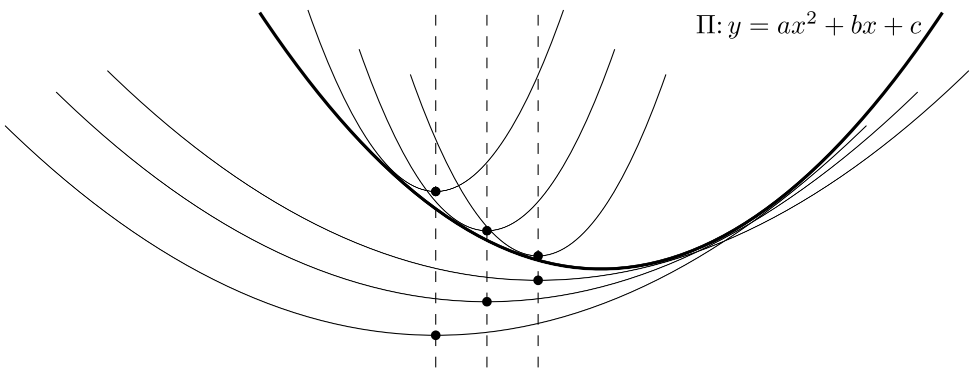

Figure 7 Upper and lower approximation of the parabola Π by "integer" parabolas.

The upper average speed $v_+ = v_+(C)$ *is defined similarly:*

$$v_+ = \limsup_{m\to\infty} \frac{\inf\{\,\gamma \mid f^{(m)}(C) \le C + \gamma\,\}}{m}$$

If v_- and v_+ coincide, we call it simply the average speed $v = v(C)$.

By approximating the parabola $y = ax^2/2 + bx + c$ from above and below by appropriate grid parabolas (see Figure 7), to which we apply Theorem 2, we arrive at the following result:

▶ **Theorem 10.**

1. *If $\frac{1}{H_t} < a < \frac{1}{H_{t-1}}$ and t is odd (or $a > \frac{1}{H_1} = 1$), then $v = 1/t$.*
2. *If $\frac{1}{H_t} < a < \frac{1}{H_{t-1}}$ and t is even, then $\frac{1}{t+1} \le v_- \le v_+ \le \frac{1}{t-1}$.*
3. *If $a = \frac{1}{H_t}$ and t is odd, then $\frac{1}{t+2} \le v_- \le v_+ \le \frac{1}{t}$.*
4. *If $a = \frac{1}{H_t}$ and t is even, then $\frac{1}{t+1} \le v_- \le v_+ \le \frac{1}{t-1}$.*

Numerical experiments, which are reported in Appendix D of the full paper, suggest that the first statement also holds for even t, and moreover, for $a = \frac{1}{2H_t}$, v exists always and lies in the range $\frac{1}{t+1} \le v \le \frac{1}{t}$.

▶ **Proposition 11.** *Let C be the graph of a convex function on $\mathbb{R}$. The upper and lower average speed is not changed by*

- *horizontal translation by an integer distance,*
- *or arbitrary vertical translation.*

Proof. It is clear that a translation by an integer vector does not change anything.

Consider an arbitrary vertical translation $C + \gamma$. Then (1) The integer translates $C_0 = C + \lfloor\gamma\rfloor$ and $C_1 = C + \lfloor\gamma\rfloor + 1$ have the same upper and lower average speeds as C, (2) they maintain a constant vertical distance of 1 during grid peeling, and (3) $C + \gamma$ is sandwiched between C_0 and C_1, and it maintains this relation during grid peeling. It follows that the upper and lower average speeds of C must agree with those of C_1 and C_2. ◀

One might be tempted to believe that a small horizontal translation should also not change the vertical speed. However, there are examples where translations cause the vertical speed to change, see Figure 15 in Appendix D.2 of the full paper.

3.4 Refined grid peeling for parabolas, proof of Theorem 1

Proof of Theorem 1. We prove our main theorem, Theorem 1 about the relation between grid peeling and ACSF for parabolas $y = ax^2/2 + bx + c$. We use (x, y) for the original coordinates, with a grid of spacing $1/n$, and $(\hat{x}, \hat{y}) = (nx, xy)$ for the scaled coordinates, with a unit grid. The curvature at the vertex of the parabola is a; thus the vertical speed of ACSF at this point (and thus everywhere, by affine invariance) is $a^{1/3}$.

At time T we have $y = ax^2/2 + bx + c + Ta^{1/3}$, and $\hat{y} = \frac{a}{n}\hat{x}^2/2 + b\hat{x} + cn + Tna^{1/3}$. Determine t such that $\frac{1}{H_t} \le \frac{a}{n} \le \frac{1}{H_{t-1}}$. So $\frac{a}{n} \approx \frac{1}{H_t}$ and Lemma 7 implies

$$\frac{n}{a} \approx H_t = \frac{2\zeta(3)t^3}{\pi^2} \cdot (1 + O(\log t/t)) = \left(\frac{t}{c_{\mathrm{g}}}\right)^3 \cdot (1 + O(\log t/t)),$$

which yields

$$t = c_{\mathrm{g}} \sqrt[3]{\frac{n}{a\bigl(1 + O(\log t/t)\bigr)}} + O(1) = c_{\mathrm{g}} \sqrt[3]{\tfrac{n}{a}} \cdot \bigl(1 + O(\log t/t)\bigr) = \Theta\Bigl(\sqrt[3]{\tfrac{n}{a}}\Bigr).$$

Here, the $O(1)$ term accounts for rounding t to an integer, and it also covers the uncertainty of Theorem 10, where t is sometimes replaced by $t-1$, $t+1$, or $t+2$. This additive error term is absorbed in the multiplicative error term $1 + O(\log t/t)$. By Theorem 10, the lower and upper average speed is

$$v \approx \frac{1}{t} = \frac{1}{c_{\mathrm{g}}} \sqrt[3]{\tfrac{a}{n}} \cdot \bigl(1 + O(\log t/t)\bigr)$$

After $m = \lfloor c_{\mathrm{g}} T n^{4/3} \rfloor$ steps, the vertical distance that the curve has moved up is therefore

$$mv + O(1) = Ta^{1/3}n\bigl(1 + O(\log t/t)\bigr) + O(1) = Ta^{1/3}n\bigl(1 + O(\sqrt[3]{\tfrac{a}{n}} \log \tfrac{n}{a})\bigr)$$

The difference to the movement of Π, which is $Tna^{1/3}$, is

$$O\bigl(Ta^{1/3}n\sqrt[3]{\tfrac{a}{n}} \log \tfrac{n}{a}\bigr) = O(Ta^{2/3}n^{2/3} \log \tfrac{n}{a}).$$

To this, we must add the distance between P_t and the reference parabola Π_t from Proposition 8, that is, $O(t^2 \log t) = O((\frac{n}{a})^{2/3} \log \frac{n}{a})$. Dividing by n, we conclude that the error term in terms of the original y-coordinates is

$$\bigl(O(Ta^{2/3}n^{2/3} \log \tfrac{n}{a}) + O((\tfrac{n}{a})^{2/3} \log \tfrac{n}{a})\bigr)/n = O\bigl((Ta^{2/3} + a^{-2/3})/n^{1/3} \log \tfrac{n}{a}\bigr). \qquad \blacktriangleleft$$

4 Proof of Theorem 2 about the period of the grid parabola

Proof of Theorem 2. When we speak of *the curve*, we mean the grid parabola P_t after some iterations of peeling.

Let $s = a/b \in S_t$ be a fixed slope. We consider the supporting line g with slope s, and we study how it evolves during the peeling process, see Figure 8 for an illustration.

▶ **Definition 12.** *The* strip *of slope s is the vertical strip that bounds the segment of slope s in P_t. It goes from $x = L_s$ to $x = R_s$. We denote by $\ell = R_s - L_s$ the* width *of the strip.*

The extended strip *of slope s includes an additional margin of $\lfloor t/2 \rfloor$ on both sides. It goes from $\bar{L}_s = L_s - \lfloor t/2 \rfloor$ to $\bar{R}_s = R_s + \lfloor t/2 \rfloor$.*

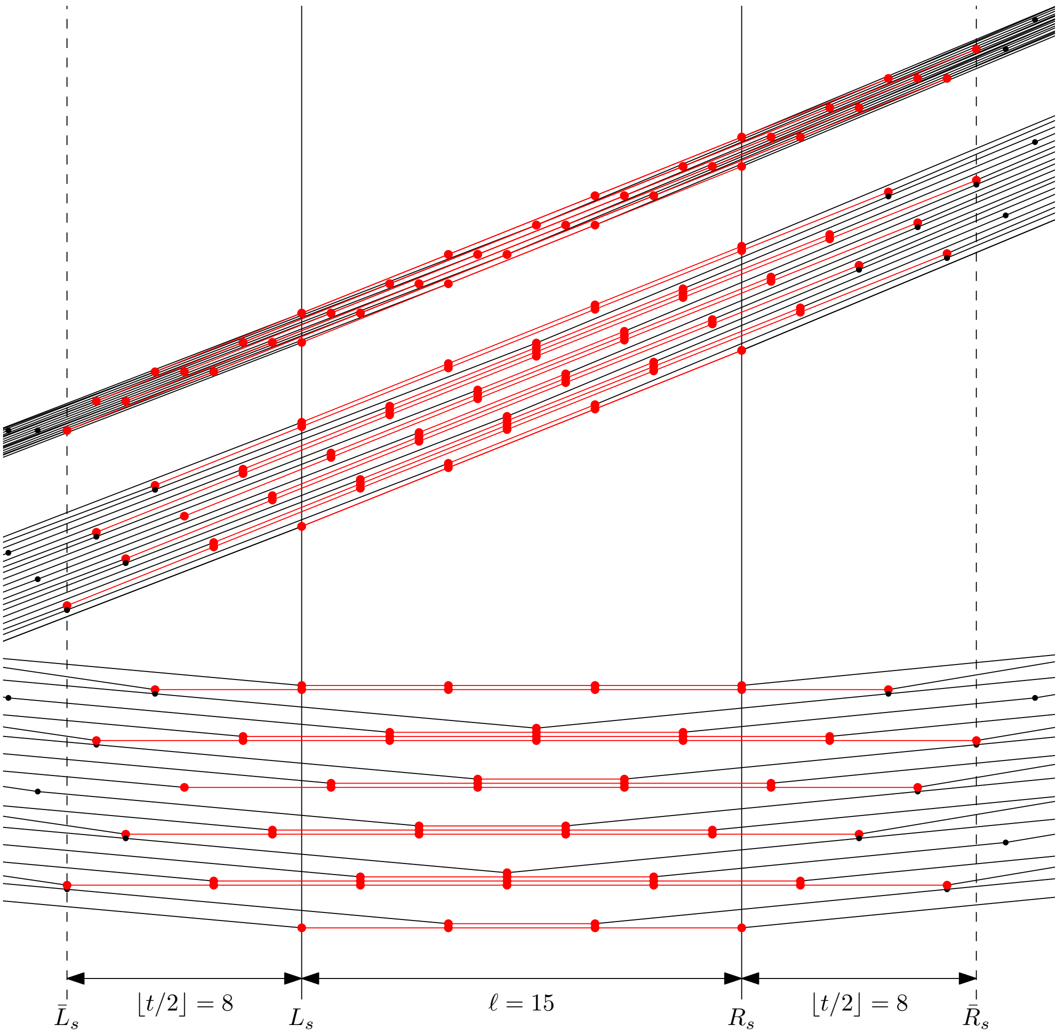

Figure 8 17 iterations of the development of slope $s = 2/5$ for $t = 16$, starting with the curve P_t. The uppermost part shows the true situation. In the middle part, the successive curves are separated for better visibility, as in Figure 5. In the lowest part, an affine transformation has been applied to make the segment of interest horizontal; this allows the different slopes to be distinguished more easily. The segment of slope s is highlighted in red on each curve. The initial segment on P_{16} has horizontal length $\ell = \lfloor \frac{16}{5} \rfloor \times 5 = 15$. The region between the dashed lines is the extended strip.

We state some obvious properties of the peeling process:

▶ **Observation 13.** *Throughout the whole peeling process:*

(i) *The supporting line g intersects the curve in a line segment or a point.*

(ii) *If the segment contains $k \geq 1$ grid points, its horizontal length is $(k-1)b$.*

(iii) *At every peeling step, the two endpoints of this line segment or the single point is peeled.*

(iv) *As long as the segment contains at least 3 grid points, the supporting line does not change, and the number k of grid points on the segment decreases by 2. In this case, we say that the segment* shrinks.

(v) *If the segment contains only 1 or 2 grid points, the supporting line changes. We say that there is a* jump *for slope s.*

We use the following terminology: The *left endpoint of slope s* is the left endpoint of the segment of slope s; in case the segment degenerates to a single point, it is that point. In other words, it is the leftmost point where the supporting line of slope s touches the curve. The *right endpoint* is defined analogously. We will only deal with *horizontal* offsets, lengths, positions, and distances, and thus we will often omit the word *horizontal*.

In the following crucial lemma, Properties 1 and 2 predict *what* happens when a jump occurs. In particular, Property 2 characterizes the possible locations of the vertices after a jump. Property 3 describes the final position of the segment before the jump. This statement allows us to predict *when* a jump occurs. Property 3 can be easily worked out, *assuming* that the initial position after the previous jump satisfies Property 2. Properties 4 and 5 describe the situation when two consecutive slopes are involved.

▶ **Lemma 14.** *The following properties hold throughout the peeling process:*

1. *(No grid line is skipped.) Whenever there is a jump, the supporting line g of slope s advances to the* next grid line *with slope s.*
2. *(The filling property.) After the supporting line g has advanced, the curve will contain precisely those grid points on g that lie within the extended strip of s. In other words, the segment fills the extended strip as much as possible. (See Figure 8 and 9.)*
3. *(Jump position) A jump of slope s occurs iff the left endpoint of slope s lies in the range*

$$L_s + \lfloor t/2 \rfloor - (b-1), \ \ldots, \ L_s + \lfloor t/2 \rfloor. \tag{4}$$

   A symmetric property holds for the right endpoint.

4. *(There are no gaps.) For any two consecutive slopes $s = a/b$ and $s' = a'/b'$ from S_t, with $s < s'$, the right endpoint of slope s coincides with the left endpoint of slope s'. In particular, no edge has an intermediate slope between s and s'. This implies that only slopes from the set S_t appear in the curves.*
5. *(Breakpoint position) The breakpoint between two consecutive slopes s, s' is in the interval between $X - t/2$ and $X + t/2$, where $X = R_s = L_{s'}$ is the boundary between the corresponding strips. (See Figure 10.)*

Properties 1–3, which were discovered experimentally, are strong enough to predict the behavior of the supporting segment of slope s during the peeling process in a purely local manner, without looking at the whole curve. The proof that this is the evolution that actually takes place amounts to checking whether these local characterizations fit together when considering different slopes. In particular, we will look at two consecutive slopes (Properties 4 and 5). This will involve checking some cases, but with the rigid structure provided by the strong properties 1–3, one cannot really avoid to come up with the proof.

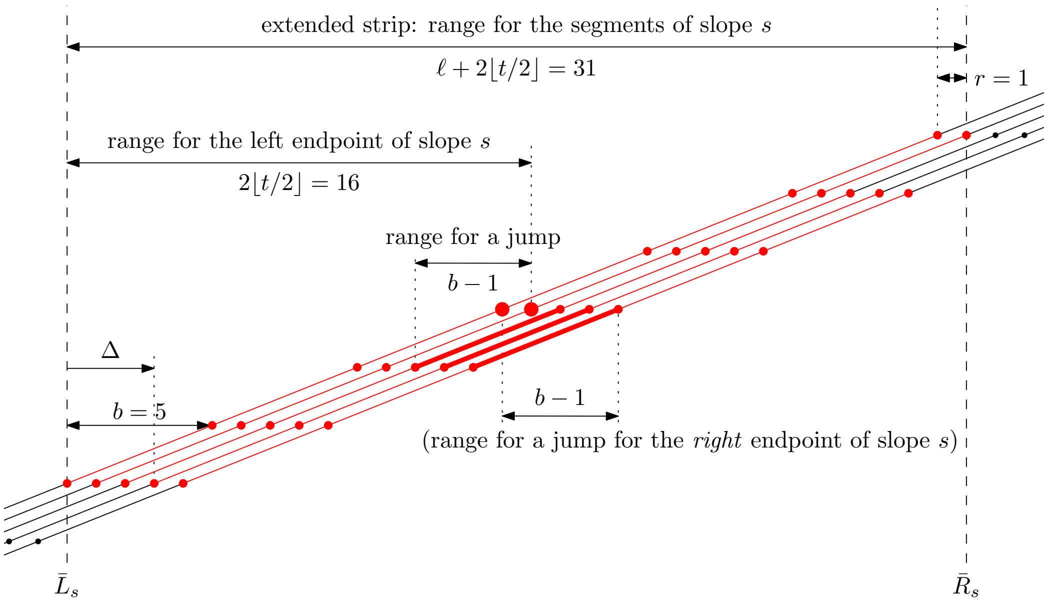

Figure 9 The $b = 5$ grid lines from Figure 8. $t = 16 = 3b + 1 = qb + r$. The initial segment on each line, immediately after the jump, is shown in red. The lines are considered according to the offset Δ of the leftmost grid point from the left edge of the extended strip. We have highlighted the last remaining single point or pair of points before the jump occurs.

Proof. We will show inductively that the claimed properties are maintained as invariants for all slopes throughout the peeling process.

We rely on the following properties of two consecutive slopes $s = \frac{a}{b}$ and $s' = \frac{a'}{b'}$, which follow from the definition of the slope set S_t (Section 2):

(I) The denominators b and b' are bounded by $b, b' \le t$, and their sum is $b + b' > t$. (Otherwise, the vector $\binom{b}{a} + \binom{b'}{a'}$ would give rise to a slope in S_t between s and s'.)

(II) The two vectors $\binom{b}{a}$ and $\binom{b'}{a'}$ form a lattice basis of the unit grid. (Otherwise, they would span a parallelogram that contains interior points, and some of these points would lead to vectors with an intermediate slope between s and s' in S_t.)

As the basis for the induction, it can be seen without computation that the "original" segment of slope s of P_t falls in the pattern of analysis leading to Property 3: Indeed, it lies centrally in the extended strip. Thus, when extending it as much as possible within the extended strip and starting the peeling process, the starting segment will appear during this process, by symmetry. Also, the endpoints of consecutive segments match on P_t by construction, establishing Property 4 and Property 5 at the beginning.

For the induction step we consider two consecutive slopes s, s' and make sure that no matter if they make a jump or not, the properties of Lemma 14 hold. Thus we have four cases. In each case we prove properties 1,2, 4, 5, and then assuming properties 1,2, 4, 5, we prove Property 3 all at once.

Case 1: s and s' both shrink. We show that this is not possible. Assume that s, s' both shrink. Since no jump occurs for s', by Property 3 the rightmost possible position of the left endpoint of s' is $L_{s'} + \lfloor t/2 \rfloor - b'$. Symmetrically, the leftmost possible position of the right endpoint of s is $R_s - \lfloor t/2 \rfloor + b$, and therefore $R_s - \lfloor t/2 \rfloor + b \le L_{s'} + \lfloor t/2 \rfloor - b'$. Since $L_{s'} = R_s$, it follows that $-\lfloor t/2 \rfloor + b \le \lfloor t/2 \rfloor - b'$, or $b + b' \le 2\lfloor t/2 \rfloor \le t$, which contradicts (I).

Case 2: s jumps and s' shrinks. We claim that the endpoint of the shrunken segment for s' arrives at the next grid line with slope s, and its position on this line matches the position for the right endpoint of s predicted by Property 2. The left endpoint of s' moves by the vector $\binom{b'}{a'}$, and since $\binom{b'}{a'}$ and $\binom{b}{a}$ form a lattice basis (II), the supporting line of slope s will indeed jump to the next grid line of slope s, establishing Property 1 in this case.

We claim that this new left endpoint of s' is indeed the rightmost grid point on this line in the extended strip for s (establishing Property 2). We show that it lies in the extended strip of s, but the point *after* this point on the grid line of slope s is already outside the extended strip.

To show the former, consider the rightmost possible position for the left endpoint of s' before shrinking. It is $L_{s'} + \lfloor t/2 \rfloor - b'$ by Property 3. After shrinking, it is $L_{s'} + \lfloor t/2 \rfloor = R_s + \lfloor t/2 \rfloor = \bar{R}_s$, and thus it lies in the extended strip of s. To show the latter, consider the leftmost possible position for the left endpoint of s' before shrinking. It is $\bar{L}_{s'} = X - \lfloor t/2 \rfloor$ by Property 3. After shrinking, it is $X - \lfloor t/2 \rfloor + b'$. The point after this point is at offset b, and $X - \lfloor t/2 \rfloor + b' + b > X - \lfloor t/2 \rfloor + t \geq X + \lfloor t/2 \rfloor = \bar{R}_s$, by (I), and thus this point is already outside the extended strip.

Therefore, the new left endpoint of s' coincides with the new right endpoint of s, establishing Property 4 and Property 5 in this case.

Case 3: s shrinks and s' jumps. The situation is symmetric to Case 2.

Case 4: s and s' both jump. By Property 3, the position x of the breakpoint $A = \binom{x}{y}$ before the jump is in the interval

$$X + \lfloor t/2 \rfloor - b' < x < X - \lfloor t/2 \rfloor + b. \tag{5}$$

Let U be the previous point on the grid line of slope s through A, and let U' be the next point on the grid line of slope s' through A. Construct the parallelogram $UAU'A^+$. Such a parallelogram is shaded in Figure 10. We know that both supporting lines of slope s and s' must advance at least to the next grid line. Those two grid lines intersect at the fourth point $A^+ = \binom{x^+}{y^+}$ of this parallelogram with $x^+ = x - b + b'$. We will show two things:

(a) The point A^+ is not peeled until after the jump. Thus it will be the common right endpoint of slope s and left endpoint of slope s'.

(b) It is indeed the rightmost grid point on the grid line of slope s in the extended strip of s. (Symmetrically, it is the leftmost point on the grid line in the extended strip of s'.)

To prove (a), assume w.l.o.g. that $b \leq b'$, so that the segment AU' lies below A^+. We are done if we show that this segment is part of the boundary when A is peeled. Assume otherwise. Then A would be not only the left endpoint of slope s', but also the right endpoint of slope s' when it is peeled. According to Property 3, this means that $x \geq R_{s'} - \lfloor t/2 \rfloor$. Let $\ell_{s'} \geq b'$ denote the length of the strip of slope s'. Then, with $b \leq b' \leq \ell_{s'}$ we get a contradiction to (5):

$$x \geq R_{s'} - \lfloor t/2 \rfloor \geq R_{s'} - \ell_{s'} - \lfloor t/2 \rfloor + b = X - \lfloor t/2 \rfloor + b > x$$

To prove (b), note that A^+ lies in the extended strip for s because $x^+ = x - b + b' < X - \lfloor t/2 \rfloor + b' \leq X - \lfloor t/2 \rfloor + t$, by the right inequality of (5), and hence $x^+ \leq X + \lfloor t/2 \rfloor$. On the other hand, the next grid point on the line of slope s has x-coordinate $x^+ + b = x + b' > X + \lfloor t/2 \rfloor$, by the left inequality of (5), and this is outside the extended strip for s.

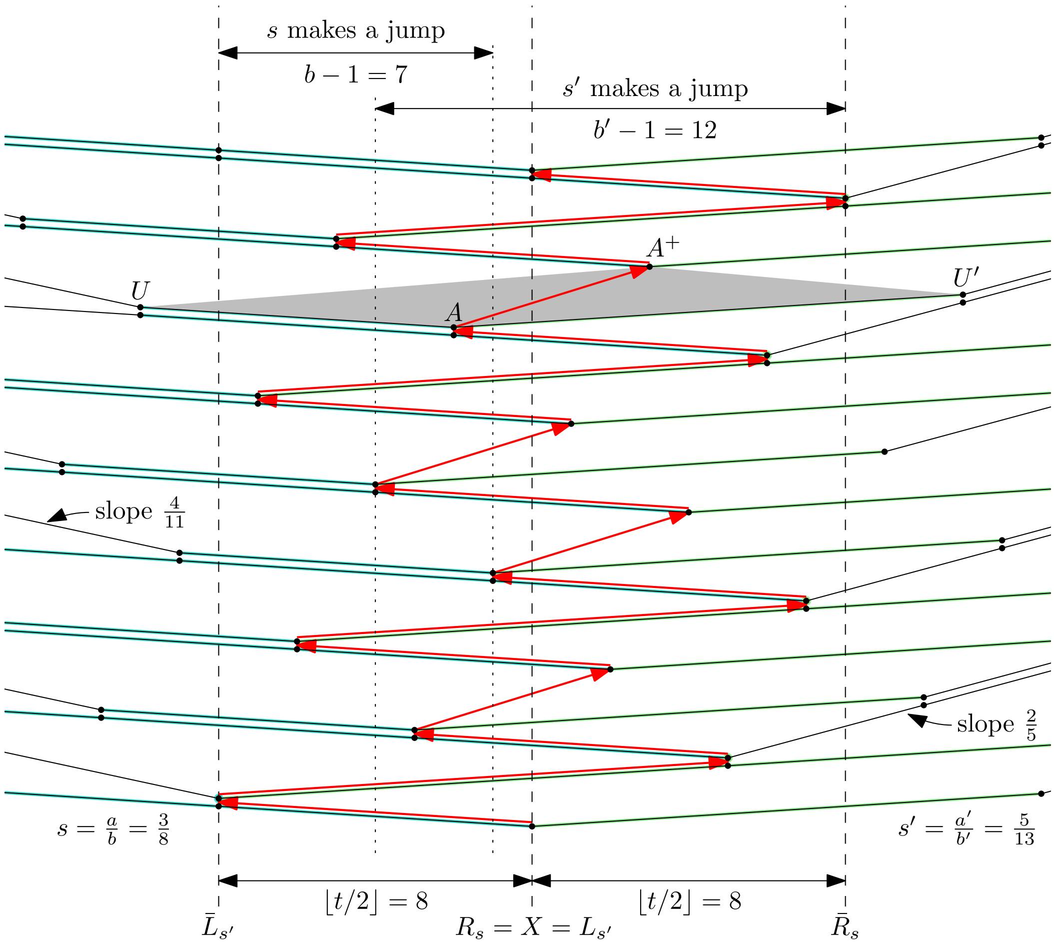

Figure 10 The transition between slope $s = 3/8$, with vector $\binom{16}{6} \in V_t$, and $s' = 5/13$, with vector $\binom{13}{5} \in V_t$, for $t = 16$. The figure is drawn after an affine transformation, following the conventions of the lower part of Figure 8. The right endpoint of slope s always coincides with the left endpoint of slope s', and it varies in the interval between $X - \lfloor t/2 \rfloor$ and $X + \lfloor t/2 \rfloor$.

Now that we have established the first four invariants in all cases, we prove Property 3, assuming Property 2 has been true so far.

Let $t = qb + r$, with $0 \le r < b$. Then $qb = \ell$ is the (horizontal) length of the vector in V_t that forms the segment of slope s on P_t. We have defined it as the width of the strip.

To illustrate Property 2, Figure 9 shows the possible cases how the segment of slope s can lie on the grid line, immediately after a jump occurs, according to this property. On every grid line, the grid points form an arithmetic progression with (horizontal) increment b. The different grid lines are distinguished by the offset Δ of the leftmost grid point from the left edge of the extended strip. There are b possibilities, $0 \le \Delta < b$.

For the sake of the following analysis, we have sorted the lines by Δ in Figure 9. (This is not the order in which they occur from bottom to top. The true order in this example is $\Delta = 0, 2, 4, 1, 3, 0, \ldots$, see Figure 8.)

For simplicity, we focus on the case when t is even [and put the odd case into brackets].

Let us start with the case $\Delta = 0$ (the topmost line in Figure 9). In a strip of width t, we can fit q segments of length b, with $q+1$ points, leaving a remainder of length r. The extended strip has width $t+\ell$ $[t-1+\ell]$. Since the extra length ℓ is filled precisely by q segments, we can fit q additional segments of length b, for a total of $2q+1$ points. [For odd t, the last claim holds only when $r > 0$.]

Since the number of points is odd, the last peeled segment on this line before the jump is a singleton, after q steps and at distance $qb = \ell$ from the left boundary $\bar{L}_s$ of the extended strip, or distance $\ell - \lfloor t/2 \rfloor$ from L_s.

We can increase Δ up to r $[r-1]$ without changing the situation:

- For $\Delta = 0, 1, \ldots, r$ $[\Delta = 0, 1, \ldots, r-1]$, the number k of points is odd, and for the last point that is peeled, the distance from L_s is in the range

$$\ell - t/2, \ldots, \ell - t/2 + r \quad [\ell - \lfloor t/2 \rfloor, \ldots, \ell - \lfloor t/2 \rfloor + r - 1].$$

Since $\ell + r = t$, this range simplifies to $\ell - \lfloor t/2 \rfloor, \ldots, \lfloor t/2 \rfloor$.

Starting from $\Delta = r+1$ $[\Delta = r]$, the situation changes. We have now an even number $2q$ of points, and the last peeled segment is a proper segment with a *pair* of points. The left peeled point is at distance $\Delta + (q-1)b = \Delta + \ell - b$ from $\bar{L}_s$, or at an offset $\Delta + \ell - b - \lfloor t/2 \rfloor$ from L_s. (This offset may be negative, in which case it denotes an offset to the left.)

- For $\Delta = r+1, \ldots, b-1$ $[\Delta = r, \ldots, b-1]$, the number k of points is even, and for left point of the last peeled pair, the distance from L_s is in the range

$$r+1+\ell-b-t/2, \ldots, b-1+\ell-b-t/2 \quad [r+\ell-b-\lfloor t/2 \rfloor, \ldots, b-1+\ell-b-\lfloor t/2 \rfloor].$$

Since $\ell + r = t$, this range simplifies to $\lfloor t/2 \rfloor - b - 1, \ldots, \ell - \lfloor t/2 \rfloor - 1$.

Combining the ranges for two cases establishes Property 3, and this concludes the proof of Lemma 14. ◀

▶ **Proposition 15.** *The left endpoint of slope s is at distance at most $\lfloor t/2 \rfloor$ from L_s (on the left or on the right), see Figure 9. Every position in this range occurs.*

Proof. Property 3 describes b possibilities before a jump, one value for each of the b residue classes modulo b, and Property 2 suggests b possibilities after a jump, namely $\Delta = 0, 1, \ldots, b-1$. Since for every Δ, the jump must occur at *some* point, the range (4) uniquely characterizes this point.

By Property 3, the left endpoint of slope s can never deviate more than $\lfloor t/2 \rfloor$ from L_s to the right, and by Property 2, it cannot deviate more than $\lfloor t/2 \rfloor$ from L_s to the left. Thus, the left endpoint of slope s is at distance at most $\lfloor t/2 \rfloor$ from L_s. In fact, since there is a grid line of slope s passing through every such point, and no grid line is skipped (Property 1), every point in this range will be peeled once as the left endpoint of slope s. ◀

There are $2\lfloor t/2 \rfloor + 1$ different offsets at distance at most $\lfloor t/2 \rfloor$ from L_s, and exactly one of them is always peeled. Therefore, after $2\lfloor t/2 \rfloor + 1$ steps the same segment of slope s repeats, one unit higher. This is true for any slope, so after $2\lfloor t/2 \rfloor + 1$ steps the same chain repeats one unit higher. It means the peeling process is periodic with period $2\lfloor t/2 \rfloor + 1$, and this concludes the proof of Theorem 2. ◀

5 Future research

The obvious open problem is to prove the relation between grid peeling and the ACSF for arbitrary convex curves. As a first challenge, one might try the case of a circle. The natural approach is to leverage Theorem 1 by locally approximating the curve by parabolas.

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